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ON THE ONE-BODY MODEL OF ALPHA RADIOACTIVITY

III. NON-S-STATE DECAY, PREFORMATION FACTORS,
MODEL VARIATIONS, GENERAL DISCUSSION

by

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ABSTRACT

It is shown by direct numerical comparison that the WKB wave functions, when the variables are in the range appropriate to alpha decay, agree with the corresponding Coulomb functions to about 0.1 % if $L(L + 1)$ is replaced in the former by $(L + 1/2)^2$. The correction to the potential energy for screening by the atomic electrons is discussed. The traditional one-body model and a many-body model are reviewed with the aid of general decay equations applicable to any model having a Coulomb barrier with a vertical inside face. It is shown that a square well one-body model with the well located at the nuclear surface has properties which might be more suitable for incorporation into alpha decay theory than those of the traditional model. A phenomenological picture of the many-body decay mechanism, to which the surface-well type of model is particularly adaptable, is found to lead to the representation of the many-body decay constant as the product of a preformation factor and a one-body decay constant.

A. INTRODUCTION

It was originally planned that Part III of the present series on the one-body model of alpha radioactivity^{1,2,3} would be restricted to a derivation of the alpha decay constant for non-S-state emission, and to the presentation of a method of analysis by which relative angular momenta could be assigned to each group of particles from those nuclei which emit several groups. The model and the method of Part IIa were to be used.

Events since that time have made it desirable to enlarge the scope and decrease the detail of the originally-planned paper. The scope is increased to include more or less detailed discussion of the accuracy of the wave functions traditionally used in the region exterior to the nucleus, the manner in which correction to the actual case of atomic (as opposed to pure nuclear) decay should be made, the nature of the so-called "preformation factor," and other forms of the model besides that used in the previous reports.¹⁻³ Other related topics, at least some of which should be given more consideration, will be briefly mentioned.

It is expected that Part IV will be as originally planned, to the extent that it will contain calculations made with observed decay data. However, the extent and nature of the calculations is not now certain. Because of this uncertainty, detailed methods of calculation for specific models will be omitted from this report. The methods finally used in Part IV will be described there.

¹G. H. Winslow and O. C. Simpson, On the One-Body Model of Alpha Radioactivity, I. Survey of the Existing Results, ANL-4841 (June, 1952).

²G. H. Winslow and O. C. Simpson, On the One-Body Model of Alpha Radioactivity, IIa. S-State Decay with the Consistent Square Well Model, ANL-4901 (December, 1952).

³G. H. Winslow and O. C. Simpson, On the One-Body Model of Alpha Radioactivity, IIb. S-State Decay with the Limiting Square Well Model, ANL-4910 (January, 1953).

B. WKB FUNCTIONS VS COULOMB FUNCTIONS

The discussion of wave functions to be given in this section will be made initially on the assumption that the potential function outside the region in which nuclear forces act is a pure Coulomb potential. Certain remarks about the effect of the atomic electrons will be made later.

The wave function in this outside region, then, can be written as

$$\Psi = (1/r) \phi(r) Y, \quad (1)$$

where Y is a spherical harmonic and the function $\phi(r)$ satisfies the equation

$$\frac{d^2\phi}{dr^2} + \frac{2M}{\hbar^2} \left[E - \frac{2Ze^2}{r} - \frac{\hbar^2}{2M} \frac{L(L+1)}{r^2} \right] \phi = 0. \quad (2)$$

If the following quantities are defined:⁴

$$k^2 \equiv 2ME/\hbar^2, \quad (2a)$$

$$\eta \equiv (2Ze^2/\hbar)(M/2E)^{1/2}, \quad (2b)$$

$$\rho \equiv kr, \quad (2c)$$

then Eq. (2) can be rewritten as

$$\frac{d^2\phi}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2} \right] \phi = 0. \quad (3)$$

This equation can be solved; one solution, the so-called regular (i.e., well-behaved at the origin) Coulomb function, is

$$F_L = z^{L+1} \exp(-z/2) {}_1F_1 \left[i\eta + L + 1; 2(L+1); z \right], \quad (4)$$

where $z = -2i\rho$ and ${}_1F_1$ is the confluent hypergeometric function of the indicated quantities. It is a property of these functions that, since $(L+1)$ is an integer, one cannot get the second solution by replacing $(L+1)$ by $(-L)$ as might be suggested by the course of the solution. The second solution, the irregular function G_L , must be obtained from

$$G_L(\rho) = F_L(\rho) \int F_L^{-2}(\rho) d\rho. \quad (5)$$

⁴In Table I of Ref. 1, the symbol F was used to designate 2η .

These functions have been discussed in great detail by others.⁵⁻¹⁰ Tables have been published^{10,11} but, as yet, they do not cover the ranges of ρ and η sufficiently well for convenient use in alpha decay. For the latter problem, as pointed out also by Bloch, et al.,⁸ η is near 20 and $\rho(R) \doteq 10$, where R is the inner radius of the Coulomb barrier and will be referred to hereafter as the channel radius.

It is certainly not worth while to attempt the extension of the tables that would be required to cover all known alpha emitters, if it can be shown that the WKB functions which have been used are not hopelessly in error, and if the size of the error is known. The surest method of investigating this point is by direct comparison of the WKB functions with the Coulomb functions.

At large values of ρ , in the region of positive kinetic energy, the WKB functions, normalized in the way generally used with the Coulomb functions, are

$$\bar{G}_L \doteq \cos(\theta_L + \pi/4), \quad (6)$$

$$\bar{F}_L \doteq \sin(\theta_L + \pi/4), \quad (6a)$$

where

$$\theta_L = \int_{\rho_E}^{\rho} \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2} \right]^{1/2} d\rho. \quad (6b)$$

Here ρ_E is the value of ρ at the exterior turning point, where the integrand is zero. The integral can be evaluated and expanded. If, in the expansion, terms in reciprocal powers of ρ are neglected compared to ρ , to $\ln \rho$ and to constant terms, the results is

⁵Yost, Wheeler, and Breit, Phys. Rev. 49, 174 (1936).

⁶J. L. Powell, Phys. Rev. 72, 626 (1947).

⁷G. Breit and M. H. Hull, Jr., Phys. Rev. 80, 392 (1950).

⁸Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Phys. Rev. 80, 553 (1950).

⁹G. Breit and M. H. Hull, Jr., Phys. Rev. 80, 561 (1950).

¹⁰Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. 23, 147 (1951).

¹¹W. D. Barfield and A. A. Broyles, Phys. Rev. 88, 892 (1952).

$$\theta_L + \pi/4 = \rho - \eta \ln 2\rho + \eta(\ln \eta - 1) + \pi/4 - [L(L+1)/(2\eta)] \\ + [L(L+1)]^2/(12\eta^3) + \dots \quad (7)$$

The asymptotic forms given¹⁰ for the Coulomb functions are

$$G_L \doteq \cos[\rho - (L\pi)/2 - \eta \ln 2\rho + \sigma_L], \quad (8)$$

$$F_L \doteq \sin[\rho - (L\pi)/2 - \eta \ln 2\rho + \sigma_L], \quad (8a)$$

where

$$\sigma_L = \arg \Gamma(L+1+i\eta). \quad (8b)$$

The quantity σ_L can be determined with the aid of an example given in Whittaker and Watson¹² to be

$$\sigma_L = \eta(\ln \eta - 1) + (L\pi)/2 + \pi/4 - [L(L+1)]/2\eta \\ + [(L+1)^3(L-1)]/(12\eta^3) + \frac{1}{2} \operatorname{Imag} \left\{ \frac{1}{2 \times 3} \sum_{r=1}^{\infty} (L+1+r+i\eta)^{-2} \right. \\ \left. + \frac{2}{3 \times 4} \sum_{r=1}^{\infty} (L+1+r+i\eta)^{-3} + \dots \right\} \quad (9)$$

The last term can be estimated to be about

$$\frac{1}{2} \operatorname{Imag} \left\{ \right\} \doteq -1/(12\eta) + (L+1)^2/(12\eta^3) + \dots \quad (9a)$$

Thus, if, in Eq. (7), $L(L+1)$ is replaced by $(L+1/2)^2$ (a substitution to be further discussed) it is found that the argument of the functions in Eqs. (8) minus the argument of the functions in Eqs. (6) is only about¹³

$$\epsilon \doteq 1/(24\eta) - [L(L+1)]/(24\eta^3) + \dots, \quad (9b)$$

and it is to be noted that $\eta \doteq 20$ for alpha decay by the heavy elements.

¹²E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, American Ed., (MacMillan Co., New York, 1946) p. 261, example 18.

¹³See also M. A. Preston, *Phys. Rev.* 75, 90 (1949).

For smaller values of ρ , in the region of negative kinetic energy, Bloch, et al.⁸ have given the ratio of the irregular Coulomb function to the corresponding WKB function for several values of ρ , L , and η . Thus, for the comparisons to be made, it is easier to start with the irregular function. The "irregular" WKB function, normalized to agree with Eq. (8), is, at these values of ρ ,

$$\bar{G}_L = \left[\frac{2\eta}{\rho} + \frac{L(L+1)}{\rho^2} - 1 \right]^{-1/4} \exp \left\{ \int_{\rho}^{\rho_E} \left[\frac{2\eta}{\rho} + \frac{L(L+1)}{\rho^2} - 1 \right]^{1/2} d\rho \right\}. \quad (10)$$

The expression for $\bar{G}_L$ obtained after performing the integration indicated in Eq. (10) will be set down here in the notation of Bloch, et al.⁸ since there appears to be either two misprints, or a misprint and an unfortunate approximation, in their Eq. (10). The latter should read:

$$\bar{G}_L = \frac{\rho^{1/2} (\ell T_0 + \eta \rho)^\ell \exp \left[\pi \eta - \eta \tan^{-1} \left(\frac{T_0 - \ell}{\eta - \rho} \right) - T_0 + \ell \right]}{\rho^\ell (T_0 - \ell)^{1/2} (\eta^2 + \ell^2)^{1/2}}, \quad (11)$$

where

$$\ell^2 = L(L+1) \quad (11a)$$

$$T_0 = \ell + (\ell^2 + 2\eta\rho - \rho^2)^{1/2}. \quad (11b)$$

Eq. (5) is a general relation between the solutions; i.e., having G_L , one can find F_L from

$$F_L(\rho) = G_L(\rho) \int G_L^{-2}(\rho) d\rho. \quad (12)$$

If $\bar{G}_L$ is used for G_L in Eq. (12), the not surprising result is that

$$\bar{F}_L = \frac{1}{2} \left[\frac{2\eta}{\rho} + \frac{L(L+1)}{\rho^2} - 1 \right]^{1/4} \exp \left\{ - \int_{\rho}^{\rho_E} \left[\frac{2\eta}{\rho} + \frac{L(L+1)}{\rho^2} - 1 \right]^{1/2} d\rho \right\}. \quad (13)$$

The functions (11) and (13) have now been inserted in certain of the recursion relations which are obeyed by Coulomb functions, namely,⁶

$$\alpha_L (F_L - 1 G_L - F_L G_L - 1) = 1, \quad (14)$$

$$\frac{dF_L}{d\rho} + \alpha_L + 1 F_L + 1 = \left(\frac{L+1}{\rho} + \frac{\eta}{L+1} \right) F_L, \quad (14a)$$

$$\frac{dG_L}{d\rho} + \alpha_L + 1 G_L + 1 = \left(\frac{L+1}{\rho} + \frac{\eta}{L+1} \right) G_L, \quad (14b)$$

where

$$\alpha_L = (1 + \eta^2/L^2)^{1/2}, \quad (14c)$$

and the results are tabulated in Table I. For these calculations values of $\rho(R) = 10$ and $\eta = 20$ were used.

TABLE I

Intercomparison of WKB Functions by Coulomb
Function Recursion Relations

L	0	1	2	3	4
R ₁	---	1.0000 ₁	1.0000 ₂	1.0000 ₂	1.0000 ₄
R ₂ (G)	1.0000 ₂	1.0000 ₁	1.0000 ₂	1.0000 ₁	---
R ₂ (F)	0.9999 ₈	0.9999 ₅	0.9999 ₃	0.9999 ₀	---

$$R_1 = \alpha_L [F_L - 1 G_L - F_L G_L - 1]$$

$$R_2(G) = \left(\frac{dG_L}{d\rho} / G_L \right) / \left[\frac{L+1}{\rho} + \frac{\eta}{L+1} - \alpha_L + 1 (G_L + 1/G_L) \right]$$

Now the ratios of the irregular Coulomb functions to the corresponding WKB functions, as given by Bloch, et al.,⁸ are very nearly independent of ρ and $\underline{L}$. Thus, if

$$G_L = \bar{G}_L (1 + \epsilon), \quad (15)$$

then it is apparent from the relationship between $\bar{G}_L$ and $\bar{F}_L$ given by Eq. (12) that

$$F_L = \bar{F}_L / (1 + \epsilon), \quad (15a)$$

if ϵ is independent of ρ . Further, if ϵ is independent of $\underline{L}$,

$$\bar{F}_L - 1 \bar{G}_L = F_L - 1 G_L, \quad (16)$$

for instance. This means that it has only been demonstrated in Table I that the interrelations between the functions $\overline{F}_L$ and $\overline{G}_L$ are, to an excellent approximation, the same as those between F_L and G_L . It has not been demonstrated that the $\overline{G}_L$ of Eq. (10) is the function whose asymptotic value for large values of ρ is that given by Eq. (6). In other words, if the expressions (8) and (8a) are used at large values of ρ , the expressions (15) and (15a) should be used where $\underline{r}$ is only slightly greater than $\underline{R}$.

This matching problem is to be considered by examining the effect of the replacement of $L(L + 1)$ by $(L + 1/2)^2$ in the WKB functions. Various arguments concerning this have been put forward;¹⁴ Langer¹⁵ has argued that it is a necessary consequence of the restricted range of the distance variable in a radial problem, i.e., that, if an appropriate change of variable to one which ranges from $-\infty$ to $+\infty$ as r ranges from zero to $+\infty$ is made in the wave equation, then the factor $(L + 1/2)^2$ quite naturally replaces $L(L + 1)$. Yost, Wheeler, and Breit⁵ made various numerical comparisons between Coulomb functions and, first, WKB functions which contain $L(L + 1)$, then, second, those (here called modified WKB functions) which contain $(L + 1/2)^2$. At the values of ρ and η where they made the comparisons, neither form would be preferred for numerical reasons, in all cases. For $\rho(R) = 10$ and $\eta = 20$, intercomparisons of the modified functions have been made here in the same way as described above for the unmodified functions. The results, tabulated in Table II, show that, while the behavior of the modified functions in this comparison is very good, it is not as good as for the unmodified functions. However, the results of Table II, when coupled with the comparisons shown in Table III of $\overline{G}_L(\text{modified})/\overline{G}_L(\text{unmodified})$ and $G_L/\overline{G}_L(\text{unmodified})$ (where the latter are taken from Bloch, et al.⁸), show that, for the alpha decay problem at least, the modified functions are equivalent, to about 0.1 % or less, to the Coulomb functions. The modified functions will be used in subsequent work.

The unmodified WKB functions were used in the discussions of S-state decay in Parts IIa² and IIb³. Since $L(L + 1)$ is zero at $L = 0$, while $(L + 1/2)^2$ is not, the modified and unmodified functions are not the same even at $L = 0$. This has already been illustrated in Table III. It is apparent then that any expressions for the decay constant obtained previously should be divided by about $(1.01)^2$, if one is trying to reduce the numerical approximations to a minimum.

More specifically, the expressions for the decay constant should be obtained by the use of Eq. (11) for G_L but with the definition of $\underline{\ell}$, Eq. (11a), replaced by

$$\underline{\ell} = L + 1/2. \quad (17)$$

¹⁴See, for instance, E. C. Kemble, The Fundamental Principles of Quantum Mechanics (McGraw-Hill Book Co., Inc., New York, 1937) p. 107.

¹⁵R. E. Langer, Phys. Rev. 51, 669 (1937).

Eq. (11), with $\underline{\ell}$ given by Eq. (17), has been put in a more convenient form for computation. The result will be given here, since this function will be common to any model using a Coulomb barrier which is cut off sharply at some point, $\underline{R}$.

TABLE II

Intercomparison of Modified WKB Functions by
Coulomb Function Recursion Relations

L	0	1	2	3	4
R_1	---	0.9993 ₇	0.9994 ₀	0.9994 ₂	0.9994 ₄
$R_2(G)$	1.0006 ₄	1.0006 ₆	1.0006 ₇	1.0006 ₇	---
$R_2(F)$	1.0006 ₁	1.0005 ₅	1.0005 ₀	1.0004 ₅	---

R_1 and R_2 are as in Table I.

TABLE III

Comparison of Ratio $\overline{G}_L(\text{modified})/\overline{G}_L(\text{unmodified})$ to
Ratio $G_L/\overline{G}_L(\text{unmodified})$.⁹

L	0	1	2	3	4
$\frac{\overline{G}_L(\text{modified})}{\overline{G}_L(\text{unmodified})}$	1.0106 ₇	1.0106 ₆	1.0106 ₂	1.0105 ₇	1.0105 ₁
$\frac{G_L}{\overline{G}_L(\text{unmodified})}$	1.010	1.0096	1.011	1.011	---

If Eq. (11) is written so that those factors which have $\underline{\ell}$ or $(\underline{\ell}/2)$ as exponents are included in the argument of the exponential, then this argument is the ω of Parts I; IIa, and IIb.¹⁻³ The first term in the expansion of ω in powers of $L(L+1)$ was given in Part I. Here, now, the expansion is given in powers of $(L+1/2)^2$. Terms which involve $(L+1/2)^2$ to higher powers than the first become more important, of course, as $\underline{L}$ gets large. Some such terms have now been determined and will be given here for the sake of completeness, though it is doubtful that they will ever be used!

Actually, the expansion of $2 \log G_L$ is given, since it is G_L^{-2} which appears in an expression for the decay constant. It is convenient to adopt and, as may be necessary, adapt the notation of the previous reports in this series.¹⁻³ Thus

$$\begin{aligned}
 2 \log G_L = & (D/E_\alpha^{1/2}) f_0(x) - 1/2 \log [(1-x)/x] \\
 & + [(L+1/2)^2/(1.15129kR|\Phi_0|^{1/2})][1-x-1/(4kR|\Phi_0|^{1/2})] \\
 & - [(L+1/2)^4/(13.8155k^3R^3|\Phi_0|^{3/2})] \\
 & [(1-x)(1+4x-8x^2)-3/(2kR|\Phi_0|^{1/2})] \\
 & + [(L+1/2)^6/(138.155k^5R^5|\Phi_0|^{5/2})] \\
 & [(1-x)(3+8x+48x^2-192x^3+128x^4) \\
 & -10/(kR|\Phi_0|^{1/2})]. \tag{18}
 \end{aligned}$$

In Eq. (18) the $\log$ is logarithm to the base 10, E_α is the observed alpha particle energy in Mev "corrected" for screening by the atomic electrons (this phrase is convenient, although logically misleading; see Section C, below), D is $1.0946_0 [ZA/(A+4)]$ (where A is the mass number, and Z is the atomic number, of the daughter), $f_0(x) = \cos^{-1}x^{1/2} - x^{1/2}(1-x)^{1/2}$, and $x = ER/2Ze^2$. In this definition of x , E is the decay energy "corrected" for screening. This number, x , can be calculated from $x = E_\alpha R/K$ where R is in units of 10^{-13} cm and $K = 2.8793 [ZA/(A+4)]$. The constants D and K have been tabulated in Part I¹ for various heavy isotopes. The use of the new values of the atomic constants given by Du Mond and Cohen¹⁶ would change D to $1.09449 [ZA/(A+4)]$ and K to $2.8797_2 [ZA/(A+4)]$. More important is the change in K which would correspond to a change in decay constant of about 2.5% for a given R or a change of about $1.4 \times 10^{-2}\%$ in R for a given decay constant.

The quantity $kR|\Phi_0|^{1/2}$ can be calculated from

$$kR|\Phi_0|^{1/2} = 1.15129(D/E_\alpha^{1/2}) [x(1-x)]^{1/2} ; \tag{19}$$

$|\Phi_0|$ is an adaptation from $|\Phi(kR)|$, where

$$|\Phi(\rho)| = [(2\eta)/\rho] + [(L+1/2)^2/\rho^2] - 1 , \tag{20}$$

obtained by dropping out the term containing L . Thus the $|\Phi_0|$ of Eqs. (18) or (19) is not $|\Phi_L = 0|$, exactly.

The orders of magnitude of the various terms in Eq. (18) have been checked for at least one representative set of values in order to see how far one should go in using them. This was done by finding the smallest

¹⁶J. W. M. Du Mond and E. R. Cohen, Revs. Modern Phys. 25, 691 (1953).

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value of L for each term which will cause the term to make a contribution to the third decimal place in $(2 \log G_L)$; this is as far as it seems necessary to carry any of the computations. (For much of the decay data, this is farther than is necessary.) With $\eta = 20$ and $\rho(R) = 10$, these values of L are found to be $L = 0$ for the first term, which gives 0.009 for the term; $L = 3$ for the $(L + 1/2)^4$ term, which gives 0.002 for that term; and $L = 6$ for the $(L + 1/2)^6$ term, which gives 0.001 for that term.

There is one final word on this particular topic. In a recent paper by Devaney¹⁷ on alpha decay theory, and in the text by Blatt and Weisskopf,¹⁸ there is given a table of the ratios of the decay constant for various values of L to that for $L = 0$, all calculated for the same energy and nuclear radius and, of course, with the assumption that the only "inhibiting" influence is the increased difficulty of barrier penetration as L is increased. These numbers, actually the ratios $(G_L/G_0)^{-2}$, are, in those tables, the squares of what they should be. These ratios have been calculated here with the use of Eq. (18), with $\eta = 20$ and with $\rho(R) = 10$. They are tabulated in Table IV.

TABLE IV

Values of $(G_L/G_0)^{-2}$ for $\eta = 20$, $\rho(R) = 10$.

L	0	1	2	3	4	5	6
$(G_L/G_0)^{-2}$	1.000	0.844	0.601	0.363	0.1854	0.0806	0.0299

¹⁷J. J. Devaney, Phys. Rev. 91, 587 (1953).

¹⁸J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics, (John Wiley and Sons, Inc., New York, 1952). See, for alpha decay theory, pp. 568 ff.

C. ATOM DECAY VS NUCLEUS DECAY

Although this section will certainly not pretend to be an exhaustive discussion of its title, it is desired to make some remarks on the subject. It is a fact that the existing theory of alpha decay is a theory about the decay of a bare nucleus, although it will be seen that the energy is not handled correctly, even so. Recently the conflict between this fact and the fact that the observations are made on decaying atoms has come under discussion. The direction this discussion has taken is the consideration of corrections to be applied to the theory of the decay of a nucleus. It is, and reasonably so, expected that the results obtained in an attempt to treat, from the start, the decay of an atom would be the same, to the first order at least, as those obtained by correcting the existing theory.

The corrections to be made are not completely independent, though it is expected that they can be treated so, again to the first order at least. Those corrections which have been given more serious consideration have to do with ejection of extra-nuclear electrons during alpha decay; with the effect of the charge change, hence of the change in total electronic binding energy on the observed decay energy; and with the "screening" of the nuclear potential function by the surrounding electron cloud.

It is probably true that direct ionization and excitation of the atom need not be seriously considered when discussing alpha decay constants and energy relations. Migdal¹⁹ has obtained values for the probability of ionization in the K, L, and M shells. While there are many more electrons than that in a heavy atom, and although Migdal's results seem to underestimate the experimental probabilities,²⁰ the ionization energy for the higher shells goes down rapidly. From the experimental evidence to date it would appear that too few alpha particles would be affected to require corrections to the decay equations because of energy robbed in this way from the emerging alpha particle.

The other two corrections, having to do with the differences in total electronic binding energy between parent and daughter and with the screening of the nuclear Coulomb potential have presented more of a puzzle. It was proposed by Ambrosino and Piatier²¹ that two corrections should be made in an equation for the alpha decay constant derived for a bare nucleus. According to these authors, the observed decay energy (the kinetic energy of the alpha particle plus that of the recoiling atom) should be increased by the difference in total electronic binding energy between the daughter and

¹⁹A. Migdal, J. Phys. (U.S.S.R.) 4, 449 (1951).

²⁰M. Rion, J. phys. radium 13, 487 (1952); W. Robinson and W. Bernstein, Phys. Rev. 86, 545 (1952).

²¹G. Ambrosino and H. Piatier, Compt. rend. 232, 400 (1951).

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the parent on the grounds that this latter energy had to be delivered to the electrons in order to readjust them to the normal configuration for the daughter. Thus, according to this argument, the total energy which the alpha particle had when it penetrated the Coulomb barrier was greater than that observed by just this binding energy difference.

In addition, since the alpha particle must pass through the electron cloud on its way out, Ambrosino and Piatier argued that the total potential energy should include that which the alpha particle has in the nuclear Coulomb field and the (negative) potential energy which it has in the electrostatic field of the electrons. They suggested that both these corrections be calculated from the Thomas-Fermi model of the atom.²² With this, or some other statistical model of the atom, it turns out that these corrections can be made by adding about 30 kev to 40 kev (depending on the exact value of Z) to the observed energy in all the functions, and an additional 30 to 40 kev in those functions applicable in the region outside the nuclear radius. The potential energy to be used in the latter functions is, after these corrections are made, just $(2Ze^2/r)$. The reception of these suggestions has been varied. The principal point of contention has not been whether they should be made, but whether both of them should be made and, if not, why not.²³

Serber and Snyder were quoted by Rasmussen, et al.²³ as having shown in a private communication to the latter that if one makes both of the corrections described above, he is correcting for the same thing twice. In their own subsequent publication, however, Serber and Snyder did not emphasize this aspect of the matter. The present authors would like to show here, for the alpha decay problem, that the only correction to be made is that to the potential, but that it can be made, in practice, by "correcting" the total energy. The notation of Serber and Snyder will be used where applicable.

Suppose the mass of the parent, M_P , is written as

$$M_P = Z M_p + (A - Z)M_n + Z m_e + E(Z) - U_P,$$

²²For a detailed discussion of various, statistical atom models see P. Gombas, Die Statistische Theorie des Atoms (Springer-Verlag, Vienna, 1949).

²³For this discussion as it applies to alpha decay, see D. Curie, J. phys. radium 12, 941 (1951); Rasmussen, Thompson, and Ghiorso, UCRL-1473 (Sept. 1951); Asaro, Reynolds, and Perlman, UCRL-1681 (Feb. 1952); F. Asaro, UCRL-2180 (June 1953); P. Benoit-Gueutal, J. phys. radium 13, 486 (1952); Ref. 17. For discussion as it applies to beta decay, see M. H. Hebb, Physica 5, 701 (1938); H. M. Schwartz, Phys. Rev. 86, 195 (1952); Freedman, Wagner, and Engelkemeir, Phys. Rev. 88, 1155 (1952). More generally, see R. Serber and H. S. Snyder, Phys. Rev. 87, 152 (1952).

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where $E(Z)$ is the electronic energy of the parent atom and U_P is the nuclear binding energy as usually written. In particular, the electrostatic term in U_P contains only the electrostatic interaction between the nuclear protons. Similarly, the mass, M_D , of the normal daughter will be

$$M_D = (Z - 2)M_p + (A - Z - 2)M_n + (Z - 2)m_e + E(Z - 2) - U_D,$$

and the mass of the product helium atom will be

$$M = 2M_p + 2M_n + 2m_e + E(2) - U_\alpha.$$

Then the decay energy, as usually defined, is found to be

$$E = (U_D - U_P) + U_\alpha + E(Z) - E(Z - 2) - E(2).$$

If some simple form for $E(Z)$ such as²⁴

$$E(Z) = -\alpha Z^n \quad (21)$$

is used, it is found that

$$E = (U_D - U_P) + U_\alpha - 2n\alpha Z^{n-1} + 2n(n-1)\alpha Z^{n-2} - E(2). \quad (22)$$

The reason for writing the second order term in $[E(Z) - E(Z - 2)]$ will appear later. With any of the usual values for α and n the two terms written for this difference are, respectively, about -40 kev and 500 ev, for heavy atoms. The energy $E(2)$ is about -80 ev. Thus it is true that the nuclear decay energy would be about 40 kev greater than the observed decay energy, and this should be recognized if one wishes to examine energy differences between bare nuclei.

Now, however, consider what would be the result if there were no electronic rearrangement during the decay of an atom. In this case the mass, M'_D , of the residue would be

$$M'_D = (Z - 2)M_p + (A - Z - 2)M_n + Zm_e + E(Z) - 2e\phi(Z) - U_D,$$

and the mass, M' , of the product alpha particle would be

$$M' = 2M_p + 2M_n - U_\alpha.$$

Here, $\phi(Z)$ is the electrostatic potential produced at the nucleus by the Z electrons arranged as in the parent, and the "electronic" energy of the residue is the same as that of the parent except for the effect of the

²⁴Here, in addition to Ref. 22, see also L. L. Foldy, Phys. Rev. 83, 397 (1951)

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decrease of charge at the nucleus by two units. The alpha particle energy (corrected for recoil) which would be observed in this case would be

$$E' = (U_D - U_P) + U_\alpha + 2e\phi(Z). \quad (23)$$

There has been no electronic rearrangement, so that E' should be the energy which the alpha particle had when it penetrated that barrier which is present when an atom decays. But

$$e\phi(Z) = dE(Z)/dZ, \quad (24)$$

so that there is no first order difference between E and E' . The principal part of the difference in electronic energy is due to the charge change at the nucleus and only a negligible part to electronic rearrangement. It should be considered now whether E' really is the energy with which the alpha particle penetrated the barrier.

The energy E' is the energy with which the alpha particle penetrated the barrier because when an alpha particle is formed by a nucleus which is surrounded by electrons some of the available nuclear energy must be used immediately to add the energy $-2e\phi(Z)$ to the residual nucleus. This part of the nuclear energy, of about 40 kev, would be available to an alpha particle formed by a bare nucleus, but it is lost, from the moment of its formation, to an alpha particle formed by an atomic nucleus.

The difference between E and E' must be found from the second order term in $[E(Z) - E(Z - 2)]$. Thus $E' - E$ is about -580 ev so that when the decay is accompanied by electron rearrangement the alpha particle actually gains a small amount of energy during its passage through the electron cloud, up to a negligible maximum of 500 ev (or 580 ev in the highly unlikely event that it escapes as a helium atom).

On the other hand it is certainly true that the alpha particle is in the electrostatic potential produced by the electrons from the moment of its formation until it completely escapes the atom. The corresponding potential energy will vary from $2e\phi(Z)$ at the nucleus to something between zero and $-4e^2/r$, depending on the behavior of the two extra electrons, as the alpha particle leaves the atom. Thus $(E - V)$ is equal to the observed alpha particle energy plus the recoil energy, when the alpha particle and the residue have become widely separated; $(E - V)$ is equal to the observed alpha particle energy plus the recoil energy minus $[2(Z - 2)e^2]/r$ (in this section only, Z is used for the atomic number of the parent; elsewhere it is used for the atomic number of the daughter) plus $2n\alpha Z^{n-1}$, when the alpha particle is very near the nucleus; when the alpha particle, in a one-body model, is "in" the nucleus, or at least behind the Coulomb barrier, $(E - V)$ is equal to the observed alpha particle energy plus the recoil energy minus the nuclear interaction energy plus $2n\alpha Z^{n-1}$. That is, the leading term in a

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series which represents the r dependence of that potential energy arising from the existence of the electron cloud is the constant $2\pi\alpha Z^{n-1}$, and this is the only term which need be included. Since this is the case, one can, in practice, accomplish the correction by adding the 40 kev (or whatever is correct for the atom under consideration) to the observed decay energy in the functions pertaining to the nuclear region and the barrier region.

It will be seen later that the product $k[G_L(R)]^{-2}$ appears in the expression for the decay constant, and it might be thought that this k should contain the true decay energy. Since, however,

$$E^{1/2} \left[\frac{2Ze^2}{ER} - \frac{0.040}{E} + \frac{(L + 1/2)^2 \hbar^2}{2MER^2} - 1 \right]^{1/2} \\ = (E + 0.040)^{1/2} \left[\frac{2Ze^2}{(E + 0.040)R} + \frac{(L + 1/2)^2 \hbar^2}{2M(E + 0.040)R^2} - 1 \right]^{1/2} \quad (25)$$

the "corrected" energy is to be used both in k and in Eq. (18) for $2 \log G_L$; it appears in the latter explicitly and, in x , implicitly also.

Actually, the way in which the energy has been handled in the past is not even correct for the treatment of the decay of a bare nucleus. In this latter case the energy itself would be greater than the decay energy observed with atoms, by about 40 kev, and there would be no correction to the potential energy. Except for the corrections of a higher order, the decay constant would actually be unchanged by removing the electrons, although there would need be no discussion of the sort given in this section since one could proceed, correctly, with the presently used alpha-nucleus interaction and the decay energy which would be observed in that case; that is, the energy as observed would be 40 kev higher than that observed when atoms decay.

D. THE GENERAL DECAY FORMULAS

In the previous parts^{1,2,3} of this series, the decay formulas were derived by methods which, though tedious, were essentially exercises in elementary quantum mechanics. Thus the wave functions and their first derivatives were separately equated at the inner radius of the barrier. Because of the advent of the method of Feshbach, Peaslee, and Weisskopf,²⁵ introduced as a foundation on which to base attempts to treat nuclear physics problems as many-body problems rather than one-body problems, the use of the logarithmic derivative of the wave function at $r = R$ has been emphasized. The method is general and is applicable to the problem of alpha decay as well as to that of nuclear reactions. Devaney¹⁷ used it in his discussion of alpha decay and Shapiro²⁶ used it to calculate cross sections for formation of a compound nucleus by charged particles. By this method the expression for the logarithmic derivative of the "outside" wave function at the inner radius of the barrier is equated to something which is expected to be more representative of conditions in the interior of the nucleus than is the logarithmic derivative of a one-body wave function. The function to which the logarithmic derivative of the outside function is equated is specified by the method. The expression for the logarithmic derivative of the outside function, however, depends only on the assumption of a one-body potential in $r \gg R$; it can be used in conjunction with any model of the nucleus. When the method of complex eigen-energies, introduced by Gamow,²⁷ is used, the two equations which must be obtained are found by equating the real and imaginary parts, respectively, of the logarithmic derivative on each side of the discontinuity at $r = R$. The discussion which leads to these two equations will be given here, briefly, by this method.

An equivalent discussion has been given by Feshbach, *et al.*,²⁵ and, for alpha decay in particular, by Devaney¹⁷ and by Preston,²⁸ for instance. It is repeated here because this series of reports is of the nature of a review, because it is convenient to use the notation of these reports (see Appendix), and because it is desired to keep more terms than did Devaney or Preston. The equations to be derived will be applicable when the inside face of the barrier is vertical, when the potential outside this face is only the sum of the Coulomb and "centripetal" potentials (a constant addition, which is what the screening correction amounts to, can be easily included by incorporating it with the real part of the energy as described in Section C)

²⁵Feshbach, Peaslee, and Weisskopf, *Phys. Rev.* 71, 145 (1947).

²⁶M. M. Shapiro, *Phys. Rev.* 90, 171 (1953).

²⁷See G. Gamow and C. L. Critchfield, Theory of Atomic Nucleus and Nuclear Energy Sources (Oxford University Press, London, 1949). See also Kemble, Ref. 12, p. 192.

²⁸M. A. Preston, *Phys. Rev.* 71, 865 (1947).

and when the alpha particle is considered to have its "outside" physical characteristics whenever it is outside the inner face of the barrier. Not all the algebra or arguments about small quantities will be given, but the salient points will be indicated.

Let the complex energy be given by

$$W = E - (i\lambda\hbar)/2. \quad (26)$$

The radial wave function must be of the form $(1/r) \phi(r)$. Far out in Region III, the region outside the barrier, ϕ must have the form

$$\phi_{III} \propto \exp(i\rho), \quad (27)$$

where

$$\rho = kr, \quad (28)$$

$$k^2 = (2MW)/\hbar^2, \quad (28a)$$

and M is the reduced mass of the alpha particle. (The use of this complex energy gives directly the proper increase of $|\phi_{III}|^2$ with r .²⁹ The expression (27) must be that approached at very large r by the more general expression³⁰

$$\phi_{III} = PG_L + QF_L. \quad (29)$$

If Eq. (29) is to approach (27) as r gets very large, it is required that

$$Q = iP. \quad (30)$$

Therefore

$$\phi_{II} = P(G_L + iF_L) \quad (31)$$

²⁹See Gamow, Ref. 27 and Ref. 2.

³⁰Actually the situation is still a little hazy here. The functions G_L and F_L are Coulomb functions. In Region II, the region within the barrier, the Coulomb functions are applicable since the screening there amounts only to the addition of a constant to the energy. On through the atom, however, the potential energy changes from $(2Ze^2/r - \text{constant})$, becoming something of the order of $(-4e^2/r)$ (for either a nonadiabatic transition or one in which the two electrons are ejected with low velocity) as r approaches infinity. The present authors have no remarks on this problem and shall proceed, in the usual way, with the Coulomb or, equivalently here, the modified WKB functions.

where, for alpha decay at least, the modified WKB functions described in Section B can be used for G_L and F_L .

If, by definition,

$$f^{II}/kR \equiv [(\phi^{II})^{-1} (d\phi^{II}/d\rho)]_{kR}, \quad (32)$$

then

$$f^{II} = -kR |\Phi|^{1/2} \frac{(1 + \gamma) - (i/2)(1 - \gamma) \exp(-2\omega)}{1 + (i/2) \exp(-2\omega)}, \quad (33)$$

where $|\Phi| [= |\Phi(kR)|]$ is given by Eq. (20) with $\underline{E}$ [see Eqs. (2)] replaced by $\underline{W}$ and ρ replaced by kR ,

$$\omega = \int_{kR}^{\rho_E} |\Phi(\xi)|^{1/2} d\xi, \quad (33a)$$

and

$$\gamma = (1/4) |\Phi|^{-3/2} [d|\Phi(\rho)|/d\rho]_{kR}. \quad (33b)$$

The upper limit in Eq. (33a) is the value of ρ at the classical exterior turning point, i.e., $|\Phi(\rho_E)| = 0$.

If, now, the expression on the right side of Eq. (26) is substituted for $\underline{W}$ in Eq. (33), and separation is made into real and imaginary parts to the first power of λ or of $\exp(-2\omega)$, the result is

$$f^{II} = -kR |\Phi|^{1/2} (1 + \gamma) + i \left\{ kR |\Phi|^{1/2} \exp(-2\omega) - [MR\lambda(1 - \gamma)]/[2\hbar k |\Phi|^{1/2} (1 + \gamma)] \right\}. \quad (34)$$

In Eq. (34), the same nomenclature is used as in Eq. (33), but every quantity is now real, i.e., wherever the energy appears it is $\underline{E}$, rather than $\underline{W}$.

The corresponding logarithmic derivative of the wave function behind the barrier, ϕ^I , will certainly be energy-dependent also. If, following Feshbach, et al.,²⁵ it also is expanded to the first power of λ , the result for f^I is

$$f^I = f^I(E) - (i/2) \lambda \hbar [\partial f^I(W)/\partial W]_E. \quad (35)$$

Since these logarithmic derivatives must be equal, the following equations, applicable for any model for which the potential at $r \gg R$ is as described above, are found:

$$f^I(E) = -kR|\Phi|^{1/2}(1 + \gamma); \quad (36)$$

$$\lambda = \frac{2\hbar}{M} \frac{k|\Phi|^{1/2}\exp(-2\omega)}{-[\hbar^2/(MR)][\partial f^I(W)/\partial W]_E + (1 - \gamma)/[k|\Phi|^{1/2}(1 + \gamma)]} \quad (36a)$$

The right side of Eq. (36) is not an expression for $f^I(E)$; Eq. (36) is not an identity. Rather, once an expression is chosen for $f^I(E)$ it can be used in Eq. (36) to find the decay energy and in Eq. (36a) to find the corresponding decay constant. Eq. (36) corresponds to Devaney's¹⁷ equation (2.14) and Eq. (36a) corresponds to his equations (2.12) and (2.8b). [Apparently the $\underline{S}$ on the right side of (2.12) should be $\underline{s}$]. When making the latter comparison it should be realized that Devaney omitted the term corresponding to the second one in the imaginary part of the present Eq. (34).

E. APPLICATION TO SPECIFIC MODELS

1. The Traditional Model

Most of the past work on alpha decay theory has been done with a simple square well representing the potential behind the barrier. This model will be designated here the traditional model. The fact that the wave functions in such a well are the Bessel functions of order half an odd integer has long been realized, of course,³¹ but the implications with regard to the dependence of λ on L , if the solution is carried out strictly, were first pointed out by Preston.²⁸

The wave function behind the barrier can be written

$$\phi I = (\kappa r)^{1/2} J_{L+1/2}(\kappa r), \quad (37)$$

where

$$\kappa^2 = (2M/\hbar^2)(W - V_L), \quad (37a)$$

V_L being the constant interior potential. The subscript is used on the latter since it is found, if the model is applied strictly, that the value of this potential calculated from observed decay data depends on L . If the operations indicated in Section D are carried out it is found that

$$f^I(E) \equiv L + 1 - \kappa R [J_{L+3/2}(\kappa R) J_{L+1/2}^{-1}(\kappa R)] \quad (38)$$

and

$$\begin{aligned} & -[\hbar^2/(MR)] [\partial f^I(W)/\partial W]_E \\ & = (1/\kappa) [\kappa R J_{L+3/2}^2(\kappa R) J_{L+1/2}^{-2}(\kappa R) - (2L+1) J_{L+3/2}(\kappa R) J_{L+1/2}^{-1}(\kappa R) + \kappa R]. \end{aligned} \quad (38a)$$

The κ which appears on the right sides of Eqs. (38) and (38a) now, of course, contains the real part of the energy only. Thus the use of the traditional model requires the solution of

$$\kappa R J_{L+3/2}(\kappa R) = (g + L + 1) J_{L+1/2}(\kappa R) \quad (39)$$

and

$$\lambda_0^L = \frac{2\hbar}{M} \frac{\kappa^2 \exp(-2\omega)}{\kappa R |\Phi|^{1/2} (1+\gamma)^2 S_T}, \quad (39a)$$

³¹See, for instance, Gamow, Ref. 27.

where

$$S_T = 1 + g^{-1} + g^{-2} [\kappa^2 R^2 - L(L+1)] + g^{-3} \kappa^2 R^2 (1 - \gamma) \quad (39b)$$

and

$$g = kR |\Phi|^{1/2} (1 + \gamma). \quad (39c)$$

A subscript zero on λ will be used to designate a one-body decay constant; the relative angular momentum quantum number will be indicated with a superscript.

2. The Devaney Model¹⁷

By following the arguments of Feshbach, et al.,²⁵ it is assumed that one can write

$$f^I = \kappa R \cot Z(W), \quad (40)$$

where κ is the wave number just inside the nuclear boundary and $Z(W)$ is some unknown function of W . The form of Eq. (40) is similar to that for f^I with $L = 0$ in the traditional model where $Z(W)$ becomes κR . It is still assumed that $Z^2(E)$ varies linearly with E and, indeed, that³²

$$\kappa^2(E) = (2M/\hbar^2)(E + E_0). \quad (41)$$

It is also assumed that one can write, in the neighborhood of a resonance energy, i.e., an energy which would correspond to a solution of Eq. (39) with the traditional model, that

$$dZ(E)/dE = \pi/D^*, \quad (42)$$

where D^* is an average spacing between levels and is expected to be of the order of the level spacing, D . Since Eq. (40) is applied for every L , the manner in which the interior solutions affect the dependence of L on λ is felt only through D^* , i.e., D^* must be the spacing between alpha energy levels of equal outgoing angular momentum. For further discussion reference should be made to Devaney.¹⁷

With this model, then, the equations to be solved are

$$\kappa(E) \cot Z(E) = -k |\Phi|^{1/2} (1 + \gamma) \quad (43)$$

and

$$\lambda_\alpha^L = \frac{2D^*}{\pi \hbar} \frac{\kappa \exp(-2W)}{k |\Phi|^{1/2} (1 + \gamma)^2 S_D} \quad (43a)$$

³²See Eq. (37a).

where

$$S_D = 1 + \kappa^2 R^2 g^{-2} + [MD^*/(\pi \hbar^2)] [1 + (1 - \gamma) \kappa^2 R^2 g^{-2}] R \kappa^{-1} g^{-1}. \quad (43b)$$

Since $Z(E)$ is not known, the value of κ must be estimated on the basis of other considerations about the kinetic energy of an alpha particle in the nucleus, and D^* must be estimated from experimental observations. The subscript α is used on the decay constant in Eq. (43a) since this method is an attempt to get a many-body decay constant.

3. A Surface Well Potential Model

Certain ideas relative to a more detailed picture of the mechanism of the alpha decay process than now seems to be in the literature will be discussed in Section F. These ideas assign a somewhat more important role to one-body aspects of the decay process than does current many-body theory. They also suggest (but do not require) a type of square well one-body potential which might allow a more natural separation of the one-body and many-body phases of the problem than would the use of a one-body model of the traditional type. Although wider application is believed possible, the present discussion will be restricted to alpha decay.

The description given in this section will be that of a strict one-body model based on the following presumptions: (a) an alpha particle, as such, does not exist in a nucleus; (b) in the absence of the external Coulomb field there would be an attractive interaction between the daughter nucleus and the alpha particle which would be very strong at the instant of the appearance of the latter at the nuclear surface, but which would approach zero very rapidly as the two separated.

A potential of the type just indicated has been referred to as a Jastrow potential by Greenberg.³³ Jastrow³⁴ used it, with an exponential form for the nuclear interaction, to discuss nucleon-nucleon scattering. Its use in more general applications is apparently being made by Greenberg.³³ In the present discussion the net potential will be approximated by a square well at the nuclear surface surrounded by the potential to which the discussion in Section D applies. In spite of this simplification the model is believed to be a better representation of the one-body aspects of the problem of alpha decay than is the traditional model.

³³J. M. Greenberg, Phys. Rev. 87, 209 (1952).

³⁴R. Jastrow, Phys. Rev. 79, 389 (1950); 81, 165 (1951); 91, 749 (1953).
See also M. M. Lévy, Phys. Rev. 86, 806 (1952) and Ref. 18.

The potential to be used, then, is shown in Fig. 1. With this model the function ϕ^I , the one-body wave function in the well, is

$$\phi^I = (\kappa r)^{1/2} \left[J_{L+1/2}(\kappa r) J_{-(L+1/2)}(\kappa R_0) - J_{-(L+1/2)}(\kappa r) J_{L+1/2}(\kappa R_0) \right], \quad (44)$$

where, as shown in Fig. 1, R_0 represents the nuclear radius rather than the so-called channel radius. The latter, the R of previous sections, is loosely defined in the traditional model of alpha decay as the sum of the nuclear radius and the radius of the alpha particle, or the sum of the radius and the interaction distance.³⁵ With the potential of Fig. 1, the interaction distance is introduced explicitly from the beginning.

From Eq. (44) it is found that³⁶

$$f^I(E) \equiv L + 1 - \kappa R(v/u), \quad (45)$$

and

$$-[\hbar^2/(MR)] [\partial f^I(W)/\partial W]_E$$

$$= R \left\{ 1 - [(2L+1)/(\kappa R)] (v/u) + (v/u)^2 - [2/(\pi \kappa R u)]^2 \right\}, \quad (45a)$$

where

$$v = J_{L+3/2}(\kappa R) J_{-(L+1/2)}(\kappa R_0) + J_{-(L+3/2)}(\kappa R) J_{L+1/2}(\kappa R_0), \quad (46)$$

and

$$u = J_{L+1/2}(\kappa R) J_{-(L+1/2)}(\kappa R_0) - J_{-(L+1/2)}(\kappa R) J_{L+1/2}(\kappa R_0). \quad (46a)$$

With a little manipulation the equations to be solved can be put in the form

$$\kappa R v = (g + L + 1) u, \quad (47)$$

and

$$\lambda_0^L = \frac{2\hbar}{M} \frac{k|\Phi|^{1/2} \exp(-2\omega)}{R S_S} \quad (47a)$$

where

$$S_S = 1 + (\kappa R)^{-2} \left\{ g(g+1) - L(L+1) - [2/(\pi u)^2] \right\} + (1-\gamma)g^{-1}. \quad (47b)$$

³⁵ See page 574 of Ref. 18 and H. A. Bethe, Rev. Modern Phys. 9, 69 (1937), bottom of second column, p. 167.

³⁶ The quantity $\left[J_{L+1/2}(x) J_{-(L+3/2)}(x) + J_{-(L+1/2)}(x) J_{L+3/2}(x) \right]$ appears, during the derivation, on the right side of Eq. (45a). This quantity can be shown to be the Wronskian $W[J_{L+1/2}(x), J_{-(L+1/2)}(x)]$; further, it can be shown that xW is a constant. Evaluation of the constant by letting x be large gives $W = (-1)^{L+1} [2/(\pi x)]$. See also Ref. 12, p. 380, example 13.

Fig. 1

The potential function for the surface well model. R_0 is taken to be the radius of the alpha daughter, R is the inside radius of the barrier and $\Delta R = R - R_0$. The exterior classical turning point is at r_E .

F. A MANY-BODY THEORY

1. The Preformation Factor

The traditional application of alpha decay theory has been the calculation of nuclear radii from observed decay data with the use, until very recently, of some form of the traditional one-body model.³⁷ The problem treated by a one-body model is the rate of "leakage" of an existing alpha particle through a broad, high potential barrier. While the radii so calculated with decay data pertaining to the main groups of even-even isotopes have seemed reasonable, the results of attempted treatments of other aspects of alpha decay, such as the weaker groups of even-even isotopes and, in particular, alpha decay of non-even-even isotopes, have not seemed so reasonable. The trouble has been laid to the most obvious source, and this is probably the source of most of it, i.e., the assumption of an existing alpha particle. Thus, there have been proposals, in one form or another, for the inclusion of what might be called a preformation factor,³⁸ i.e., a factor which reflects the belief that alpha particles are not fundamental constituents of nuclei.

The concept of the preformation factor to be presented here is reached by adding the following presumptions to those of Section E3: (c) in the absence of the external Coulomb field there would still be a non-infinite probability per second for the emission of alpha particles; (d) the presence of the barrier, which is the net effect of the Coulomb and nuclear interactions, makes it possible that an alpha particle, once formed, might become unformed and reabsorbed rather than be emitted through the barrier. In addition, if it is desired to apply the concept to the traditional model, it is necessary to amplify somewhat - essentially to modify - the previous presumption (a) that alpha particles do not exist in nuclei. Use is made here of the suggestion by Bethe³⁹ that "there might be some slight tendency for the preformation of alpha particles in nuclei due to their great stability." This suggestion is applied literally where the corresponding one-body picture is the traditional one. In the case of the surface well model it is considered that there is a tendency for preformation of alpha particles at the nuclear surface.

³⁷The most common forms have been reviewed in Ref. 1. The approximations and errors which have led to so many different forms, and the resulting large differences in radii calculated with the same decay data, have been discussed in Refs. 2 and 3.

³⁸See H. A. Bethe, Ref. 35, p. 163, §67; Perlman, Ghiorso, and Seaborg, Phys. Rev. 75, 1096 (1949); 77, 26 (1950); B. L. Cohen, Phys. Rev. 80, 105 (1950). The Devaney model,¹⁷ based on the method of Feshbach, Peaslee, and Weisskopf²⁵ is, as has been indicated, an attempt to get around this one-body - many-body difficulty. It is a more precise formulation of the form of the many-body decay constant used by Bethe and Cohen.

³⁹H. A. Bethe, Ref. 35, second column, p. 167.

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Pending further discussion, particularly of presumption (c) above, it is assumed for the moment that there are periods during which one-body wave functions apply behind the barrier. If this is so, then some one-body model (not necessarily, of course, a square well model) applies during these periods, and the total alpha decay constant can be written

$$\lambda_{\alpha}^L = p_L \lambda_0^L, \quad (48)$$

where p_L is the preformation factor for an alpha particle of relative angular momentum L . The difference between the traditional model and surface well model is the difference in the locale of the alpha particle during those periods when it is in existence behind the Coulomb barrier.

It will be assumed here that the preformation factor as defined by Eq. (48) is less than unity, although it is not inconceivable that it might be greater than unity for an isotope which has enough particles, of the right kind, outside some shell (as at Pb^{208}) or subshell, to make up more than one alpha particle. If the preformation factor, p_L , is less than unity, then it follows that the value of R calculated from Eq. (48) would be larger than that calculated from the corresponding one-body model. The use of such a preformation factor corresponds to using, with the one-body model, a decay constant which is greater than that observed. A corollary of this point will be discussed further in Section G.

The following argument is now made to show, in somewhat greater detail, the character of the preformation factor that would follow from the concept being presented here. It is supposed that, when the alpha particle is in existence behind the Coulomb barrier, the net interaction between the alpha particle and the daughter nucleus can be represented by some sort of well surrounded by that barrier. It is also supposed that there is only one alpha particle in existence behind the barrier at a time, i.e., it is supposed that there is one species with no alpha particle in the well (hence, no well or alpha-daughter Coulomb barrier) and a second species with one alpha particle in the well. Let λ_1^L be the decay constant of the first species to the second, i.e., let it be, or very nearly be, the alpha decay constant in the absence of an external Coulomb field. Let λ_2^L represent the decay constant of the second species to the first, i.e., let it be the probability per second that an alpha particle at the nucleus (or the nuclear surface) will break up and its constituents be reabsorbed into the nucleus. Thus the second species branches by decaying to the first species with the probability λ_2^L and by emitting alpha particles with the probability λ_0^L . If the generic relationships between these two species are set up the effective alpha decay constant can be determined. With the reasonable assumptions that $\lambda_0^L \ll \lambda_1^L, \lambda_2^L$, it turns out that

$$\lambda_{\alpha}^L = \lambda_1^L \lambda_0^L / (\lambda_1^L + \lambda_2^L) \quad (49)$$

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Thus the preformation factor should be $\left[\lambda_1^L / (\lambda_1^L + \lambda_2^L) \right]$. The use of a one-body model with a preformation factor of unity amounts to assuming that $\lambda_1^L \gg \lambda_2^L$, whereas $\lambda_2^L > \lambda_1^L$ would seem to be more reasonable.

To go into greater detail as to the expected value of the preformation factor [aside from calculations of what value it must have to satisfy Eq. (48) with a particular one-body model] would require the discussion of a nuclear physics problem. This will not be attempted here. It will be noted, however, that the concept is similar to that leading to the use of a sticking coefficient, ξ , as used by Bethe,⁴⁰ Peaslee,⁴¹ Weisskopf and Ewing,⁴¹ or, more particularly, to the probability, P , discussed by Feshbach, Porter, and Weisskopf,⁴² both relating to the ease of coagulation of the incident particle and the target nucleus to form the compound nucleus, in the problem of nuclear reactions. Characteristics of the structure of any incident deuteron or alpha particle or of an emitted alpha particle, would have to be considered when calculating numbers like λ_1^L and λ_2^L in addition to the sort of consideration which would apply to elementary particles.

2. Discussion

The Devaney¹⁷ many-body model of alpha decay might be said to follow from a strict interpretation of Bethe's suggestion⁴³ that the alpha particle is formed only at the moment of its emission. This means that it is not formed unless it is emitted, and implies that the Coulomb barrier itself, which only exists when the alpha particle exists, plays a controlling role in the determination of the probability of the formation of an alpha particle as such. Indeed, this is seen to be the case when the discussion of the level spacing $\underline{D}$ as given by Weisskopf⁴⁴ is examined. In Devaney's model it is the Coulomb barrier itself which ensures the existence of the well-defined states whose spacing is $\underline{D}$. It is alternately proposed here that these states would still be reasonably well-defined in the absence of the Coulomb barrier.

It is fairly common to make such a statement as "a well-defined nuclear surface will be assumed" when the use to which the statement is put shows that the primary emphasis is on "a well-defined channel radius will be assumed." Here it is actually the first of these which is used, i.e.,

⁴⁰H. A. Bethe, Ref. 35, Part D of §54, p. 96.

⁴¹D. C. Peaslee, Phys. Rev. 74, 1001 (1948); V. F. Weisskopf and D. H. Ewing, Phys. Rev. 57, 472 (1940).

⁴²Feshbach, Porter, and Weisskopf, Phys. Rev. 87, 188 (1952).

⁴³H. A. Bethe, Ref. 35, top of first column, p. 164.

⁴⁴V. F. Weisskopf, Helv. Phys. Acta, 23, 187 (1950); Ref. 18, p. 386.

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it is assumed that the nuclear surface of the parent is well-defined and constitutes a strong and sudden change of potential apart from the latent existence of the Coulomb interaction between an alpha particle and the daughter nucleus. If this is correct it follows that, in the absence of the Coulomb barrier, there would be a finite decay constant equal, say, to $[D^*/(2\pi\hbar)]$ multiplied by some number less than unity. The principal role of the Coulomb barrier, in this picture, is to inhibit the complete separation of the two entities: alpha particle and daughter nucleus. This latter inhibition is very strong, and it is presumed that recombination is likely to occur before the alpha particle "leaks" through the barrier.

Whether an existing alpha particle is imagined to move throughout $r < R$, as in the traditional model, or is restricted to $R_0 < r < R$, as in the surface well model, it would be expected that its absorption to re-form the parent would also be delayed. Such a phenomenon has already been envisaged⁴² as applying even to neutrons. Thus the present picture of the alpha decay process poses two specific problems. The fraction of the time that there is an existing alpha particle is a nuclear problem, the solution of which will require consideration of nuclear properties. The rate of leakage of an existing alpha particle through the barrier must be determined after first deciding what one-body potential best represents the situation. It may well turn out that this decision will have to await detailed consideration of the first problem, although some arguments in support of a model of the surface well type will be made later.

If it should turn out, as is assumed to be the case here, that the mean life of an existing alpha particle prior to its escape through the barrier is not only much less than the mean life for such escape, but is also large compared to the period of its motion behind the Coulomb barrier, then the approximation represented by Eqs. (48) and (49) should be good.

G. CERTAIN NUMERICAL COMPARISONS

The principal part of the numerical work to be done as part of this series of reports on alpha decay theory will be deferred to Part IV. Here it is only desired to make a few comparisons which will illustrate some of the broader aspects of the differences between the traditional model and the surface well model.

It is more difficult to give numerical discussions of Devaney's model since the estimates required are not cut and dried. Reference should be made to Devaney's paper¹⁷ and to the text by Blatt and Weisskopf.¹⁸ Devaney has promised a second paper which will contain numerical results. The text mentioned contains a table of some calculated channel radii which are credited to Devaney. No description of the exact source of the level spacings, D^* , on which such calculations must be based appears to be given, however, nor are the values of κ which were used.

Actually it would be illogical to compare Devaney's results directly with those to be given in this section, since the latter are for one-body models. The separate discussion of the preformation factor in terms of nuclear properties, which would be required to complete the many-body problem as it has been pictured here, has not been made. Nevertheless, it is true that the channel radii, as published by Blatt and Weisskopf and calculated with Devaney's model, are not significantly different from those found by the use of one of the more exact formulations of the traditional model. The same problem of "abnormally small" radii occurs, for instance. Furthermore, the subtraction of an "effective" alpha particle radius of 1.2×10^{-13} cm from the channel radius to give a nuclear radius is an additional operation that could be performed just as logically on the "traditional" channel radii.

First, there are shown values which the preformation factor, p_L , must have for a particular example and for various values of the radii involved in order that Eq. (48) be satisfied. This method of comparison allows one to examine the different properties of the two one-body models conveniently, and at the same time allows one to show the indicated difference between the results of each one-body model and those which would be required of a many-body model.

The calculations have been made with $L = 0$, with Po^{212} as an example,⁴⁵ and, as indicated, with both the traditional and the surface well models. The required values of $\log p_0$ are shown in Fig. 2a for the traditional model and in Fig. 2b for the surface well model. The depths of the various wells have been calculated also and are shown in Table V.

⁴⁵Data for the examples have been taken from the Table of Isotopes, Hollander, Perlman, and Seaborg, Rev. Mod. Phys. 25, 469 (1953). Use of this convenient table should not detract from the credit due the original authors, to whom reference is given in the table.

TABLE V

Comparison of Well Depths (in Mev) in Po^{212} Decay For
Different Radii (in 10^{-13} cm).

ΔR R	Surface Well Model			Traditional
	0.5	1.5	2.5	Any
7.75	-78.10	-5.20	3.01	8.22
8.25	-76.85	-4.99	3.08	8.31
8.75	-75.66	-4.78	3.15	8.38
9.25	-74.53	-4.58	3.23	8.44

The surface well model does not necessarily involve more parameters than does the traditional model. It would only do so if it seemed to be still incorrect to associate the R_0 of the surface well model directly with the nuclear radius. Rather it is hoped that R_0 can be associated with the nuclear radius and ΔR with the interaction distance. It is assumed that there must be a preformation factor associated with each model. Thus the difference is that only the sum of the nuclear radius and the interaction distance appear in the traditional model, whereas each of these plays a separate role in the surface well model.

Fig. 2a has been drawn to show this. The required value of $\log p_0$ depends only on R , the channel radius. For comparison with Fig. 2b, however, the R of Fig. 2a has been separated into an interaction distance and a nuclear radius. The abscissa is the nuclear radius parameter, r_0 , i.e., the nuclear radius divided by $(208)^{1/3}$.

In Fig. 2b the fact that $\log p_0$ depends separately on the nuclear radius and the interaction distance, i.e., the width of the well, is clearly indicated. It is also seen that the surface well one-body model ($p_0 = 1$) would yield, for reasonable values of the radii, values of λ_0^O which are much larger than observed decay constants. Thus, values of a preformation factor which are less than unity are called for, whereas, with the

Fig. 2a Values of ratios, p_0 , of experimentally observed decay constant of Po^{212} to those calculated with the traditional model for various values of $\underline{R}$ and ΔR as a function of r_0 [$= (R - \Delta R)(208)^{-1/3}$]. All distances are in units of 10^{-13} cm.

Fig. 2b

Values of ratios, p_0 , of experimentally observed decay constant of Po^{212} to those calculated with the surface well model for various values of R and ΔR as a function of r_0 [$= (R - \Delta R)(208)^{-1/3}$]. All distances are in units of 10^{-13} cm.

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traditional model, values of the radii which are a little small but could still be called reasonable would require the use of a preformation factor which is greater than unity.

As would be expected, the narrow well used in the surface well model turns out to be much deeper, in general, than the wide well of the traditional model. It is unfortunate that some of the calculations were not made just at $\Delta R = 1.2 \times 10^{-13}$ cm, the interaction distance used by Blatt and Weisskopf.¹⁸ It is easily seen, however, that the values of V_0 would be negative (estimated to be $V_0 \approx -11$ Mev), which seems much more reasonable than the rather large positive values given by the traditional model.

The next item of interest is the dependence of the decay constant on L . Since no discussion has been given of the values of λ_1^L and λ_2^L which would be expected on the basis of some detailed model of the nucleus, only the dependence of the one-body λ_0^L on L can be shown here. Details of the computations will not be given, but the general features can be easily described.

If, for the traditional model, Eq. (39) is solved for the first non-zero values of (κR) , it is found that this value increases rather rapidly with L . But κ^2 appears as a factor in Eq. (39a) for λ_0^L , and it is found that for small values of L , κ^2 increases faster than $\exp(-2\omega)$ decreases, as L is increased.²⁸ Values of $\log(\lambda_0^L/\lambda_0^0)$ have been calculated for constant energy and radius [$\eta = 20$, $\rho(R) = 10$, and $R = 9 \times 10^{-13}$ cm; see Eqs. (2a), (2b), and (2c)]. The results are plotted against $L(L+1)$ in Curve a of Fig. 3, where it is seen that the increase in κ^2 causes λ_0^L to go through a maximum at $L \approx 2$.

On the other hand, although Eqs. (47) and (47a) for the surface well model look much more complicated, a little computation soon shows that the values of (κR) which solve Eq. (47) increase much more slowly than is the case with the traditional model. Further, S_g remains nearly constant (for constant energy and radii) and the result is that λ_0^L monotonically decreases with L . Values of $\log(\lambda_0^L/\lambda_0^0)$ for the same parameters as used with the traditional model, plus $\Delta R = 1.2 \times 10^{-13}$ cm, have been plotted against $L(L+1)$ as Curve b of Fig. 3. It is further found that Curve b would lie very nearly on top of a curve of $\log(G_L/G_0)^{-2}$. At $L = 6$, for instance, Curve b goes through -1.537; $\log(G_L/G_0)^{-2}$ is -1.525.

Thus the Gamow treatment of the effect of the angular momentum on the decay constant, which represents an arbitrary departure from the traditional model, is the exact result of the surface well model.

The next comparison is that of the way in which the kinetic energy in the well of the two models changes as L is changed. The exact calculations of the change in kinetic energy from that at $L = 0$ is given as Curve a

Fig. 3 Comparison of $\log(\lambda_0^L/\lambda_0^0)$ for constant energy and radii. Curve a is for the traditional model and Curve b is for the surface well model.

for the traditional model, and Curve b for the surface well model, in Fig. 4. It is seen that the latter difference increases linearly with $L(L + 1)$ whereas the former does not. The calculations were made with the same examples used to compare the changes in decay constant as L is changed.

Since total energy was held constant for these computations, these changes in kinetic energy are actually increases in the depth of the wells. Equations can be worked out for these changes in energy for the surface well model by making use of the fact that if ΔR is not too large, the change in energy is small compared to the kinetic energy at $L = 0$. If the total energy is held constant it is found that

$$(\Delta V)_L \doteq [\hbar^2/2MR_0][L(L + 1)] \frac{(\kappa_0^2 R^2 + g_0^2)[1 + R_0/(g_0 \Delta R)]}{[\kappa_0^2 R^2 + g_0^2 + (Rg_0)/\Delta R]}. \quad (50)$$

If the depth of the well is held constant it is found that

$$(\Delta E)_L \doteq [\hbar^2/2MR_0][L(L + 1)] \frac{[1 + R_0/(g_0 \Delta R)]}{[1 + R/(g_0 \Delta R)]}. \quad (51)$$

In Eqs. (50) and (51) $\kappa_0 R$ is the solution of Eq. (47) for $L = 0$, and g_0 is the value of g at $L = 0$.

This comparison is accompanied by the following suggestion. Suppose that two successive potential functions are sketched as in Fig. 5, so that the base line of the parent potential is coincident with the line which represents the decay energy of the daughter to the ground state of the granddaughter. The argument is now made that there are many nucleon configurations associated with the ground state of the parent, one of which corresponds to there being an alpha particle in the ground level of the well in the first potential function. Similar arguments apply to the ground level and to excited levels in the well of the second potential function. Thus the transition PD should represent that to the ground state of the daughter and PD* to an excited state.

If this suggestion is adopted, tentatively at least, then one must consider whether the various levels in such a well agree with experimental observations on the excited states of nuclei. It is seen that while the energy changes are smaller for the surface well model than for the traditional model and that while the dependence on L is that proposed by Bohr and Mottleson,⁴⁶ nevertheless they are still larger than the level spacings observed experimentally, particularly for the very heavy isotopes. On the other hand, the linear dependence on $L(L + 1)$ is interesting and it is further suggested that the numerical difficulty might be resolved by the use of a more complete Hamiltonian.

⁴⁶A. Bohr and B. Mottleson, Phys. Rev. 89, 316 (1953).

Fig. 4 The increase in kinetic energy in the well as L is increased. Curve a is for the traditional model and Curve b is for the surface well model. Since $\underline{E}$ was held constant the curves represent the increase in the depths of the wells.

Fig. 5 Suggested sketch relating alpha particle levels with nuclear levels of the parent and daughter. The transition PD represents that from the ground state of the parent to the ground state of the daughter and PD* represents that to an excited state of the daughter.

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Finally, some numerical calculations pertaining to the main group and first weak group of alpha particles emitted by even-even isotopes have been made. It was assumed that $L = 0$ for the main group and $L = 2$ for the weaker group. The values $r_0 = \Delta R = 1.2 \times 10^{-13}$ cm were picked arbitrarily and the ratio (p_0/p_2) was calculated. Some preliminary calculations showed that not only was S_S insensitive to changes in L , but it was also insensitive to the small energy differences involved here and to changes in the values of r_0 and ΔR . This last result has been shown, actually, only for $r_0 = \Delta R$. Thus S_S was assumed to be the same for each group from the same isotope, which made it unnecessary to calculate any values of κ for the surface well model.

It was, of course, necessary to find κ in order to calculate (p_0/p_2) for the traditional model, since it is just the large changes in S_T with changes in L which cause that numerical difference between the models which is to be demonstrated here. The same values of r_0 and ΔR were used as with the surface well model. Insensitivity of the traditional model to values of R was not shown here by direct calculation, but it can be shown by comparison with some numbers calculated by Asaro.⁴⁷

Asaro, with the use of Kaplan's⁴⁸ approximation to Preston's²⁸ result, calculated radii with data for the main groups of several even-even isotopes and then used these radii to find values of a so-called τ theoretical (theoretical half-life) for the weaker group emitted by each isotope. This second calculation was again made with Kaplan's equation, i.e., no term which is explicitly dependent on L was included. Asaro then tabulated values of $(\tau_{\text{experimental}}/\tau_{\text{theoretical}})$.

Because of the insensitivity of (p_0/p_2) to the radii (providing, of course, that the same radii are used with each group from a given isotope) it is reasonable, for the moment at least, to refer to these ratios as $(\tau_{\text{experimental}}/\tau_{\text{theoretical}})$ for the weaker group, in the sense used by Asaro. The results are given in Table VI. No arguments can be given for the preference of one model over the other on the basis of the values of these ratios themselves. The drop by a factor of two (roughly) as one proceeds down the table can probably not be explained by any model of this type, i.e., a one-body model with complete spherical symmetry, with a barrier having a vertical inner face, and in which the distances are given literal meanings, i.e., in which the same radii are used for each group. It is suggested, however, that when the principal interest is alpha decay theory as opposed to the empirical use of alpha decay systematics, the effect of L on the exponential term, at least, should be included in such ratios wherever possible. That is, it should be expected that such terms must appear in a complete expression for the alpha decay constant. Thus, here, the assignment of $L = 0$ to the main group and of $L = 2$ to the first weak group seems reasonable.

⁴⁷F. Asaro, UCRL=2180 (June, 1953), Table 19.

⁴⁸I. Kaplan, Phys. Rev. 81, 962 (1951).

TABLE VI

($\tau_{\text{experimental}}/\tau_{\text{theoretical}}$) for the First Weak Group

Parent	a	b	c	d
Cm ²⁴⁴	3.32	1.05	1.73	1.8
Cm ²⁴²	3.22	1.01	1.68	1.7
Pu ²⁴⁰	3.14	0.99	1.66	1.7
Pu ²³⁸	3.41	1.07	1.80	1.9
Pu ²³⁶	4.33	1.36	2.28	2.2
U ²³⁸	2.70	0.85	1.45	1.5
U ²³⁶	2.07	0.65	1.11	1.1
U ²³⁴	2.48	0.78	1.33	1.3
U ²³²	1.89	0.60	1.01	1.1
U ²³⁰	2.78	0.88	1.48	1.5
Th ²³²	1.85	0.58	1.00	1.0
Th ²³⁰	1.80	0.57	0.97	1.0
Th ²²⁸	1.57	0.50	0.84	0.9
Th ²²⁶	1.98	0.63	1.06	1.1
Ra ²²⁶	1.54	0.49	0.84	0.9
Ra ²²⁴	2.28	0.73	1.24	1.25

Column a: Traditional model, $L = 2$.

Column b: Surface well model, $L = 2$. These numbers would be essentially those obtained from the traditional model if only the effect of $\underline{L}$ on the exponential term were included.

Column c: The results if no $\underline{L}$ terms are included. They were obtained by elimination of such terms from the results in Column b and are to be compared with the results obtained by the use of Kaplan's approximation to Preston's formulation of the traditional model, Column d.

Column d: Asaro, Ref. 47.

H. DISCUSSION

1. The Surface Well Many-Body Model

The only characteristic distance appearing in any derivation of the alpha decay constant, which is now in the literature, is the one designated $\underline{R}$. It is presently called the channel radius if one is speaking carefully, but the habit of speaking of it loosely as the nuclear radius persists even in discussions in which it is also differentiated from the nuclear radius. This latter differentiation customarily takes the form of writing $R = R_0 + \Delta R$ at the end of the derivation. If the place $r = R_0$ is sufficiently well-defined that such a relation can be written at the end of the derivation, then it should have been written at the beginning of the derivation. If there were not some sort of change as r passes from $r < R_0$ to $r > R_0$ which, it must be presumed a priori, would have some effect on the course of the derivation, then the place $r = R_0$ would not be sufficiently well-defined to ever allow the consideration of the relation $R = R_0 + \Delta R$. Further, in a model like Devaney's for instance, a strict interpretation of the level spacing D^* in the framework of its definition by Weisskopf⁴⁴ would require that $\underline{R}$ be interpreted as the nuclear radius of the parent (the difference between R_0 for parent and daughter is much less than ΔR , if $R_0 \doteq 1.3 \times 10^{-13} A^{1/3}$ cm and $\Delta R \doteq 1.2 \times 10^{-13}$ cm) since it is, in such a model, the Coulomb barrier of inner radius $\underline{R}$ which prevents disintegration at the first appearance of the alpha particle group at, i.e., just within, the "nuclear" surface.

The surface well many-body model as described here represents an attempt to include the assumption of both a well-defined channel radius and a well-defined nuclear radius in the derivation itself. The present description is certainly approximate. The region in which the alpha particle is assumed to be an entity includes $R_0 < r < R$, for instance. On the other hand, this might not be too bad since the binding energy of the alpha particle is so great.

The boundary condition used at $r = R_0$ represents another approximation in the one-body part of this model. It is, however, an approximation which is consistent with the supposition that the mean life of an existing alpha particle is reasonably long compared to the period of its motion in the region available to it behind the Coulomb barrier. The approximation does not lead to any results known to be invalid; rather it leads to ones which are now used or are otherwise attractive. The monotonic decrease of λ_0^L , the one-body factor in the decay constant, with increasing $\underline{L}$ appears as an exact result; the surface well is deep, corresponding to a large nuclear attraction between the daughter and the alpha particle; and the energy levels in the surface well depend linearly on $L(L + 1)$.

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It is suggested that the Feshbach, Peaslee, Weisskopf method is more properly applicable at $r = R_0$, and that this should be the first method for reconsideration of the boundary condition at the nuclear radius. Without going into detail here, one can at least note that if "formal" resonance energies are defined by making the logarithmic derivative infinite at $r = R_0$ (the approximation of this report) rather than zero, then the argument of the trigonometric factor in the "inside" logarithmic derivative is only shifted by $\pi/2$. One can still apply the discussion of Weisskopf⁴⁴ to write, for instance, that

$$\lambda_1^0 = [D^*/(2\pi\hbar)] \xi, \quad (52)$$

where ξ is a transmission coefficient at the nuclear surface.

Now suppose it is assumed, at least for the ground states of even-even nuclei, perhaps, that

$$\lambda_2^0 = [v/(2\Delta R)] \xi, \quad (53)$$

where v is the velocity in the well. Generally, $[v/(2\Delta R)] \gg [D^*/(2\pi\hbar)]$, so that

$$P_0 \doteq [D^*/(2\pi\hbar)] [(2\Delta R)/v]. \quad (54)$$

If only the leading terms in Eq. (47b) are used in Eq. (47a) for λ_0^0 , it is found that the results for λ_α^0 is identical with Devaney's Eq. (2.21). However, one has now gone through a process by which the levels whose average spacing is D^* are those arising from configurations within a nuclear radius. The effect of treating both R_0 and R as well-defined has been shown. It has been shown that it is consistent to consider R_0 and R as well-defined and, at the same time, to use the customary Gamow retardation of the escape of an existing alpha particle by the "centrifugal" barrier. Nor have the assumptions and approximations which led, by this path, to Devaney's expression for the decay constant destroyed the existence of the deep surface well and the linear dependence of the levels therein on $L(L + 1)$.

2. Concluding Remarks

This work was undertaken when it was discovered that there were so many equations, in the literature and in texts, which related nuclear radii, alpha particle energies and alpha decay constants. It was found that if these equations were put to their usual use, the calculation of nuclear radii, the maximum difference in radius obtained with a single good set of input data was considerably greater than the error in any one of them which could be expected because of the errors in the input data. It was also found that there was sufficient good data to warrant the belief that the time had come to try to bring a little order out of this chaos.

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This primary objective has certainly been accomplished, i.e., it has been demonstrated by several methods of derivation that these various equations, all derived on the basis of the traditional square well nuclear potential surrounded by a sharply cut-off Coulomb potential, were not the result of different theories but rather were the result of different approximations and errors. Indeed it was also found that an equation calculated with WKB connection formulas throughout, i.e., with the square well inserted at the end of the derivation rather than at the beginning, gave the same calculated radius for a given set of data as the equation derived directly for the square well.

The fact that this traditional model does not predict a monotonic decrease in decay constant with increasing relative angular momentum between the alpha particle and the daughter, first pointed out by Preston,²⁸ was also confirmed. Thus it was shown that even if it finally turned out that the Gamow²⁷ treatment of the effect of changing angular momentum correctly represented the ratios of the decay constants for different values of L , there was involved an arbitrary, illogical departure from the traditional model. The implication would be that a different model was called for.

It was originally planned that this work would be wound up with a calculation of radii of alpha emitters (the calculated radius being associated with that of the daughter nucleus) and attempts at the assignment of spin changes on the basis of the consistent solution of the one-body problem with the traditional model. As might have been expected, the authors have gone into various by-ways having to do with: (a) more minor corrections to the traditional one-body model; (b) the many-body problem; and (c) variations in the one-body model. By now the exact form of any calculations which it might seem valuable to make is rather vague.

In connection with the by-ways the following has been done. A model in which there is the same wide well as in the traditional model, but with a nonvertical boundary, has been briefly investigated for S-state decay. The approximations made render the result valid only for very steep walls, but it was shown that the equations smoothly approach those for the square well as the boundary of the well approaches the vertical.

The problem of the wave functions in the external Coulomb field was considered. It was found that, for alpha decay, the replacement of $L(L + 1)$ by $(L + 1/2)^2$ in the usual WKB functions gave functions which were excellent approximations to the Coulomb functions; the latter are the actual solutions of the wave equation involved. This conclusion was reached by actual numerical comparisons of the so modified WKB functions and the Coulomb functions for several values of L and for an energy and radius example which was typical of alpha decay.

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A discussion of the arguments of Serber and Snyder²³ (see also Freedman, et al.²³) was given to show how correction was to be made so that the derived equations should apply to the alpha decay of atoms rather than of bare nuclei. It was concluded that this is done using the observed alpha particle energy plus the recoil energy as the total energy in equations in which the exterior potential is the screened nuclear potential. The difference in electronic binding energy between the parent and daughter atoms is not to be added to the total energy. Effectively, however, the screening correction can be made by simply adding a number, which is equal to the difference in electronic binding energy, to the observed decay energy. On the other hand it was concluded that if one wishes to investigate nuclear energy differences he should use the observed decay energy plus the electron binding energy difference since nuclear mass formulas contain only the Coulomb interaction between protons and not that between the protons and the surrounding electron cloud.

A new one-body square well model has been proposed in which the motion of the alpha particle, when it is behind the Coulomb barrier, is restricted to a narrow well at the surface of the daughter nucleus. This model has the following properties, all of which are more or less contrary to those of the traditional model when reasonable distances are used. The "more or less" depends on the definition of "reasonable." The predicted decay constants are larger than those observed, thus requiring the use of a preformation factor which is less than unity. The decay constant is a monotonically decreasing function of the relative angular momentum. The bottom of the well is at a negative potential energy. The energy levels in a well of constant depth increase linearly with $L(L + 1)$.

In connection with this model a discussion of a preformation factor, i.e., of a factor by which one might go from the one-body to the many-body problem, was given. The concept of the many-body problem presented in connection with the discussion of this preformation factor seems to be different from that now in the literature. The latter seems to envisage alpha decay as a single process, i.e., an alpha particle is not formed unless it is emitted. That presented here, however, assumes that there are periods during which there is an alpha particle in existence behind the barrier, and that one-body considerations apply during these periods. Some support was quoted for this assumption, but the nuclear physics problem of determining the value of the preformation factor on the basis of some theory of nuclear structure was not considered.

Many of the above items are of little value in some aspects of so-called alpha decay systematics, for instance in attempts to guess half lives and alpha decay energies of artificial isotopes. On the other hand, they are expected to be important in any attempt to extend the usefulness of alpha decay data in the study of nuclear structure. In this connection there are at least the following additional items of interest in alpha decay theory,

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some of which are probably minor, some important. The degree to which they have been discussed in the literature varies from none, or a bare mention, to the publication of several papers on the subject. The present authors have only recently come upon them, or have only recently become conscious of them, now that some of what appeared to be more pressing problems are either disposed of or definitely put aside. What, if any, is done about them in the preparation of Part IV remains for the future to decide.

The first two are probably of only minor importance. One of these has already been mentioned in Ref. 30, to which reference is made here. It is the problem of connecting the Coulomb functions for potential energy ($2Ze^2/r$ - constant) close to the barrier to those for potential energy ($-4e^2/r$) up to zero in the region outside the atom. The other is the problem of ionization of the daughter which has been treated theoretically by Migdal¹⁹ and on which experimental work is in hand.²⁰

The next item is related to the screening correction and was suggested by a remark of Curie's.⁴⁹ The screening corrections generally applied are calculated on the basis of a statistical model of the atom, i.e., on the basis of an average electron distribution, or an average potential at each point due to the electrons. But since the decay constant depends so strongly on the barrier, accurate work might require that one take into account the probability distribution of the screening correction. For instance a variation of the screening correction from zero up to twice the average would correspond to a change in decay constant by about a factor of three.

Of considerable importance in the decay of non-even-even isotopes, or even in the decay to excited states of even-even isotopes is the effect of the nuclear shape factor associated with nuclear quadrupole moments. This has been discussed briefly by Hill and Wheeler⁵⁰ in connection with the short half-life of Sm^{147} , and by Rasmussen.⁵¹

Preston⁵² has published two papers on a topic which seems to have come from a suggestion by Gamow, passed on to Preston by Peierls. It has also been discussed by Dancoff.⁵³ This is a non-central electrostatic interaction between an alpha particle in the process of being emitted and the protons in the daughter nucleus and is thus a general discussion of a

⁴⁹D. Curie, Ref. 23, Footnote (1).

⁵⁰D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1952), p. 1133.

⁵¹J. O. Rasmussen, Jr., UCRL-2431 (December 1953).

⁵²M. A. Preston, Ref. 13; Phys. Rev. 82, 515 (1951).

⁵³S. M. Dancoff, AECD-2853 (Declassified May, 1950).

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topic, one specific aspect of which is the quadrupole moment interaction.^{50,51} One can think of this interaction as being a perturbing potential which causes transitions after the alpha particle has been emitted in a given state, although it turns out that ordinary perturbation theory cannot be used when the states involved are as close together as, for instance, the ground state and the first excited state of the heavy even-even alpha emitters. Presumably this interaction would be a factor in any alpha decay; Preston proposes, in particular, in the case of the very low intensity maximum energy transitions observed in some odd isotopes, that the direct formation of the high energy alpha particle is completely forbidden and that the few which are observed occur as a result of this interaction.

Preston has so written his results that, given (essentially) a relative intensity ratio between two groups (to the present he has only worked with two levels) of alpha particles at the nucleus he can find the intensity ratio at infinity, i.e., the observed intensity ratio. The relative intensity ratio at the nucleus would be given by the relative preformation factors. On the other hand, as Preston says, it appears to be impossible as yet to separate, with any certainty, "experimental" intensity ratios at the nucleus (deduced from some reasonably simple theory and the observed ratios) into those factors due separately to angular momentum, "formation" prohibition, and non-central electrostatic interaction.

Consideration should be given to the proposal of Chaudhury⁵⁴ that a better approximation to the shape of the barrier than a sharply cut-off Coulomb repulsion is necessary. The barrier must actually be the net result of the long-range Coulomb repulsion and a short range nuclear attraction. Two types of approximation which would achieve his proposed reduction in "cross-sectional area" of the barrier, and which could be used separately or together, are to make the top of the barrier flat from R out to some (parametric) distance, and to use a well with a nonvertical boundary such as was discussed briefly in Part IIb.³ Such a reduction in the size of the barrier, if used with a model in which the whole region behind the barrier is accessible to the alpha particle, as in the traditional model, would then require the use of a preformation factor less than unity in conjunction with distances which have been called reasonable in this report. One would probably still find, however, a dependence of λ_0^L on L similar to that which occurs in the traditional model. It should be remembered that the equations of Section D would not apply where the barrier has been altered as described in this paragraph.

Finally, to back-track a moment, it is believed that the most important problem to be solved now is that of the preformation factor. The present authors do not believe the method of Feshbach, Peaslee and Weisskopf²⁵ as applied by Devaney¹⁷ to be the only alternative. Indeed, the former put forth certain details of their method with the admonition that they could only be considered a rough approximation to the true state of affairs.

⁵⁴M. L. Chaudhury, Phys. Rev. 88, 137 (1952).

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APPENDIX

It might be in order to review the notation, particularly since there are minor variations in this report from that used in the previous reports of this series.¹⁻³ The notation is a mixture of that used by Bloch, *et al.*,¹⁰ Bethe,⁵⁵ and Preston.²⁸ Bethe's $\underline{C}$ has been replaced by Preston's ω on the grounds that $\underline{C}$ is generally used as a constant of some sort. Bethe wrote the quantity $|\Phi|$ as $|\Phi(\rho)|$ in certain places, but did not bring η into evidence. It is believed to be preferable to use the ρ as a notation for $\underline{kr}$, and to reserve the symbol $\underline{x}$, which was used for $\underline{kr}$ by Preston and for $\underline{kR}$ by Devaney,¹⁷ for its older meaning as the ratio of the energy to the barrier height.

Thus, the radial wave function in $r > R$ is written as

$$\psi(r) = \phi(r)/r \quad (55)$$

with

$$\phi(r) \propto G_L(\rho) + iF_L(\rho) \quad ; \quad (55a)$$

$G_L(\rho)$ and $F_L(\rho)$ are the irregular and regular Coulomb functions, respectively, of

$$\rho = kr \quad , \quad (56)$$

where

$$k = (2ME)^{1/2}/\hbar \quad . \quad (56a)$$

At very large ρ , the asymptotic behavior of G_L and F_L are

$$G_L(\rho) \rightarrow \cos [\rho - (L\pi)/2 - \eta \ln 2\rho + \sigma_L] \quad , \quad (57)$$

$$F_L(\rho) \rightarrow \sin [\rho - (L\pi)/2 - \eta \ln 2\rho + \sigma_L] \quad , \quad (58)$$

where

$$\eta = (Z'Ze^2/\hbar) [M/(2E)]^{1/2} \quad (58a)$$

and

$$\sigma_L = \arg \Gamma(L+1+i\eta) \quad . \quad (58b)$$

⁵⁵H. A. Bethe, Ref. 35, p. 161 ff.

In equation (58a), Z' is the charge number on the particle ($Z' = 2$ in this case) and Z is the charge number on the nucleus (the daughter, in this case).

As a result of the discussion in Section B it is deemed proper to write for the case when $kR \leq \rho \ll kr_E = \rho_E$,

$$G_L(\rho) = |\Phi(\rho)|^{-1/4} \exp [\omega(\rho)] , \quad (59)$$

$$F_L(\rho) = 1/2 |\Phi(\rho)|^{-1/4} \exp [-\omega(\rho)] . \quad (60)$$

Here r_E is the location of the classical turning point on the outside face of the barrier,

$$|\Phi(\rho)| = \frac{2\eta}{\rho} + \frac{(L + 1/2)^2}{\rho^2} - 1 \quad (60a)$$

$$\omega(\rho) = \int_{\rho}^{\rho_E} |\Phi(\rho)|^{1/2} d\rho . \quad (60b)$$

This present $|\Phi(\rho)|$ is just k^{-1} times the $|\Phi(r)|$ used previously.¹⁻³ Thus, whereas the previous $|\Phi(r)|^{1/2}$ had units of reciprocal distance, the present $|\Phi(\rho)|^{1/2}$ is unitless.

For computation, Eq. (36a) can be written as

$$\lambda = 4E(\hbar kRS')^{-1} [G_L(kR)]^{-2} , \quad (61)$$

with

$$S' = [-\hbar^2/(MR^2)] [\partial f^I(W)/\partial W]_E + (1 - \gamma) g_L^{-1} . \quad (61a)$$

Eq. (61) can be further rewritten as⁵⁶

$$\log \tau = 2 \log [G_L(kR)] + \log \left[\frac{ARS'}{(A + 4)E_{\alpha}^{1/2}} \right] - 22.30185 , \quad (61b)$$

⁵⁶The numerical constants given in this Appendix have been based on the results of Du Mond and Cohen,¹⁶ which see for further necessary references. The constants tabulated for various heavy isotopes in part I¹ were based on Du Mond and Cohen, Rev. Mod. Phys. 20, 82 (1948). The difference amounts to about 2.5 per cent in the decay constant for a given R.

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where, in the second term, $\underline{A}$ is the mass number of the daughter, $\underline{R}$ is in units of 10^{-13} cm, and E_α is the observed alpha particle energy in Mev, "corrected" as described in Section C. The first term of Eq. (61b) is given by Eq. (18) in which

$$D = 1.09449 [ZA/(A + 4)] \quad , \quad (62)$$

$$x = E_\alpha R/K \quad , \quad (62a)$$

$$f_0(x) = \cos^{-1} x^{1/2} - [x(1 - x)]^{1/2} \quad (62b)$$

$$\begin{aligned} kR |\Phi_0|^{1/2} &= kR \left[\frac{2\eta}{kR} - 1 \right]^{1/2} \\ &= 1.15129 (D/E_\alpha^{1/2}) [x(1 - x)]^{1/2} \quad . \end{aligned} \quad (62c)$$

Again, in Eqs. (18) and (62) to (62c), E_α is the "corrected" alpha particle energy in Mev, $\underline{R}$ is in units of 10^{-13} cm and $\underline{A}$ is the mass number of the daughter. The function $f_0(x)$, plotted in Part I,¹ is, of course, unaffected by any changes in the value of the fundamental constants.⁵⁶

The number g_L which appears in Eq. (61a) (and elsewhere) is defined in Eq. (39c) and can be computed from

$$\begin{aligned} g_L^2 &= (1 + \gamma)^2 k^2 R^2 |\Phi| \\ &= (1 + \gamma)^2 [k^2 R^2 |\Phi_0| + (L + 1/2)^2] \quad , \end{aligned} \quad (63)$$

where, as in Eq. (61a) (and elsewhere)

$$\gamma = -[4kR |\Phi|^{1/2} (1 - x)]^{-1} [1 + (k^2 R^2 |\Phi|)^{-1} (1 - 2x)(L + 1/2)^2] \quad . \quad (63a)$$

This number, γ , is generally about -0.02 (and, hence, is frequently omitted); the correction term in Eq. (63a) is only about 0.05 at $L = 5$.