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Negative Temperature States
for the Two-Dimensional
Guiding-Centre Plasma

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ABSTRACT

Theoretical development and numerical simulation are presented for the two-dimensional electrostatic guiding centre plasma with positive total interaction energy. This is a system for which equilibrium statistical mechanics predicts that no spatially homogeneous thermal equilibrium state will exist. The non-existence of the homogeneous thermal equilibrium state is associated with the phenomenon of "negative temperatures." Quasi-stable spatially inhomogeneous states are shown to form, and are characterized by macroscopic spatially-separated vortex structures.

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1. Introduction

Some interest has arisen lately in the equilibrium and non-equilibrium statistical mechanics of an electrostatic guiding-centre plasma in two dimensions (see, e.g., Taylor & McNamara 1971; Vahala & Montgomery 1971; Montgomery 1972a). The mathematical description of this system is identical to that of a collection of "line" vortices in two-dimensional ideal, inviscid, incompressible hydrodynamics, and we may draw on the literature of that subject for indications of interesting effects. In particular, this present paper explores several implications of the prediction (Onsager 1949) that above a certain value of the interaction energy, no uniform, quiescent, thermodynamic equilibrium state should exist. This prediction is associated with the phenomenon of "negative temperatures." We are concerned here with an effort to uncover, by analytical and numerical methods, the details of these negative temperature states.

§ 2 reviews previous analytical work on the subject. § 3 reports the results of a series of numerical simulations. § 4 presents an analytic framework which enables one to calculate some features of the negative temperature states. The results are briefly discussed in § 5.

The numerical work is primarily due to Joyce, and the analytical work to Montgomery.

2. Vortex Motion and Negative Temperatures

(a) Statistical-mechanical Prediction of Negative Temperatures

The electrostatic guiding-centre plasma in two dimensions may be imagined as a collection of very long uniformly charged rods that move so that they remain aligned parallel to a uniform constant magnetic field $\vec{B}$. $\vec{B}$ may be imagined to point in the z-direction. The j^{th} rod, with charge e_j and position $\vec{x}_j$, moves with the guiding-centre drift velocity

$$\vec{v}_j = \frac{d\vec{x}_j}{dt} = \frac{c\vec{E}(\vec{x}_j, t) \times \vec{B}}{B^2} = \frac{c\vec{B} \times \nabla\phi(\vec{x} = \vec{x}_j, t)}{B^2} \quad (1)$$

in the x and y directions. The electric field is purely electrostatic and the scalar potential $\phi(\vec{x}, t)$ is self-consistently determined from the two dimensional Poisson equation

$$\nabla^2\phi = -4\pi \sum_j (e_j/\ell) \delta(\vec{x} - \vec{x}_j) \quad (2)$$

The length of the very long rods is ℓ . The $\sum_j$ runs over N rods of charge $+e$ and N more of charge $-e$ in a spatial volume $V = L^2$. The vectors $\vec{x}_j$, $\vec{v}_j$, and $\vec{E} = -\nabla\phi$ are all vectors with x and y components only.

The above dynamics can be readily put in Hamiltonian form. If the cartesian coordinates of the j^{th} rod are $\vec{x}_j = (x_j, y_j)$, the canonical Hamiltonian variables are

$$(q_j, p_j) = |K_j|^{1/2} (x_j, y_j \operatorname{sgn} K_j) , \quad (3)$$

where $K_j \equiv -4\pi c e_j / \ell B$. The Hamiltonian function is

$$H = -\frac{1}{2\pi} \sum_{i < j} K_i K_j \ln |\vec{x}_i - \vec{x}_j| . \quad (4)$$

It can be readily verified that Hamilton's equations

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} , \quad \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} \quad (5)$$

express the same dynamics as equations (1) and (2). Since H is not explicitly time dependent, it is a constant of the motion, and up to a multiplicative constant, is simply the potential energy of interaction among the $2N$ Coulomb line charges. There is no kinetic energy in this problem, in the usual sense.

Equations (3), (4), and (5) also describe the motion of a collection of discrete "line" vortices in two-dimensional, inviscid,

incompressible hydrodynamics. The quantity K_j is to be interpreted as the "vortex strength" of the j^{th} vortex. We are thus able to use language appropriate to the two systems interchangeably.

Now consider the equilibrium statistical mechanics of the $2N$ charges inside the box of volume $V = L^2$. The interesting feature of the system which contrasts with most classical Hamiltonian systems considered in statistical mechanics is that it has only a finite amount of total phase space available (Onsager 1949). If we define the "structure function" $\Omega(\mathcal{E})$ (e.g., Khinchin 1949) as the phase space volume per unit interaction energy $\mathcal{E}$, then

$$\Phi(\mathcal{E}) \equiv \int_{-\infty}^{\mathcal{E}} d\mathcal{E}' \Omega(\mathcal{E}')$$

increases monotonically from zero to $\prod_j (|K_j| L^2)$ as $\mathcal{E}$ goes from $-\infty$ to $+\infty$. The product is over all the charges (vortices). $\Phi(\mathcal{E})$ is the total phase space volume with values of $H < \mathcal{E}$. $\Omega(\mathcal{E}) = d\Phi(\mathcal{E})/d\mathcal{E}$ will therefore have a maximum value at some $\mathcal{E} = \mathcal{E}_m$, say. Now it can be shown from either the canonical or the grand canonical ensemble that the equilibrium thermodynamic entropy S and temperature T to be associated with a macroscopic system of energy $\mathcal{E}_0$ and structure function $\Omega(\mathcal{E})$ are

$$S = k \ln \Omega(\epsilon_0) + \text{constant}$$

$$\frac{1}{kT} = \frac{1}{\Omega(\epsilon_0)} \frac{d\Omega(\epsilon_0)}{d\epsilon_0} \quad (6)$$

(k is Boltzmann's constant.)

For a system with a maximum $\Omega(\epsilon_m)$, equations (6) predict that for $\epsilon_0 > \epsilon_m$, the temperature T will be negative. Landau and Lifshitz (1969; see pp. 35-37) show on quite general grounds that for a system whose component parts are free to move relative to each other, no spatially-uniform, quiescent, thermal equilibrium state can be expected for a system with $kT < 0$. The derivation which leads to equations (6) has broken down. The question naturally arises: What characteristics will the collection of charges (vortices) exhibit when their energy is greater than ϵ_m ?

Some indication can be obtained from considering the limit of very large ϵ (Montgomery 1972b). The highest energy states are clearly those in which the charges of positive sign are crowded as close together as possible and those of negative sign are crowded together as far away from the positives as possible. The macroscopic, large-scale configuration this corresponds to is that of two large counter-rotating vortices physically well separated in the box. The electric field in the neighborhood of each one will be radial, and the fluid velocity azimuthal. Though the fluid velocity will be non-zero, it is not inconsistent that it might be time-independent in the limit of very large N.

For states with $\mathcal{E}$ only slightly greater than $\mathcal{E}_m$, the separation will be neither so complete nor so dramatic, but there will still be a tendency for the vortices of like sign to group themselves together on a macroscopic scale. This means we should abandon any hope of fitting the situation with any of the various fluid theories of "homogeneous turbulence" (e.g., Batchelor 1953) in which correlation functions of velocities depend only upon the spatial separations of points, but not upon their absolute spatial locations. This should be recognized as representing something of a break with most classical quantitative theories of hydrodynamic turbulence, which make the assumption of spatial homogeneity at the outset and take it for granted that all the important features of the turbulence are contained in the spectral density function.

(b) Taylor's Calculation of $\mathcal{E}_m$

Taylor (1972) has given a calculation for $\mathcal{E}_m$ which will be briefly remarked upon here. The procedure is to calculate an approximate expression for $\Omega(\mathcal{E})$ by the method of stationary phase, and to ask at what value of $\mathcal{E}$ the expression $d(\ln \Omega(\mathcal{E}))/d\mathcal{E}$ goes from positive to negative. One important point to be aware of in Taylor's (1972) calculation is that his expression for $\mathcal{E}$ differs from Onsager's (1949) and ours (equation (4)) by an additive constant which corresponds to the infinite Coulomb self-energy of the $2N$ line charges. (This fact is responsible for the necessity for

a maximum wave number in his Fourier transforms.) We have repeated Taylor's calculation after subtracting off this infinite additive constant, and find the result

$$\epsilon_m = 0. \quad (7)$$

This result is in fact, fully in agreement with Taylor's equation (8), which is nothing but the total self-energy of the line charges.

This result is extraordinary in that it says all states with positive interaction energy will correspond to the negative temperature regime. We may imagine (Vahala & Montgomery 1971; Montgomery 1972a) a "Vlasov limit" of equations (1) and (2) in which the discrete charges are allowed to become a continuous fluid whose (smooth) number densities $n_{\pm}(\vec{x}, t)$ obey the equations

$$\frac{\partial n_{\pm}}{\partial t} + \frac{c\vec{B} \times \nabla \varphi}{B^2} \cdot \nabla n_{\pm} = 0 \quad (8)$$

with

$$\nabla^2 \varphi = \frac{-4\pi e}{\ell} (n_+ - n_-) \quad (9)$$

In this limit, the interaction energy per particle in the spatially uniform state $n_+ = n_- = N/V$ vanishes, in the usual way. (This limit is equivalent to ideal incompressible hydrodynamics in two dimensions with zero viscosity.) The interaction energy for any state which is perturbed from spatial uniformity in this "Vlasov" limit is then proportional to $\int d\vec{x} (\nabla\phi)^2$, which is a positive-definite functional of the displacements. Thus any finite perturbation at all can be expected to put us on the negative temperature side of the boundary for sufficiently low values of the plasma parameter (discreteness parameter).

A picture of typical negative temperature states can be imagined as follows. There will in general be no steady state, but there will be concentrations of charge (vorticity) in stable or nearly-stable aggregations. These may sometimes roam about the volume, and probably charged clumps of the same sign will have a tendency to coalesce with each other, at least under some circumstances. (The phase space volume corresponding to a given total energy is probably increased by decreasing the number of vortices among which it has to be shared.) Indeed it was evidence of the correctness of this picture generated from computer solutions of the two-dimensional continuum Navier-Stokes equations at high Reynolds numbers (Deem & Zabusky 1971; Tappert & Hardin 1971) that re-awakened interest in the concept of negative temperatures (Montgomery 1972b; Joyce & Montgomery, 1972; Taylor, 1972).

In the next section, some computer simulation results for the discrete-vortex model are presented. An analytical discussion of the negative temperature states is deferred until § 4.

3. Numerical Simulation

The guiding-centre plasma equations (1) and (2) can be put in dimensionless form. In a set of dimensionless variables which are described in the Appendix, the equation of motion of the j^{th} charge (or vortex) can be written as

$$\frac{d\vec{x}_j}{dt} = \vec{E}_j \times \hat{e}_z \quad . \quad (10)$$

$\vec{E}_j = \vec{E}(\vec{x}_j, t)$ is the electrostatic field, evaluated at the location of the charge. It can be determined from $\vec{E} = -\nabla\phi$, where Poisson's equation is

$$\nabla^2\phi = -2\pi D(n_+ - n_-) \quad , \quad (11)$$

and the exact number densities $n_{\pm}$ for the two signs of charge are

$$n_{\pm} = \sum_{j=1}^N \delta(x - x_j) \delta(y - y_j) \quad . \quad (12)$$

$2\pi D$ is a constant which is essentially determined by the magnitude of the charge (vortex strength).

Equations (10), (11), and (12) have been solved numerically on an IBM 360/65 using nearly standard simulation techniques. The algorithm used was a second-order scheme in two steps:

$$\begin{aligned}\vec{x}_j^{n+\frac{1}{2}} &= \vec{x}_j^n + \frac{1}{2} \Delta t \vec{E}(\vec{x}_j^n) \times \hat{e}_z \\ \vec{x}_j^{n+1} &= \vec{x}_j^n + \Delta t \vec{E}(\vec{x}_j^{n+\frac{1}{2}}) \times \hat{e}_z, \end{aligned} \quad (13)$$

where j is, as before, the particle index and n labels the time step. The values of the electric field were determined from Poisson's equation (Fourier-transformed) by Fast Fourier Transform (F.F.T.) methods. The density assignments for Poisson's equation were accomplished by particle-in-cell (P.I.C.) area-weighting techniques. The code is in many ways similar to the VORTEX code of Christiansen (1969, 1971).

The particles were confined to a rectangle of 16×32 mesh points with perfectly reflecting boundaries. (Perfect reflection means that the tangential component of $\vec{E}$ vanishes at the boundary; or for the hydrodynamic problem, that the boundary is a streamline.) Perfect reflection was achieved by replacing the effect of the walls by an infinite doubly-periodic array of image charges with a period twice the box dimension. Thus, if we had a square box of edge L and there were a positive charge at $(x,y) = (a,b)$, there would be positive

images at $(2nL-a, 2nL-b)$ and negative images at $(2nL-a, b)$ and $(a, 2nL-b)$, where n runs from $-\infty$ to $+\infty$ in integer steps. (The generalization to a rectangle is obvious.) If these symmetries exist at $t = 0$, the equations of motion guarantee that they hold for all time, and we may forget about the boundary. The total number of particles in the basic box was $2N = 4008$. The image system was imposed, in accordance with the above symmetry principles, at each time step.

The electric field Fourier-analyzed appears as

$\vec{E} = \sum_{\vec{k}} \vec{E}_{\vec{k}} \exp i\vec{k} \cdot \vec{x}$, where $\vec{k} = \pi (n_x/L_x, n_y/L_y)$, where n_x and n_y are integers and L_x and L_y are the x and y dimensions of the box. The normalized total energy of the system is calculated from the formula

$$W = \frac{\sum_{\vec{k}} \left| \frac{\vec{E}_{\vec{k}}}{k} \right|^2}{\text{S.E.}} \quad (14)$$

where S.E. stands for self energy. If particle positions are loaded completely randomly within the box, the expectation value of W should be unity. We determined S.E. as the average over 100 sets of initial sets of such random numbers for the (x_j, y_j) . Such a large sample was required, because, as is pointed out in the work of Taylor & McNamara (1971), the expectation value of the fluctuation $\langle (W - \langle W \rangle)^2 \rangle / \langle W \rangle^2$ does not become small for large N .

Changes in W are effected by systematic changes of the initial spatial loading of the particles. Note that, from the remarks in § 2 and Taylor's calculation (1972), we should be in the unstable regime ($T < 0$) when $W > 1$. Thus the initial loading is one of the most important aspects of the simulation.

Two different loading methods were used to achieve $W > 1$ and $W < 1$. For the high energy (large W) runs, the particles were loaded into 16 regularly positioned squares, each one containing the same number of either positive or negative charges. The signs of the charges in the squares alternated. The positions inside the squares were chosen randomly. By decreasing the sizes of the squares, we may bring charges of like signs closer together, thus raising the energy. By varying the square size, it has been possible to vary W by about a factor of 100. When the size of the square was of the order of the box size, the configuration approached a random loading, and the energy of interaction was nearly zero ($W \approx 1$).

The above method will not produce negative interaction energies ($W < 1$). These energies were obtained by randomly loading pairs of positive and negative charges. The minimum value of W is zero and is achieved when the distance between the pairs is zero (since in the P.I.C. model, particles at the same point inside the same cell give no electric field when the signs are opposite). When the distance between the pair becomes of the order of the box size, W approaches 1 from below. Runs for other loading schemes were

carried out, but the results appear to be independent of the particular loading scheme used.

The evolution of the collections of vortices are in general agreement with the predictions of § 2. For $W \gg 1$, the system separates into two spatially well-separated groups of charge of opposite sign. This can be seen by looking at the spatial plots of particle coordinates in figures 1a and 1b. (Figure 1 is done for a rectangular box, with the length twice the width.) In figures 1a and 1b which is the higher-energy case, the initial pattern broke up before $t = 10$, and the large vortices were formed by $t = 50$, remaining largely unchanged thereafter.

As W is lowered to values only slightly greater than 1, the spatial separation is not distinct enough to be visually discernible (figures 1c and 1d). More subtle diagnostics are required to explore the neighborhood of $W = 1$. The clearest indication of the nature of the configuration is provided by $|\vec{E}(k_x, k_y)|^2$, which is proportional to the amount of energy in the various k_x, k_y modes. $\mathcal{E}(k_x, k_y)$ is this quantity, normalized to unit total volume. The maximum-energy state is one with all the particles of one sign at the point $(x, y) = (8, 8)$ and all those of the other sign at $(x, y) = (8, 24)$. [The corners of the box are $(0, 0)$, $(16, 0)$, $(16, 32)$ and $(0, 32)$.] The energies of these charges and their images are exactly calculable analytically, and have a characteristic Fourier spectrum which is shown in figure 2. It is convenient to plot the normalized energy per unit wave number

$\mathcal{E}(k_x, k_y)$ as a function of (n_x, n_y) , where

$$(k_x, k_y) = \left(\frac{2\pi}{2L_x} n_x, \frac{2\pi}{2L_y} n_y \right) = \frac{\pi}{16} \left(n_x, \frac{n_y}{2} \right).$$

The basic mode which, together with its harmonics, contains most of the energy in the maximum-energy state is $(n_x, n_y) = (1, 2)$, as seen in figure 2. For comparison, the Fourier spectrum corresponding to the random distribution, with the same numbers of particles, is shown in figure 3.

Figures 4, 5, and 6 show spectra for $W = 99.0$, 16.7 , and 2.3 , respectively. All share with the maximum-energy state the feature of a prominent peak at $(n_x, n_y) = (1, 2)$, which contains a significant fraction of the energy. It should be clear that this feature, which corresponds to a similarly spatially non-uniform state in each case, will not be removed by ensemble averaging. A homogeneous, isotropic formulation of the theory of turbulence would miss the essential features of this state.

A typical state with $W < 1$, which should correspond to $T > 0$, appears in figure 7. The most interesting differences we have noted in the spectra with $W > 1$ and $W < 1$ are that in the latter, the latter, the large peak at $(n_x, n_y) = (1, 2)$ is absent, and the energy is much more broadly distributed in $\vec{k}$ space. The capacity of the eye to discern differences in the arrangements of particles in (x, y) -space is lost well above $W = 1$. (See figure 8 for a thermal equilibrium spectrum.)

4. Analytic Models of Negative Temperature States

We now present the results of an attempt to construct a quantitative picture of the negative temperature states. The first serious attempt to construct an equilibrium statistical mechanics for a spatially non-uniform medium was apparently van Kampen's (1964) model of the condensation of a van der Waals gas. The onset of negative temperatures in our system is superficially similar to a condensation phenomenon, in that it involves a transition from a state which is macroscopically spatially uniform to a state which is not. Like van Kampen, we arrive at a time-independent integral equation for the density which we are unable to solve, but which is amenable to an approximate treatment. We utilize van Kampen's cellular technique to compute phase space volumes, but the details of our calculation are, of course, very different.

We imagine a spatial volume $V = L^2$ subdivided into very small cells of volume Δ , $\Delta \ll V$. The cells are large enough, however, to contain many particles. We call N_i^+ and N_i^- the number of positive and negative charges inside cell i . The dynamical state of the system can be specified completely by giving all the N_i^+ and N_i^- .

We do not attempt to distinguish between the locations of particles within the same cell. The cell size will eventually shrink to zero. We write, for the total energy,

$$\mathcal{E} = \frac{1}{2} \sum_{i,j} (N_i^+ - N_i^-) \varphi_{ij} (N_j^+ - N_j^-) \quad (15)$$

$\varphi_{ij} = -\frac{2e^2}{l} \ln |\vec{r}_i - \vec{r}_j| + \text{const.}$, and is the Coulomb interaction. $\vec{r}_i$ and $\vec{r}_j$ locate the centers of the i^{th} and j^{th} cells. Without loss of generality, we can imagine the additive constant in φ_{ij} to be so large that $\varphi_{ij} > 0$ for all i and j within the box. Equation (15) is invariant under addition of a constant to φ_{ij} . We may also imagine the Coulomb potential to be cut off at some short distance so that $\varphi_{ii} < \infty$. Defining $\rho_i \equiv N_i^+ - N_i^-$, we see that $\mathcal{E} = \frac{1}{2} \sum_{i,j} \rho_i \varphi_{ij} \rho_j$ is a positive definite functional of the ρ_i , and that the spatially uniform state ($N_i^+ = \frac{N\Delta}{V}$, all $\rho_i = 0$) has zero energy which is also the minimum allowable energy. All positive-energy states will necessarily be non-uniform ones involving non-zero ρ_i 's. Negative-energy states cannot be constructed in this model.

Using the elementary combinatorial method of Boltzmann, we now ask for the most probable values of the N_i^+ and N_i^- , subject to the constraints that $\mathcal{E}$ in equation (15) shall have a prescribed positive value and that

$$\sum_i N_i^+ = N = \text{const.} \quad (16)$$

$$\sum_i N_i^- = N = \text{const.} \quad (17)$$

For the a priori probability of any given N_i^+ configuration, we take the conventional expression (see e.g., ter Haar 1966):

$$\mathcal{W} = \left\{ N! \prod_i \frac{\Delta^{N_i^+}}{N_i^+!} \right\} \left\{ N! \prod_i \frac{\Delta^{N_i^-}}{N_i^-!} \right\} \quad (18)$$

where the products run over all the cells. We assume the N_i^+ to be so large that Stirling's formula applies to the factorials in (18). Finding the stationary states of $\mathcal{W}$ is the same as finding the stationary states of

$$\begin{aligned} \ln \mathcal{W} \cong & 2(N \ln N - N) \\ & + \sum_i \{ N_i^+ \ln \Delta - N_i^+ \ln N_i^+ + N_i^+ \\ & + N_i^- \ln \Delta - N_i^- \ln N_i^- + N_i^- \} \end{aligned} \quad (19)$$

The constraints (15), (16), and (17) are to be taken into account by the method of Lagrange multipliers in the usual way, treating the variations of N_i^+ and N_i^- as independent. The results of setting the coefficients of δN_i^+ equal to zero are

$$\ln N_i^+ + \alpha^+ + \beta \sum_j \varphi_{ij} (N_j^+ - N_j^-) = 0$$

$$\ln N_i^- + \alpha^- - \beta \sum_j \varphi_{ij} (N_j^+ - N_j^-) = 0. \quad (20)$$

The three constants (α^+ , α^- , and β) in equations (20) are to be determined by imposing the constraints (15), (16), and (17), with given values of N and $\mathcal{E}$.

Adding equations (20) together gives the relation

$$N_i^+ N_i^- = \exp(-\alpha^+ - \alpha^-) = \text{const.}, \quad (21)$$

all i . Subtracting them gives:

$$\begin{aligned} N_i^+ - N_i^- &= \exp\{-\alpha^+ - \beta \sum_j \varphi_{ij} (N_j^+ - N_j^-)\} \\ &\quad - \exp\{-\alpha^- + \beta \sum_j \varphi_{ij} (N_j^+ - N_j^-)\} \end{aligned} \quad (22)$$

These last relations can now be replaced by an integral equation by passing to the limit of zero cell size (continuum limit). The discretization was mainly useful in order to accomplish the combinatorics.

We let

$$\Delta \rightarrow 0$$

$$N_i^+ - N_i^- = \rho_i \rightarrow \sigma(\vec{r})\Delta$$

$$\frac{\exp(-\alpha^\pm)}{\Delta} \rightarrow n_\pm$$

$$\pm \beta \sum \varphi_{ij} \rho_j \rightarrow \pm \beta \int d\vec{r}' \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}')$$

where $\varphi(r) = -2(e^2/\ell) \ln r + \text{const.}$, now. Without loss of generality, the constant can be set equal to zero. This gives the integral equation

$$\begin{aligned} \sigma(\vec{r}) = & n_+ \exp \left\{ -\beta \int d\vec{r}' \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}') \right\} \\ & - n_- \exp \left\{ +\beta \int d\vec{r}' \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}') \right\} \end{aligned} \quad (23)$$

which is perhaps the most basic relation of the problem.

The above integral equation can, alternatively, be written as a partial differential equation by defining

$$\psi(\vec{r}) \equiv e\Phi(\vec{r}) \equiv \int d\vec{r}' \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}') \quad (24)$$

$$\nabla^2 \psi = -(4\pi e^2/\ell) \sigma(\vec{r}) \quad (25)$$

to get

$$\nabla^2 \psi = -(4\pi e^2/\ell) \{ n_+ e^{-\beta\psi} - n_- e^{\beta\psi} \} . \quad (26)$$

$\Phi \equiv \psi/e$ has the physical interpretation of being the scalar potential, and $(e/\ell)n_{\pm} \exp(\mp \beta \psi)$ represents the positive (negative) electrical charge density. Note that since $(\vec{B} \times \nabla \psi) \cdot \nabla \sigma = 0$, any solution of (23) or (26) is automatically a time-independent one. $n_{\pm}$ are positive constants that are to be determined by the requirements that

$$n_{\pm} \int d\vec{r} \exp(\mp \beta \psi) = N. \quad (27)$$

The constant β is to be determined from the requirement that the total energy has the value $\mathcal{E}$:

$$\ell \int d\vec{r} \frac{(\nabla \psi)^2}{8\pi e^2} = \mathcal{E}. \quad (28)$$

$\mathcal{E} > 0$, but notice that we do not yet know the sign of β . Since it enters in the way $1/kT$ usually does, we suspect it may be negative.

If β were positive, (26) would be similar to the familiar Debye-Poisson equation in two dimensions. Its general solution is not known. In the present context, it is not clear for either sign of β what would be appropriate boundary data in order to determine a unique solution of (26) that would satisfy the constraints (27) and (28). For the time being, approximate methods are indicated.

A fairly detailed treatment can be given for the case of low energies, $\beta^2 \psi^2 \ll 1$. For this purpose it is convenient to return to the integral equation formulation, (23), which can be

written, when $\beta^2 \psi^2 \ll 1$, as:

$$\begin{aligned} \sigma(\vec{r}) = n_+ \{ 1 - \beta \int d\vec{r}' \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}') \} \\ - n_- \{ 1 + \beta \int d\vec{r}' \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}') \} . \end{aligned} \quad (29)$$

The solutions will be periodic with equal average number densities, so we may set $n_+ = n_- = n_0 \equiv N/V$, and write (29) finally as

$$\sigma(\vec{r}) = -2n_0 \beta \int d\vec{r}' \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}') \quad (30)$$

Multiplying (30) by $\sigma(\vec{r})$ and integrating over $\vec{r}$ gives

$$-\beta = \frac{1}{2n_0} \frac{\int \sigma^2(\vec{r}) d\vec{r}}{\int d\vec{r} \int d\vec{r}' \sigma(\vec{r}) \varphi(\vec{r} - \vec{r}') \sigma(\vec{r}')} . \quad (31)$$

Since the right-hand side of (31) is positive-definite, it is clear that β must be negative.

Equation (30) is now a linear integral equation. Solutions periodic in a large box of volume $V = L^2$ can be found of the form

$$\sigma(\vec{r}) = \sum_{\vec{k}} \sigma(\vec{k}) e^{i\vec{k} \cdot \vec{r}} \quad (32)$$

where the sum is over vectors $\vec{k} = 2\pi (m, n)/L$, with m and n integers. The vanishing of the tangential component of the electric field at the walls can only be achieved if neither m nor n is zero. We also have $\sigma(\vec{k}) = \sigma^*(-\vec{k})$.

The Fourier transformed version of (30) is

$$\sigma(\vec{k})[1 + 2n_0 \beta \varphi(\vec{k})] = 0 \quad (33)$$

where $\varphi(\vec{k}) = 4\pi e^2 / \ell k^2$. Equation (33) can have $\sigma(\vec{k}) \neq 0$ only if

$$k^2 = 8\pi n_0 |\beta| e^2 / \ell \equiv K^2 \quad (34)$$

If $k^2 = 4\pi^2 (m^2 + n^2)/L^2$ satisfies (34), then $\sigma(\vec{k})$ is arbitrary.

The fact that m and n are constrained to be integers means that for a small box, not all values of the temperature $kT = -|\beta|^{-1}$ are allowed; but the allowed values become dense as the volume gets very large. The fact that more than one pair of integers m, n may satisfy (34) means that the $\sigma(\vec{k})$ are not unique but may be any set of numbers for which the energy relation

$$\begin{aligned} \frac{\mathcal{E}}{\mathcal{L}^2} &= \sum_{\text{allowed } \vec{k}} \frac{2\pi e^2 |\sigma(\vec{k})|^2}{\ell^2 k^2} = \\ &= \frac{1}{4n_0 |\beta| \ell} \sum_{\text{allowed } \vec{k}} |\sigma(\vec{k})|^2, \end{aligned} \quad (35)$$

and $\sigma(\vec{k}) = \sigma^*(-\vec{k})$, are satisfied.

There appears to be no unique negative temperature state unless it happens that only one pair of integers (m, n) is allowed. Rather, there are a finite number of states, any linear superposition of which is stationary (though not necessarily being a state which gives an absolute maximum in $\mathcal{H}$). It is not known to what extent this non-uniqueness would be removed if we could do the full non-linear problem, but we believe that to the extent that stationary negative-temperature states exist, they will be contained in the solutions to (23). They bear a qualitative resemblance to the results of § 3.

We may estimate the range of validity of the linear equation (29). If m, n are of order unity, (34) provides the estimate $\beta^2 \sim \pi^2 \ell^2 / n_0^2 e^4 L^4$. ψ can be estimated in terms of $\mathcal{E}$. Thus the condition $\beta^2 \psi^2 \ll 1$ is $(2\pi \mathcal{E} \ell / n_0^2 e^2 L^4) \ll 1$, or

$$\frac{1}{2\pi} \left(\frac{n_0 L^2 e}{\ell} \right) (n_0 L^2 e) \gg \mathcal{E} \quad (36)$$

To interpret (36), note that $n_0 L^2 e$ is the total charge of one sign present. Call it Q . Then $\Phi_Q = -2(e n_0 L^2 / \ell) \ln R$ is the potential it produces at a distance R . Thus the left-hand side of (36) is basically $Q \Phi_Q$, if the logarithms are estimated as ~ 10 . This is the interaction energy that would result from the concentration of

the charges into two line charges about a box dimension apart. This condition is generally satisfied in all the computer simulations.

To summarize the results of this section, we have shown how, by using the elementary combinatorial method of Boltzmann and a cellular model which ultimately becomes a continuum, an integral equation may be derived whose solutions contain possible negative temperature states. This equation is solved in the limit of small positive energies, and a class of negative-temperature, periodic, steady states are constructed. These states are macroscopically non-uniform, and involve non-zero fluid velocities. They are, however, time-independent. Their lack of uniqueness prohibits a detailed comparison with the simulation results, but their qualitative similarity is not in doubt.

5. Discussion

We can usefully summarize, by reference to figure 9, the various possible energy states of a two-dimensional electrostatic guiding-centre plasma. Figure 9 is a schematic plot of the structure function $\Omega(\mathcal{E})$ as a function of the interaction energy $\mathcal{E}$.

Region I, with $\mathcal{E}$ large and negative, is the region of collapse or "negative pressures" (Salzberg & Prager 1963; May 1967; Knorr 1968; Deutsch & Lavaud 1972). It is associated with plasma parameters of order unity. Region II is a region in which presumably a thermodynamic equilibrium state exists, but where the reciprocal of the number of rods per Debye square is not small enough to be used as the basis for a perturbation expansion. About this region we know little. Region III, $\mathcal{E}$ small and negative, is the region plasma physics usually concerns itself with. It is the only region in which the various recent calculations of transport coefficients apply. Region IV is the region of interest for this article, the region of positive interaction energy and negative temperature.

Region IV, involving as it does a strong spatial non-uniformity and macroscopic local concentrations of vorticity, can be thought of without too much exaggeration as another "phase" of a plasma. Numerical simulations of the structure of this phase have been presented in § 3, and an analytic framework within which it can be discussed appears in § 4.

It is perhaps worth remarking that the motions discussed here are included among the (as yet poorly-understood) motions of the three-dimensional guiding-centre Vlasov plasma (Montgomery 1972a). We have resisted the temptation, as well, to attempt to extract any premature conclusions concerning Navier-Stokes turbulence with finite viscosity from this ideal (i.e., non-dissipative) formulation.

APPENDIX

The equations of motion for either the guiding centre plasma or the line vortex system can be written in the form of equations (10) and (11) in appropriate dimensionless variables. We outline here one such set of units for both systems.

(a) The Guiding Centre Plasma

If we measure lengths in units of λ_D , the Debye length, and times in units of $1/\omega_p$, the inverse plasma frequency, then equations (1) and (10) become identical if we measure the electric field strength in units of $E_0 = B\lambda_D\omega_p/c$. Then equations (2) and (11) become identical if and only if

$$-2\pi D = \frac{\omega_p}{\omega_c} \frac{1}{n_0 \lambda_D^2}$$

where $\omega_c = eB/mc$ is the gyrofrequency of an electron and $n_0 = N/V$.

Energy densities are then conveniently expressed in units of $E_0^2/4\pi$, which will be a small fraction of the magnetic energy density in thermal equilibrium. Since we express all energies in units of the electrostatic self energies, these are of not much significance.

(b) The Line Vortex System

If we measure lengths in units of the box size L , and times in units of the time T required for two unit vortices a unit length apart to circle once around each other, then

$$2\pi D = \frac{|K|T}{L^2},$$

where $\pm|K|$ is the strength of the two species of vortices.

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FIGURE CAPTIONS

FIGURE 1. (Color plate) Spatial locations of positive (green) and negative (red) charges at $t = 0$ and at a later time. (a) is for $W = 99$ at $t = 0$. (b) is for $W = 99$ at $t = 50$. (c) is for $W = 2.3$ at $t = 0$. (d) is for $W = 2.3$ at $t = 160$. A yellow dot implies a positive and a negative within a visually unresolvable proximity.

FIGURE 2. Energy spectrum $\mathcal{E}(n_x, n_y)$ for the maximum energy state. An overall normalization of the boxes is always chosen so that their total volume sums to unity. Smaller values of $\mathcal{E}(n_x, n_y)$ than $\mathcal{E}(n_x, n_y) = 0.023$ are omitted from the graph for visual clarity.

FIGURE 3. Energy spectrum for the case in which the particles are placed in the box completely randomly. This should be contrasted with figure 2. (Values < 0.017 are not shown.)

FIGURE 4. Energy spectrum at $t = 50.0$ for $W = 99.0$. Note the qualitative similarity to figure 2. Note also that the essential features of this spectrum would be lost by an integration over angle in Fourier space. (Values < 0.018 are not shown.)

FIGURE 5. Energy spectrum at $t = 64.0$ for $W = 16.7$. Note that the energy is still concentrated in the (1, 2) mode. (Values < 0.02 are not shown.)

FIGURE 6. Energy spectrum at $t = 160$ for $W = 2.3$. Though for smaller W than figures 3 and 4, this is still predicted to lie in the $T < 0$ region. (Values < 0.035 are not shown.)

FIGURE 7. Energy spectrum for $W = 0.43 < 1$ at $t = 300$. Though not a thermal equilibrium spectrum, this value of W should lie in a regime characterized by positive thermal equilibrium temperatures. Note the much broader distribution than figures 2, 4, 5 and 6. (Values < 0.003 are not shown.)

FIGURE 8. Thermal equilibrium spectrum, calculated from a small plasma parameter expansion of the equilibrium Gibbs distribution (not a simulation result). (Values < 0.005 are not shown.)

FIGURE 9. Schematic plot of the structure function $\Omega(\mathcal{E})$ showing the four regions of basically different macroscopic behavior as a function of interaction energy $\mathcal{E}$. Region IV is the subject matter of this paper, and region III is the region plasma physics is usually concerned with.

FIGURE 1. (Color plate)

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Figure 2

A - G72 - 466

Figure 3

A-G72-399

Figure 4

A-G72-400

Figure 5

A-G72-401

Figure 6

A-G72-402

Figure 7

Figure 8

Figure 9

A-672-479