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ELECTROMAGNETIC WAVE PROPAGATION IN MEDIA OF
PERIODICALLY TIME-VARYING PERMITTIVITY

by

Dieter Ernst Holberg

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JANUARY 1965

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ELECTROMAGNETIC WAVE PROPAGATION IN MEDIA
OF PERIODICALLY TIME-VARYING PERMITTIVITY

by

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University Park, New Mexico

January 1965

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ABSTRACT

This paper presents a study of the propagation of electromagnetic plane waves in a nonconducting, charge-free medium with a periodically time-varying permittivity given by $\epsilon(t) = \epsilon/(1 - 2p\cos 2\Omega t)$. The permittivity is considered to be "modulated" by an external agency, detailed consideration of which is outside the scope of this work. With the assumed permittivity, the wave equation for the electric displacement can be separated into Mathieu's equation for the time dependence and Helmholtz' equation for the space dependence of the field. The separation constant $-k^2$ defines the wave number k . Plane waves of a single wave number can be written as products of Mathieu functions and exponentials $\exp(\pm ikx)$, the properties of the Mathieu functions depending on the choice of k .

After a brief discussion of means to realize the assumed permittivity modulation, the properties and computational determination of Mathieu functions are reviewed. Written in the form of Floquet, the Mathieu functions are combined with the associated spatial factors $\exp(ikx)$ to form expressions for single- k plane waves. Depending on the choice of k , these waves may have envelopes that vary periodically or exponentially in time.

After showing that in an infinite space of periodically time-varying permittivity waves with amplitudes increasing exponentially in time can be maintained, it is investigated whether such waves may also occur in a permittivity-modulated slab oriented normal to the propagation vector. The existence of increasing waves is in the latter case

restricted by the fact that the electric and magnetic field must be continuous at the boundaries of the slab. This requirement is shown to be met by a superposition of single- k waves increasing at a common rate, provided the index of the permittivity modulation $2p$ exceeds a critical value $2p_0$. This value depends on the slab width and on the relation of the permittivities within and outside the slab. A slab with a permittivity-modulation index greater than $2p_0$ is considered an unstable system for electromagnetic wave propagation since certain waves, once excited, grow exponentially with time. On the other hand, if the modulation index is less than $2p_0$, the slab is a stable system unable to support such exponentially increasing waves. In the latter case, if a plane wave of frequency ω illuminates the slab, plane waves having a frequency spectrum given by $\omega \pm 2n\Omega$ ($n = 0, 1, 2, \dots$) are reflected and transmitted from the slab. Here 2Ω is the modulation frequency. The amplitudes of the various waves depend on the quotient ω/Ω , the modulation index $2p$, and the electrical width of a slab. Such dielectric slabs may be designed to act as tuned amplifiers and frequency converters.

The critical modulation index $2p_0$ is found in closed form for the case $2p_0 \ll 1$. If the modulation index is greater than $2p_0$, the rate of increase of the field is also given in closed form. For the case of a modulation index less than $2p_0$, the reflection and transmission properties of a slab for an incident wave of a frequency near half the modulation frequency are determined. For the general case of larger modulation indices, the equivalent results are obtained using iterative procedures.

An independent approach to problems involving a permittivity-modulated slab is made by employing partial difference equations. This method, applied to exponentially increasing waves in a dielectric slab, yielded results in good agreement with those obtained by using Mathieu functions. This method is not limited to the particular permittivity variation assumed above, which gives rise to Mathieu's equation.

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CHAPTER I

Introduction

Propagation of electromagnetic waves in inhomogeneous media has been investigated considerably. Especially the case of layered media, i.e., media with properties depending only on one coordinate, has been discussed in detail. The somewhat related problem of wave propagation in media with time-varying properties has attracted comparatively little attention. This may, at least in part, be due to the fact that a variation of the constitutive parameters with time does not arise quite as naturally as their dependence on position.

In this study only the electric permittivity is assumed to vary with time, while the conductivity is supposed to be zero. Such independent values for the permittivity and conductivity can be assumed only for frequencies small compared to the absorption frequencies of the medium. Otherwise, the two parameters would be related by the Kronig-Kramer dispersion formulas. The permittivity $\epsilon(t)$ is assumed to vary periodically at a frequency* 2Ω about a fixed value ϵ ; more specifically, its inverse is assumed to vary as

$$\frac{1}{\epsilon(t)} = \frac{1}{\epsilon} (1 - 2p \cos 2\Omega t) \quad (1.1)$$

* Throughout this work, the term "frequency" is used in the sense of circular frequency.

Solutions to Maxwell's equations will be sought with the permittivity variation of Equation (1.1) given. Thus the permittivity is assumed to be "modulated" by an external agency, which is not affected by the electromagnetic fields propagating in the medium and arising mathematically as solutions to Maxwell's equations. This view point suggests the terms "modulation index" and "modulation frequency" for the quantities $2p$ and 2Ω of (1.1), respectively, where the factors 2 are introduced merely for later convenience.

The object of this study is not a description of suitable agencies for producing the assumed permittivity variations, but a discussion of electromagnetic fields in a medium that has the permittivity variation given by (1.1). Nevertheless, some remarks as to where and how periodic permittivity variations may arise seem to be in order. The macroscopically defined permittivity of a medium is a consequence of the polarizability of its molecules. This polarizability, relating the average dipole moment of the molecules to the local electric field, is in principle a function of field strength, but for most materials the polarizability is a constant over the range of electric field strengths that can be practically applied. There are however a few materials, the so-called ferroelectrics, with polarizabilities depending strongly on the electric field strength in certain temperature ranges. Ferroelectricity and nonlinear dielectrics have been an object of considerable research during the past decade.^{1,2,3} There are three distinct classes of ferroelectrics: the tartrates including Rochelle salt, the alkali and ammonium phosphates represented by potassium

dihydrogen phosphate, and the so-called oxygen octahedron class,⁴ whose most widely known member is barium titanate. For microwave applications, barium-strontium-titanate mixtures seem to be most suitable. They have comparatively large permittivities* (of the order of 10^3), which can be reduced by thirty to fifty percent by applying an electric field of the order of 10^6 volts/meter,^{5,6} Such a ferroelectric medium, subjected to a strong electric field of the form

$$E_0(t) = E_1 + E_2 \cos 2\Omega t \quad (1.2)$$

where E_1 and E_2 are constant vectors, would then indeed present a permittivity similar to the one assumed in (1.1) to an electromagnetic wave of field strength small compared to E_0 . Thus a suitable agency to cause the assumed permittivity variation is a strong harmonically time-varying electric field, which is of constant amplitude and phase throughout the region in which the permittivity is to be modulated. This requirement introduces certain restrictions on the size and geometry of dielectric structures with the type of permittivity variation given in (1.1).

The permittivities of many ferroelectrics have been investigated experimentally.^{1,2,3} For example, the permittivity of

* The term permittivity is here used in the sense of differential permittivity, defined as $\epsilon = dD/dE$.

strontium titanate has been experimentally shown to be nondispersive up to 37 Gc/sec.⁷ Its dielectric loss angle, which has been measured up to 22 Gc/sec, increases with frequency and also with the strength of the applied field, but remains below 10^{-2} , as long as one avoids temperatures too close to the Curie point.⁸ Thus the assumption of zero conductivity should be justified. The relaxation time, which sets the upper limit to the frequency at which the permittivity may be varied, has apparently not been experimentally determined. However, since barium-strontium mixtures have been successfully tested as nonlinear elements for parametric-circuit application,⁹ one may conclude that a periodic permittivity variation at frequencies up to at least 10 Gc/sec can be achieved.

Experimental evidence thus indicates that the assumed permittivity variation may occur, at least approximately, in a ferroelectric medium subjected to a strong electric field. While this is probably the most obvious approach to realizing the assumed permittivity variation, it should be mentioned that there are other means of periodically affecting the permittivity of a medium. A plasma subjected to a strong magnetic field is one possibility. Ionospheric cross-modulation, i.e., nonlinear mixing of two radio signals in the ionosphere, was observed already in the early days of broadcasting.¹⁰ Also mechanical stress applied to a piezoelectric material by acoustic waves changes its dielectric constant. Finally, if a population inversion, as occurring in masers and lasers, can be controlled periodically, a corresponding permittivity modulation would result.

As was mentioned before, the modulation agency, i.e., the strong electric, magnetic, or acoustic field producing the permittivity modulation, is not the object of this study. Rather, the propagation of comparatively weak electromagnetic waves of origin independent of the strong field will be discussed. These weak waves may be used to probe the dielectric properties of a medium subjected to a strong perturbation. Or, one may consider the use of a piece of ferroelectric material in a strong field as a parametric device. The inputs and outputs of this device would then be described by the weak fields studied in this paper.

For the weak fields, the constitutive equation

$$D(t, \vec{r}) = \epsilon(t)E(t, \vec{r}) \quad (1.3)$$

is a linear relationship between the electric field and the electric displacement. In general, the permittivity is a tensor because of the crystal structure of the ferroelectrics. However, in the following treatment only plane waves propagating along a single axis are considered, so that the permittivity becomes a scalar function of time. With zero conductivity, constant permeability, and no free charge present, Maxwell's equations may be written in terms of the magnetic field strength H and the electric displacement D as

$$\nabla \times H = \frac{\partial D}{\partial t} \quad (1.4)$$

$$\nabla \times D = -\mu_0 \epsilon(t) \frac{\partial H}{\partial t} \quad (1.5)$$

$$\nabla \cdot H = 0 \quad (1.6)$$

$$\nabla \cdot D = 0 \quad (1.7)$$

For transverse plane waves, the two divergence equations are automatically satisfied. The two curl equations may be combined to yield

$$\nabla \times \nabla \times D = \nabla(\nabla \cdot D) - \nabla^2 D = \mu_0 \epsilon(t) \frac{\partial^2 D}{\partial t^2} \quad (1.8)$$

Using (1.7) this yields the wave equation for D

$$\nabla^2 D - \mu_0 \epsilon(t) \frac{\partial^2 D}{\partial t^2} = 0 \quad (1.9)$$

If propagation is restricted to the direction of the x axis, Equation (1.9) reduces to the one-dimensional wave equation

$$\frac{\partial^2 D}{\partial x^2} = \mu_0 \epsilon(t) \frac{\partial^2 D}{\partial t^2} \quad (1.10)$$

For those modes of propagation for which the electric displacement can be expressed as a product of a space function and a time function, namely

$$D(t, x) = T(t)X(x) \quad (1.11)$$

the wave equation can be separated into the two ordinary differential equations

$$\frac{d^2X}{dx^2} + k^2X = 0 \quad (1.12)$$

and

$$\frac{d^2T}{dt^2} + \frac{k^2}{\mu_0 \epsilon(t)} T = 0 \quad (1.13)$$

With the permittivity variation specified in (1.1), Equation (1.13) becomes Mathieu's differential equation

$$\frac{d^2T}{dt^2} + \frac{k^2}{\mu \epsilon} (1 - 2p \cos 2\Omega t) T = 0 \quad (1.14)$$

Since both (1.12) and (1.14) have two independent solutions, the displacement field for a particular k can be written in four terms, each weighted by an arbitrary complex constant, thus

$$\begin{aligned} D(t, x; k) = & A_1 T_1(t, k) e^{ikx} + A_2 T_1(t, k) e^{-ikx} \\ & + A_3 T_2(t, k) e^{ikx} + A_4 T_2(t, k) e^{-ikx} \end{aligned} \quad (1.15)$$

It will be shown that a field for one particular wave number k contains an infinite frequency spectrum of the form $\omega + r2\Omega$, where the integer r ranges from $-\infty$ to $+\infty$. The wave number k and the frequency ω are connected by a "dispersion relation," which

depends on the quantities $\mu_0 \epsilon$, $2p$, and 2Ω and follows mathematically from Mathieu's equation. There is an infinite spectrum of wave numbers giving rise to a given frequency spectrum $\omega + r2\Omega$. An arbitrary wave with a frequency spectrum $\omega + r2\Omega$ can therefore be synthesized as a superposition of waves of the form (1.15) using all wave numbers that are connected with the given frequency spectrum, namely

$$D(t,x) = \sum_n D(t,x;k_n) \quad (1.16)$$

The most general solution to the wave equation (1.10) is an integration over the fields associated with all wave numbers k and is thus of the form

$$D(t,x) = \int_0^\infty D(t,x;k) dk \quad (1.17)$$

where the complex constants of (1.15) are now continuous functions of k . However, for the problem treated in this study the superposition of waves with a discrete wave-number spectrum, as given in (1.16), will be sufficient.

CHAPTER II

Waves in an Infinite Medium of Time-varying Electric Permittivity

The complete solution to the wave equation (1.10) for a given wave number k has been shown in (1.15) to consist of four waves written as products of $e^{\pm ikx}$ and the two solutions, T_1 and T_2 to Mathieu's differential equation (1.14). Depending on the relation of the wave number k to the quantity $K = \Omega\sqrt{\mu_0\epsilon}$, where 2Ω is the frequency modulating the inverse of the permittivity ϵ , and the modulation index $2p$, the waves are either propagating waves whose amplitudes and instantaneous phase velocities vary periodically in time or standing waves with amplitudes varying exponentially in time. Since the space function e^{ikx} is common to both types of waves, the particular behavior of a wave depends only on the character of the Mathieu function $T(t,k)$. This chapter begins, therefore, with a treatment of Mathieu functions, but this treatment is limited to stating the properties of the various types of Mathieu functions to the extent required for the subsequent discussion of the waves in a modulated medium. Details, proofs, and remarks regarding practical computations are relegated to the appendices. A special notation is devised to reflect the character of each Mathieu function. This notation contains all information needed for effective manipulation of these Mathieu functions, which are relatively unfamiliar to most investigators. In Section 2.2, expressions for the various types of electric displacement waves

are developed, while Section 2.3 gives expressions for the corresponding electric fields.

2.1 Properties and Designation of Mathieu Functions

The differential equation for the time function

$$\frac{d^2 T}{dt^2} + \frac{k^2}{\mu_0 \epsilon} (1 - 2p \cos 2\Omega t) T(t) = 0 \quad (2.1)$$

is changed into a more familiar form for Mathieu's equation

$$y'' + \kappa^2 (1 - 2p \cos 2z) y(z) = 0 \quad (2.2)^*$$

with the substitutions

$$\Omega t = z \quad (2.3)$$

and

$$k^2 / \Omega^2 \mu_0 \epsilon = k^2 / K^2 = \kappa^2 \quad (2.4)$$

Customarily, the quantity κ^2 , which may be complex in this development, is called the *characteristic value* or the *characteristic number* of Mathieu's equation. Here the first term will be used to denote κ^2 , while the term *characteristic number* will be applied to κ , defined as the root of κ^2 with positive real part. The characteristic number κ is thus proportional to the wave number k .

* Two even more frequently used forms are $y'' + (a - 2q \cos 2z)y = 0$, and $y'' + (\lambda - 2h^2 \cos 2z)y = 0$ with $a = \lambda = \kappa^2$ and $q = h^2 = \kappa^2 p$.

Mathieu's equation (2.2) has two independent solutions which are functions of z with properties depending on the parameters κ and p . This dependency may be explicitly shown by writing the solutions as $y(z; \kappa, p)$. If $p = 0$,

$$y_a(z; \kappa, 0) = \cos wz \quad (2.5)$$

$$y_b(z; \kappa, 0) = \sin wz \quad (2.6)$$

are solutions to (2.2) provided

$$\kappa = w \quad (2.7)$$

Writing $wz = w\Omega t$ as ωt with the understanding that ω is the angular frequency of a wave in the modulated medium indicates that

$$w = \omega/\Omega \quad (2.8)$$

is a normalized frequency. This is similar to $\kappa = k/K$, which is a normalized wave number. With the help of (2.4) and (2.8), Equation (2.7) is seen to be equivalent to the customary relation between the propagation constant and the frequency of a wave, namely:

$$k(\mu_0 \epsilon)^{-1/2} = \omega \quad (2.9)$$

A solution to Mathieu's equation for p small can be developed in a

power series in p and written

$$y_a(z; \kappa, p) = \cos wz + \sum_{i=1}^{\infty} p^i f_i(z) \quad (2.10)$$

provided that κ and w are properly related.¹¹ It can be shown that the relation between κ and w can also be expressed by a power series, namely

$$\kappa = \kappa_a = w(1 + \sum_{i=1}^{\infty} \alpha_i(w) p^i) \quad (2.11)$$

The coefficients $\alpha_i(w)$ are defined in Appendix 1. It is here assumed that p is sufficiently small so that the expansions converge. With $p \rightarrow 0$ Equations (2.10) and (2.11) clearly go over into (2.5) and (2.7).

Alternately a solution can be developed in the form

$$y_b(z; \kappa, p) = \sin wz + \sum_{i=1}^{\infty} p^i g_i(z) \quad (2.12)$$

requiring that κ and w satisfy a relation similar to (2.11),

$$\kappa = \kappa_b = w(1 + \sum_{i=1}^{\infty} \beta_i(w) p^i) \quad (2.13)$$

When w is not an integer, the coefficients α_i and β_i are the same, making κ_a and κ_b equal, and $g_i(z)$ in (2.12) follows from $f_i(z)$ in (2.10) by replacing all cosines by sines. In this case, one can also use the complex solutions

$$y_1(z; \kappa, p) = e^{i\omega z} + \sum_{i=1}^{\infty} p^i F_i(z) \quad (2.14)$$

and

$$y_2(z; \kappa, p) = y_1(-z; \kappa, p) \quad (2.15)$$

When $w = m$, m being an integer, then $\alpha_i \neq \beta_i$ for $i > m$, and hence $\kappa_a \neq \kappa_b$. In this case the solutions $y_a(z; \kappa, p)$ and $y_b(z; \kappa, p)$ cannot coexist because they belong to different characteristic numbers. These solutions are called integral Mathieu functions and denoted by $ce_m(z, p) = y_a(z; \kappa, p)$ and $se_m(z, p) = y_b(z; \kappa, p)$. It is customary to denote the *characteristic values* leading to such solutions by $\kappa_a^2 = \kappa_a^2 = a_1, a_2, \dots$, and by $\kappa_b^2 = \kappa_b^2 = b_1, b_2, \dots$, respectively. Here this notation will be applied to the characteristic numbers (which are the roots of the characteristic values). Thus the *characteristic numbers* giving rise to the solutions $ce_m(z, p)$ are $\kappa = \kappa_a = a_m$, while those giving rise to the solutions $se_m(z, p)$ are $\kappa = \kappa_b = b_m$.

The relation between κ , w , and p can be displayed by contour curves of constant w over a plane with coordinates κ (real) and p , henceforth called κp -plane, as shown in Figure 2.1. All curves start from $\kappa = w$ for $p = 0$. The curves for $w = m$ are split at $p = 0$. The left-hand branches marked b_m ($m = 1, 2, \dots$) represent those pairs of values (κ, p) associated with se_m , while the right-hand branches, marked a_m ($m = 1, 2, \dots$) represent those associated

Figure 2.1

CONTOUR CURVES OF CONSTANT NORMALIZED FREQUENCY w

with ce_m . The contour curves for complex w , which cannot be obtained from (2.11) and (2.13), are included for future reference. If the expansions for the Mathieu functions given in (2.10), (2.12), and (2.14) are carried out, it is seen that Mathieu functions generally contain a spectrum of normalized frequencies $w + 2r$, the integer r ranging from $-\infty$ to $+\infty$. Only in the special case $p = 0$, does a Mathieu function consist of only one spectral term at w . Otherwise "sidebands" appear at $w + 2r$, which increase in magnitude with increasing p . It was determined from a table¹² that, at least for the range $1 < \kappa < 10$, the spectral component at w exceeds all other components as long as $p < 2/\kappa$.

For the problems under study, the Mathieu functions are expediently written in the form given by Floquet

$$y_1(z; \kappa, p) = e^{ivz} \sum_r c_{2r} e^{i2rz} \quad (2.16)$$

$$y_2(z; \kappa, p) = e^{-ivz} \sum_r c_{2r} e^{-i2rz} \quad (2.17)$$

Floquet solutions are thus seen to be products of a function of period π and an exponential term involving the so-called *characteristic exponent* v , which will be shown to be connected with the normalized frequency w . Here, as in all later equations, a summation sign without limits indicates a sum over the range $-\infty$ to ∞ . Insertion of (2.16) or (2.17) into Mathieu's differential equation (2.2) yields the recurrence relation for the Fourier coefficients c_{2r} , namely

$$-p\kappa^2 c_{2r+2} + [\kappa^2 - (2r+v)^2]c_{2r} - p\kappa^2 c_{2r-2} = 0 \quad (2.18)$$

The recurrence relation constitutes an infinite set of linear homogenous equations for the unknowns c_{2r} . For the expansions (2.16) and (2.17) to be physically meaningful, one must require that $c_{2r} \rightarrow 0$ as $r \rightarrow \pm\infty$. It is stated here without proof that the infinite set of equations corresponding to (2.18) has a physically meaningful nontrivial solution for the unknowns c_{2r} only if the infinite determinant Δ formed of the coefficients of the unknowns c_{2r} vanishes. This presupposes that the infinite determinant has a limit, as further explained in Appendix II. All Fourier coefficients are then determined in terms of one Fourier coefficient, which can be arbitrarily chosen. This defines the Floquet solutions (2.16) and (2.17) except for an arbitrary constant. This arbitrariness may be eliminated by a suitable normalization of the Fourier coefficients. For the applications in this study, the arbitrary Fourier coefficient is chosen so that the arguments of the remaining Fourier coefficients obey certain relations, which will be specified later. Normalization of the moduli of the Fourier coefficients is here not necessary.

Since the determinant Δ is a function of v , κ , and p , the condition

$$\Delta(v, \kappa; p) = 0 \quad (2.19)$$

provides an implicit relation between v , κ , and p . As shown in Appendix II, the condition (2.19) can be brought into the form

$$\sin^2 v \frac{\pi}{2} = \Delta(0, \kappa; p) \sin^2 \kappa \frac{\pi}{2} \quad (2.20)$$

As (2.16) and (2.20) indicate, the characteristic exponent is multiple valued since ν and $\nu + 2r$, where r is an integer, are clearly equivalent. In the following, ν stands for that value lying in the range

$$0 \leq \text{Re}(\nu) < 2 \quad (2.21)$$

This value is termed the *principal value* of the characteristic exponent. In addition to the principal value, a second choice for the characteristic exponent is the *normalized frequency* w related to ν by

$$w = 2n + \nu \quad (2.22)$$

For sufficiently small p , the integer n is defined by the requirement that c_{2n} is the largest Fourier coefficient.

In a three-dimensional space with coordinates $\text{Re}(\kappa)$, $\text{Im}(\kappa)$, p , referred to as κp -space, Equation (2.20) assigns a value of ν to each point of the space. Complex κ 's generally lead to complex ν 's. These will be written

$$\nu = \beta - i\mu \quad (2.23)$$

where β and μ are real numbers. The minus sign has been chosen in order to have

$$e^{i\nu z} = e^{(\mu + i\beta)z} \quad (2.24)$$

Thus with a positive μ , the solution $y_1(z; \kappa, p)$ given by (2.16) is an exponentially increasing function.

If κ is restricted to real values, $\sin^2 v \pi/2$ in (2.20) will be real. Therefore, in view of (2.21), v must be of one of the three forms

$$v = \beta; \quad 0 \leq \beta < 2 \quad (2.25)$$

$$v = -i\mu \quad (2.26)$$

$$v = 1-i\mu \quad (2.27)$$

By means of (2.20), programmed on a computer, v was determined for a sufficient number of pairs (k, p) to prepare contour curves of constant v over a portion of the κp -plane (κ real), as shown in Figure 2.2. Similar charts covering part of the range of Figure 2.2 and remarks regarding the preparation of the charts may be found in ^{13,14}.

The contour curves $v = 0$ and $v = 1$ divide the plane into regions in which v is everywhere either real, imaginary, or of the form $1-i\mu$.

In Figure 2.1 the curves labeled $a_1, b_1, a_3, b_3, a_5, b_5, \dots$ correspond to $v = 1$ and those labeled $a_2, b_2, a_4, b_4, \dots$ correspond to $v = 0$. The regions between curves a_m and b_m for $m = 1, 2, \dots$ are called unstable regions because v in these regions has a non-zero imaginary part μ and the solution $y_1(z; \kappa, p)$ given by (2.16) increases as $e^{\mu z}$. The unstable regions are numbered 1, 2, 3 by the indices of the curves enclosing them, thus the m^{th} unstable region is bounded by a_m and b_m , which come into contact at $k = m$ when $p = 0$. The remaining regions are called stable regions because both solutions y_1 and y_2 are bounded.

Figure 2.2 CONTOUR CURVES OF CONSTANT CHARACTERISTIC EXPONENT $\nu = \beta - i\mu$ FOR REAL CHARACTERISTIC NUMBERS K . (STABILITY CHART FOR MATHIEU FUNCTIONS)

Figure 2.3 CONTOUR CURVES OF CONSTANT β AND μ FOR COMPLEX CHARACTERISTIC NUMBERS K IN THE PLANE $2p=1$

Therefore, Figure 2.2 is commonly referred to as the stability chart for the Mathieu functions.

Since, except for the integral Mathieu functions, there exists no standard notation for the solutions of Mathieu's equation, a notation particularly suitable for treating problems involving a permittivity-modulated slab surrounded by a medium of constant permittivity has been devised. At the boundaries of the slab the tangential electric and magnetic fields must be continuous at all times, which requires that the fields inside and outside the slab must contain the same frequency spectrum. Rather than specifying a Mathieu function by its characteristic number κ , it is then convenient to specify its normalized frequency w , which is the major component of its frequency spectrum, as long as p is sufficiently small. The Mathieu functions are therefore designated by a two-letter symbol beginning with M, followed by an argument $wz = (2n + v)z$, or $\omega t = (2n + v)\Omega t$, respectively. For example, the symbol $Ma(2 - 0.1i)z$ denotes a Mathieu function whose major spectral term is $\exp i(2 - 0.1i)z = \exp(0.1 + 2i)z$, provided p is small enough. The second letter of the two-letter symbol is related to the various regions in the κp -space, where κ may be complex, and thus characterizes the type of Mathieu function. The characteristic number κ , or the wave number k , are carried as subscripts only when necessary. The modulation index $2p$ is considered a constant and hence not expressed in the notation. The following paragraphs define the notation and essential properties of the Mathieu functions in the various regions.

In the unstable regions numbered $2n$ ($n = 1, 2, \dots$), which are enclosed by the contours $v = 0$ marked b_{2n} and a_{2n} , the characteristic

exponent is of the form $v = -i\mu$. The normalized frequency is then $w = 2n - i\mu$, where $2n$ is the number of the unstable region. As Figures 2.1 and 2.2 indicate, if p is held constant, μ reaches a maximum $\mu_{\max}$ in each unstable region, and a given $\mu < \mu_{\max}$ belongs to two different values of κ . To distinguish between the two functions with the same μ , the notation

$$y_1(z, \kappa) = Ma(2n - i\mu)z = \sum_r c_{2r}^{2n} e^{i(2r - i\mu)z}; c_{-2r}^{2n} = c_{2r}^{2n*} \quad (2.28)$$

$$y_2(z, \kappa) = Ma[-(2n - i\mu)z] = \sum_r c_{2r}^{2n} e^{-i(2r - i\mu)z} \quad (2.29)$$

is used for the functions whose characteristic number κ is closer to a_{2n} , while

$$y_1(z, \kappa) = Mb(2n - i\mu)z = \sum_r d_{2r}^{2n} e^{i(2r - i\mu)z}; d_{-2r}^{2n} = d_{2r}^{2n*} \quad (2.30)$$

$$y_2(z, \kappa) = Mb[-(2n - i\mu)z] = \sum_r d_{2r}^{2n} e^{-i(2r - i\mu)z} \quad (2.31)$$

denotes the functions with the characteristic number closer to b_{2n} .

The number $2n$, used in the argument of the Mathieu functions and as a superscript to the Fourier coefficients, indicates the number of the unstable region in which the characteristic number is located. The functions Ma and Mb with positive argument increase exponentially, while those with negative argument decrease exponentially. For $\mu \rightarrow \mu_{\max}$, p being held constant,

$$Ma[\pm(2n-i\mu)z] \rightarrow Mb[\pm(2n-i\mu)z] \quad (2.32)$$

while for $\mu \rightarrow 0$,

$$Ma(2n-i\mu)z \rightarrow Ma2nz = ce_{2n}(z); \kappa \rightarrow a_{2n} \quad (2.33)$$

$$Ma[-(2n-i\mu)z] \rightarrow Ma(-2nz) = ce_{2n}(-z) = ce_{2n}(z) \quad (2.34)$$

and

$$Mb(2n-i\mu)z \rightarrow Mb2nz = ise_{2n}(z); \kappa \rightarrow b_{2n} \quad (2.35)$$

$$Mb[-(2n-i\mu)z] \rightarrow Mb(-2nz) = ise_{2n}(-z) = -ise_{2n}(z) \quad (2.36)$$

Thus with $\kappa \rightarrow a_{2n}$, the functions $Ma(wz)$ and $Ma(-wz)$ both approach the even function $ce_{2n}(z)$, whose derivative vanishes at $z = 0$. An independent odd solution to Mathieu's equation for $\kappa = a_{2n}$ can be shown to be

$$fe_{2n}(z) = \lim_{\mu \rightarrow 0} \frac{Ma(2n-i\mu)z - Ma[-(2n-i\mu)z]}{\frac{d}{dz}[Ma(2n-i\mu)z - Ma[-(2n-i\mu)z]] \Big|_{z=0}} \quad (2.37)$$

where the denominator makes the derivative of fe_{2n} at $z = 0$ unity, thus preventing the derivative from vanishing.* Similarly, with $\kappa \rightarrow b_{2n}$, $Mb(wz)$ and $-Mb(-wz)$ approach the odd solution $ise_{2n}(z)$. Here an independent even solution is

* McLachlan, ref. cit.,¹¹ p. 141.

$$ge_{2n}(z) = \lim_{\mu \rightarrow 0} \frac{Mb(2n-i\mu)z + Mb[-(2n-i\mu)z]}{\{Mb(2n-i\mu)z + Mb[-(2n-i\mu)z]\}_{z=0}} \quad (2.38)$$

where the denominator insures that $ge_{2n}(0)$ does not vanish. Written in the form of Floquet, the integral Mathieu functions are

$$ce_{2n}(z) = \sum_r c_{2r}^{2n} e^{i2rz}; \quad c_{-2r}^{2n} = c_{2r}^{2n}; \quad \kappa = a_{2n} \quad (2.39)$$

and

$$se_{2n}(z) = \sum_r s_{2r}^{2n} e^{i2rz}; \quad s_{-2r}^{2n} = -s_{2r}^{2n}; \quad \kappa = b_{2n} \quad (2.40)$$

where all Fourier coefficients can be chosen real. By substituting

$d_{2r}^{2n} = i s_{2r}^{2n}$, the limit (2.35) is

$$ise_{2n}(z) = \sum_r d_{2r}^{2n} e^{i2rz}; \quad d_{-2r}^{2n} = d_{2r}^{2n*} \quad (2.41)$$

If p is sufficiently small, the largest Fourier coefficients are the two corresponding to $r = \pm n$. The functions $fe_{2n}(z)$ and $ge_{2n}(z)$ contain linearly increasing terms and can therefore not be written in the form of Floquet. They cannot be associated with a given normalized frequency w and do therefore not naturally arise in solutions to the problems discussed in this study.

In the unstable regions numbered $2n+1$ ($n = 1, 2, \dots$), which are inclosed by the curves b_{2n+1} and a_{2n+1} corresponding to $\nu = 1$, the characteristic exponent and the normalized frequency are of the form $\nu = 1-i$ and $w = 2n+1+i\mu$, where $2n+1$ is the number of the unstable region. From a notational standpoint, one finds it convenient to substitute in (2.16) and (2.17)

$$\nu = 1 + \nu' \quad (2.42)$$

and to change the subscripts of the Fourier coefficients from $2r$ to $2r+1$. The general form of the Floquet solutions is then

$$y_1(z, \kappa) = e^{i\nu'z} \sum_r c_{2r+1} e^{i(2r+1)z} \quad (2.43)$$

$$y_2(z, \kappa) = e^{-i\nu'z} \sum_r c_{2r+1} e^{-i(2r+1)z} \quad (2.44)$$

rather than that given in (2.16) and (2.17). With $\nu' = i\mu$, the solutions take a form similar to Equations (2.28) to (2.31), namely for κ closer to a_{2n+1}

$$y_1(z, \kappa) = Ma(2n+1-i\mu)z = \sum_r c_{2r+1}^{2n+1} e^{i(2r+1-i\mu)z};$$

$$c_{-2r-1}^{2n+1} = c_{2r+1}^{2n+1*} \quad (2.45)$$

$$y_2(z, \kappa) = Ma[-(2n+1-i\mu)z] = \sum_r c_{2r+1}^{2n+1} e^{-i(2r+1-i\mu)z} \quad (2.46)$$

and for κ closer to b_{2n+1}

$$y_1(z, \kappa) = Mb(2n+1-i\mu)z = \sum_r d_{2r+1}^{2n+1} e^{i(2r+1-i\mu)z};$$

$$d_{-2r-1}^{2n+1} = d_{2r+1}^{2n+1*} \quad (2.47)$$

$$y_2(z, \kappa) = Mb[-(2n+1-i\mu)z] = \sum_r d_{2r+1}^{2n+1} e^{-i(2r+1-i\mu)z} \quad (2.48)$$

where $2n+1$ is the number of the unstable region containing the characteristic point (κ, p) . When (κ, p) falls on one of the boundaries of the regions, the solutions are one of the integral Mathieu functions

$$ce_{2n+1}(z) = \sum_r c_{2r+1}^{2n+1} e^{i(2r+1)z}; \quad c_{-2r-1}^{2n+1} = c_{2r+1}^{2n+1};$$

$$\kappa = a_{2n+1} \quad (2.49)$$

or

$$se_{2n+1}(z) = \sum_r s_{2r+1}^{2n+1} e^{i(2r+1)z}; \quad s_{-2r-1}^{2n+1} = -s_{2r+1}^{2n+1};$$

$$\kappa = b_{2n+1} \quad (2.50)$$

with all Fourier coefficients real. Again, the limit of $M_b(2n+1-i\mu)z$ when $\mu \rightarrow 0$ can be written as

$$\text{ise}_{2n+1}(z) = \sum_r d_{2r+1}^{2n+1} e^{i(2r+1)z}; \quad d_{-2r-1}^{2n+1} = d_{2r+1}^{2n+1*} \quad (2.51)$$

if one substitutes $d_{2r+1}^{2n+1} = i s_{2r+1}^{2n+1}$. For sufficiently small p , the largest Fourier coefficients are the two corresponding to $r = \pm n$.

In the stable regions, the characteristic exponent is real of the form $v = \beta$, where β is restricted to the range $0 \leq \beta < 2$. The normalized frequency is $w = 2n+\beta$, where $2n$ may be considered defined as the number of the closest even-numbered unstable region lying to the left of the stable region in which κ is located. The solutions, commonly called fractional Mathieu functions, are then written

$$y_1(z, \kappa) = \text{Me}(2n+\beta)z = \sum_r c_{2r}^{2n} e^{i(2r+\beta)z} \quad (2.52)$$

$$y_2(z, \kappa) = \text{Me}[-(2n+\beta)z] = \sum_r c_{2r}^{2n} e^{-i(2r+\beta)z} \quad (2.53)$$

If one combined unstable and stable regions into areas numbered 0, 1, 2, . . . , as illustrated in Figure 2.4, one could define the number n for the stable Mathieu functions in the form (2.52) and (2.53) as well as for the unstable Mathieu functions in the form (2.28) and (2.29) as the number of the area containing the characteristic point (κ, p) . If one wants to write the stable Mathieu functions in the primed notation given in (2.43) and (2.44), one would write $w = 2n+1+\beta'$, restricting β'

Figure 2.4

AREAS DEFINING INDEX n OF MATHIEU FUNCTIONS

to the range $-1 < \beta' \leq 1$. Again, n can be defined as the number of the area containing the characteristic point. The stable Mathieu functions now appear in the form

$$y_1(z, \kappa) = \text{Me}(2n+1+\beta')z = \sum_r c_{2r+1}^{2n+1} e^{i(2r+1+\beta')z} \quad (2.54)$$

$$y_2(z, \kappa) = \text{Mc}[-(2n+1-\beta')z] = \sum_r c_{2r+1}^{2n+1} e^{-i(2r+1+\beta')z} \quad (2.55)$$

where the number n is the number of the area of Figure 2.5 containing the characteristic point. For example, if $w = 0.8$, the function $\text{Me}(wz)$ would be written $\text{Me}(0+0.8)z$ in the form (2.52) or $\text{Me}(1-0.2)z$ in the form (2.54), where in either case $n = 0$. Since the recurrence relations for the Fourier coefficients of the stable Mathieu functions contain no complex constants, all Fourier coefficients may be chosen real; consequently

$$\text{Me}[-(2n+\beta)z] = [\text{Me}(2n+\beta)z]^* \quad (2.56)$$

For sufficiently small p , the functions have only one dominant Fourier coefficient, namely the one with $r = n$, provided the characteristic point (κ, p) does not lie in the immediate vicinity of the unstable regions. In the latter case, the functions have two dominant coefficients, namely the two with $r = \pm n$, where $|c_{-2n}^{2n}| \leq |c_{2n}^{2n}|$ and $|c_{-2n+1}^{2n+1}| \leq |c_{2n+1}^{2n+1}|$, respectively.

All functions defined up to this point belong to real characteristic numbers κ . The integral and the fractional Mathieu functions form a complete set for expressing a periodic function. For example, if $w = 2R$, where R is an integer, one may express the exponential function e^{iwz} in the form

$$e^{iwz} = e^{i2Rz} = \sum_{n=0}^{\infty} [A_{2n} ce_{2n}(z) + i B_{2n} se_{2n}(z)] \quad (2.57)$$

Here the integral Mathieu functions are superimposed so as to cancel all spectral components e^{i2rz} except the one with $r = R$. Likewise, if $w = 2R + \beta$, one may write

$$\begin{aligned} e^{iwz} &= e^{i(2R+\beta)z} = \sum_{n=0}^{\infty} \{A_{2n} \text{Me}(2n+\beta)z \\ &\quad + B_{2n} \text{Me}[-(2n-\beta)z]\} \\ &= \sum_{n=0}^{\infty} \sum_r [A_{2n} c_{2r}^{2n} e^{i(2r+\beta)z} \\ &\quad + B_{2n} d_{2r}^{2n} e^{-i(2r-\beta)z}] \\ &= \sum_r e^{i(2r+\beta)z} \sum_{n=1}^{\infty} (A_{2n} c_{2r}^{2n} + B_{2n} d_{-2r}^{2n}) \end{aligned} \quad (2.58)$$

On the other hand, one cannot express a term $e^{i(1-i\mu)z}$ with the set of unstable Mathieu functions defined so far. As the stability chart indicates, with a fixed modulation index $2p$, a given μ is attained only by a finite number of functions or may even not be reached at all. The set of unstable Mathieu functions is made complete by including the solutions for complex characteristic numbers. Letting

$$\kappa = \kappa_r - i\kappa_i \quad (2.59)$$

the associated solutions are written either in the form of (2.16) and (2.17)

$$y_1(z, \kappa) = \text{Mi}(2n+\beta-i\mu)z = \sum_r c_{2r}^{2n} e^{i(2r+\beta-i\mu)z}; \quad -\infty < \beta \leq 2 \quad (2.60)$$

$$y_2(z, \kappa) = \text{Mi}[-(2n+\beta-i\mu)z] = \sum_r c_{2r}^{2n} e^{-i(2r+\beta-i\mu)z} \quad (2.61)$$

where, however, $v = \beta - i\mu$ is no longer restricted to the three cases β , $-i\mu$, $1-i\mu$; or alternately in the form of (2.43) and (2.44), with $-1 < \beta \leq 1$,

$$y_1(z, \kappa) = \text{Mi}(2n+1+\beta'-i\mu)z = \sum_r c_{2r+1}^{2n+1} e^{i(2r+1+\beta'-i\mu)z} \quad (2.62)$$

$$y_2(z, \kappa) = \text{Mi}[-(2n+1+\beta'-i\mu)z] = \sum_r c_{2r+1}^{2n+1} e^{-i(2r+1+\beta'-i\mu)z} \quad (2.63)$$

The function y_1 increases exponentially, while the function y_2 decreases exponentially. The Fourier coefficients are complex, but c_{-2r}^{2n} and c_{2r}^{2n} ,

or c_{-2r-1}^{2n+1} and c_{2r+1}^{2n+1} , are generally not complex conjugates. If $y_1(z, \kappa)$ and $y_2(z, \kappa)$ are the solutions to Mathieu's equation in the form (2.2), the solution to

$$Y''(z) + (\kappa^*)^2 (1 - 2p \cos 2z) Y(z) = 0 \quad (2.64)$$

are

$$Y_1(z, \kappa^*) = [y_1(z, \kappa)]^* = M_i^*(2n + \beta - i\mu)z \quad (2.65)$$

$$Y_2(z, \kappa^*) = [y_2(z, \kappa)]^* = M_i^*[-(2n + \beta - i\mu)z] \quad (2.66)$$

With the help of the Mathieu functions belonging to complex characteristic numbers, one may, for example, express $e^{i(1-i\mu)z}$ in the form

$$\begin{aligned} e^{i(1-i\mu)z} = & \sum_{n=0}^N [A_{2n+1} Ma(2n+1-i\mu)z + B_{2n+1} Mb(2n+1-i\mu)z] \\ & + \sum_{n=N+1}^{\infty} [A_{2n+1} Mi(2n+1-i\mu)z + B_{2n+1} M_i^*(2n+1-i\mu)z] \end{aligned} \quad (2.67)$$

where $2N+1$ is the number of the last odd-numbered unstable region in which the characteristic exponent attains the value μ at the given p .

By evaluating Eq. (2.20) for a number of complex κ 's contour curves for constant β and μ in the plane $2p = 1$ of the κp -space were prepared (Figure 2.3). As the curves indicate, the function $v(\kappa)$, given by (2.20), is analytic except at the branch points $\kappa = b_1, a_1, b_2$.

$a_2, \dots$, which are the characteristic numbers for the integral Mathieu functions.

2.2 Electric Displacement Waves

The most general expression for waves of a single wave number k was given in (1.15) as the real part of

$$D(t, x; k) = A_1 T_1(t, k) e^{ikx} + A_2 T_1(t, k) e^{-ikx} \\ + A_3 T_2(t, k) e^{ikx} + A_4 T_2(t, k) e^{-ikx} \quad (2.68)$$

involving four complex constants to provide for arbitrary amplitudes and phases. The time functions $T(t, k)$ are, with the substitutions $z = \Omega t$ and $\kappa = k/K$, equivalent to the functions $y(z, \kappa)$. Thus $T(t, k)$ appears in one of the forms $ce_m(\Omega t)$, $se_m(\Omega t)$, $Ma(m-i\mu)\Omega t$, $Mb(m-i\mu)\Omega t$, $Me(m+\beta)\Omega t$, or $Mi(m+\beta-i\mu)\Omega t$, where $m = 2n$ or $m = 2n+1$. In the specially developed notation, the argument of the function

$$\omega t = (m+\nu)\Omega t = (m+\beta-i\mu)\Omega t \quad (2.69)$$

contains the dominant, possibly complex, frequency component ω , provided the modulation index $2p$ is sufficiently small. The associated wave number k is sometimes given as a subscript to make the specification of the function more definite.

Since the physical field is represented by the real part of the right-hand side of (2.68), the complex conjugate may be taken of any portion of (2.68) without affecting the physical quantity being represented. Taking the complex conjugate of portions of (2.68) will in some cases simplify the expressions for the electric

displacement waves. These expressions will assume distinct forms depending on the location of the characteristic point (κ, p) , as will be shown below. All examples are chosen with odd-order Mathieu functions resulting from the form given in (2.43) and (2.44) with the primed characteristic exponent.

If (κ, p) , with κ real, falls in a stable region of the stability chart, the time functions are complex conjugates according to (2.56). It is then sufficient to take only the first two terms of (2.68) and write

$$D(t, x; k) = \text{Me}_{2n+1+\beta'}(\Omega t) [Ae^{ikx} + Be^{-ikx}] \quad (2.70)$$

By virtue of (2.52), the first term of (2.70), namely

$$D_1(t, x; k) = A \sum_r c_{2r+1}^{2n+1} e^{i[(2r+1+\beta')\Omega t + kx]} \quad (2.71)$$

is a superposition of partial waves, which are by themselves not solutions to the differential equation, moving at different phase velocities in both the positive and the negative direction. Since the dominant term is $c_{2n+1}^{2n+1} e^{i[(2n+1+\beta')\Omega t + kx]}$, the major part of the wave moves to the left, i.e., toward negative x . Similarly,

$$D_2(t, x; k) = B \sum_r c_{2r+1}^{2n+1} e^{i[(2r+1+\beta')\Omega t - kx]} \quad (2.72)$$

is a superposition of partial waves with the major part of the wave

moving to the right. Waves belonging to wave numbers in the stable regions are therefore propagating waves. In the immediate vicinity of the boundaries of the stable regions, where $\beta \rightarrow 0$ or $\beta \rightarrow 1$ ($\beta' \rightarrow 0$), one has $c_{-2n}^{2n} \rightarrow \pm c_{2n}^{2n}$ or $c_{-2n-1}^{2n+1} \rightarrow \pm c_{2n+1}^{2n+1}$, so that the wave becomes more and more nearly a standing wave.

On the other hand, if (κ, p) , where κ is real, falls in an unstable region, for example between the contours b_{2n+1} and a_{2n+1} in Figure 2.2, Equation (2.68) may be written

$$D(t, x; k) = Ma(2n+1-i\mu)\Omega t [A_1 e^{ikx} + B_1 e^{-ikx}] \quad (2.73)$$

$$+ Ma[-(2n+1-i\mu)\Omega t] [A_2 e^{ikx} + B_2 e^{-ikx}]$$

The first half of (2.73) may be expanded in the form

$$D_1(t, x; k) = \sum_r c_{2r+1}^{2n+1} e^{i(2r+1-i\mu)\Omega t} [A_1 e^{ikx} + B_1 e^{-ikx}] \quad (2.74)$$

Since $c_{-2r-1}^{2n+1*} = c_{2r+1}^{2n+1}$ according (2.45), taking the complex conjugates of all terms in (2.74) containing Fourier coefficients with negative subscripts leads to

$$D_1(t, x; k) = \sum_{r=0}^{\infty} c_{2r+1}^{2n+1} e^{i(2r+1-i\mu)\Omega t} [(A_1 + B_1^*) e^{ikx}$$

$$+ (A_1^* + B_1) e^{-ikx}] \quad (2.75)$$

Writing $A_1 + B_1^* = \frac{1}{2} C_1 e^{i\gamma_1}$ with real C_1 , changes (2.75) into

$$D_1(t, x; k) = C_1 \sum_{r=0}^{\infty} c_{2r+1}^{2n+1} e^{i(2r+1-i\mu)\Omega t} \cos(kx + \gamma_1) \quad (2.76)$$

Thus D_1 is a standing wave whose amplitude increases exponentially with time. The same operations applied to the second term of (2.73) yield

$$D_2(t, x; k) = C_2 \sum_{r=0}^{\infty} c_{2r+1}^{2n+1} e^{-i(2r+1-i\mu)\Omega t} \cos(kx + \gamma_2) \quad (2.77)$$

indicating that D_2 is a standing wave exponentially decreasing with time.

If (κ, p) , with κ real, falls on a boundary separating a stable from an unstable region, the time part of the wave is either an even or an odd integral Mathieu function. Since $ce_m(\Omega t) = Ma(m\Omega t)$ and $ise_m(\Omega t) = Mb(m\Omega t)$, as explained in (2.33) to (2.36), the waves associated with the integral Mathieu functions appear as a special case of the exponentially varying standing waves discussed in the last paragraph. For example, if $\kappa = a_{2n+1}$, i.e. if (κ, p) falls on one of the curves $a_1, a_3, a_5, \dots$ of Figure 2.2, one has

$$D(t, x; k) = Ma(2n+1)\Omega t [Ae^{ikx} + Be^{-ikx}] \quad (2.78)$$

which can be converted, with the same operations used in the last paragraph, into

$$D(t, x; k) = C \sum_{r=1}^{\infty} c_{2r+1}^{2n+1} e^{i(2r+1)\Omega t} \cos(kx + \gamma) \quad (2.79)$$

where C is a real constant. On the other hand, if $\kappa = b_{2n+1}$,

$$D(t, x; k) = Mb(2n+1)\Omega t [Ae^{ikx} + Be^{-ikx}] \quad (2.80)$$

$$= C \sum_{r=1}^{\infty} d_{2r+1}^{2n+1} e^{i(2r+1)\Omega t} \cos(kx + \delta)$$

Thus the waves containing integral Mathieu functions are standing waves of constant amplitude.

The last type of waves to discuss is the one associated with complex wave numbers. Letting $k = k_r - ik_i$, one may write

$$\begin{aligned} D(t, x; k) = & Mi(2n+1+\beta' - i\mu)\Omega t [A_1 e^{ikx} + A_2 e^{-ikx}] \quad (2.81) \\ & + Mi[-(2n+1+\beta' - i\mu)\Omega t] [A_3 e^{ikx} + A_4 e^{-ikx}] \end{aligned}$$

which represents four independent waves. The first term

$$D_1(t, x; k) = A_1 \sum_r c_{2r+1}^{2n+1} e^{i[(2r+1+\beta')\Omega t + k_r x]} e^{\mu\Omega t + k_i x} \quad (2.82)$$

consists of partial waves moving in both directions with the envelope $e^{k_1 x}$ and the dominant partial wave moving to the left.

The second term,

$$D_2(t, x; k) = A_2 \sum_r c_{2r+1}^{2n+1} e^{i[(2r+1+\beta')\Omega t - k_r x]} e^{\mu \Omega t - k_1 x} \quad (2.83)$$

is a wave with the envelope $e^{k_1 x}$ and the dominant partial wave moving to the right. Similarly, D_3 is a wave whose envelope $e^{k_1 x}$ and whose dominant partial wave move to the right, while D_4 has an envelope $e^{-k_1 x}$ and a dominant partial wave which move to the left.

2.3 Waves in Terms of the Electric Field

All the types of waves discussed up to this point are electric-displacement waves. The electric field is related to the electric displacement in the following way:

$$E(t, x) = \frac{D(t, x)}{\epsilon(t)} = \frac{D(t, x)}{\epsilon} (1 - 2p \cos 2\Omega t) \quad (2.84)$$

When the time part of the electric-displacement field is a Mathieu function, the time part of the corresponding electric field is the same Mathieu function multiplied by $1 - 2p \cos 2\Omega t$. If the time part $D(t)$ is any one of the Mathieu functions treated earlier, written in agreement with (2.16) as

$$D(t) = \sum_r c_{2r}^{2n} e^{i(2r+\nu)\Omega t} \quad (2.85)$$

then the time part $E(t)$ of the electric field $E(t, x)$ is by (2.84)

$$\begin{aligned} E(t) &= \frac{1}{\epsilon} \sum_r c_{2r}^{2n} e^{i(2r+v)\Omega t} [1 - p(e^{i2\Omega t} + e^{-i2\Omega t})] \\ &= \frac{1}{\epsilon} \sum_r [c_{2r}^{2n} - p(c_{2r+2}^{2n} + c_{2r-2}^{2n})] e^{i(2r+v)\Omega t} \end{aligned} \quad (2.86)$$

Using the recurrence relation (2.18), which can be written

$$\left[1 - \frac{(2r+v)^2}{\kappa^2}\right] c_{2r}^{2n} - p(c_{2r+2}^{2n} + c_{2r-2}^{2n}) = 0 \quad (2.87)$$

one may express (2.86) as

$$E(t) = \frac{1}{\epsilon} \sum_r \frac{(2r+v)^2}{\kappa^2} c_{2r}^{2n} e^{i(2r+v)\Omega t} \quad (2.88)$$

Thus the electric field follows from the displacement field by multiplying each Fourier coefficient c_{2r}^{2n} by $(2r+v)^2/\kappa^2$.

CHAPTER III

Wave Propagation in a Dielectric Slab with Small Permittivity Modulation

As shown in the previous chapter, plane waves in an infinite space of periodically time varying permittivity have amplitudes which are either periodically or exponentially modulated in time depending on whether their wave number k falls in a stable or unstable region, respectively. The stability chart indicates that, at least for modulation indices $2p$ smaller than $\frac{1}{2}$, the fastest rate of increase occurs in the first unstable region when $\kappa \approx 1$, or $k \approx K = \Omega \sqrt{\mu_0 \epsilon}$. No matter how small the modulation index, if a wave of a wave number $k \approx K$ is excited at an arbitrarily small amplitude, after sufficient time this wave would dominate over all other waves previously existing. For the propagation of electromagnetic waves, an infinite structure of periodically time varying permittivity would be considered an unstable system. In the absence of a mechanism preventing such excitations, an infinite physical system whose permittivity is modulated at a frequency 2Ω would ultimately support standing waves of fundamental frequency Ω increasing as $e^{\mu_{\max} \Omega t}$, where $\mu_{\max}$ is the largest μ attainable at the given modulation index.

The situation is somewhat different for a finite space of time-varying permittivity surrounded by an infinite space of constant permittivity. Depending on the extent of the permittivity-modulated space and the reflection coefficient at the boundary between the two spaces, exponentially increasing waves will be shown to exist only

if the modulation index $2p$ exceeds a critical value $2p_0$. Only then would a finite space of time-varying permittivity be considered an unstable system. On the other hand, if $2p < 2p_0$, waves can exist in the modulated space only if they are generated by some source inside or outside the modulated space.

In this and the following chapter, the modulated space will be of the form of a dielectric slab bounded by the planes $x = \pm \ell$ or, in some cases, by the planes $x = 0$ and $x = L$. This space, which is of width $L = 2\ell$, will be referred to as the slab or the interior space. Accordingly, the field in the slab will be called the interior field. The slab is nonconducting and has a permittivity

$$\epsilon(t) = \epsilon \frac{1}{1 - 2p \cos 2\Omega t} \quad (3.1)$$

where ϵ is a constant and $\epsilon(t) = \epsilon$ when $p = 0$. The permeability of the slab is μ_0 , which should not be confused with the imaginary part μ of the characteristic exponent ν giving the rate of increase of the field. It is convenient to introduce the quantity $V = (\mu_0 \epsilon)^{-\frac{1}{2}}$, which is a constant equal to the phase velocity in the slab when $p = 0$. The remaining space, referred to as the exterior space, is also nonconducting and has a constant permittivity ϵ_1 and a permeability equal to that of the slab. The phase velocity in the external space is $v_1 = (\mu_0 \epsilon_1)^{-\frac{1}{2}}$. For the following development it will be convenient to define a dimensionless quantity

$$v = V/v_1 = (\epsilon_1/\epsilon)^{\frac{1}{2}} \quad (3.2)$$

to be considered as a normalized phase velocity.

3.1 Stability Criterion for a Dielectric Slab with Small Permittivity Modulation

In this section, the critical modulation index $2p_0$, which is the minimum modulation index permitting exponentially increasing waves in a permittivity-modulated slab, is computed for cases where $2p_0 \ll 1$. The slab extends from $x = -\ell$ to $x = \ell$. The investigation will be restricted to plane waves with propagation vectors normal to the slab. The critical modulation index $2p_0$ will be shown to depend on the width of the slab $L = 2\ell$, as well as the quantities $K = \Omega(\mu_0\epsilon)^{\frac{1}{2}}$ and $v = (\epsilon_1/\epsilon)^{\frac{1}{2}}$. For $2p > 2p_0$ the maximum rate of increase will be computed.

For small $2p$, the largest exponential increase with time occurs when κ falls approximately in the center of the first unstable region of the stability chart (see Figure 2.2), where μ reaches its maximum. In an infinite modulated space, the wave with the fastest exponential increase is, as seen by reference to (2.73), (2.76), and (2.32),

$$\begin{aligned} D(t, x) &= Ma_k (1 - i\mu_{\max}) \Omega t \left[C e^{ikx} + D e^{-ikx} \right] \\ &= A \sum_r \frac{1}{c_{2r+1}} e^{i(2r+1 - i\mu_{\max}) \Omega t} \cos(kx + \gamma) \end{aligned} \quad (3.3)$$

On the other hand, for a finite space modulated at the same index $2p$, one would expect that the interior field increases at a rate $\mu < \mu_{\max}$ because of the continual losses resulting from transmission of waves through the boundaries into the exterior space. One would thus be led to suppose that for $|x| < \ell$ the field is of the form

$$D(t, x) = Ma_{k_a} (1 - i\mu)\Omega t \left[C e^{ik_a x} + D e^{-ik_a x} \right] \\ + Mb_{k_b} (1 - i\mu)\Omega t \left[E e^{ik_b x} + F e^{-ik_b x} \right] \quad (3.4)$$

Here Ma and Mb are the two unstable Mathieu functions of common characteristic exponent $\nu' = -i\mu$ and corresponding to points in the first unstable region. These are the functions given in (2.45) and (2.47) for $n = 0$. Their characteristic numbers $\kappa_a = k_a/K$ and $\kappa_b = k_b/K < \kappa_a$ are defined by the two intersections of the contour curve $\mu = \text{constant}$ with the ordinate $2p = \text{constant}$. According to (2.76) and (2.77), the field can be written with two real constants A and B

$$D(t, x) = \sum_{r=0}^{\infty} e^{i(2r+1 - i\mu)\Omega t} \left[A c_{2r+1}^1 \cos(k_a x + \gamma) \right. \\ \left. + B d_{2r+1}^1 \cos(k_b x + \delta) \right] \quad (3.5)$$

In this chapter, the modulation index $2p$ is assumed small enough to permit the approximation

$$E(t, x) = \frac{D(t, x)}{\varepsilon(t)} = \frac{D(t, x)}{\varepsilon} (1 - 2p \cos 2\Omega t) \approx \frac{D(t, x)}{\varepsilon} \quad (3.6)$$

Moreover, for small p , the only significant Fourier coefficients are those with $2r+1 = \pm(2n+1)$, therefore the only Fourier coefficients to be considered in this case are c_1^1 , c_{-1}^1 and d_1^1 , d_{-1}^1 . Since the sum in (3.5) extends only over positive r , the interior electric field is

$$E(t, x) = e^{(i + \mu)\Omega t} \left[A c_1^1 \cos(k_a x + \gamma) + B d_1^1 \cos(k_b x + \delta) \right] \quad (3.7)$$

where the real constants A and B have absorbed the factor $1/\varepsilon$ of (3.6).

In general, μ , k_a , k_b , and the Fourier coefficients c_{2r+1}^{2n+1} and d_{2r+1}^{2n+1} are related through the infinite determinant and the recurrence relation, but in the first unstable region for small p simpler relationships may be derived. This derivation, which is contained in Appendix 3, was prompted by a treatment of wave propagation in a medium with a progressive sinusoidal disturbance.^{15,16} Letting $\kappa = 1 + \zeta$, where $|\zeta| < 1$, it follows from Appendix 3 that

$$\zeta^2 + \mu^2 = \left(\frac{p}{2}\right)^2 \quad (3.8)$$

One may therefore write

$$\kappa_a = 1 + \zeta_a = 1 + \alpha \quad (3.9a)$$

$$\kappa_b = 1 + \zeta_b = 1 - \alpha \quad (3.9b)$$

where $\alpha = +[(\frac{p}{2})^2 - \mu^2]^{\frac{1}{2}}$. Since Mathieu functions of the type Ma and Mb belong to real characteristic numbers κ , one has $\mu \leq p/2$. The quotient of the Fourier coefficients is, according to Appendix 3,

$$\frac{c_{-1}^1}{c_1^1} = \frac{\alpha + i\mu}{p/2} \quad (3.10a)$$

for the function Ma , and

$$\frac{d_{-1}^1}{d_1^1} = \frac{-\alpha + i\mu}{p/2} \quad (3.10b)$$

for the function Mb . Writing

$$\frac{\alpha + i\mu}{p/2} = \frac{\alpha + i\mu}{[\alpha^2 + (p/2)^2]^{\frac{1}{2}}} = e^{i\phi} \quad (3.11a)$$

and

$$\frac{-\alpha + i\mu}{p/2} = \frac{-\alpha + i\mu}{[\alpha^2 + (p/2)^2]^{\frac{1}{2}}} = e^{i\psi} = e^{i(\pi-\phi)} \quad (3.11b)$$

the quotient of the Fourier coefficients becomes

$$\frac{c_{-1}^1}{c_1^1} = e^{i\phi} \quad (3.12a)$$

and

$$\frac{d_{-1}^1}{d_1^1} = e^{i\psi} = e^{i(\pi-\phi)} \quad (3.12b)$$

respectively. Since α and μ are positive quantities, the angle ϕ lies in the first quadrant while the angle ψ lies in the second quadrant. The angle ϕ may be expressed in terms of μ and p , or α and p , as

$$\phi = \sin^{-1} \frac{\mu}{p/2} = \cos^{-1} \frac{\alpha}{p/2} \quad (3.13)$$

The relationship between p , α , μ , and ϕ is illustrated in Figure 3.1. For $\mu \rightarrow 0$ one has $\alpha \rightarrow p/2$, $\phi \rightarrow 0$, $\psi \rightarrow \pi$, $Ma \rightarrow ce_1$, $Mb \rightarrow se_1$; while for $\mu \rightarrow \mu_{\max}$ one has $\alpha \rightarrow 0$, $\phi \rightarrow \pi/2$, $\psi \rightarrow \pi/2$, $Ma \rightarrow Mb$. Figure 3.1 shows the cases $\mu = 0$, $\mu < \mu_{\max}$, $\mu \sim \mu_{\max}$, and $\mu = \mu_{\max}$. Since the Fourier coefficients c_1^1 and c_{-1}^1 , as well as d_1^1 and d_{-1}^1 , are complex conjugates according to (2.45) and (2.47), one may write

$$c_1^1 = e^{-i\phi/2}, \quad c_{-1}^1 = e^{i\phi/2} \quad (3.14a)$$

$$d_1^1 = e^{-i\psi/2}, \quad d_{-1}^1 = e^{i\psi/2} \quad (3.14b)$$

thus choosing $|c_1^1| = |d_1^1| = 1$. Using (3.14) in (3.7), the electric field in the interior space may be written as

$$E(t, x) = e^{(i + \mu)\Omega t} \left[A e^{-i\phi/2} \cos(k_a x + \gamma) + B e^{-i\psi/2} \cos(k_b x + \delta) \right] \quad (3.15)$$

The field in the exterior space is assumed to be produced entirely by partial transmission of the interior field through the boundaries

Figure 3.1 RELATIONSHIP BETWEEN p, α, ϕ, μ , AND THE FOURIER COEFFICIENTS IN THE FIRST UNSTABLE REGION

$x = \pm \ell$. Like the interior field, the exterior field consists of plane waves with propagation vectors normal to the boundaries. Since wave propagation must be away from the slab, the field for $x > \ell$ must be a function of $t - x/v_1$ while the field for $x < -\ell$ must be a function of $t + x/v_1$, where v_1 is the phase velocity in the exterior space. Continuity at the tangential component of the electric field at the boundaries $x = \pm \ell$ at all times requires that the external waves also contain the time function $\exp(i + \mu)\Omega t$. Thus the external field may be written in the form

$$E(t, x) = T e^{(i + \mu)\Omega(t - \frac{x - \ell}{v_1})} \quad x > \ell \quad (3.16)$$

$$E(t, x) = R e^{(i + \mu)\Omega(t + \frac{x + \ell}{v_1})} \quad x < -\ell \quad (3.17)$$

where T and R are arbitrary complex constants.

With a particular value of μ , Equations (3.15), (3.16), and (3.17) constitute a solution for the field in the entire space only if the μ -independent arbitrary quantities A , B , γ , and δ in (3.15), as well as T and R in (3.16) and (3.17), can be selected so as to make the electric field and its space derivative continuous at the boundaries of the slab. It will be shown that this requirement can be met only for certain values of μ . Thus, the condition of continuity of the field is used to determine the possible values of the exponent μ .

Suppose now that a particular value of μ has been found for which there exists a solution $E_1(t, x)$ which is continuous at the

boundaries $x = \pm \ell$. According to (3.16) and (3.17), the field at the boundaries is

$$E_1(t, \ell) = Te^{(i + \mu)\Omega t} ; E_1(t, -\ell) = Re^{(i + \mu)\Omega t} \quad (3.18)$$

with the constant T and R determined. Then, because of the physical symmetry, a second continuous solution $E_2(t, x)$ can be found by substituting $-x$ for x . For the second solution, one has

$$E_2(t, \ell) = Re^{(i + \mu)\Omega t} ; E_2(t, -\ell) = Te^{(i + \mu)\Omega t} \quad (3.19)$$

Clearly, the sum and difference of the fields E_1 and E_2 are also solutions satisfying the requirement of continuity at the boundaries. The sum of E_1 and E_2 results in a symmetric field, for which

$$E(t, \ell) = E(t, -\ell) = (T + R)e^{(i + \mu)\Omega t} \quad (3.20)$$

while the difference $E_1 - E_2$ leads to an antisymmetric field, for which

$$E(t, \ell) = -E(t, -\ell) = (T - R)e^{(i + \mu)\Omega t} \quad (3.21)$$

It follows that one needs to look only for symmetric (even) and antisymmetric (odd) fields. For, if there exists a field $E_1(t, x)$ satisfying the requirement of continuity with arbitrary constants T and R , then

there exist an even and an odd field, both associated with the same time function $e^{(i + \mu)\Omega t}$. Conversely, T and R can be arbitrarily chosen only if there exist an even and an odd solution for the field, both solutions containing the same time function. Usually, the even and the odd fields will be associated with different values of μ . Then the physically interesting solution is the one increasing at the faster rate since it will eventually dominate over the other solution.

In the following treatment, even and odd fields are considered separately. If the interior field is to be even, Equation (3.15) reduces to

$$E_{ev}(t,x) = e^{(i + \mu)\Omega t} \left[A e^{-i\phi/2} \cosh k_a x + B e^{-i\psi/2} \cosh k_b x \right] \quad (3.22)$$

while, to yield an odd field, (3.15) would be written

$$E_{od}(t,x) = e^{(i + \mu)\Omega t} \left[A e^{-i\phi/2} \sinh k_a x + B e^{-i\psi/2} \sinh k_b x \right] \quad (3.23)$$

where A and B are real constants. Since $p \ll 1$, it follows from (3.8) that also $\alpha \ll 1$ and $\mu \ll 1$. The derivatives with respect to x may then be approximated as

$$\frac{d}{dx} \cosh k_a x = \frac{d}{dx} \cosh K(1 + \alpha)x \approx -K \sinh K(1 + \alpha)x \quad (3.24)$$

and

$$\frac{d}{dx} e^{(i + \mu)\Omega \frac{x}{v_1}} \approx i \frac{\Omega}{v_1} e^{(i + \mu)\Omega \frac{x}{v_1}} = i v K e^{(i + \mu)\Omega \frac{x}{v_1}} \quad (3.25)$$

where the identity

$$\frac{\Omega}{v_1} = \frac{V}{v_1} \frac{\Omega}{V} = vK \quad (3.26)$$

containing the dimensionless normalized phase velocity defined in (3.2) was used. With the above approximations, continuity of the even electric field and its space derivative is expressed by the equations

$$E(\ell) = Ae^{-i\phi/2} \cos k_a \ell + Be^{-i\psi/2} \cos k_b \ell = T + R \quad (3.27)$$

$$\frac{dE}{dx}(\ell) = -KAe^{-i\phi/2} \sin k_a \ell - KBe^{-i\psi/2} \sin k_b \ell = -ivK(T + R) \quad (3.28)$$

where the time function $e^{(i + \mu)\Omega t}$ has been divided out. Eliminating the unknown constant $T + R$ leads to the continuity condition for the even field

$$Ae^{-i\phi/2} (\cos k_a \ell + \frac{i}{v} \sin k_a \ell) = -Be^{-i\psi/2} (\cos k_b \ell + \frac{i}{v} \sin k_b \ell) \quad (3.29)$$

The continuity condition for the field is satisfied when the two sides of (3.29) are equal in modulus and argument or, if one speaks in engineering terms, in amplitude and phase. Since A and B are real, the phase of the two sides is equal or opposite when

$$\tan^{-1} \left(\frac{1}{v} \tan k_a \ell \right) - \frac{\phi}{2} = \tan^{-1} \left(\frac{1}{v} \tan k_b \ell \right) - \frac{\psi}{2} + M\pi \quad (3.30)$$

where M is an integer. This condition, referred to as the phase condition, will suffice to determine the exponent μ , because after meeting (3.30), a pair of real constants A and B satisfying (3.29) can always be found. Similarly, starting from (3.23), continuity of the odd field is expressed by the equations

$$E(\ell) = Ae^{-i\phi/2} \sinh_a \ell + Be^{-i\psi/2} \sinh_b \ell = T - R \quad (3.31)$$

$$\frac{dE}{dx}(\ell) = KAe^{-i\phi/2} \cosh_a \ell + KBe^{-i\psi/2} \cosh_b \ell = -ivK(T - R) \quad (3.32)$$

from which $T - R$ is eliminated to yield the continuity condition for the odd field

$$Ae^{-i\phi/2} (\cosh_a \ell + iv \sinh_a \ell) = -Be^{-i\psi/2} (\cosh_b \ell + iv \sinh_b \ell) \quad (3.33)$$

Again, the exponent μ can be determined by the requirement that the phase of the two sides of (3.33) is either equal or opposite, which is met if

$$\tan^{-1}(v \tanh_a \ell) - \frac{\phi}{2} = \tan^{-1}(v \tanh_b \ell) - \frac{\psi}{2} + M\pi \quad (3.34)$$

This equation is referred to as the phase condition for the odd field.

The function $y = \tan^{-1}(v \tanh x)$, shown in Figure 3.2 for several values of v , is here to be considered a single-valued, continuous,

$$y = \tan^{-1}(v \tan Kx)$$

Figure 3.2 SINGLE-VALUED FUNCTION $\tan^{-1}(v \tan Kx)$

monotonically increasing function. This interpretation follows from the special case $v = 1$, for which $y = kx$. As Figure 3.2 indicates, if $kx = n\pi + \xi$, where $0 \leq \xi \leq \pi/2$, then $y = n\pi + \eta$, where $\eta = \tan^{-1}(v \tan \xi)$ is also restricted to the range $0 \leq \eta \leq \pi/2$. One would thus write

$$\tan^{-1} \left[v \tan(n\pi + \xi) \right] = n\pi + \tan^{-1}(v \tan \xi) \quad (3.35)$$

From (3.35) follows that

$$\begin{aligned} \tan^{-1} \{ v \tan[(n + \frac{1}{2})\pi + \xi] \} &= \tan^{-1} \{ v \tan[(n + 1)\pi - (\frac{\pi}{2} - \xi)] \} \\ &= (n + 1)\pi - \tan^{-1} [v \tan(\frac{\pi}{2} - \xi)] \\ &= (n + 1)\pi - \tan^{-1}(v \cot \xi) \\ &= (n + 1)\pi - [\frac{\pi}{2} - \cot^{-1}(v \cot \xi)] \end{aligned} \quad (3.36a)$$

and

$$\begin{aligned} \tan^{-1} \{ v \tan[(n + \frac{1}{2})\pi - \xi] \} &= n\pi + \tan^{-1} [v \tan(\frac{\pi}{2} - \xi)] \\ &= n\pi + \tan^{-1}(v \cot \xi) \\ &= n\pi + [\frac{\pi}{2} - \cot^{-1}(v \cot \xi)] \end{aligned} \quad (3.36b)$$

The last two equations can be combined to yield

$$\tan^{-1}\{v \tan[(n + \frac{1}{2})\pi \pm \xi]\} = (n + \frac{1}{2})\pi \pm \tan^{-1}(\frac{1}{v} \tan \xi) \quad (3.37)$$

where the identity

$$\cot^{-1}(v \cot \xi) = \tan^{-1}(\frac{1}{v} \tan \xi) \quad (3.38)$$

has been used. By referring to Figure 3.2, the relations (3.35) and (3.37) can be seen to be valid even if ξ is not restricted to the range $0 \leq \xi \leq \pi/2$.

Since $k_a > k_b$ and since $\tan^{-1}(v \tan kx)$ is monotonically increasing,

$$\tan^{-1}(v \tan k_a \ell) - \tan^{-1}(v \tan k_b \ell) > 0 \quad (3.39)$$

Writing the phase condition for the odd field (3.34) as

$$\tan^{-1}(v \tan k_a \ell) - \tan^{-1}(v \tan k_b \ell) = M\pi - \frac{\psi}{2} + \frac{\phi}{2} \quad (3.40)$$

it follows from (3.39) that $M = 1, 2, \dots$ since $\psi > \phi$, as determined by (3.11) and illustrated in Figure 3.1. Since $\psi = \pi - \phi$ by (3.11), one has $\pi - \psi/2 + \phi/2 = \phi - \pi/2$. It is then convenient to replace M by $m + 1$ and to write the phase condition for the odd field as

$$\tan^{-1}(v \tan k_a \ell) - \tan^{-1}(v \tan k_b \ell) = \phi + \frac{\pi}{2} + m\pi \quad (3.41)$$

where $m = 0, 1, 2, \dots$. Similarly, the phase condition for the even field (3.30) can be written

$$\tan^{-1}\left(\frac{1}{v} \tan k_a \ell\right) - \tan^{-1}\left(\frac{1}{v} \tan k_b \ell\right) = \phi + \frac{\pi}{2} + m\pi \quad (3.42)$$

with $m = 0, 1, 2, \dots$. Since the exponent μ follows from ϕ , p , and k_a or k_b by (3.9) and (3.13), the above equations connect the rate of exponential increase with the width of the permittivity-modulated space.

For the special case $v = 1$, i.e., $\epsilon_1 = \epsilon$, the phase conditions (3.41) and (3.42) for both the even and the odd fields become

$$k_a \ell - k_b \ell = \phi + \frac{\pi}{2} + m\pi \quad (3.43)$$

or, with $k_a = K(1 + \alpha)$ and $k_b = K(1 - \alpha)$,

$$2K\alpha\ell = \phi + \frac{\pi}{2} + m\pi \quad m = 0, 1, 2 \quad (3.44)$$

Using (3.8) and (3.13), the last equation can be rewritten to connect the electric width of the modulated space $2K\ell$, the modulation index $2p$, and the rate of increase $\mu\Omega$ directly as

$$K\ell p = \frac{\phi + \frac{\pi}{2} + m\pi}{\cos \phi} = \frac{\sin^{-1} \frac{\mu}{p/2} + \frac{\pi}{2} + m\pi}{\left[1 - \left(\frac{\mu}{p/2}\right)^2\right]^{\frac{1}{2}}} \quad (3.45)$$

If $K\ell p < \pi/2$, there is no solution for the interior field in terms of the Mathieu functions M_a and M_b , as was assumed at the outset in (3.4), because ϕ must be positive and $m = 0, 1, 2, \dots$. Therefore, exponentially increasing fields cannot be maintained, and the permittivity-

modulated space may be considered a stable system. When $K\ell p = \pi/2$, Equation (3.45) is satisfied with $\mu = 0$ and $m = 0$, implying that a field may be maintained at a constant amplitude. When $K\ell p > \pi/2$, there is at least one μ satisfying (3.45), namely for $m = 0$. For larger $K\ell p$, there may exist more than one μ , but the one for $m = 0$ is the largest and would hence eventually determine the rate of increase. The discussion can therefore be restricted to $m = 0$. The maximum μ obtainable with a given p , namely $\mu = p/2$, is attainable only if the modulated region is infinite in extent. The relationship between $K\ell p$, μ , and ϕ for $K\ell p > \pi/2$ is depicted in Figure 3.3. The critical modulation index $2p_0$ is defined by

$$2p_0 = \frac{\pi}{K\ell} \quad (3.46)$$

If $2p < 2p_0$, the modulated space behaves as a stable system, otherwise it should be considered unstable because exponentially increasing waves can be maintained.

In the general situation $v \neq 1$, or $\epsilon_1 \neq \epsilon$, the exponent μ must be determined by satisfying the phase condition (3.41) or (3.42). For the odd field, Equation (3.41) must be met, of which only $m = 0$ will be considered since it leads to the largest exponent μ . The relationship between $k_a \ell$, $k_b \ell$, and $\phi + \pi/2$, as expressed by the phase condition (3.41), is illustrated in Figure 3.4 for the case $v = 3$, typical for the general case $v > 1$. For a given difference $k_a \ell - k_b \ell =$

Figure 3.3 RELATION BETWEEN $K\ell_p$, ϕ , AND μ FOR THE SPECIAL CASE $v=1$, AS GIVEN BY EQUATION (3.45)

$2K\ell$, the resulting angle $\phi + \pi/2$ is largest when $\frac{1}{2}(k_a\ell + k_b\ell) = K\ell = n\pi$, as shown in the left-hand part of Figure 3.4, while the angle $\phi + \pi/2$ is smallest when $\frac{1}{2}(k_a\ell - k_b\ell) = K\ell = (n + \frac{1}{2})\pi$, as shown in right-hand part of Figure 3.4. Since μ increases with increasing ϕ , the case $K\ell = n\pi$ leads to the largest value for μ . In the example shown in Figure 3.4, ϕ is positive for $K\ell = n\pi$, indicating that a solution of the assumed form (3.4) for waves increasing as $e^{\mu\Omega t}$ exists. On the other hand, for $K\ell = (n + \frac{1}{2})\pi$, ϕ is seen to be negative, hence a solution of the form (3.4) does not exist, which implies that exponentially increasing waves cannot be maintained. For $v < 1$ the situation is exactly opposite, because then $\tan^{-1}(v \tan kx)$ has the form of the dashed curves in Figure 3.2. Then the most favorable condition for increasing waves is $K\ell = (n + \frac{1}{2})\pi$, while the least favorable condition is $K\ell = n\pi$.

For the even field, the phase condition (3.42) must be satisfied. Comparison of (3.42) with (3.41) indicates that the even solution for $v < 1$ has the same properties as the odd solution for $v > 1$, and vice versa.

To summarize, $K\ell = n\pi$ is the condition favorable for exponentially increasing waves for the odd field if $v > 1$, and for the even field if $v < 1$; while $K\ell = (n + \frac{1}{2})\pi$ is the favorable condition for the odd field if $v < 1$, and for the even field if $v > 1$. Stated in another way, the field increasing at the fastest rate is odd if $K\ell = n\pi$, $v > 1$ or $K\ell = (n + \frac{1}{2})\pi$, $v < 1$; and is even if $K\ell = n\pi$, $v < 1$ or $K\ell = (n + \frac{1}{2})\pi$, $v > 1$.

For $K\ell = n\pi$, the phase equation for the odd field (3.41) becomes

$$\tan^{-1}(v \tan K\ell) = \tan^{-1}[v \tan(-K\ell)] = \phi + \frac{\pi}{2} \quad (3.47)$$

$$y = \tan^{-1}(3 \tan Kx)$$

Figure 3.4

PHASE CONDITION FOR THE ANTISYMMETRIC FIELD,
AS GIVEN IN EQUATION (3.41)

or

$$2\tan^{-1}(v \tan K\alpha\ell) = \phi + \frac{\pi}{2} \quad (3.48)$$

while the phase equation for the even field (3.42) becomes

$$2\tan^{-1}\left(\frac{1}{v} \tan K\alpha\ell\right) = \phi + \frac{\pi}{2} \quad (3.49)$$

For $K\ell = (n + \frac{1}{2})\pi$, the phase equation for the odd field becomes, with the help of (3.37),

$$\begin{aligned} & \tan^{-1}\{v \tan[(n + \frac{1}{2})\pi + K\alpha\ell]\} - \tan^{-1}\{v \tan[(n + \frac{1}{2})\pi - K\alpha\ell]\} \\ & = 2 \tan^{-1}\left(\frac{1}{v} \tan K\alpha\ell\right) = \phi + \frac{\pi}{2} \end{aligned} \quad (3.50)$$

and similarly for the even field

$$2\tan^{-1}(v \tan K\alpha\ell) = \phi + \frac{\pi}{2} \quad (3.51)$$

Thus, as to be expected, the phase condition with $K\ell = n\pi$ for the anti-symmetric field, and with $K\ell = (n + \frac{1}{2})\pi$ for the symmetric field, which is in either case the favorable condition for exponentially increasing waves when $v > 1$, is the same, namely

$$2\tan^{-1}(v \tan K\alpha\ell) = \phi + \frac{\pi}{2} \quad (3.52)$$

or

$$2K\alpha\ell = 2\tan^{-1}\left[\frac{1}{v} \tan\frac{1}{2}\left(\phi + \frac{\pi}{2}\right)\right] \quad (3.53)$$

An equation connecting $K\ell$, p , and μ results from (3.53) with the help of (3.8) and (3.13), namely

$$K\ell p = \frac{2\tan^{-1}\left[\frac{1}{v} \tan\frac{1}{2}\left(\phi + \frac{\pi}{2}\right)\right]}{\cos\phi} = \frac{2\tan^{-1}\left[\frac{1}{v} \left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^{\frac{1}{2}}\right]}{\cos\phi}$$

$$= \frac{2\tan^{-1}\left[\frac{1}{v} \left(\frac{1 + \frac{\mu}{p/2}}{1 - \frac{\mu}{p/2}}\right)^{\frac{1}{2}}\right]}{\left[1 - \left(\frac{\mu}{p/2}\right)^2\right]^{\frac{1}{2}}}$$

$$\begin{aligned} K\ell &= n\pi \text{ (odd field)} \\ K\ell &= (n + \frac{1}{2})\pi \text{ (even field)} \end{aligned} \quad (3.54)$$

An approximation for large v is

$$K\ell p = \frac{2}{v(1 - \frac{\mu}{p/2})} \quad (3.55)$$

The stability criterion follows from (3.54) with $\mu = 0$, thus

$$k\ell p < K\ell p_0 = 2 \tan^{-1} \frac{1}{v} \quad \begin{aligned} K\ell &= n\pi \text{ (odd field)} \\ K\ell &= (n + \frac{1}{2})\pi \text{ (even field)} \end{aligned} \quad (3.56)$$

On the other hand, the phase condition with $K\ell = n\pi$ for the even solution, and with $K\ell = (n + \frac{1}{2})\pi$ for the odd solution, which is in both cases the favorable condition for increasing waves when $v < 1$, is

$$2 \tan^{-1} \left(\frac{1}{v} \tan k\alpha l \right) = \phi + \frac{\pi}{2} \quad (3.57)$$

Then

$$K\ell p = \frac{2 \tan^{-1} \left[v \left(\frac{1 + \frac{\mu}{p/2}}{1 - \frac{\mu}{p/2}} \right)^{\frac{1}{2}} \right]}{\left[1 + \left(\frac{\mu}{p/2} \right)^2 \right]^{\frac{1}{2}}} \quad \begin{array}{l} K\ell = n\pi \text{ (even field)} \\ K\ell = (n + \frac{1}{2})\pi \text{ (odd field)} \end{array} \quad (3.58)$$

and the stability criterion becomes

$$K\ell p < K\ell p_0 = 2 \tan^{-1} v \quad \begin{array}{l} K\ell = n\pi \text{ (even field)} \\ K\ell = (n + \frac{1}{2})\pi \text{ (odd field)} \end{array} \quad (3.59)$$

The relationship between $K\ell p$, ϕ , and μ for the two cases $K\ell = n\pi$ and $K\ell = (n + \frac{1}{2})\pi$, as given by (3.54) and (3.58), is illustrated in Figure 3.5. The more v differs from unity (i.e., the larger the difference of the permittivities at the boundaries), the larger becomes μ for a given $K\ell p$. This is to be expected because less energy of the interior field is lost by transmission into the exterior space. From (3.54) and (3.58) follows that the value of μ remains unchanged if v is replaced by $1/v$. The constant- v curves are therefore annotated by two numbers, which are inverses of each other. In each case, the curves refer to the condition most favorable for exponentially increasing waves. Comparing (3.54) and (3.58), one finds that, if $v > 1$, the smaller $K\ell p$ for a given μ results from (3.54), or conversely a given $K\ell p$ leads with (3.54) to the larger μ . On the other hand, if $v < 1$, the

Figure 3.5

RELATION BETWEEN $K\ell p$, ϕ , AND μ FOR VARIOUS
VALUES OF ν AND $K\ell = n\pi$ OR $K\ell = (n+1/2)\pi$

smaller $K\ell$ for a given μ results from (3.58), or conversely a given $K\ell$ leads with (3.58) to the larger μ . Thus, the first of the numbers annotating the constant- v curves refers to (3.54), while the second refers to (3.58). The symmetry properties of the field depend on the width of the slab: For $v > 1$ the field is odd if $K\ell = n\pi$, and even if $K\ell = (n + \frac{1}{2})\pi$. For $v < 1$ the field is even if $K\ell = n\pi$, and odd if $K\ell = (n + \frac{1}{2})\pi$.

It has been shown for the cases $K\ell = n\pi$ and $K\ell = (n + \frac{1}{2})\pi$ that, unless $v = 1$, the even field and the odd field increase at different rates $\mu\Omega$. This is also true for general $K\ell$ with the two exceptions

$$k_a \ell - k_b \ell = 2K\ell\alpha = n\pi \quad (3.60)$$

and

$$k_a \ell + k_b \ell = 2K\ell = (n + \frac{1}{2})\pi \quad (3.61)$$

If one of these two conditions is met, the left-hand sides of (3.41) and (3.42) are equal, so that the same ϕ and μ result for the odd and the even fields. With the condition (3.60) satisfied, both phase conditions (3.41) and (3.42) lead to $n\pi = \phi + \pi/2 + m\pi$, which implies $m = n - 1$, $\phi = \frac{\pi}{2}$, and $\psi = \frac{\pi}{2}$, the latter following from (3.11). The continuity conditions (3.29) and (3.33) then both require $B = -A(-1)^n$ making the expressions for the field and its space derivative at the boundaries, as given by (3.27) and (3.28), as well as (3.31) and (3.32), identically equal to zero. The second condition given by (3.61), implies

that the width of the slab is equal to an odd multiple of a quarter of a wavelength $\Lambda = 2\pi V/\Omega$. Then, neither the even nor the odd field is favored. This case is illustrated for $v = 3$ and $k_a \ell + k_b \ell = (2n + \frac{1}{2})\pi$ in Figure 3.6, where the solid curve gives the relationship between $K\alpha\ell$ and ϕ for the odd field while the dashed curve gives the corresponding relationship for the even field.

3.2 Reflection and Transmission Properties of a Slab with Small Permittivity Modulation for Frequencies Near Half the Modulation Frequency

It is now assumed that $K\ell p < 2 \tan^{-1} \frac{1}{v}$ if $v > 1$ and $K\ell p < 2 \tan^{-1} v$ if $v < 1$, so that exponentially increasing waves cannot be maintained, in accordance with (3.56) and (3.59). One may then consider the reflection and transmission properties of a permittivity-modulated space for a normally incident wave of frequency $\omega \approx \Omega$, i.e., a frequency ω close to half the modulation frequency 2Ω . It will be shown that the space, which here is assumed to extend from $x = 0$ to $x = L = 2\ell$, reflects and transmits waves of frequencies ω and $2\Omega - \omega$, as illustrated in Figure 3.7. Writing $\omega = (1 + \beta')\Omega$, the exponential function $\exp(i\omega t)$ may be expressed in terms of the fractional Mathieu functions $\text{Me}(2n + 1 + \beta')\Omega t$ and $\text{Me}[-(2n + 1 - \beta')\Omega t]$ as indicated in (2.58). Since p and β' are assumed small, each Mathieu function has only two significant components, namely c_{2n+1}^{2n+1} and c_{-2n-1}^{2n+1} , as was stated in the paragraph following Equation (2.56). Then $\exp[i(1 + \beta')\Omega t]$ can be expressed by super-

Figure 3.6 CONDITION FOR EQUAL EXPONENTIAL INCREASE OF THE SYMMETRIC AND ANTISYMMETRIC FIELD

Figure 3.7

**TRANSMISSION AND REFLECTION PROPERTIES
OF A PERMITTIVITY-MODULATED SLAB**

imposing only the two fractional Mathieu functions $Me(1 + \beta')\Omega t$ and $Me[-(1 - \beta')\Omega t]$. It is therefore justified to assume the field in the modulated space to be of the form

$$D(t, x) = Me_{k_1}(1 + \beta')\Omega t (Ae^{ik_1 x} + Be^{-ik_1 x}) \\ + Me_{k_2}[-(1 - \beta')\Omega t] (Ce^{ik_2 x} + De^{-ik_2 x}) \quad (3.62)$$

Including only the two significant Fourier coefficients, namely c_1^1 and c_{-1}^1 of Me_{k_1} , as well as d_1^1 and d_{-1}^1 of Me_{k_2} , one may write the field as

$$D(t, x) = (c_1^1 e^{i(1 + \beta')\Omega t} + c_{-1}^1 e^{i(-1 + \beta')\Omega t}) (Ae^{ik_1 x} + Be^{-ik_1 x}) \\ + (d_1^1 e^{-i(1 - \beta')\Omega t} + d_{-1}^1 e^{-i(-1 - \beta')\Omega t}) (Ce^{ik_2 x} + De^{-ik_2 x}) \quad (3.63)$$

which may be rewritten as

$$D(t, x) = e^{i(1 + \beta')\Omega t} (Ac_1^1 e^{ik_1 x} + Bc_1^1 e^{-ik_1 x} + Cd_{-1}^1 e^{ik_2 x} + Dd_{-1}^1 e^{-ik_2 x}) \\ + e^{-i(1 - \beta')\Omega t} (Ac_{-1}^1 e^{ik_1 x} + Bc_{-1}^1 e^{-ik_1 x} + Cd_1^1 e^{ik_2 x} + Dd_1^1 e^{-ik_2 x}) \quad (3.64)$$

Since $\beta' \ll 1$, the points (κ_1, p) and (κ_2, p) are located in the immediate vicinity of the boundaries of the first unstable region. As shown in Appendix 3, the quantities β' , κ_1 , κ_2 , and the Fourier coefficients are

then related as follows:

$$\zeta^2 = \beta'^2 = (p/2)^2 \quad (3.65)$$

hence

$$\zeta_{1,2} = \pm [(p/2)^2 + \beta'^2]^{\frac{1}{2}} = \pm \alpha \quad (3.66)$$

$$\kappa_1 = 1 + \zeta_1 = 1 + \alpha \quad (3.67a)$$

$$\kappa_2 = 1 + \zeta_2 = 1 - \alpha \quad (3.67b)$$

$$\frac{c_{-1}^1}{c_1^1} = \frac{\alpha - \beta'}{p/2} \approx 1 - \frac{\beta'}{p/2} \quad (3.68a)$$

$$\frac{d_{-1}^1}{d_1^1} = - \frac{\alpha - \beta'}{p/2} \approx - (1 - \frac{\beta'}{p/2}) \quad (3.68b)$$

where the approximation on the right-hand side of (3.68a) and (3.68b) is valid only if in addition $\beta' < \frac{p}{2} < 1$. Above relationships are graphically displayed in Figure 3.8. Letting $c_1^1 = d_1^1 = 1$, one can write

$$c_{-1}^1 = - d_{-1}^1 = \frac{\alpha - \beta'}{p/2} = c \quad (3.69)$$

Since p is small, the electric field is proportional to the displacement field, therefore, with the constants absorbing the ϵ , the electric

Figure 3.8 RELATION BETWEEN p, α, β' , AND THE FOURIER COEFFICIENTS FOR THE FUNCTIONS $\text{Me}(1+\beta')Z$ AND $\text{Me}(1-\beta')Z$ IF $\beta' < p/2 \ll 1$

field in the modulated space may be written

$$E(t,x) = e^{i(1+\beta')\Omega t} \left[A e^{iK(1+\alpha)x} + B e^{-iK(1+\alpha)x} - C e^{iK(1-\alpha)x} - D e^{-iK(1-\alpha)x} \right] + e^{-i(1-\beta')\Omega t} \left[A e^{iK(1+\alpha)x} + B e^{-iK(1+\alpha)x} - C e^{iK(1-\alpha)x} - D e^{-iK(1-\alpha)x} \right] \quad (3.70)$$

Its space derivative, within the approximation $k_1 \approx k_2 \approx K$, is

$$\frac{\partial E}{\partial x}(t,x) \approx e^{i(1+\beta')\Omega t} iK \left[A e^{iK(1+\alpha)x} - B e^{-iK(1+\alpha)x} - C e^{iK(1-\alpha)x} + D e^{-iK(1-\alpha)x} \right] + e^{-i(1-\beta')\Omega t} iK \left[A e^{iK(1+\alpha)x} - B e^{-iK(1+\alpha)x} - C e^{iK(1-\alpha)x} + D e^{-iK(1-\alpha)x} \right] \quad (3.71)$$

The incident wave in the space $x < 0$ is assumed to be of unit amplitude

$$E_i(t,x) = e^{i(1+\beta')\Omega(t - x/v_1)} \quad (3.72)$$

Its space derivative is sufficiently approximated by

$$\frac{\partial E_i}{\partial x}(t,x) \approx -i \frac{\Omega}{v_1} e^{i(1+\beta')\Omega(t - x/v_1)} \quad (3.73)$$

Since the field in the modulated space contains the frequencies

$\omega = (1+\beta')\Omega$ and $2\Omega - \omega = (1-\beta')\Omega$, the reflected field, existing in

the space $x < 0$, and its space derivative may be written in the form

$$E_r(t, x) = R_1 e^{i(1 + \beta')\Omega(t + x/v_1)} + R_{-1} e^{-i(1 - \beta')\Omega(t + x/v_1)} \quad (3.74)$$

and

$$\begin{aligned} \frac{\partial E_r}{\partial x}(t, x) \approx i \frac{\Omega}{v_1} \left[R_1 e^{i(1 + \beta')\Omega(t + x/v_1)} \right. \\ \left. - R_{-1} e^{-i(1 - \beta')\Omega(t + x/v_1)} \right] \end{aligned} \quad (3.75)$$

where R_1 and R_{-1} are the complex reflection coefficients. Likewise, the transmitted field and its space derivative should be of the form

$$E_t(t, x) = T_1 e^{i(1 + \beta')\Omega(t - \frac{x - L}{v_1})} + T_{-1} e^{-i(1 - \beta')\Omega(t - \frac{x - L}{v_1})} \quad (3.76)$$

and

$$\begin{aligned} \frac{\partial E_t}{\partial x}(t, x) \approx i \frac{\Omega}{v_1} \left[-T_1 e^{i(1 + \beta')\Omega(t - \frac{x - L}{v_1})} \right. \\ \left. + T_{-1} e^{-i(1 - \beta')\Omega(t - \frac{x - L}{v_1})} \right] \end{aligned} \quad (3.77)$$

At the boundaries of the slab $x = 0$ and $x = L$, the electric field and its space derivative must be continuous. At $x = 0$, the interior field must be equal to the sum of the incident and reflected waves. Thus, by (3.70), (3.72), and (3.74)

$$\begin{aligned}
& e^{i(1 + \beta')\Omega t} (A + B - Cc - Dc) + e^{-i(1 - \beta')\Omega t} (Ac + Bc + C + D) \\
& = e^{i(1 + \beta')\Omega t} + R_1 e^{i(1 + \beta')\Omega t} + R_{-1} e^{-i(1 - \beta')\Omega t} \quad (3.78)
\end{aligned}$$

Because of the linear independence of the exponential functions involved, it follows from (3.78) that

$$A + B - Cc - Dc - R_1 = 1 \quad (3.79)$$

and

$$Ac + Bc + C + D - R_{-1} = 0 \quad (3.80)$$

Moreover, the space derivatives of the interior field and of the sum of the incident and reflected waves must be equal, which requires by (3.71), (3.73), and (3.75) that

$$\begin{aligned}
& e^{i(1 + \beta')\Omega t} (A - B - Cc + Dc) + e^{-i(1 - \beta')\Omega t} (Ac - Bc + C - D) \\
& = -\frac{1}{K} \frac{\Omega}{v_1} e^{i(1 + \beta')\Omega t} + \frac{1}{K} \frac{\Omega}{v_1} \left[R_1 e^{i(1 + \beta')\Omega t} - R_{-1} e^{-i(1 - \beta')\Omega t} \right] \quad (3.81)
\end{aligned}$$

With

$$\frac{1}{K} \frac{\Omega}{v_1} = \frac{V}{\Omega} \frac{\Omega}{v_1} = \frac{V}{v_1} = v \quad (3.82)$$

it follows from (3.81) that

$$A - B - Cc + Dc - vR_1 = -v \quad (3.83)$$

and

$$Ac - Bc + C - D + vR_{-1} = 0 \quad (3.84)$$

The condition of continuity of the field and its space derivative at $x = 0$ is therefore expressed by a set of four equations, namely (3.79), (3.80), (3.83), and (3.84), which connect the coefficients A, B, C, D with R_1 and R_{-1} . Continuity of the field and its space derivative at $x = L$ yields another four equations connecting the coefficients A, B, C, D with T_1 and T_{-1} . Thus, the condition of continuity of the field at the two boundaries is expressed by a set of eight equations for the eight unknown coefficients $A, B, C, D, R_1, R_{-1}, T_1, T_{-1}$. The continuity condition is therefore suitably expressed by the matrix equation

$$\begin{bmatrix} 1 & 1 & -c & -c & -1 \\ 1 & -1 & -c & c & -v \\ c & c & 1 & 1 & -1 \\ c & -c & 1 & -1 & v \\ e^{iK(1+\alpha)L} & e^{-iK(1+\alpha)L} & -ce^{iK(1-\alpha)L} & -ce^{-iK(1-\alpha)L} & -1 \\ e^{iK(1+\alpha)L} & -e^{-iK(1+\alpha)L} & -ce^{iK(1-\alpha)L} & ce^{-iK(1-\alpha)L} & v \\ ce^{iK(1+\alpha)L} & ce^{-iK(1+\alpha)L} & e^{iK(1-\alpha)L} & e^{-iK(1-\alpha)L} & -1 \\ ce^{iK(1+\alpha)L} & -ce^{-iK(1+\alpha)L} & e^{iK(1-\alpha)L} & -e^{-iK(1-\alpha)L} & -v \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ R_1 \\ R_{-1} \\ T_1 \\ T_{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ -v \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.85)$$

Eliminating the unknown coefficients A, B, C, D of the Mathieu functions from the set of equations represented by (3.85) leads to a set of four

equations for the reflection and transmission coefficients R_1 , R_{-1} , T_1 , and T_{-1} . These four equations can be represented by the matrix equation

$$\begin{bmatrix} -c(1-v) & 1+v & c(1+v)e^{iK(1-\alpha)L} & -(1-v)e^{iK(1-\alpha)L} \\ -c(1+v) & 1-v & c(1-v)e^{-iK(1-\alpha)L} & -(1+v)e^{-iK(1-\alpha)L} \\ -(1-v) & -c(1+v) & (1+v)e^{iK(1+\alpha)L} & c(1-v)e^{iK(1+\alpha)L} \\ -(1+v) & -c(1-v) & (1-v)e^{-iK(1-\alpha)L} & c(1+v)e^{-iK(1+\alpha)L} \end{bmatrix} \begin{bmatrix} R_1 \\ R_{-1} \\ T_1 \\ T_{-1} \end{bmatrix} = \begin{bmatrix} c(1+v) \\ c(1-v) \\ 1+v \\ 1-v \end{bmatrix} \quad (3.86)$$

The determinant of the square matrix in (3.86) can be shown to be

$$\Delta = (1 + c^2)^2 \left[\left(v + \frac{1}{v} \right)^2 \cos^2 KaL - \left(v - \frac{1}{v} \right)^2 \cos^2 KL \right] - 4(1 - c^2)^2 \sin^2 KaL \\ + i2(1 - c^4) \left(v + \frac{1}{v} \right) \sin 2KaL \quad (3.87)$$

The determinant is written using the quantities $v - 1/v$ and $1 - c^2$ for later convenience when discussing the special cases $v = 1$ and $c = 1$. Provided $\Delta \neq 0$, a unique solution exists for the unknowns R_1 , R_{-1} , T_1 , and T_{-1} , which may then be determined by Kramer's rule and written in the form

$$R_1 \Delta = -(1 + c^2)^2 \left(v^2 - \frac{1}{v^2} \right) (\sin^2 KaL - \sin^2 KL) \\ + i \left(v - \frac{1}{v} \right) \left[(1 + c^2)^2 \sin 2KL + (1 - c^4) \sin 2KaL \right] \quad (3.88)$$

$$R_{-1}\Delta = i2c(1 + c^2)(v + \frac{1}{v})\sin 2KaL - 8c(1 - c^2)\sin^2 KaL \quad (3.89)$$

$$T_1\Delta = 2(1 + c^2)^2 \left[2e^{-iKL} - i(\sqrt{v} - \frac{1}{\sqrt{v}})^2 \sin KL \right] \cos KaL \\ + i2(1 - c^4) \left[2e^{-iKL} + (\sqrt{v} - \frac{1}{\sqrt{v}})^2 \cos KL \right] \sin KaL \quad (3.90)$$

$$T_{-1}\Delta = -i4c(1 + c^2)(v - \frac{1}{v})\cos KL \sin KaL \quad (3.91)$$

where also the quantities $v = 1/v$ and $1 - c^2$ have been used. The determinant as given in (3.87), cannot vanish unless $c = 1$. For, in order to have the imaginary part of Δ vanish with $c \neq 1$, it is required that $KaL = n\pi$ or $(n + \frac{1}{2})\pi$. For $KaL = n\pi$, the real part of Δ reduces to the bracket, which cannot vanish since $\cos^2 KaL = 1$ and $(v + 1/v)^2 - (v - 1/v)^2 \cos KL > 0$. On the other hand, if $KaL = (n + \frac{1}{2})\pi$, then $\cos^2 KaL = 0$ and $\sin^2 KaL = 1$. The real part of Δ then becomes $-\{(1 + c^2)^2(v - 1/v)^2 + 4(1 - c^2)^2\}$, which cannot vanish unless $c = 1$.

The case $c = 1$ deserves some special consideration. Employing (3.66) and (3.69), $c = 1$ is seen to result when $\beta' = 0$. Physically, this case corresponds to illuminating one side of the slab with a wave of that angular frequency which the slab could spontaneously emit if the modulation index was raised to the critical value $2p_0$. Replacing L by $2L$ to achieve correspondence with the previous section, the determinant given by (3.87) becomes with $c = 1$, after dividing by 4,

$$\Delta = (v + \frac{1}{v})^2 \cos^2 K\ell p - (v - \frac{1}{v})^2 \cos^2 2K\ell \quad (3.92)$$

since $\alpha = p/2$ by (3.66) when $\beta' = 0$. The determinant vanishes if

$$\cos K\ell p = \left| \frac{v - 1/v}{v + 1/v} \cos 2K\ell \right| \quad (3.93)$$

where $K\ell p$ and $K\ell$ are positive quantities. Restricting $K\ell p$ to values between 0 and $\pi/2$, the determinant cannot vanish with $v > 1$ as long as

$$\cos K\ell p > \left| \frac{v - 1/v}{v + 1/v} \cos 2K\ell \right| \quad (3.94)$$

For $v < 1$, the determinant cannot vanish as long as

$$\cos K\ell p > \left| \frac{1/v - v}{1/v + v} \cos 2K\ell \right| \quad (3.95)$$

If (3.94) and (3.95) are satisfied, a unique solution exists for the reflection and transmission coefficients corresponding to an incident wave of unit amplitude and angular frequency $\omega = \Omega$. Since $\beta' = 0$ for $c = 1$, the two reflected waves, as given by (3.74), combine to a single wave, which may be written

$$E_r(t, x) = (R_1 + R_{-1}^*) e^{i\Omega(t + x/v_1)} \quad (3.96)$$

without affecting the physical quantity which is represented by the real part. The same is true for the transmitted wave, which is given

by (3.76). For the case $c = 1$, the transmitted wave is therefore written

$$E_t(t, x) = (T_1 + T_{-1}^*) e^{i\Omega(t - \frac{x - L}{v_1})} \quad (3.97)$$

The combined reflection coefficient $R = R_1 + R_{-1}^*$ follows from (3.88) and (3.89) with $c = 1$, after dividing by 4, as

$$R\Delta = -(v^2 - \frac{1}{v^2})(\sin^2 K\ell p - \sin^2 KL) - i(v - \frac{1}{v})\sin 2KL - i(v + \frac{1}{v})\sin 2K\ell p \quad (3.98)$$

The combined transmission coefficient $T = T_1 + T_{-1}^*$ is obtained from (3.90) and (3.91) with $c = 1$, after dividing by 4, as

$$T\Delta = 2 \left[2e^{-iKL} - i(v^{\frac{1}{2}} - v^{-\frac{1}{2}})^2 \sin KL \right] \cos K\ell p + i2(v - \frac{1}{v}) \cos KL \sin K\ell p \quad (3.99)$$

In general, Equations (3.98), and (3.99) do not vanish when the determinant vanishes. Thus R and T generally approach infinity when Δ approaches zero. This means that, with an incident wave of unit amplitude, the amplitudes of the reflected wave and the transmitted wave become infinite. A better physical interpretation would be that waves are reflected and transmitted from the slab although the amplitude of the incident wave approaches zero. When $\Delta = 0$, the slab may emit waves spontaneously, i.e., without any incident wave. The conditions (3.94) and (3.95) consequently constitute the stability

criterion for wave propagation through a permittivity-modulated slab of arbitrary width. The stability criterion is evidently most stringent for the cases $K\ell = n\pi$ and $(n + \frac{1}{2})\pi$. Stability criteria for these two cases have been derived in Section 3.1. They are now shown to be in agreement with (3.94) and (3.95). For $v > 1$, the stability criterion given in (3.56) as $K\ell p < 2\tan^{-1} \frac{1}{v}$ would apply. This may be stated as

$$\tan \frac{K\ell p}{2} < \frac{1}{v} \quad v > 1 \quad (3.100)$$

implying, since $0 < K\ell p < \pi/2$,

$$\cos \frac{K\ell p}{2} > \left(1 + \frac{1}{v^2}\right)^{-\frac{1}{2}} \quad (3.101)$$

and therefore

$$\cos K\ell p > -1 + \frac{2}{1 + \frac{1}{v^2}} = \frac{v - \frac{1}{v}}{v + \frac{1}{v}} \quad (3.102)$$

which agrees with (3.94) evaluated at $K\ell = n\pi$ or $(n + \frac{1}{2})\pi$. Similarly, the stability criterion $K\ell p < 2\tan^{-1} v$, which is given in (3.59) and applies to $v < 1$, may be written

$$\tan \frac{K\ell p}{2} < v \quad (3.103)$$

which leads to

$$\cos K\ell p > \frac{\frac{1}{v} - v}{\frac{1}{v} + v} \quad (3.104)$$

in agreement with (3.95) for $K\ell = n\pi$ and $(n + \frac{1}{2})\pi$. At the beginning of this section it was assumed that $K\ell p$ is smaller than both $2\tan^{-1} \frac{1}{v}$ and $2\tan^{-1} v$. Consequently, the inequalities (3.94) and (3.95) are satisfied for all values of $K\ell$. Then, even for the case $c = 1$ ($\beta' = 0$), the determinant Δ cannot vanish.

The expressions describing the reflection and transmission coefficients R and T for the case $c = 1$ ($\beta' = 0$), as given in (3.98) and (3.99), simplify if the width of the slab is restricted such that $KL = 2n\pi$ or $(2n + 1)\pi$. One obtains then from (3.98)

$$R\Delta = -(v^2 - \frac{1}{v^2})\sin^2 K\ell p - i(v + \frac{1}{v})\sin 2K\ell p \quad (3.105)$$

and from (3.99)

$$T\Delta = \pm [4\cos K\ell p + i2(v - \frac{1}{v})\sin K\ell p] \quad (3.106)$$

where here and in the following equations the upper sign refers to $KL = 2n\pi$, while the lower sign refers to $KL = (2n + 1)\pi$. The determinant becomes, for $KL = 2n\pi$ and $KL = (2n + 1)\pi$,

$$\Delta = (v + \frac{1}{v})^2 \cos^2 K\ell p - (v - \frac{1}{v})^2 \quad (3.107)$$

Finally, when also $v = 1$,

$$\Delta = 4\cos^2 K\ell p \quad (3.108)$$

$$R = -\frac{i2\sin 2K\ell p}{4\cos^2 K\ell p} = -i \tan K\ell p \quad (3.109)$$

$$T = +\frac{4\cos K\ell p}{4\cos^2 K\ell p} = +\frac{1}{\cos K\ell p} \quad (3.110)$$

For the case $c = 1$, $v = 1$, and $KL = 2n\pi$ and $(2n + 1)\pi$, the magnitudes of the reflection and transmission coefficients, as given by (3.109) and (3.110), are shown in Figure 3.9.

Formulas for the reflection and transmission coefficients for the case $\beta' \neq 0$ were given by (3.88) to (3.91). These equations are valid when $\beta' \ll 1$ and $p/2 \ll 1$. If the frequency of the incident wave is very close to the permittivity-modulation frequency, so that in addition $\beta' \ll p/2$, simple approximations for the terms involving the quantity c may be used. According to (3.68a) and (3.69), one may then write

$$c \approx 1 - 2\beta'/p, \quad c^2 \approx 1 - 4\beta'/p, \quad c^4 \approx 1 - 8\beta'/p \quad (3.111)$$

and use the following approximations:

$$\begin{aligned} 1 + c^2 &\approx 2 & 1 + c^4 &\approx 2 & (1 + c^2)^2 &\approx 4 \\ 1 - c^2 &\approx 2\frac{\beta'}{p/2} & 1 - c^4 &\approx 4\frac{\beta'}{p/2} & (1 - c^2)^2 &\approx 0 \end{aligned} \quad (3.112)$$

Figure 3.9 REFLECTION COEFFICIENT R AND TRANSMISSION COEFFICIENT T , AS GIVEN BY EQUATIONS (3.109) AND (3.110)

For the case $KL = 2n\pi$ or $(2n + 1)\pi$, the equations for the general case (3.88) to (3.91) can then be reduced, after dividing by 4, to

$$\Delta = (v + \frac{1}{v})^2 \cos^2 KaL - (v - \frac{1}{v})^2 + i \frac{2\beta'}{p/2} (v + \frac{1}{v}) \sin 2KaL \quad (3.113)$$

$$R_1 \Delta = -(v^2 - \frac{1}{v^2}) \sin^2 KaL + i \frac{\beta'}{p/2} (v - \frac{1}{v}) \sin 2KaL \quad (3.114)$$

$$R_{-1} \Delta = i(v + \frac{1}{v}) \sin 2KaL - \frac{4\beta'}{p/2} \sin^2 KaL \quad (3.115)$$

$$T_1 \Delta = \pm 4 \cos KaL \pm i \frac{2\beta'}{p/2} (v + \frac{1}{v}) \sin KaL \quad (3.116)$$

$$T_{-1} \Delta = \mp i 2(v - \frac{1}{v}) \sin KaL \quad (3.117)$$

where the upper and lower sign apply to $KL = 2n\pi$ and $KL = (2n + 1)\pi$, respectively. In the particular case $v = 1$, R_1 and T_{-1} vanish, while, after again dividing by 4,

$$\begin{aligned} \Delta &= \cos^2 KaL + i \frac{\beta'}{p/2} \sin 2KaL \\ &= \cos KaL [\cos KaL + i \frac{2\beta'}{p/2} \sin KaL] \end{aligned} \quad (3.118)$$

$$R_{-1} \Delta = i \sin KaL [\cos KaL + i \frac{\beta'}{p/2} \sin KaL] \quad (3.119)$$

$$T_1 \Delta = \pm \cos KaL \pm i \frac{\beta'}{p/2} \sin KaL \quad (3.120)$$

hence

$$R_{-1} = i \tan KaL \left[1 - \frac{i \frac{\beta'}{p/2} \sin KaL}{\cos KaL + i \frac{2\beta'}{p/2} \sin KaL} \right] \quad (3.121)$$

and

$$T_1 = \frac{1}{\cos KaL} \left[1 - \frac{i \frac{\beta'}{p/2} \sin KaL}{\cos KaL + i \frac{2\beta'}{p/2} \sin KaL} \right] \quad (3.122)$$

With $KaL = KL[(p/2)^2 + \beta'^2]^{\frac{1}{2}}$ close to, but smaller than $\pi/2$, the transmitted wave, being of the same angular frequency $\omega = \Omega + \beta'$ as the incident wave, is considerably stronger than the incident wave. The reflected wave is also much stronger than the incident wave, but the frequency of the reflected wave is

$$\omega_r = (1 - \beta')\Omega = 2\Omega - \omega \quad (3.123)$$

which is the difference of the modulation frequency and the frequency of the incident wave. Thus a permittivity-modulated slab has the basic properties of a tuned amplifier and a frequency converter.

CHAPTER IV

Wave Propagation in a Dielectric Slab with Large
Permittivity Modulation

The use of sets of Mathieu functions will be shown to provide the complete solution to three kinds of problems:

1. The stability criterion. For example, given the electrical width KL , find the minimum modulation index $2p_0$ which makes the system unstable.
2. Determination of the rate of exponential increase of electromagnetic waves if $p > p_0$.
3. The reflection and transmission problem. Given that the system is stable, i.e., that $p < p_0$, determine the transmission and reflection properties.

4.1 The Stability Criterion

The stability criterion for wave propagation in a permittivity-modulated slab can be found in two ways: One may either start from an exponentially increasing field and determine the critical modulation index $2p_0$ as that modulation index which leads to $\mu \rightarrow 0$, as was done in Section 3.1. Or, one may consider the reflection and transmission properties of such a slab and determine the condition for which the reflection and transmission coefficients become infinite, as was done in Section 3.2. Here, the first approach is used. The Mathieu functions to be used for the expansion of the field in the slab are therefore functions of the type Ma and Mb with $\mu \rightarrow 0$, which go over into the

integral Mathieu functions $Ma(m\Omega t) = ce_m(\Omega t)$ and $Mb(m\Omega t) = ise_m(\Omega t)$, as was shown in (2.33) and (2.35). The field in the modulated slab, also called the interior field, is then a superposition of waves associated with the integral Mathieu functions, as given by (2.78), namely

$$D(t, x) = \sum_{m=1}^{\infty} \{ Ma_{k_m^a}(m\Omega t) [A_m e^{ik_m^a x} + B_m e^{-ik_m^a x}] + Mb_{k_m^b}(m\Omega t) [C_m e^{ik_m^b x} + D_m e^{-ik_m^b x}] \} \quad (4.1)$$

Since the Mathieu functions of order $m = 2n + 1$ contain only the frequencies $\Omega, 3\Omega, 5\Omega, \dots$ and the Mathieu functions of order $m = 2n$ contain only the frequencies $0, 2\Omega, 4\Omega, \dots$,* the field (4.1) can be written as two mutually independent sums of waves, each of the form (2.79) and (2.80), namely

$$D(t, x) = \sum_{r=0}^{\infty} e^{i(2r+1)\Omega t} \sum_{n=0}^{\infty} [A_{2n+1} c_{2r+1}^{2n+1} \cos(k_{2n+1}^a x + \gamma_{2n+1}) + B_{2n+1} d_{2r+1}^{2n+1} \cos(k_{2n+1}^b x + \delta_{2n+1})] + \sum_{r=0}^{\infty} e^{i2r\Omega t} \sum_{n=1}^{\infty} [A_{2n} c_{2r}^{2n} \cos(k_{2n}^a x + \gamma_{2n}) + B_{2n} d_{2r}^{2n} \cos(k_{2n}^b x + \delta_{2n})] \quad (4.2)$$

* The electric field has no component at $\omega = 0$ because of (2.88), where $r = 0$ and $v = 0$.

where the c's and d's are the Fourier coefficients for the functions M_a and M_b , respectively. Because of their mutual independence, the fields represented by the two sums can be considered separately. As was pointed out in Section 3.1, due to the physical symmetry, either field may be written as a superposition of a symmetric (even) and an anti-symmetric (odd) component, which can also be treated separately. In order to fully evaluate a given slab as to its stability, one thus needs to consider four different modes, i.e., the even and the odd field containing the frequencies Ω , 3Ω , 5Ω , . . . and the even and odd fields with the frequencies 2Ω , 4Ω , 6Ω , For each of the four fields one would then determine a critical modulation index $2p_0$. The slab would act as a stable system if the modulation index $2p$ is smaller than the smallest of the four critical modulation indices.

In the following discussion, the symmetric field containing the frequencies Ω , 3Ω , 5Ω , . . . is treated as an example for the four calculations to be carried out to determine whether a given configuration is stable. Since the solution for the field is to be even, all angles γ and δ in (4.2) must be zero. The displacement field in the slab may then be written

$$D(t,x) = \sum_{r=0}^{\infty} e^{i(2r+1)\Omega t} \sum_{n=0}^{\infty} [A_{2n+1} c_{2r+1}^{2n+1} \cos k_{2n+1}^a x + B_{2n+1} d_{2r+1}^{2n+1} \cos k_{2n+1}^b x] \quad (4.3)$$

where the A's and B's are real constants. The corresponding electric field is, according to (2.88),

$$E(t, x) = \sum_{r=0}^{\infty} (2r+1)^2 e^{i(2r+1)\Omega t} \sum_{n=0}^{\infty} \left[\frac{A_{2n+1} c_{2r+1}^{2n+1}}{(\kappa_{2n+1}^a)^2} \cos k_{2n+1}^a x + \frac{B_{2n+1} d_{2r+1}^{2n+1}}{(\kappa_{2n+1}^b)^2} \cos k_{2n+1}^b x \right] \quad (4.4)$$

where the κ 's are the characteristic numbers for the integral Mathieu functions, namely $\kappa_{2n+1}^a = a_{2n+1}$ and $\kappa_{2n+1}^b = b_{2n+1}$, which are real quantities. Instead of the arbitrary constants A_{2n+1} and B_{2n+1} , one may use

$$X_{2n+1} = \frac{A_{2n+1}}{(\kappa_{2n+1}^a)^2} = \frac{A_{2n+1}}{(a_{2n+1})^2} \quad (4.5a)$$

and

$$Y_{2n+1} = \frac{B_{2n+1}}{(\kappa_{2n+1}^b)^2} = \frac{B_{2n+1}}{(b_{2n+1})^2} \quad (4.5b)$$

For further economy of notation, $2r+1$ and $2n+1$ are changed into r and n , where now $r, n = 1, 3, 5, \dots$. This is indicated by writing $r = 1, 3$ and $n = 1, 3$ under the respective summation signs. With the above changes, the interior field and its space derivative appear in the form

$$E(t, x) = \sum_{r=1,3}^{\infty} r^2 e^{ir\Omega t} \sum_{n=1,3}^{\infty} (X_n c_r^n \cos k_n^a x + Y_n d_r^n \cos k_n^b x) \quad (4.6)$$

and

$$\frac{\partial E}{\partial x}(t, x) = - \sum_{r=1,3}^{\infty} r^2 e^{ir\Omega t} \sum_{n=1,3}^{\infty} (X_n c_r^n k_n^a \sin k_n^a x + Y_n d_r^n k_n^b \sin k_n^b x) \quad (4.7)$$

Assuming that the slab extends from $x = -\ell$ to $x = \ell$, the exterior field and its space derivative may be suitably written as

$$E(t, x) = \sum_{r=1,3}^{\infty} T_r e^{ir\Omega(t - \frac{x - \ell}{v_1})} \quad x > \ell \quad (4.8)$$

$$\frac{\partial E}{\partial x}(t, x) = -i \frac{\Omega}{v_1} \sum_{r=1,3}^{\infty} r T_r e^{ir\Omega(t - \frac{x - \ell}{v_1})} \quad (4.9)$$

where Ω/v_1 may be replaced by vK , as shown in (3.26). The external field for $x < -\ell$ is defined by the fact that the field is to be an even function. Since continuity of the electric field and its derivative at the boundaries $x = \pm \ell$ requires equality of the coefficients for each frequency, the equations

$$r^2 \sum_{n=1,3}^{\infty} (X_n c_r^n \cos k_n^a \ell + Y_n d_r^n \cos k_n^b \ell) = T_r \quad (4.10)$$

and

$$r^2 \sum_{n=1,3}^{\infty} (X_n c_r^n k_n^a \sin k_n^a \ell + Y_n d_r^n k_n^b \sin k_n^b \ell) = irvKT_r \quad (4.11)$$

must be satisfied for $r = 1, 3, 5, \dots, \infty$. After eliminating the unknown transmission coefficient T_r between (4.10) and (4.11), the boundary condition takes the form

$$r^2 \sum_{n=1,3}^{\infty} \left[X_n c_r^n (\cos k_n^a \ell + i \frac{k_n^a}{rvK} \sin k_n^a \ell) + Y_n d_r^n (\cos k_n^b \ell + i \frac{k_n^b}{rvK} \sin k_n^b \ell) \right] = 0 \quad (4.12)$$

where $r = 1, 3, 5, \dots$. In the above equations all quantities are real except the Fourier coefficients d_r^n which are purely imaginary according to (2.35), (2.40), and (2.41). After substituting is_r^{n+} for d_r^n , y for $K\ell$, a_n for $k_n^a = k_n^a/K$ and b_n for $k_n^b = k_n^b/K$, and dividing by r^2 , the real and imaginary part of (4.12) can be written

$$\sum_{n=1,3}^{\infty} \left[(c_r^n \cos a_n y) X_n - \left(\frac{b_n}{rv} s_r^n \sin b_n y \right) Y_n \right] = 0 \quad (4.13)$$

and

$$\sum_{n=1,3}^{\infty} \left[\left(\frac{a_n}{rv} c_r^n \sin a_n y \right) X_n + (s_r^n \cos b_n y) Y_n \right] = 0 \quad (4.14)$$

With

$$c_r^n \cos a_n y = a_{rn} \quad - \frac{b_n}{rv} s_r^n \sin b_n y = a_{r+1,n} \quad (4.15a)$$

$$\frac{a_n}{rv} c_r^n \sin a_n y = a_{r,n+1} \quad s_r^n \cos b_n y = a_{r+1,n+1}$$

+ The s_r^n are the real Fourier coefficients of the integral Mathieu functions se_n , as defined in (2.50) and (2.51).

and

$$X_n = x_n$$

$$Y_n = x_{n+1}$$

(4.15b)

the Equations (4.13) and (4.14) can be written in matrix form as

$$\sum_{i,j} a_{ij} x_j = 0 \quad (4.16)$$

where $i = 1, 2, \dots, \infty$ results from $i = r$ or $i = r + 1$, and $j = 1, 2, \dots, \infty$ results from $j = n$ and $j = n + 1$. It can be shown that the infinite determinant a_{ij} can be made to converge by a proper normalization of the Fourier coefficients c_r^n and d_r^n . One should then expect to obtain a nontrivial solution for the unknowns x_j if

$$\det a_{ij} = 0 \quad (4.17)$$

After (4.17) is satisfied, the unknowns x_j can be determined within an arbitrary constant. If the solution is to be physically meaningful, the amplitudes of the waves in the slab, A_n and B_n , which are related to the unknowns x_j by (4.5a,b) and (4.15b), should decrease after the odd integer n exceeds a certain value. Also, the amplitudes in the external space, T_r , which can be computed by (4.10) or (4.11), should decrease after the odd integer r exceeds a certain value.

In practice, one would restrict oneself to a finite determinant a_{ij} of order N , where N would be an even number. One would assume a small p , find a_n and b_n either by the Method of False Position described in Appendix 2 or by the expansions (2.11) and (2.13), and determine the Fourier coefficients c_r^n and s_r^n from the recurrence relations

$$[(a_n)^2 - r^2]c_r^n - p(a_n)^2 (c_{r+2}^n + c_{r-2}^n) = 0 \quad (4.18)$$

and

$$[(b_n)^2 - r^2]s_r^n - p(b_n)^2 (s_{r+2}^n + s_{r-2}^n) = 0 \quad (4.19)$$

where $n, r = 1, 3, 5, \dots$. One would then evaluate the determinant a_{ij} , whose elements are defined by (4.15). One would determine the critical modulation index by increasing the value of p in small steps until the determinant vanishes for the first time. One may then compute the wave amplitudes A_n , B_n and T_r for $n = 1, 3, \dots, N - 1$, and $r = 1, 3, \dots, N - 1$. If the order of the determinant was chosen large enough, the wave amplitudes should become insignificantly small when n and r approach $N - 1$.

4.2 Exponentially Increasing Waves

Assuming now p large enough that exponentially increasing fields can be maintained, the rate of increase of the dominating wave and its harmonic content can be determined. If p is sufficiently large, there are again four mutually unrelated solutions: a symmetric and an antisymmetric field containing frequencies whose real part is Ω , 3Ω , 5Ω , . . . , as well as a symmetric and an antisymmetric field containing frequencies whose real part is 2Ω , 4Ω , 6Ω , For smaller p , any one, two, three, or all four solutions may not exist as an increasing wave. The following discussion treats only the symmetric field containing odd frequencies; it is assumed that p is sufficiently large so that this solution exists. The modulated space is again supposed to extend from $x = -\ell$ to $x = \ell$.

The external field is assumed to be of the form

$$E(t, x) = \sum_{r=0}^{\infty} T_{2r+1} e^{i(2r+1 - i\mu)\Omega(t - \frac{x - \ell}{v_1})} \quad x > \ell \quad (4.20)$$

which defines also the electric field for $x < -\ell$ because the field is to be symmetric. The internal field, inside the modulated space, may then contain only those Mathieu functions that also have the frequency spectrum $e^{i(2r+1 - i\mu)\Omega t}$, $r = . . . , -2, -1, 0, 1, 2, . . .$. As stated in Section 2.1, there are two types of Mathieu functions with the required frequency spectrum, namely the type $Ma(2n+1 - i\mu)\Omega t$ and $Mb(2n+1 - i\mu)\Omega t$ with real characteristic numbers and the type $Mi(2n + 1 - i\mu)\Omega t$ with complex characteristic numbers. The

stability chart (Figure 2.2), which applies to real κ , indicates that a given μ is attained at a fixed p only in a finite number of the unstable regions. Therefore, for a given μ , the functions $Ma(2n+1 - i\mu)\Omega t$ and $Mb(2n+1 - i\mu)\Omega t$ exist only up to a finite n , say $n = N$. For $N > n$, the given μ results from complex characteristic numbers and hence the Mathieu function is of the form $Mi(2n+1 - i\mu)\Omega t$. The interior field may therefore be written

$$\begin{aligned}
 D(t,x) = & \sum_{n=0}^N \left[Ma(2n+1 - i\mu)\Omega t (A_{2n+1} e^{ik_{2n+1}^a x} + B_{2n+1} e^{-ik_{2n+1}^a x}) \right. \\
 & \left. + Mb(2n+1 - i\mu)\Omega t (C_{2n+1} e^{ik_{2n+1}^b x} + D_{2n+1} e^{-ik_{2n+1}^b x}) \right] \\
 & + \sum_{n=N+1}^{\infty} \left[Mi(2n+1 - i\mu)\Omega t (E_{2n+1} e^{ik_{2n+1} x} + F_{2n+1} e^{-ik_{2n+1} x}) \right. \\
 & \left. + Mi^*(2n+1 - i\mu)\Omega t (G_{2n+1} e^{ik_{2n+1}^* x} + H_{2n+1} e^{-ik_{2n+1}^* x}) \right]
 \end{aligned} \tag{4.21}$$

where k_{2n+1} is complex for $n > N$. Replacing, for notational economy, $2n+1$ and $2r+1$ by n and r , where $n, r = 1, 3, 5, \dots$, the first summation may be written, with the terms in the form of (2.79) and (2.80), as

$$D_1(t,x) = \sum_{r=1,3}^{\infty} e^{i(r-i\mu)\Omega t} \sum_{n=1,3}^N \left[A_n c_r^n \cos(k_n^a x + \gamma_n) + B_n d_r^n \cos(k_n^b x + \delta_n) \right] \tag{4.22}$$

with new real constants A_n and B_n . Since the field is to be symmetric, γ_n and δ_n are zero. The corresponding electric field is then, according to (2.85) and (2.88),

$$E_1(t, x) = \sum_{r=1,3}^{\infty} (r - i\mu)^2 e^{i(r-i\mu)\Omega t} \sum_{n=1,3}^{\infty} \left[X_n c_r^n \cos k_n^a x + Y_n d_r^n \cos k_n^b x \right] \quad (4.23)$$

where $X_n = A_n / (\kappa_n^a)^2$ and $Y_n = B_n / (\kappa_n^b)^2$ are again real constants.

The second summation in (4.21) can be simplified by taking the complex conjugate of the terms involving G_n and H_n ; then

$$D_2(t, x) = \sum_{n=N+2}^{\infty} Mi(n - i\mu)\Omega t \left[(E_n + H_n^*) e^{ik_n x} + (F_n + G_n^*) e^{-ik_n x} \right] \quad (4.24)$$

where n takes on only odd values. Since the field is to be an even function, one must have $E_n + H_n^* = F_n + G_n^* = I_n$. Denoting the Fourier coefficients of $Mi(n - i\mu)\Omega t$ by g_r^n , one may write (4.24) in the form

$$D_2(t, x) = \sum_{n=N+2}^{\infty} \sum_r g_r^n e^{i(r-i\mu)\Omega t} I_n \cos k_n x \quad (4.25)$$

The corresponding electric field is, according to (2.88),

$$E_2(t, x) = \sum_r (r - i\mu)^2 e^{i(r-i\mu)\Omega t} \sum_{n=N+2}^{\infty} Z_n g_r^n \cos k_n x$$

where $Z_n = I_n / (\kappa_n)^2$ and the κ_n are complex. Dividing the summation over r into a part with positive r and a part with negative r and then taking the complex conjugates of all waves with negative r yields an expression compatible with (4.23), namely

$$E_2(t, x) = \sum_{r=1,3}^{\infty} (r - i\mu)^2 e^{i(r-i\mu)\Omega t} \sum_{n=N+2}^{\infty} Z_n g_r^n \cos \kappa_n x + Z_n^* g_{-r}^{n*} \cos \kappa_n^* x \quad (4.26)$$

The internal electric field is the sum of (4.23) and (4.26).

Equating the external and internal field and their derivatives at the boundary $x = \ell$, eliminating the unknown transmission coefficients T_r , and substituting y for $K\ell$, yields a set of homogeneous equations for the unknowns X , Y , Z with $r = 1, 3, 5, \dots$

$$\begin{aligned} & \sum_{n=1,3}^N \left[c_r^n (\cos \kappa_n^a y + \frac{i\kappa_n^a}{v(r-i\mu)} \sin \kappa_n^a y) X_n + d_r^n (\cos \kappa_n^b y + \frac{i\kappa_n^b}{v(r-i\mu)} \sin \kappa_n^b y) Y_n \right] \\ & + \sum_{n=N+2}^{\infty} \left[g_r^n (\cos \kappa_n y + \frac{i\kappa_n}{v(r-i\mu)} \sin \kappa_n y) Z_n \right. \\ & \left. + g_{-r}^{n*} (\cos \kappa_n^* y + \frac{i\kappa_n^*}{v(r-i\mu)} \sin \kappa_n^* y) Z_n^* \right] = 0 \end{aligned} \quad (4.27)$$

where X_n and Y_n are real, but Z_n is complex. Writing $Z_n = X_n + iY_n$, one can express (4.27) as

$$\sum_{n=1,3}^N A_r^n X_n + B_r^n Y_n + \sum_{n=N+2}^{\infty} C_r^n (X_n + iY_n) + D_r^n (X_n - iY_n) = 0 \quad (4.28)$$

or

$$\sum_{n=1}^N A_r^n X_n + B_r^n Y_n + \sum_{n=N+2}^{\infty} (C_r^n + D_r^n) X_n + i(C_r^n - D_r^n) Y_n = 0 \quad (4.29)$$

Letting

$$\begin{aligned} \operatorname{Re} A_r^n &= a_{rn} & \operatorname{Im} A_r^n &= a_{r+1,n} \\ \operatorname{Re} B_r^n &= a_{r,n+1} & \operatorname{Im} B_r^n &= a_{r+1,n+1} \end{aligned} \quad (4.30)$$

and also

$$\begin{aligned} \operatorname{Re} (C_r^n + D_r^n) &= a_{rn} & \operatorname{Im} (C_r^n + D_r^n) &= a_{r+1,n} \\ \operatorname{Im} (C_r^n - D_r^n) &= -a_{r,n+1} & \operatorname{Re} (C_r^n - D_r^n) &= a_{r+1,n+1} \end{aligned} \quad (4.31)$$

as well as

$$X_n = x_n \quad Y_n = x_{n+1} \quad (4.32)$$

one may write the continuity condition (4.27) in the form of a real matrix equation

$$\sum_{i,j=1}^{\infty} a_{ij} x_j = 0 \quad (4.33)$$

Convergence of $\det a_{ij}$ can be assured by a proper normalization of the

Fourier coefficients. One should obtain a nontrivial solution for the x_n with $x_n \rightarrow 0$ as $n \rightarrow \infty$ if

$$\det a_{ij} = 0 \quad (4.34)$$

In practice, one would approximate the determinant of (4.34) by one of finite order. Since the waves associated with the complex Mathieu functions have, according to (2.82) and (2.83), envelopes of the form $e^{\mu\Omega t - k_i x}$ and $e^{\mu\Omega t + k_i x}$, where k_i is the imaginary part of the wave number, continuity of the field at both ℓ and $-\ell$ cannot be satisfied by these waves alone. There must be at least one pair of Mathieu functions associated with real characteristic numbers κ_n^a and κ_n^b , and the largest μ that could possibly let the determinant (4.34) vanish is the maximum μ obtainable with real characteristic numbers κ for the given p , as shown by the stability chart. One would therefore start with the $\mu_{\max}$ just defined and determine κ_n^a , κ_n^b , and the complex κ_n by the Method of False Position described in Appendix 2. One would then compute the Fourier coefficients c_r^n , d_r^n , g_r^n , g_{-r}^n from the recurrence relation and evaluate the determinant. By stepwise decreasing μ until the determinant vanishes the first time, the exponent μ giving rise to the dominant increasing wave would be determined.

4.3 Transmission and Reflection Properties of a Permittivity-Modulated Slab

When $2p$ is smaller than the smallest of the critical modulation indices $2p_0$ for the four different modes mentioned after Equation (4.2),

waves in the modulated slab can exist only if excited by external means.

It is now assumed that a wave of a single frequency $\omega = (2m+1 + \beta')\Omega$,

$$E_i(t, x) = e^{i\omega(t - \frac{x}{v_1})} = e^{i(2m+1+\beta')\Omega(t - \frac{x}{v_1})} \quad x < 0 \quad (4.35)$$

is normally incident from the left on a permittivity-modulated slab extending from $x = 0$ to $x = L$. Within the slab all waves containing a spectral term at the frequency ω may be excited. Thus the interior field is written, as a generalization of (3.62), in the form

$$\begin{aligned} D(t, x) = & \sum_{n=0}^{\infty} \text{Me}(2n + 1 + \beta')\Omega t (A_{2n+1} e^{ik_{2n+1}^+ x} + B_{2n+1} e^{-ik_{2n+1}^+ x}) \\ & + \text{Me}[-(2n + 1 - \beta')\Omega t] (C_{2n+1} e^{ik_{2n+1}^- x} + D_{2n+1} e^{-ik_{2n+1}^- x}) \end{aligned} \quad (4.36)$$

where κ_{2n+1}^+ and κ_{2n+1}^- are the characteristic numbers for the functions $\text{Me}(2n + 1 + \beta')\Omega t$ and $\text{Me}(2n + 1 - \beta')\Omega t$. Using again $n = 1, 3, 5$ instead of $2n+1$ and $r = 1, 3, 5$ instead of $2r+1$, Equation (4.36) is expanded into

$$\begin{aligned} D(t, x) = & \sum_{n=1,3}^{\infty} \left\langle c_r^n e^{i(r+\beta')\Omega t} \left[A_n e^{ik_n^+ x} + B_n e^{-ik_n^+ x} \right] \right. \\ & \left. + d_r^n e^{-i(r-\beta')\Omega t} \left[C_n e^{ik_n^- x} + D_n e^{-ik_n^- x} \right] \right\rangle \end{aligned} \quad (4.37)$$

where the Fourier coefficients c_r^n and d_r^n are real quantities.

After changing in the second part of the sum r into $-r$, the corresponding electric field is, according to (2.88),

$$E(t, x) = \sum_r (r + \beta')^2 e^{i(r + \beta')\Omega t} \sum_{n=1,3}^{\infty} \left\langle c_r^n \left[W_n e^{ik_n^+ x} + X_n e^{-ik_n^+ x} \right] + d_{-r}^n \left[Y_n e^{ik_n^- x} + Z_n e^{-ik_n^- x} \right] \right\rangle \quad (4.38)$$

with the new complex constants $W_n = A_n / (\kappa_n^+)^2$, $X_n = B_n / (\kappa_n^+)^2$, $Y_n = C_n / (\kappa_n^-)^2$, and $Z_n = D_n / (\kappa_n^-)^2$. The incident wave is assumed, with odd integral m , as

$$E_i(t, x) = e^{i(m + \beta')\Omega(t - \frac{x}{v_1})} \quad x < 0 \quad (4.39)$$

Then the reflected and transmitted waves must be of the forms

$$E_r(t, x) = \sum_r R_r e^{i(r + \beta')\Omega(t - \frac{x}{v_1})} \quad x < 0 \quad (4.40)$$

and

$$E_t(t, x) = \sum_r T_r e^{i(r + \beta')\Omega(t - \frac{x-L}{v_1})} \quad x > L \quad (4.41)$$

The continuity conditions for the field and its space derivative are at $x = 0$

$$(r + \beta')^2 \sum_{n=1,3}^{\infty} \left[c_r^n (W_n + X_n) + d_{-r}^n (Y_n + Z_n) \right] = R_r + \delta_{rm} \quad (4.42)$$

$$i(r + \beta')^2 \sum_{n=1,3}^{\infty} \left[c_{r,n}^n k_n^+ (W_n - X_n) + d_{-r,n}^n k_n^- (Y_n - Z_n) \right] \\ = i(r + \beta') \frac{\Omega}{v_1} R_r - i(m + \beta') \frac{\Omega}{v_1} \delta_{rm} \quad (4.43)$$

and at $x = L$

$$(r + \beta')^2 \sum_{n=1,3}^{\infty} \left[c_{r,n}^n (W_n e^{ik_n^+ L} + X_n e^{-ik_n^+ L}) + d_{-r,n}^n (Y_n e^{ik_n^- L} + Z_n e^{-ik_n^- L}) \right] = T_r \quad (4.44)$$

$$i(r + \beta')^2 \sum_{n=1,3}^{\infty} \left[c_{r,n}^n k_n^+ (W_n e^{ik_n^+ L} - X_n e^{-ik_n^+ L}) + d_{-r,n}^n k_n^- (Y_n e^{ik_n^- L} - Z_n e^{-ik_n^- L}) \right] \\ = -i(r + \beta') \frac{\Omega}{v_1} T_r \quad (4.45)$$

for $r = \dots, -3, -1, 1, 3, \dots$. After dividing (4.43) and (4.45) by $i(r + \beta')\Omega t$ and substituting Kv for Ω/v_1 , κ for k/K , and y for KL , continuity of E and $\partial E/\partial x$ at $x = 0$ may be expressed by

$$(r + \beta')^2 \sum_{n=1,3}^{\infty} \left[c_{r,n}^n (W_n + X_n) + d_{-r,n}^n (Y_n + Z_n) \right] - R_r = \delta_{rm} \quad (4.46)$$

$$\frac{r + \beta'}{v} \sum_{n=1,3}^{\infty} \left[c_{r,n}^n k_n^+ (W_n - X_n) + d_{-r,n}^n k_n^- (Y_n - Z_n) \right] - R_r = -\delta_{rm} \quad (4.47)$$

Similarly, the continuity conditions for E and $\partial E'/\partial x$ at $x = L$ become

$$\begin{aligned}
(r + \beta')^2 \sum_{n=1,3}^{\infty} \left[c_r^n (w_n e^{i\kappa_n^+ y} + x_n e^{-i\kappa_n^+ y}) \right. \\
\left. + d_{-r}^n (y_n e^{i\kappa_n^- y} + z_n e^{-i\kappa_n^- y}) \right] - T_r = 0
\end{aligned} \tag{4.48}$$

$$\begin{aligned}
\frac{r + \beta'}{v} \sum_{n=1,3}^{\infty} \left[c_r^n (w_n e^{i\kappa_n^+ y} - x_n e^{-i\kappa_n^+ y}) \right. \\
\left. + d_{-r}^n (y_n e^{i\kappa_n^- y} - z_n e^{-i\kappa_n^- y}) \right] + T_r = 0
\end{aligned} \tag{4.49}$$

Since β' is given, the characteristic numbers κ_n^+ and κ_n^- may be readily computed by the expansion (2.11) or by the Method of False Position described in Appendix 2. The Fourier coefficients follow then from the recurrence relations. The set of equations may then be solved for the complex R_r , T_r , w_n , x_n , y_n , z_n for $r = \underline{+1}, \underline{+3}, \underline{+5}, \dots, \underline{+R}$ and $n = 1, 3, 5, \dots, R$ on a computer.

CHAPTER V

Use of Difference Equations in the Treatment
of Propagation in a Modulated Medium

An alternate approach to studying wave propagation in finite permittivity-varying media is the use of partial difference equations, which may be readily handled on digital computers. This method is applicable to arbitrary permittivity variations in time and permits also the specification of initial conditions. It is again assumed that a permittivity-modulated slab bounded by the planes $x = -\ell$ to $x = +\ell$ is surrounded by an infinite space of constant permittivity ϵ_1 . In order to relate the results to those previously obtained, the permittivity modulation is assumed to be the same as that used in the preceding chapters.

5.1 The Wave Equation in the Form of a Difference Equation

The wave equation in the exterior space of constant permittivity may be written as

$$v_1^2 \frac{\partial^2 D}{\partial x^2} = \frac{\partial^2 D}{\partial t^2} \quad |x| > \ell \quad (5.1)$$

involving the phase velocity $v_1 = (\mu\epsilon_1)^{-1/2}$. The wave equation in the permittivity-modulated slab (also referred to as the interior space) may be written in similar form, namely

$$g(t) \frac{\partial^2 D}{\partial x^2} = \frac{\partial^2 D}{\partial t^2} \quad |x| < \ell \quad (5.2)$$

which follows from (1.10) with*

$$g(t) = \frac{1}{\mu \epsilon(t)} = \frac{1 - 2p \cos 2\Omega t}{\mu \epsilon} = v^2 (1 - 2p \cos \Omega t) \quad (5.3)$$

where ϵ and hence also $V = (\mu \epsilon)^{-1/2}$ are constants, as defined at the beginning of Chapter 3.

The difference equation equivalent to the wave equation in the exterior space (5.1) is

$$v_1^2 \frac{D(t, x+h_x) - 2D(t, x) + D(t, x-h_x)}{h_x^2} = \frac{D(t+h_t, x) - 2D(t, x) + D(t-h_t, x)}{h_t^2} \quad (5.4)$$

where h_x and h_t are the increments in space and time. Solved for $D(t+h_t, x)$, the difference equation appears in the form

$$D(t+h_t, x) = \frac{v_1^2}{h_x^2/h_t^2} \left[D(t, x+h_x) - 2D(t, x) + D(t, x-h_x) \right] + 2D(t, x) - D(t-h_t, x) \quad (5.5)$$

Similarly, the difference equation equivalent to the wave equation in the interior space may be written

* Note that ϵ and $\epsilon(t)$ stand for different quantities.

$$D(t+h_t, x) = \frac{g(t)}{h_x^2/h_t^2} \left[D(t, x+h_x) - 2D(t, x) + D(t, x-h_x) \right] \\ + 2D(t, x) - D(t-h_t, x) \quad (5.6)$$

To solve a partial difference equation, one needs a fixed relation between the space increment h_x and the time increment h_t . As the difference equations indicate, $D(t+h_t, x)$ depends only on $D(t, x+h_x)$, $D(t, x)$, $D(t, x-h_x)$, and $D(t-h_t, x)$. An effect taking place at time t and location x can therefore be felt at time $t+h_t$ only at the locations $x+h_x$, x , and $x-h_x$. Thus, the difference equation permits the effect to propagate at most at the velocity h_x/h_t . It can be shown that the solution of the difference equation does not approach the solution to the corresponding wave equation if the quotient h_x/h_t is chosen smaller than the velocity at which a disturbance would be propagated.¹⁷ For the difference equation (5.5), one would clearly have to choose

$$\frac{h_x}{h_t} \geq v_1 \quad (5.7)$$

because a disturbance would be propagated in the exterior space at a speed equal to the phase velocity v_1 . In a permittivity-modulated slab, the propagation velocity is not a clearly defined quantity. Writing a wave in the form

$$D(t, x; k) = \sum_r c_{2r} e^{i[(2r+\beta)\Omega t - kx]} = A(t) e^{i[\phi(t) - kx]} \quad (5.8)$$

one may define an instantaneous phase velocity

$$V(t;k) = \frac{1}{k} \frac{d\phi}{dt} = \frac{1}{k} \frac{d}{dt} \left[\tan^{-1} \frac{\sum_r c_{2r} \sin(2r+\beta)\Omega t}{\sum_r c_{2r} \cos(2r+\beta)\Omega t} \right] \quad (5.9)$$

which is a periodic function if β is a rational fraction. If waves of different k are superimposed, the instantaneous phase velocity becomes also a function of position. Nevertheless, for small p , the instantaneous phase velocity should not differ too much from $V = (\mu\epsilon)^{-\frac{1}{2}}$, which is the value of the phase velocity when $p = 0$.

It has not been investigated whether the restriction for the choice of h_x/h_t would apply in the given case to the phase velocity, group velocity, or their time averages. Besides, computation of the group velocity from (5.9) would be a difficult matter. Rather, the quotient h_x/h_t is chosen appreciably larger than V , in order to assure that the instantaneous phase velocity does not exceed h_x/h_t .

For the example to be discussed here, it is assumed that $2p = .32$ and $\epsilon = 4\epsilon_1$, so that $v_1 = 2V$. One should then expect that the instantaneous phase velocity in the modulated slab does never exceed the phase velocity v_1 in the exterior space. Then a particularly convenient choice is

$$\frac{h_x}{h_t} = v_1 \quad (5.10)$$

because it reduces the difference equation for the exterior space (5.5) to

$$D(t+h_t, x) = D(t, x+h_x) + D(t, x-h_x) - D(t-h_t, x) \quad (5.11)$$

With the choice (5.10), the difference equation for the slab may be written

$$\begin{aligned} D(t+h_t, x) = f(t) [D(t, x+h_x) - 2D(t, x) + D(t, x-h_x)] \\ + 2D(t, x) - D(t-h_t, x) \end{aligned} \quad (5.12)$$

where

$$f(t) = \frac{g(t)}{v_1^2} = \frac{v^2}{v_1^2} (1 - 2p \cos 2\Omega t) = v^2 (1 - 2p \cos 2\Omega t) \quad (5.13)$$

In order to obtain a particular solution to a wave equation like (5.1) or (5.2), one may specify the initial conditions $D(0, x)$ and $\frac{\partial D}{\partial t}(0, x)$ for the range of x for which the wave equation is valid. Likewise, to obtain a particular solution to a partial difference equation as (5.11) or (5.12), one may specify as initial conditions the values $D(0, mh_x)$ and $D(h_t, mh_x)$, for all integers m to which the difference equation applies. The particular solution will then furnish the values of the displacement at $t = nh_t$, where $n = 2, 3, \dots$, and $x = mh_x$. With the notation $Z_n^m = D(nh_t, mh_x)$ for the electric displacement in the exterior space and $Y_n^m = D(nh_t, mh_x)$ for the electric displacement in the interior space, the difference equations (5.11) and (5.12) may be written

$$Z_{n+1}^m = Z_n^{m+1} + Z_n^{m-1} - Z_{n-1}^m \quad (5.14)$$

and

$$Y_{n+1}^m = f_n(Y_n^{m+1} - 2Y_n^m + Y_n^{m-1}) + 2Y_n^m - Y_{n-1}^m \quad (5.15)$$

with

$$f_n = f(nh_t) \quad (5.16)$$

where the superscripts refer to space, while the subscripts refer to time.

5.2 Joining the Solutions at the Boundaries $x = \pm \ell$

The modulated slab extending from $x = -\ell$ to $x = \ell$ is now divided into $2M$ layers of width h_x , implying $h_x = \ell/M$. Centered about $x = mh_x = 0$ or simply $m = 0$, the slab includes all points $x = mh_x$ with $|m| \leq M$ or simply all points $|m| \leq M$. The unmodulated, or exterior, space consists of all points $x = mh_x$ with $|m| > M$ or simply all points $|m| > M$. The boundaries $x = \pm Mh_x$ or $m = \pm M$ are considered to belong to both spaces. Thus, the values for the electric displacement in the interior space Y_n^m range over the points $|m| \leq M$, while the values for the electric displacement in the exterior space Z_n^m range over the points $|m| \geq M$. At the boundaries $m = \pm M$, the electric displacement values Y and Z must be interpreted as

$$Y_n^M = D(nh_t, l-) \quad Y_n^{-M} = D(nh_t, -l+) \quad (5.17)$$

$$Z_n^M = D(nh_t, l+) \quad Z_n^{-M} = D(nh_t, -l-) \quad (5.18)$$

where $l-$ stands for $l - \delta$, $\delta < h_x$. In the difference equations (5.14) and (5.15) the electric displacement values at $m = \underline{+M}$ are used on the right-hand side in order to compute Y_{n+1}^{M-1} , Y_{n+1}^{-M+1} , Z_{n+1}^{M+1} , and Z_{n+1}^{-M-1} , but the difference equations do not furnish the boundary values Y_{n+1}^M , Y_{n+1}^{-M} , Z_{n+1}^M , and Z_{n+1}^{-M} required for the next iteration. The latter values follow from the requirement that the electric field and its space derivative must be continuous at the boundaries $x = \underline{+l}$. Written in terms of the electric displacement, continuity of the electric field implies

$$D(t, l-) = \frac{\epsilon_1}{\epsilon(t)} D(t, l+) = f(t) D(t, l+) \quad (5.19)$$

and

$$D(t, -l+) = \frac{\epsilon_1}{\epsilon(t)} D(t, -l-) = f(t) D(t, -l-) \quad (5.20)$$

In terms of the discrete displacement values Y and Z , continuity of the electric field is expressed as

$$Y_n^M = f_n Z_n^M \quad Y_n^{-M} = f_n Z_n^{-M} \quad (5.21)$$

Continuity of the space derivative of the electric field requires

$$\frac{\partial D}{\partial x} (\ell^-, t) = f(t) \frac{\partial D}{\partial x} (\ell^+, t) \quad (5.22)$$

$$\frac{\partial D}{\partial x} (-\ell^+, t) = f(t) \frac{\partial D}{\partial x} (-\ell^-, t) \quad (5.23)$$

With the simplest approximation for the derivative, Equation (5.22) becomes

$$\frac{1}{h_x} (Y_n^M - Y_n^{M-1}) = f_n \frac{1}{h_x} (Z_n^{M+1} - Z_n^M) \quad (5.24)$$

leading by virtue of the first equation of (5.21) to

$$Y_n^M = \frac{1}{2} (f_n Z_n^{M+1} + Y_n^{M-1}) \quad (5.25)$$

Similarly, one obtains

$$Y_n^{-M} = \frac{1}{2} (Y_n^{-M} + f_n Z_n^{-M-1}) \quad (5.26)$$

The values Z_n^M and Z_n^{-M} follow from (5.21).

The principal error in the approximation for the derivative used in (5.24) results from $\frac{1}{h_x} (Y_n^M - Y_n^{M-1})$ approximating the derivative at $m = M - \frac{1}{2}$ rather than at $m = M$. In order to avoid this error, the following derivative approximation, which is centered about $m = M$, was used. Instead of (5.26) one may use

$$\frac{1}{2h_x} (Y_n^{M+1} - Y_n^{M-1}) = f_n \frac{1}{2h_x} (Z_n^{M+1} - Z_n^{M-1}) \quad (5.27)$$

where however Y_n^{M+1} and Z_n^{M-1} do not exist since they are outside their respective spaces. If there is justification for assuming that Y and Z are essentially of second order, so that their third-order differences become small, the nonexistent values can be eliminated from (5.27). Thus, supposing the forward third-order difference $\Delta^3 Z_n^{M-1}$ is zero, so that

$$\Delta^3 Z_n^{M-1} = Z_n^{M+2} - 3Z_n^{M+1} + 3Z_n^M - Z_n^{M-1} = 0 \quad (5.28)$$

and the backward third-order difference $\nabla^3 Y_n^{M+1}$ is zero, so that

$$\nabla^3 Y_n^{M+1} = Y_n^{M+1} - 3Y_n^M + 3Y_n^{M-1} - Y_n^{M-2} = 0 \quad (5.29)$$

one may subtract $f_n \Delta^3 Z_n^{M-1} / 2h_x$ from the right-hand side of (5.27) and $\nabla^3 Y_n^{M+1} / 2h_x$ from the left-hand side of (5.27) to obtain

$$3Y_n^M - 4Y_n^{M-1} + Y_n^{M-2} = f_n (-Z_n^{M+2} + 4Z_n^{M+1} - 3Z_n^M) \quad (5.30)$$

This leads with the help of the first equation of (5.21) to

$$Y_n^M = \frac{1}{6} (-f_n Z_n^{M+2} + 4f_n Z_n^{M+1} + 4Y_n^{M-1} - Y_n^{M-2}) \quad (5.31)$$

One obtains similarly

$$Y_n^{-M} = \frac{1}{6} (-Y_n^{-M+2} + 4Y_n^{-M+1} + 4f_n Z_n^{-M-1} - f_n Z_n^{-M-2}) \quad (5.32)$$

The values Z_n^M and Z_n^{-M} follow from (5.21).

5.3 Application of the Difference-Equation Method

The method discussed in the preceding two sections was applied to supplement the study of exponentially increasing waves in a permittivity-modulated slab, as treated in Sections 3.1 and 4.2. As already mentioned, the value ϵ was chosen equal to four times the external permittivity ϵ_1 , implying $2V = v_1$ or $v = \frac{1}{2}$. The electrical width $2K\ell$ was chosen as 6π , ~~or three~~ wavelengths λ of a wave of angular frequency Ω in a medium of permittivity ϵ . This width is the favorable condition for exponential increase of symmetric fields. The critical modulation index for a symmetric field in a slab of electric width $K\ell = 3\pi$ is, according to (3.59)

$$2p_0 = \frac{4 \tan^{-1} v}{K\ell} = \frac{4 \tan^{-1} \frac{1}{2}}{3\pi} = 0.196 \quad (5.33)$$

while the minimum modulation index for maintaining an antisymmetric field is, according to (3.56)

$$2p = \frac{4 \tan^{-1} \frac{1}{v}}{K\ell} = \frac{4 \tan^{-1} 2}{3\pi} = 0.473 \quad (5.34)$$

The modulation index was chosen as $2p = 0.32$, i.e., large enough to maintain an exponentially increasing even field, but too small to maintain an odd field.

The space increment was chosen to provide a resolution of twenty points per wavelength $\Lambda = 2\pi/K$ in the modulated slab. Since the width of the slab is equal to three wavelengths Λ , the slab contains 61 space points ranging from $m = -30$ to $m = 30$. Thus, the space increment is

$$h_x = \frac{\ell}{30} = \frac{\Lambda}{20} \quad (5.35)$$

The time increment is related to the space increment by (5.10); therefore

$$h_t = \frac{h_x}{v_1} = \frac{\Lambda}{20v_1} = \frac{2\pi/K}{20V/v} = \frac{v2\pi}{20\Omega} = \frac{vT}{20} = \frac{T}{40} \quad (5.36)$$

where T is the period corresponding to Ω . In terms of the period of the modulation frequency

$$h_t = \frac{T_{\text{mod}}}{20} \quad (5.37)$$

since the modulation frequency is equal to 2Ω .

Three different initial conditions were specified:

$$\text{Case 1: } Y_0^m = Z_0^m = 0 \quad (5.38)$$

$$Y_1^m = Z_1^m = 0 \quad m \neq 0, |m| \neq M \quad (5.39)$$

$$Y_1^0 = 1 \quad (5.40)$$

$$\text{Case 2: } Y_0^m = Z_0^m = 0 \quad (5.41)$$

$$Y_1^m = Z_1^m = 0 \quad |m| \neq 15, |m| \neq M \quad (5.42)$$

$$Y_1^{15} = -Y_1^{-15} = 1 \quad (5.43)$$

$$\text{Case 3: } Y_0^m = \sin 2\pi \frac{m}{20} \quad (5.44)$$

$$Z_0^m = 0 \quad (5.45)$$

$$Y_1^m = \sin 2\pi \frac{m-1}{20} \quad |m| \neq M \quad (5.46)$$

$$Z_1^m = 0 \quad |m| \neq M \quad (5.47)$$

The routine begins with computing the boundary values Y_1^M, Z_1^M, Y_1^{-M} , and Z_1^{-M} by (5.31), (5.32), and (5.21). Then the difference Equation (5.15) is employed to compute $Y_2^m, |m| < M$, and the difference Equation (5.14) to compute $Z_2^m, |m| > M$, whereafter the boundary values at M are computed as shown above. If no waves originate in the external space, the difference Equation (5.14) reduces to

$$Z_{n+1}^m = Z_n^{m-1} \quad m > M \quad (5.48)$$

$$Z_{n+1}^m = Z_n^{m+1} \quad m < -M \quad (5.49)$$

The initial condition of Cases 1 and 2 insure the excitation of an even and odd field, respectively. The initial condition of Case 3

represents a wave of the form $\sin(Kx - \Omega t)$; therefore, Case 3 is initially neither even nor odd. After a different initial development, both Cases 1 and 3 lead to the same symmetric field, which increases with time at a rate $\mu\Omega = 0.04\Omega$. Computation of μ by (3.58) yields $\mu = 0.036$. The field obtained for Case 2 vanishes, as should be anticipated.

The iteration was carried to approximately $n = 1000$ which corresponds to fifty periods of the modulation frequency. The electric field $E(nh_t, x)$ resulting from Case 1 is shown in Figure 5.1 for $n = 1045, 1050, 1053$, and 1055 . The permittivity variation and the values of the instantaneous permittivity at the times for which the field is shown in Figure 5.1 are given in Figure 5.2.

CURVE 1 : $n = 1045$
 CURVE 2 : $n = 1050$
 CURVE 3 : $n = 1053$
 CURVE 4 : $n = 1055$

▲ Figure 5.1

EXPONENTIALLY INCREASING ELECTRIC FIELD RESULTING FROM AN EXCITATION BY AN IMPULSE OCCURRING AT $t=0+$ ($n=1$) AND $x=0$ ($m=0$)

53RD PERIOD OF MODULATION FREQUENCY AFTER EXCITATION

◀ Figure 5.2

VALUES OF THE RECIPROCAL INSTANTANEOUS PERMITTIVITY $\epsilon^{-1}(t)$ AT THE TIMES t_1 , t_2 , t_3 , AND t_4 , FOR WHICH THE FIELDS WERE PLOTTED IN FIGURE 5.1

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APPENDIX I

Expansion of the Characteristic Number
as a Power Series

Considering Mathieu's equation in the form

$$y''(z) + (a - 2q \cos 2z)y(z) = 0 \quad (\text{A1.1})$$

where $a = \kappa^2$ and $q = \kappa^2 p$, one may expand the characteristic value a in the form

$$a = w^2 + \sum_{i=1}^{\infty} A_i(w) q^i \quad (\text{A1.2})$$

The coefficients $A_i(w)$ have been calculated up to at least $i = 6$ for the characteristic values a that yield as solutions of (A1.1) the integral Mathieu functions ce_0 to ce_6 and se_1 to se_6 .^{*} (Note that the letters a_m and b_m are customarily used for the *characteristic values* of ce_m and se_m , respectively.) In addition, the coefficients $A_i(w)$ have been computed in general up to $i = 6$ for the characteristic values a yielding as solutions of (A1.1) the fractional Mathieu functions of the type Me .[†] In the notation used in this work, the expansion (A1.2) becomes, after replacing a by κ^2 and q by $\kappa^2 p$,

$$\kappa^2 = w^2 + \sum_{i=1}^{\infty} A_i(w) [\kappa^2 p]^i \quad (\text{A1.3})$$

^{*} McLachlan, ref. cit.,¹¹ p. 16.

[†] McLachlan, ref. cit.,¹¹ p. 20.

By inserting the above expansion for κ on the right-hand side of (A1.3), raising the bracket to the i^{th} power, and rearranging the terms as to the power of p , one may rewrite (A1.3) in the form

$$\kappa^2 = w^2 + \sum_{i=1}^{\infty} B_i(w)p^i \quad (\text{A1.4})$$

The coefficients B_i , computed up to $i = 6$, are

$$\begin{aligned} B_1 &= A_1 w^2 \\ B_2 &= A_1^2 w^2 + A_2 w^4 \\ B_3 &= A_1^3 w^2 + 3A_1 A_2 w^4 + A_3 w^6 \\ B_4 &= A_1^4 w^2 + 6A_1^2 A_2 w^4 + (4A_1 A_3 + 2A_2^2) w^6 + A_4 w^8 \\ B_5 &= A_1^5 w^2 + 10A_1^3 A_2 w^4 + (10A_1^2 A_3 + 10A_1 A_2^2) w^6 \\ &\quad + (5A_1 A_4 + 5A_2 A_3) w^8 + A_5 w^{10} \\ B_6 &= A_1^6 w^2 + 15A_1^4 A_2 w^4 + (20A_1^3 A_3 + 30A_1^2 A_2^2) w^6 \\ &\quad + (15A_1^2 A_4 + 30A_1 A_2 A_3 + 5A_2^3) w^8 \\ &\quad + (6A_1 A_5 + 6A_2 A_4 + 3A_3^2) w^{10} + A_6 w^{12} \end{aligned} \quad (\text{A1.5})$$

Since all coefficients are divisible by w^2 , one may write (A1.4) in the form

$$\kappa^2 = w^2 \left[1 + \sum_{i=1}^{\infty} C_i(w) p^i \right] \quad (\text{A1.6})$$

where $C_i = B_i/w^2$.

In order to obtain an expansion for the characteristic number κ , the root of the bracket in (A1.6) may be taken in form of a Taylor series. Letting

$$Y(p) = 1 + \sum_{i=1}^{\infty} C_i(w) p^i \quad (\text{A1.7})$$

the expansion for the characteristic number becomes

$$\kappa = w \left[Y^{\frac{1}{2}}(0) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j Y^{\frac{1}{2}}}{dp^j}(0) p^j \right] \quad (\text{A1.8})$$

or

$$\kappa = w \left[1 + \sum_{j=1}^{\infty} \alpha_j(w) p^j \right] \quad (\text{A1.9})$$

Computed up to $j = 6$, the coefficients α_j in terms of the original coefficients A_i are

$$\alpha_1 = \frac{1}{2} A_1$$

$$\alpha_2 = \frac{3}{8} A_1^2 + \frac{1}{2} A_2 w^2$$

$$\alpha_3 = \frac{5}{16} A_1^3 + \frac{5}{4} A_1 A_2 w^2 + \frac{1}{2} A_3 w^4$$

$$\alpha_4 = \frac{35}{128} A_1^4 + \frac{35}{16} A_1^2 A_2 w^2 + \frac{7}{8} (2A_1 A_3 + A_2^2) w^4 + \frac{1}{2} A_4 w^6$$

$$\alpha_5 = \frac{63}{256} A_1^5 + \frac{105}{32} A_1^3 A_2 w^2 + \frac{63}{16} (A_1^2 A_3 + A_1 A_2^2) w^4$$

$$+ \frac{9}{4} (A_1 A_4 + A_2 A_3) w^6 + \frac{1}{2} A_5 w^8$$

$$\alpha_6 = \frac{231}{1024} A_1^6 + \frac{1155}{256} A_1^4 A_2 w^2 + \frac{231}{64} (3A_1^2 A_2^2 + 2A_1^3 A_3) w^4$$

$$+ \frac{33}{16} (3A_1^2 A_4 + 6A_1 A_2 A_3 + A_2^3) w^6$$

$$+ \frac{11}{8} (2A_2 A_4 + 2A_1 A_5 + A_3^2) w^8 + \frac{1}{2} A_6 w^{10} \quad (A1.10)$$

APPENDIX II

General Relation Between the Characteristic Exponent ν
and the Characteristic Number κ

Insertion of a solution in the form of Floquet

$$y(z; \kappa, p) = \sum_r c_{2r} e^{i(2r+\nu)z} \quad (\text{A2.1})$$

into Mathieu's equation

$$y''(z) + \kappa^2 (1 - 2p \cos 2z) y(z) = 0 \quad (\text{A2.2})$$

yields the recurrence relation

$$-\frac{p\kappa^2}{\kappa^2 - (2r+\nu)^2} c_{2r-2} + c_{2r} - \frac{p\kappa^2}{\kappa^2 - (2r+\nu)^2} c_{2r+2} = 0 \quad (\text{A2.3})$$

for the Fourier coefficients c_{2r} in (A2.1). This can be written

$$d_r c_{2r-2} + c_{2r} + d_r c_{2r+2} = 0 \quad (\text{A2.4})$$

with

$$d_r = -\frac{p\kappa^2}{\kappa^2 - (2r+\nu)^2} \quad (\text{A2.5})$$

The infinite set of equations expressed by (A2.4) may be written in matrix form as

$$\begin{bmatrix} \dots & \dots & \dots & & & & \\ & d_{-1} & 1 & d_{-1} & & & \\ & & d_0 & 1 & d_0 & & \\ & & & d_1 & 1 & d_1 & \\ & & & & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \dots \\ c_{-2} \\ c_0 \\ c_2 \\ \dots \end{bmatrix} = 0 \quad (\text{A2.6})$$

In order to obtain a nontrivial solution for the Fourier coefficients c_{2r} , which represents a physically meaningful Fourier expansion, the infinite determinant

$$\Delta = \begin{vmatrix} \dots & \dots & \dots & & & & \\ & d_{-2} & 1 & d_{-2} & & & \\ & & d_{-1} & 1 & d_{-1} & & \\ & & & d_0 & 1 & d_0 & \\ & & & & d_1 & 1 & d_1 \\ & & & & & d_2 & 1 & d_2 \\ & & & & & & \dots & \dots & \dots \end{vmatrix} \quad (\text{A2.7})$$

must vanish. A determinant of infinite order, also referred to as a *multilinearant*, has a value only if the sequence of the finite determinants of increasing order with a common central element has a limit. Then the value of the infinite determinant is the limit of the values of finite determinants of increasing order. Considering a sequence of determinants of order 1, 3, 5, . . ., one may thus write

$$\Delta = \lim_{r \rightarrow \infty} \Delta_{2r+1} \quad (\text{A2.8})$$

where Δ_{2r+1} is the determinant of order $2r+1$. If the rows and columns of the infinite determinant are numbered $\dots -2, -1, 0, 1, 2, \dots$ such that row 0 and column 0 intersect in its center, the finite determinants Δ_{2r+1} may be said to consist of the rows and columns $-r$ to r of Δ . For example, one would have for $r = 0$

$$\Delta_1 = 1 \quad (\text{A2.9})$$

and for $r = 1$

$$\Delta_3 = \begin{vmatrix} 1 & d_{-1} & 0 \\ d_0 & 1 & d_0 \\ 0 & d_1 & 1 \end{vmatrix} \quad (\text{A2.10})$$

The sequence Δ_{2r+1} is said to converge, if for every positive quantity δ there corresponds a positive integer R such that

$$|\Delta_{2r+2s+1} - \Delta_{2r+1}| < \delta \quad (\text{A2.11})$$

for all integers r greater than R and for all positive integers s .

This condition has been shown to be fulfilled if the product of the diagonal terms and the sum of the nondiagonal terms of the sequence Δ_{2r+1} are absolutely convergent.¹⁸⁻²¹ It has also been shown that, if Δ converges, it is immaterial which diagonal element of Δ is taken as the central element.¹⁹ The product of the diagonal elements of any Δ_{2r+1} is clearly unity. Since for sufficiently large $|R|$

$$d_R = \frac{p\kappa^2}{(2R+\nu)^2 - \kappa^2} \rightarrow \frac{p\kappa^2}{4R^2} \quad (A2.12)$$

the sum of the nondiagonal elements d_r converges provided $\kappa \neq \pm(2n+\nu)$, where $n = \dots, -2, -1, 0, 1, 2, \dots$. Therefore $\lim_{r \rightarrow \infty} \Delta_{2r+1}$ exists except when $\kappa = \pm(2n+\nu)$.

Since the determinants Δ_{2r+1} are functions of ν, κ , and p , the condition

$$\Delta(\nu; \kappa, p) = \lim_{r \rightarrow \infty} \Delta_{2r+1}(\nu; \kappa, p) = 0 \quad (A2.13)$$

connects the characteristic exponent ν with the characteristic number κ and the parameter p . Viewed as a function of the complex variable ν , the infinite determinant has the following properties:* The only singularities of $\Delta(\nu)$ are simple poles, which occur when $\kappa = \pm(2n+\nu)$. $\Delta(\nu)$ is an even function of ν since a change from ν into $-\nu$ changes d_r into d_{-r} and thus only inverts the order of the rows and columns in (A2.7). $\Delta(\nu)$ is periodic in ν with period 2 since a replacement of ν by $\nu+2n$ transforms d_r into d_{r+n} , which merely changes the indices of rows and columns from r to $r+n$. This implies also that the residues at all poles are equal. Because $\Delta(\nu)$ is an even function and is periodic, its behavior is deducible from that in the infinite strip

* McLachlan, ref. cit.,¹¹ p. 68

$0 \leq \text{Re}(\nu) \leq 1$ of the ν plane. If $\text{Im}(\nu)$ approaches infinity, the non-diagonal elements d_r approach zero. Therefore

$$\lim_{\text{Im}(\nu) \rightarrow \infty} \Delta(\nu) = 1 \quad (\text{A2.14})$$

The function

$$\chi(\nu) = \frac{1}{\cos \nu\pi - \cos \kappa\pi} \quad (\text{A2.15})$$

is also even and periodic in ν with period 2 and has also simple poles at $\kappa = \pm(2n+\nu)$. When $\text{Im}(\nu)$ approaches infinity, its limit is

$$\lim_{\text{Im}(\nu) \rightarrow \infty} \chi(\nu) = 0 \quad (\text{A2.16})$$

It follows then that the function

$$\Xi(\nu) = \Delta(\nu) - C\chi(\nu) \quad (\text{A2.17})$$

will have no poles if C is suitably chosen, so by Liouville's theorem Ξ must be a constant. To determine Ξ , Equation (A2.17) is evaluated with $\text{Im}(\nu)$ approaching infinity. By virtue of the limits (A2.14) and (A2.16) it follows that $\Xi = 1$. The constant C may then be determined by evaluating (A2.17) at $\nu = 0$, which yields

$$C = \frac{\Delta(0) - \Xi}{\chi(0)} = [\Delta(0) - 1] [1 - \cos \kappa\pi] \quad (\text{A2.18})$$

The condition $\Delta(v) = 0$, which provides the connection between v , κ , and p , can then be expressed with the help of (A2.17) as

$$\frac{\Xi}{\chi(v)} = -C \quad (\text{A2.19})$$

or

$$\cos v\pi - \cos \kappa\pi = -[\Delta(0) - 1] [1 - \cos \kappa\pi] \quad (\text{A2.20})$$

This equation may be rewritten as

$$1 - \cos v\pi = \Delta(0; \kappa, p) [1 - \cos \kappa\pi] \quad (\text{A2.21})$$

or

$$\sin^2 v\pi/2 = \Delta(0; \kappa, p) \sin^2 \kappa\pi/2 \quad (\text{A2.22})$$

Above equation is indeterminate for $\kappa = 2n$ because $\Delta(0; \kappa, p)$ has poles there, while $\sin^2 \kappa\pi/2$ becomes zero. For all other values of κ , the above equation is suitable for determining v for a given pair (κ, p) provided an effective scheme for evaluating the infinite determinant is available. For $v = 0$, the elements d_r and d_{-r} become equal, so

that the infinite determinant becomes centrosymmetric, namely

$$\Delta(0) = \begin{vmatrix} \dots & \dots & \dots & & & & & & & \\ & d_2 & 1 & d_2 & & & & & & \\ & & d_1 & 1 & d_1 & & & & & \\ - & - & - & d_0 & 1 & d_0 & - & - & - & \\ & & & & d_1 & 1 & d_1 & & & \\ & & & & & d_2 & 1 & d_2 & & \\ & & & & & & \dots & \dots & \dots & \end{vmatrix} \quad (\text{A2.23})$$

Centrosymmetric determinants can be expressed as a product of two subdeterminants.* Adding column -1 to column 1, column -2 to column 2, etc. yields

$$\Delta(0) = \begin{vmatrix} \dots & \dots & \dots & & & & \dots & \dots & \dots \\ & d_2 & 1 & d_2 & & & d_2 & 1 & d_2 \\ & & d_1 & 1 & d_1 & & d_1 & & \\ - & - & - & d_0 & 1 & 2d_0 & - & - & - \\ & & & & d_1 & 1 & d_1 & & \\ & & & & & d_2 & 1 & d_2 & \\ & & & & & & \dots & \dots & \dots \end{vmatrix} \quad (\text{A2.24})$$

* Muir, ref. cit.,²¹ p. 365.

Then subtracting row 1 from row-1, row 2 from row-2 etc., leads to

$$\Delta(0) = \left| \begin{array}{cccc|cccc} \dots & \dots & \dots & & & & & \\ & d_2 & 1 & d_2 & & & & \\ & & d_1 & 1 & & & & \\ \hline - & - & - & d_0 & 1 & 2d_0 & - & - & - \\ & & & & d_1 & 1 & d_1 & & \\ & & & & & d_2 & 1 & d_2 & \\ & & & & & & \dots & \dots & \dots \end{array} \right| \quad (A2.25)$$

By Laplace's Theorem, since all elements in the upper right-hand corner are zero, the above determinant can be expressed as the product of the subdeterminants in the upper left-hand corner and the lower right-hand corner. By referring to the special recurrence relations for the Fourier coefficients of the integral Mathieu functions* it is seen that the upper subdeterminant contains the elements of the matrix of the recurrence relation for the Fourier coefficients of the odd integral Mathieu functions se_{2n} . Likewise the lower subdeterminant is seen to contain the matrix elements of the recurrence relation for the even Mathieu functions ce_{2n} . Conversely, it has been shown by Schäfer, that the characteristic exponent ν can be conveniently determined from the recurrence relations for the integral Mathieu

* McLachlan, ref. cit.,¹¹ p. 28. McLachlan uses $A_0/2$ and A_{2r} instead of c_{2r} for the Fourier coefficients of ce_{2n} , and B_{2r} instead of d_{2r} for the Fourier coefficients of se_{2n} .

functions.²² While Schäfke makes no reference to the infinite determinant $\Delta(0)$, his recurrence formulas are suitable expansions of the subdeterminants. In the form (A2.25), the finite determinant Δ_{2r+1} , which is of order $2r+1$, is the product of an upper subdeterminant of order r , called β_r , and a lower subdeterminant of order $r+1$, called γ_{r+1} . Thus

$$\Delta_{2r+1}(0) = \beta_r \gamma_{r+1} \quad (\text{A2.26})$$

and therefore

$$\Delta(0; \kappa, p) = \lim_{r \rightarrow \infty} \beta_r \gamma_{r+1} \quad (\text{A2.27})$$

For example, for $r = 4$

$$\beta_4 = \begin{vmatrix} 1 & d_4 & & & \\ d_3 & 1 & d_3 & & \\ & d_2 & 1 & d_2 & \\ & & d_1 & 1 & \end{vmatrix} \quad (\text{A2.28})$$

Expanding β_4 by the elements of its upper row shows that

$$\beta_4 = \begin{vmatrix} 1 & d_3 & & \\ d_2 & 1 & d_2 & \\ & d_1 & 1 & \end{vmatrix} - d_4 d_3 \begin{vmatrix} 1 & d_2 \\ d_1 & 1 \end{vmatrix} \quad (\text{A2.29})$$

so that

$$\beta_4 = \beta_3 - d_4 d_3 \beta_2 \quad (\text{A2.30})$$

or in general for $r > 1$

$$\beta_{r+1} = \beta_r - d_{r+1} d_r \beta_{r-1} \quad (\text{A2.31})$$

By expanding γ_{r+1} for $r > 1$ one finds that also

$$\gamma_{r+1} = \gamma_r - d_{r+1} d_r \gamma_{r-1} \quad (\text{A2.32})$$

The initial conditions for the recurrence relations (A2.31) and (A2.32) are found by considering Δ_{2r+1} for $r = 0$ and $r = 1$. For $r = 0$ one has

$$\Delta_1(0) = \beta_0 \gamma_1 = 1 \quad (\text{A2.33})$$

and since $\gamma_1 = 1$, one needs to choose $\beta_0 = 1$ (where β_0 has no meaning as a subdeterminant). For $r = 1$ one has

$$\Delta_3(0) = \beta_1 \gamma_2 = \begin{vmatrix} 1 & & \\ & 1 & 2d_0 \\ & d_2 & 1 \end{vmatrix} \quad (\text{A2.34})$$

where obviously $\beta_1 = 1$ and $\gamma_2 = 1 - 2d_1 d_0$. In order to be able to compute γ_2 from the recurrence relation (A2.32) with $\gamma_1 = 1$, one

suitable chooses $\gamma_0 = 2$. With the initial conditions

$$\gamma_0 = 2 \quad (A2.35)$$

$$\beta_0 = \beta_1 = \gamma_1 = 1 \quad (A2.36)$$

the recurrence relations (A2.31) and (A2.32) are valid for all positive r . With the help of (A2.5), where $v = 0$, the recurrence relations are

$$\beta_{r+1} = \beta_r - \frac{\kappa^4 p^2}{[\kappa^2 - (2r+2)^2][\kappa^2 - (2r)^2]} \beta_{r-1} \quad (A2.37)$$

and

$$\gamma_{r+1} = \gamma_r - \frac{\kappa^4 p^2}{[\kappa^2 - (2r+2)^2][\kappa^2 - (2r)^2]} \gamma_{r-1} \quad (A2.38)$$

for $r = 1, 2, \dots$. For the case $\kappa = 2n$, which has been excluded up to this point, the recurrence relations may be suitably modified to eliminate the term

$$\frac{\sin^2 \kappa\pi/2}{[\kappa^2 - (2n)^2]}$$

which makes (A2.22) indeterminate.²³

With $\Delta(0; \kappa, p)$ computable by (A2.27), the relation between v and κ given by (A2.22) becomes

$$\sin^2 v\pi/2 = \lim_{r \rightarrow \infty} \beta_r \gamma_{r+1} \sin^2 \kappa\pi/2 \quad (\text{A2.39})$$

or

$$\begin{aligned} v &= \pm \frac{2}{\pi} \sin^{-1} \left(\sqrt{\lim_{r \rightarrow \infty} \beta_r \gamma_{r+1}} \sin \kappa\pi/2 \right) \\ &= \pm \frac{2}{\pi} \sin^{-1} z(\kappa, p) \end{aligned} \quad (\text{A2.40})$$

where, for real κ , z is either real or imaginary depending on the sign of $\lim_{r \rightarrow \infty} \beta_r \gamma_{r+1}$, while for complex κ , z is generally complex. Assuming the general case of complex z , it follows²⁴ that

$$v = \pm \frac{2}{\pi} [-i \log(iz + \sqrt{1-z^2})] \quad (\text{A2.41})$$

Letting $iz + \sqrt{1-z^2} = R e^{i\phi}$, one obtains

$$v = \pm \frac{2}{\pi} [-i \log R + \phi \pm n\pi] = \pm \left[\frac{2\phi}{\pi} - \frac{i}{\pi} \log R^2 \right] \pm 2n \quad (\text{A2.42})$$

The method described above was programmed for the preparation of Figures 2.1, 2.2, and 2.3. For the range of values covered in the figures at least five-place accuracy was obtained with three to six iterations.

In order to determine κ for a given pair of values (v, p) , one needs to determine the root of the equation

$$F(\kappa) = \lim_{r \rightarrow \infty} \beta_r \gamma_{r+1} \sin^2 \kappa\pi/2 - \sin^2 v\pi/2 = 0 \quad (\text{A2.43})$$

In most cases, this can be conveniently accomplished on a computer by the Method of False Position (Regula Falsi). After selecting two approximate values κ_0 and κ_1 for the root of (A2.43) from the stability chart (Figure 2.2) or a chart similar to Figure 2.3, one computes

$$\kappa_2 = \kappa_1 - \frac{\kappa_1 - \kappa_0}{F(\kappa_1) - F(\kappa_0)} F(\kappa_1) \quad (\text{A2.44})$$

evaluates $F(\kappa_2)$, recomputes

$$\kappa_3 = \kappa_2 - \frac{\kappa_2 - \kappa_0}{F(\kappa_2) - F(\kappa_0)} F(\kappa_2) \quad (\text{A2.45})$$

and continues in this way until the correction becomes sufficiently small. Some care must be taken in selecting κ_0 and κ_1 in order to assure that the recurrence formula converges.

APPENDIX III

Relation Between ν and κ for Small p in the
Vicinity of $\kappa = 1$

Using the alternate form of the Floquet solutions, as given by
(2.43),

$$y = \sum_r c_{2r+1} e^{i(2r+1 + \nu')z} \quad (\text{A3.1})$$

the recurrence relations for the Fourier coefficients may be written
in a form similar to (A2.4), namely

$$d'_r c_{2r-1} + c_{2r+1} + d'_r c_{2r+3} = 0 \quad (\text{A3.2})$$

where

$$d'_r = - \frac{p\kappa^2}{\kappa^2 - (2r + 1 + \nu')^2} \quad (\text{A3.3})$$

The set of Equations (A3.2) corresponding to various integral values
of r can be written in matrix form as

$$\begin{bmatrix}
 \dots & \dots & \dots & & & \\
 & d'_{-2} & 1 & d'_{-2} & & \\
 & & d'_{-1} & 1 & d'_{-1} & \\
 & & & d'_0 & 1 & d'_0 \\
 & & & & d'_1 & 1 & d'_1 \\
 & & & & & \dots & \dots & \dots
 \end{bmatrix}
 \begin{bmatrix}
 \dots \\
 c_{-3} \\
 c_{-1} \\
 c_1 \\
 c_3 \\
 \dots
 \end{bmatrix}
 = 0 \quad (A3.4)$$

For $p \ll 1$ and $\kappa = 1 + \zeta$ with $|\zeta| \ll 1$ since $\kappa \approx 1$, the only significant elements of the matrix are d'_{-1} and d'_0 . As Figure 2.1 indicates, the normalized frequency w is close to unity, hence v' is close to zero [see (2.22) for $n = 0$], and the only significant Fourier coefficients are c_{-1} and c_1 . Then an approximate solution to the matrix equation is

$$\begin{bmatrix}
 1 & d'_{-1} \\
 d'_0 & 1
 \end{bmatrix}
 \begin{bmatrix}
 c_{-1} \\
 c_1
 \end{bmatrix}
 = 0 \quad (A3.5)$$

The matrix elements may be approximated by

$$d'_0 = - \frac{p(1 + \zeta)^2}{(1 + \zeta)^2 - (1 + v')^2} \approx - \frac{p(1 + 2\zeta)}{(1 + 2\zeta) - (1 + 2v')} \approx - \frac{p/2}{\zeta - v'} \quad (A3.6)$$

$$d'_{-1} = - \frac{p(1 + \zeta)^2}{(1 + \zeta)^2 - (-1 + v')^2} \approx - \frac{p(1 + 2\zeta)}{(1 + 2\zeta) - (1 - 2v')} \approx - \frac{p/2}{\zeta + v'} \quad (\text{A3.7})$$

For a nontrivial solution, the determinant of (A3.5) must vanish, hence,

$d'_{-1} d'_0 = 1$, or from (A3.6) and (A3.7)

$$(\zeta - v')(\zeta + v') = \zeta^2 - v'^2 = \left(\frac{p}{2}\right)^2 \quad (\text{A3.8})$$

If κ , and hence ζ , is real, so is v'^2 . Then v' is either real or pure imaginary. In the latter case with $v' = -i\mu$, Equation (A3.8) becomes

$$\zeta^2 + \mu^2 = \left(\frac{p}{2}\right)^2 \quad (\text{A3.9})$$

Then the characteristic point is located in the first unstable region between b_1 and a_1 . In the first case with $v' = \beta'$, Equation (A3.8) becomes

$$\zeta^2 - \beta'^2 = \left(\frac{p}{2}\right)^2 \quad (\text{A3.10})$$

The characteristic point is then located in one of the stable regions adjacent to the first unstable region. For $v' = 0$, one has

$$\zeta = \pm \frac{p}{2} \quad (\text{A3.11})$$

which determines the curves for a_1 and b_1 for small p . For $\zeta = 0$,

that is $\kappa = 1$,

$$\mu = \frac{p}{2} \quad (\text{A3.12})$$

Values of $\mu > p/2$ can be attained only with complex values for ζ . Contour curves for constant β' and μ are shown in Figure A3.1. For small p and in the vicinity of $\kappa = 1$, the curves match those of the stability chart, Figure 2.2.

Having ascertained that the 2×2 determinant vanishes, the relation between the Fourier coefficients follows from either one of the two equations represented by (A3.5). Choosing the second equation yields

$$\frac{c_{-1}}{c_1} = \frac{1}{d_0'} = \frac{\zeta - v'}{p/2} \quad (\text{A3.13})$$

In the first case with $v' = \beta'$

$$\frac{c_{-1}}{c_1} = \frac{\zeta - \beta'}{p/2} \quad (\text{A3.14})$$

while in the second case with $v' = -i\mu$

$$\frac{c_{-1}}{c_1} = \frac{\zeta + i\mu}{p/2} \quad (\text{A3.15})$$

Figure A3.1 CONTOUR CURVES OF CONSTANT $\nu = \beta' - i\mu$
FOR SMALL p IN THE VICINITY OF THE FIRST
UNSTABLE REGION

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13. ABSTRACT This paper presents a study of the propagation of electromagnetic plane waves in a nonconducting, charge-free medium with a periodically time-varying permittivity given by $\epsilon(t) = \epsilon/(1 - 2p\cos 2\Omega t)$. The permittivity is considered to be "modulated" by an external agency, detailed consideration of which is outside the scope of this work. With the assumed permittivity, the wave equation for the electric displacement can be separated into Mathieu's equation for the time dependence and Helmholtz equation for the space dependence of the field. The separation constant $-k^2$ defines the wave number k . Plane waves of a single wave number can be written as products of Mathieu functions and exponentials $\exp(+ikx)$, the properties of the Mathieu functions depending on the choice of k . After a brief discussion of means to realize the assumed permittivity modulation, the properties and computational determination of Mathieu functions are reviewed. Written in the form of Floquet, the Mathieu functions are combined with the associated spatial factors $\exp(ikx)$ to form expressions for single- k plane waves. Depending on the choice of k , these waves may have envelopes that vary periodically or exponentially in time. After showing that in an infinite space of periodically time-varying permittivity waves with amplitudes increasing exponentially in time can be maintained, it is investigated whether such waves may also occur in a permittivity-modulated slab			

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oriented normal to the propagation vector. The existence of increasing waves is in the latter case restricted by the fact that the electric and magnetic field must be continuous at the boundaries of the slab. This requirement is shown to be met by a superposition of single- k waves increasing at a common rate, provided the index of the permittivity modulation $2p$ exceeds a critical value $2p_0$. This value depends on the slab width and on the relation of the permittivities within and outside the slab. A slab with a permittivity-modulation index greater than $2p_0$ is considered an unstable system for electromagnetic wave propagation since certain waves, once excited, grow exponentially with time. On the other hand, if the modulation index is less than $2p_0$, the slab is a stable system unable to support such exponentially increasing waves. In the latter case, if a plane wave of frequency ω illuminates the slab, plane waves having a frequency spectrum given by $\omega + 2n\Omega$ ($n = 0, 1, 2, \dots$) are reflected and transmitted from the slab. Here 2Ω is the modulation frequency. The amplitudes of the various waves depend on the quotient ω/Ω , the modulation index $2p$, and the electrical width of a slab. Such dielectric slabs may be designed to act as tuned amplifiers and frequency converters.

The critical modulation index $2p_0$ is found in closed form for the case $2p_0 \ll 1$. If the modulation index is greater than $2p_0$, the rate of increase of the field is also given in closed form. For the case of a modulation index less than $2p_0$, the reflection and transmission properties of a slab for an incident wave of a frequency near half the modulation frequency are determined. For the general case of larger modulation indices, the equivalent results are obtained using iterative procedures.

An independent approach to problems involving a permittivity-modulated slab is made by employing partial difference equations. This method, applied to exponentially increasing waves in a dielectric slab, yielded results in good agreement with those obtained by using Mathieu functions. This method is not limited to the particular permittivity variation assumed above, which gives rise to Mathieu's equation.