SOME FUNCTIONAL INEQUALITIES,
WITH APPLICATION TO GREENS' FUNCTIONS

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ABSTRACT

Some elementary inequalities for functionals with functional derivatives of given sign are presented and proved. These are then applied to the study of Greens' functions for diffusion processes in a medium with sources and sinks (or alternately Schrödinger operator Greens' functions). The resulting inequalities are shown to include as quite special cases the super - and sub - additive inequalities of potential theory. One consequence of the generalization is that the scattering length is also shown to have sub - additive properties.

1. INTRODUCTION

In this paper we establish what seems to be a natural basis for certain types of "superadditive" inequalities among the Greens' functions of potential theory.

It will be shown that these inequalities are special cases of certain quite elementary inequalities for functionals when the Greens' functions are represented by path (Wiener) integrals. In addition, the point of view established here enables us to very considerably generalize the scope of such inequalities.

Typically, the (usual) superadditive inequalities are of the following form. Call $g(P, Q | D)$ the potential theoretic Greens' function for a domain D with a source at the point Q , satisfying the boundary condition $g = 0$ if P is a point on the boundary of D . Let D_1 and D_2 be two (not necessarily disjoint) domains in space, $D_1 \cup D_2$ their union, $D_1 \cap D_2$ their intersection. Then g has the "super-additive" property¹.

$$g(P, Q | D_1 \cup D_2) + g(P, Q | D_1 \cap D_2) \geq g(P, Q | D_1) + g(P, Q | D_2) \quad (1.1)$$

(In reference 1, $g(P, Q | D)$ is defined to be zero if P and Q are not in the domain D .) There are similar inequalities for the stress function and torsional rigidity of domains² and for capacitive potential and capacity³.

Our results generalize (1.1) in several directions. First, instead of the Greens' function of potential theory in a domain, we consider the Greens' function for heat flow in a medium with distributed sources or sinks (or, alternatively, we replace the Laplace operator by the Schrödinger operator of a particle moving in a potential). The usual results for potential theory in a domain are obtained by integration over

time and specialization of the potential. One consequence of this generalization is that Choquet's results on the subadditivity of the capacity are generalized to subadditivity of the scattering length. Second, the nature of the proof shows that such inequalities arise in even more general situations. In fact, they arise whenever the Laplace operator is replaced by an operator whose Greens' function can be thought of as a probability density for a diffusion process. So, for example, if the Laplace operator is replaced by the Laplace - Beltrami operator we are considering diffusion on a Riemannian manifold, and we again have this type of inequality.

This paper is organized as follows. In section 2 the general functional inequalities are proved, and in section 3 they are applied to the Greens' function to obtain our inequalities.

2. FUNCTIONAL INEQUALITIES

Consider a real valued functional $F[f]$ of real function $f(\tau)$, where τ is a real variable confined to the interval $[a, b]$. Further, let F have the property that its first M functional derivatives alternate in sign. That is

$$(-1)^n \frac{\delta^n F[f]}{\delta f(\tau_1) \delta f(\tau_2) \dots \delta f(\tau_n)} \equiv (-1)^n F_{\tau_1 \tau_2 \dots \tau_n} [f] \geq 0 \quad (n = 1, 2, \dots, M) \quad (2.1)$$

Call such a functional an alternating functional of order M . We shall show that such functionals satisfy some rather simple inequalities. The first set of inequalities is very trivial. Consider the ordinary function $\theta(\lambda_1, \lambda_2, \dots, \lambda_n)$ of real variables λ_i ($0 \leq \lambda_i \leq 1$) defined by

$$\theta(\lambda_1, \lambda_2, \dots, \lambda_n) = F \left[f + \lambda_1 h_1 + \lambda_2 h_2 + \dots + \lambda_n h_n \right] \quad (2.2)$$

where all the $h_i(\tau) \geq 0$. Clearly

$$\begin{aligned} (-1)^n \frac{\partial^n \theta(\lambda_1, \dots, \lambda_n)}{\partial \lambda_1 \dots \partial \lambda_n} &= \\ &= (-1)^n \int_a^b \dots \int_a^b d\tau_1 \dots d\tau_n F_{\tau_1 \dots \tau_n} \left[f + \lambda_1 h_1 + \dots + \lambda_n h_n \right] h_1(\tau_1) \dots h_n(\tau_n) \end{aligned} \quad (2.3)$$

Using (2.1), the right hand side of (2.3) is non-negative, for $n = 1, 2, \dots, M$.

Therefore we have at once

$$(-1)^n \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 \dots \int_0^1 d\lambda_n \frac{\partial^n \theta(\lambda_1 \dots \lambda_n)}{\partial \lambda_1 \dots \partial \lambda_n} \geq 0 \quad n = 1, 2, \dots, M \quad (2.4)$$

Evaluating these terms successively, we obtain ⁴

$$F[f] - F[f + h_1] \geq 0 \quad (2.5)$$

$$F[f] - (F[f + h_1] + F[f + h_2]) + F[f + h_1 + h_2] \geq 0 \quad (2.6)$$

$$\begin{aligned} F[f] - \sum_{i=1}^M F[f + h_i] + \sum_{j>i \geq 1}^M F[f + h_i + h_j] - \dots \\ + \sum_{k>j>i \geq 1}^M F[f + h_i + h_j + h_k] + \dots + (-1)^M F[f + h_1 + h_2 + \dots + h_M] \geq 0 \end{aligned} \quad (2.7)$$

where

$$h_i(\tau) \geq 0 \quad (2.8)$$

There is also another set of inequalities, just a little more difficult to prove, which we shall also discuss. To see what is involved, consider the (ordinary) function

$F(\lambda)$ ($0 \leq \lambda \leq 1$) defined by

$$F(\lambda) = F[h_\lambda] \quad (2.9)$$

$$h_\lambda(\tau) \equiv \lambda f(\tau) + (1 - \lambda) g(\tau) \quad (2.10)$$

From the mean value theorem we have

$$F(1) = F(0) + \left(\frac{dF(\lambda)}{d\lambda} \right)_{\lambda=\bar{\lambda}} (0 \leq \bar{\lambda} \leq 1) \quad (2.11)$$

However,

$$\begin{aligned} \frac{dF(\lambda)}{d\lambda} &= \int_a^b F_{\tau} [h_{\lambda}] \frac{\partial h_{\lambda}(\tau)}{\partial \lambda} d\tau \\ &= \int_a^b F_{\tau} [h_{\lambda}] [f(\tau) - g(\tau)] d\tau \end{aligned} \quad (2.12)$$

Therefore (2.11) becomes

$$F[f] = F[g] + \int_a^b F_{\tau} [h_{\lambda}] [f(\tau) - g(\tau)] d\tau \quad (2.13)$$

(This may be thought of as the mean value theorem for functionals.)

Now suppose we choose

$$g(\tau) = \text{Max} [f(\tau), f_1(\tau)] \equiv \widehat{ff}_1(\tau) \quad (2.14)$$

Clearly, at each point τ

$$\widehat{ff}_1 \geq f \quad (2.15)$$

Assuming

$$F_{\tau} \leq 0 \quad (2.16)$$

(special case of (2.1) for $n = 1$), (2.13) tells us that

$$F[f] \geq F[\widehat{ff}_1] \quad (2.17)$$

This is the first of the series of increasingly strong inequalities which $F[f]$ will satisfy when (2.1) is true.

This process is not difficult to continue. Let us define a sequence of

functionals $H^{(1)}, \dots, H^{(M)}$ by

$$\begin{aligned} H^{(1)}[f] &= F[f] - F[\widehat{ff}_1] \\ H^{(2)}[f] &= H^{(1)}[f] - H^{(1)}[\widehat{ff}_2] \\ H^{(M)}[f] &= H^{(M-1)}[f] - H^{(M-1)}[\widehat{ff}_M] \end{aligned} \quad (2.18)$$

Consider $F_{\tau}[f]$ and apply the mean value theorem (2.13) to it, again taking

$$g = \widehat{ff}_1$$

$$F_{\tau}[f] = F_{\tau}[\widehat{ff}_1] + \int_a^b F_{\tau_1 \tau_2}[\widehat{h_2}] [f(\tau') - \widehat{ff}_1(\tau')] d\tau' \quad (2.19)$$

Assuming $F_{\tau_1} \geq 0$ ((2.1) for $n = 2$), (2.19) gives

$$F_{\tau}[f] \leq F_{\tau}[\widehat{ff}_1] \quad (2.20)$$

It is clear that this process may be continued till (2.1) is exhausted. That is, we have

$$\begin{aligned} (-1)^n \left[F_{\tau_1 \tau_2 \dots \tau_n}[f] - F_{\tau_1 \tau_2 \dots \tau_n}[\widehat{ff}_1] \right] &\geq 0 \\ (n = 1, 2, \dots, M-1) \end{aligned} \quad (2.21)$$

Now we are in a position to discuss the functional derivatives of $H^{(1)}$. It is obvious that

$$(-1)^n \frac{\delta^n H^{(1)}[f]}{\delta f(\tau_1) \delta f(\tau_2) \dots \delta f(\tau_n)} = (-1)^n \left[F_{\tau_1 \dots \tau_n}[f] - F_{\tau_1 \dots \tau_n}[\widehat{ff}_1] \right]$$

if $f(\tau_i) \geq f_1(\tau_i)$
($i = 1, \dots, n$)

$$= (-1)^n F_{\tau_1 \dots \tau_n} [f] \quad \text{otherwise} \quad (2.22)$$

Therefore from (2.1), (2.17) and (2.22) we have

$$(-1)^n \frac{\delta^n H^{(1)} [f]}{\delta f(\tau_1) \dots \delta f(\tau_n)} \geq 0 \quad (0 \leq n \leq M-1) \quad (2.23)$$

Repeating the same discussion with F replaced by $H^{(1)}$ and $H^{(1)}$ by $H^{(2)}$ we have

$$(-1)^n \frac{\delta^n H^{(2)} [f]}{\delta f(\tau_1) \dots \delta f(\tau_n)} \geq 0 \quad (0 \leq n \leq M-2) \quad (2.24)$$

Continuing this process, we obtain for $H^{(2)}$

$$(-1)^n \frac{\delta^n H^{(i)} [f]}{\delta f(\tau_1) \dots \delta f(\tau_n)} \geq 0 \quad (0 \leq n \leq M-i) \quad (1 \leq i \leq M) \quad (2.25)$$

Writing out each of these inequalities for $n = 0$ we obtain

$$H^{(2)} [f] = H^{(1)} [f] - H^{(1)} [\widehat{ff}_2] = F [f] - F [\widehat{ff}_1] - F [\widehat{ff}_2] + F [\widehat{ff}_1 \widehat{f}_2] \geq 0 \quad (2.26)$$

$$\begin{aligned} H^{(3)} [f] &= H^{(2)} [f] - H^{(2)} [\widehat{ff}_3] = F [f] - F [\widehat{ff}_1] - F [\widehat{ff}_2] - F [\widehat{ff}_3] + \dots \\ &\quad + F [\widehat{ff}_1 \widehat{f}_2] + F [\widehat{ff}_2 \widehat{f}_3] + F [\widehat{ff}_1 \widehat{f}_3] \\ &\quad - F [\widehat{ff}_1 \widehat{f}_2 \widehat{f}_3] \geq 0 \quad (2.27) \end{aligned}$$

etc.

$$\begin{aligned} H^{(M)} [f] &= F [f] - \sum_{i=1}^M F [\widehat{ff}_i] + \sum_{i>j=1}^M F [\widehat{ff}_i \widehat{f}_j] - \sum_{k>i>j=1}^M F [\widehat{ff}_i \widehat{f}_j \widehat{f}_k] + \dots \\ &\quad + (-1)^M F [\widehat{ff}_1 \widehat{f}_2 \dots \widehat{f}_M] \geq 0 \quad (2.28) \end{aligned}$$

where

$$\overbrace{f f_1 f_2 f_3 \dots} (\tau) = \text{Max } (f(\tau), f_1(\tau), f_2(\tau), f_3(\tau), \dots) \quad (2.29)$$

These results may be summarized in terms of a Theorem. An alternating functional of order M satisfies the inequality (2.28).

Each inequality in the hierarchy (2.17), (2.26), (2.27), ... (2.28) is weaker than the following one, and may be derived from it (the n^{th} is derived from the $n + 1^{\text{st}}$ by taking $f_{n+1} \geq \overbrace{f f_1 f_2 \dots f_n}$ for all τ).

If a functional satisfies (2.1) for all M and is in addition non-negative (which is a trivial restriction for the type of inequalities we are considering) then we say that it is a completely monotonic functional.⁵ For a completely monotonic functional, (2.28) is satisfied for all $M \geq 1$.

A special case of (2.28) of considerable interest for what follows is obtained when one takes

$$f(\tau) = \text{Min } (f_1(\tau), f_2(\tau), \dots, f_M(\tau)) \equiv \underbrace{f_1 f_2 \dots f_M} (\tau) \quad (2.30)$$

We then have

$$\begin{aligned} F \left[\underbrace{f_1 f_2 \dots f_M} \right] &= \sum_{i=1}^M F \left[\underbrace{f_i} \right] - \sum_{i>j>1}^M F \left[\underbrace{f_i f_j} \right] + \sum_{k>i>j>1}^M F \left[\underbrace{f_i f_j f_k} \right] - \dots \\ &\quad + (-1)^M F \left[\underbrace{f_1 f_2 \dots f_M} \right] \geq 0 \end{aligned} \quad (2.31)$$

3. GREENS' FUNCTIONS

We now consider the application of the inequalities of the previous section to Greens' functions. The simplest of these applications is the following. Denote by H the operator (in an m -dimensional euclidian space, E_m)

$$H = -\frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_m^2} \right) + \Psi(x_1, x_2, \dots, x_m) \quad (3.1)$$

Let $G_t(P, Q)$ ($P = (x_1, x_2, \dots, x_m)$, $Q = (y_1, y_2, \dots, y_m)$) be the Greens' function defined by

$$\frac{\partial G_t(P, Q)}{\partial t} + H G_t(P, Q) = 0 \quad (t > 0) \quad (3.2)$$

$$\lim_{t \rightarrow 0} G_t(P, Q) = \delta(P - Q) \quad (3.3)$$

(We may think of this as the Greens' function associated with a particle in a potential Ψ , or as that for diffusion in the presence of absorbers, for example).

It is well-known⁶ that this Greens' function has a representation as a Wiener path integral

$$G_t(P, Q) = \frac{\exp \left\{ -d^2(P, Q)/2t \right\}}{(2\pi t)^{m/2}} \left\langle \exp - \left\{ \int_0^t \Psi(r(\tau)) d\tau \right\} \middle| \begin{matrix} r(\tau) = P \\ r(0) = Q \end{matrix} \right\rangle \quad (3.4)$$

Here $d^2(P, Q) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_m - y_m)^2$, $r(\tau)$ is a curve (path) in the m -dimensional space leading from Q to P and the angular bracket represents a conditional average in the sense of Wiener over all these paths.

Now consider some definite path $r(\tau)$ leading from Q to P . For such a path $\Psi(r(\tau))$ will be a definite function of τ . The quantity

$$E[\Psi] = \exp \left\{ - \int_0^t \Psi(r(\tau)) d\tau \right\} \quad (3.5)$$

is obviously a completely monotonic functional of Ψ , and therefore the inequalities of the previous section all apply. To see what this means for the Greens' function, we make the functional dependence of $G_t(P, Q)$ on Ψ explicit by writing $G_t(P, Q | \Psi)$. Then from (3.4) and (2.7) we have at once

$$G_t(P, Q | \Psi) = \sum_{i=1}^M G_t(P, Q | \Psi + u_i) + \sum_{i>j>1}^M G_t(P, Q | \Psi + u_i + u_j) - \dots$$

$$+ (-)^M G_t(P, Q | \Psi + u_1 + u_2 + \dots + u_M) \geq 0$$

(M \geq 1) (3.6)

where the u_i are non-negative functions of position.

Similarly (3.4), (2.28) and (2.31) yield

$$G_t(P, Q | \Psi) = \sum_{i=1}^M G_t(P, Q | \widehat{\Psi \Psi_i}) + \sum_{i>j>1}^M G_t(P, Q | \widehat{\Psi \Psi_i \Psi_j}) - \dots$$

$$+ (-)^M G_t(P, Q | \widehat{\Psi \Psi_1 \Psi_2 \dots \Psi_M}) \geq 0$$

(M \geq 1) (3.7)

$$G_t(P, Q | \widehat{\Psi_1 \Psi_2 \dots \Psi_M}) = \sum_{i=1}^M G_t(P, Q | \Psi_i) + \sum_{i>j>1}^M G_t(P, Q | \widehat{\Psi_i \Psi_j}) - \dots$$

$$+ (-)^M G_t(P, Q | \widehat{\Psi_1 \Psi_2 \dots \Psi_M}) \geq 0$$

(M \geq 1) (3.8)

where the Ψ_i are functions of position, and

$$\begin{aligned} \overbrace{\varphi_1 \varphi_2 \dots}^{(r)} &= \text{Max} (\varphi_1(r), \varphi_2(r), \dots) \\ \underbrace{\varphi_1 \varphi_2 \dots}^{(r)} &= \text{Min} (\varphi_1(r), \varphi_2(r), \dots) \end{aligned} \quad (3.9)$$

The inequalities (3.6), (3.7), (3.8) may be regarded as generalizations of certain inequalities already known in potential theory. To see the connection it is easiest to start from (3.8) (though (3.6) or (3.7) can also be chosen as the starting point). In (3.8) take

$$\begin{aligned} \varphi_i(r) &= 0 \quad r \in D_i \\ &= \infty \quad r \notin D_i \end{aligned} \quad (3.10)$$

where D_i is a domain in E_m and the D_i are not necessarily disjoint.

Denote by $D_1 \cup D_2 \cup D_3 \dots$ the union of the domains $D_1, D_2, D_3, \dots$ and by $D_1 \cap D_2 \cap D_3 \dots$ the intersection of these domains. Then clearly

$$\begin{aligned} \overbrace{\varphi_1 \varphi_2 \dots \varphi_M}^{(r)} &= 0 \quad r \in D_1 \cup D_2 \cup D_3 \dots \cup D_M \\ &= \infty \quad r \notin D_1 \cup D_2 \cup D_3 \dots \cup D_M \end{aligned} \quad (3.11)$$

$$\begin{aligned} \overbrace{\varphi_1 \varphi_2 \dots \varphi_M}^{(r)} &= 0 \quad r \in D_1 \cap D_2 \cap D_3 \dots \cap D_M \\ &= \infty \quad r \notin D_1 \cap D_2 \cap D_3 \dots \cap D_M \end{aligned} \quad (3.12)$$

Using an obvious notation, (3.8) then becomes

$$\begin{aligned}
G_t(P, Q | D_1 \cup D_2 \cup D_3 \dots \cup D_M) &= \sum_{i=1}^M G_t(P, Q | D_i) + \sum_{i>j \geq 1}^M G_t(P, Q | D_i \cap D_j) - \dots \\
&+ (-1)^M G_t(P, Q | D_1 \cap D_2 \cap D_3 \dots \cap D_M) \geq 0
\end{aligned} \tag{3.13}$$

Now if we consider

$$g(P, Q | D) = \int_0^\infty dt G_t(P, Q | D) \tag{3.14}$$

it follows at once by integrating (3.2) with respect to t and using (3.3) that $g(P, Q | D)$ is the ordinary Greens' function of potential theory (in E_m) corresponding to a source of two units at Q , satisfying Laplace's equation in D and vanishing on the boundary of D . Integrating (3.13) over t we obtain

$$\begin{aligned}
g(P, Q | D_1 \cup D_2 \dots \cup D_M) &= \sum_{i=1}^M g(P, Q | D_i) + \sum_{i>j \geq 1}^M g(P, Q | D_i \cap D_j) - \dots \\
&+ (-1)^M g(P, Q | D_1 \cap D_2 \dots \cap D_M) \geq 0
\end{aligned} \tag{3.15}$$

For the special case $M = 2$, (3.15) becomes

$$g(P, Q | D_1 \cup D_2) + g(P, Q | D_1 \cap D_2) \geq g(P, Q | D_1) + g(P, Q | D_2) \tag{3.16}$$

which has been known for a long time ¹, and is usually referred to as the super-additive property of the Greens' functions of potential theory. By analogy, we may also refer to (3.13) as generalized superadditivity for the Greens' function G_t .

Other similar properties are known. For example, in problems connected with the torsion of a bar the integral of $g(P, Q | D)$ over Q is proportional to what is called the stress function and the integral over Q and P is proportional to the torsional rigidity of the bar ². By integrating (3.15) first over Q and then over P

also we get superadditive inequalities for the stress function and torsional rigidity.

These are also known, having been found for $M = 2$ by Pólya² and for general M by Aissen².

Finally, we mention that the electrostatic capacity of an ideal conductor as a function of its shape, and the "capacitory potential" have been found by Choquet³ to satisfy similar inequalities. For the capacitory potential it is again just a matter of integrating (3.15) over Q . For the capacity itself, things are a little more complicated because it is not simply given as an integral of $g(P, Q)$ with a weighting function of one sign. We can, however, make use of a result due to Spitzer⁷. Call the capacity of an ideal conductor which occupies a region B , $C(B)$. The Spitzer result is (for the three dimensional case)

$$C(B) = \lim_{t \rightarrow \infty} \frac{1}{2\pi t} \int d\underline{r} \int d\underline{r}' \left\{ G_t^{(0)}(\underline{r}, \underline{r}') - G_t(\underline{r}, \underline{r}' | B') \right\} \quad (3.17)$$

where B' is the region complementary to B and $G_t^{(0)}(\underline{r}, \underline{r}')$ is the Greens' function for empty space. Write

$$\tilde{G}_t(\underline{r}, \underline{r}' | B') \equiv G_t^{(0)}(\underline{r}, \underline{r}') - G_t(\underline{r}, \underline{r}' | B') \quad (3.18)$$

Then from (3.13), choosing $D_i = B_i'$,

$$\begin{aligned} \tilde{G}_t(\underline{r}, \underline{r}' | (B_1 \cap B_2 \cap B_3 \dots \cap B_M)') &= \sum_{i=1}^M \tilde{G}_t(\underline{r}, \underline{r}' | B_i') + \sum_{i>j \geq 1}^M \tilde{G}_t(\underline{r}, \underline{r}' | (B_i \cup B_j)') - \dots \\ &+ (-1)^M \tilde{G}_t(\underline{r}, \underline{r}' | (B_1 \cup B_2 \cup B_3 \cup \dots \cup B_M)') \leq 0 \end{aligned} \quad (3.19)$$

since

$$B_1^c \cap B_2^c = (B_1 \cup B_2)^c$$

$$B_1^c \cup B_2^c = (B_1 \cap B_2)^c$$

Now making use of (3.17) we have at once

$$C(B_1 \cap B_2 \cap \dots \cap B_M) - \sum_{i=1}^M C(B_i) + \sum_{|i| \geq 1}^M C(B_i \cup B_j) - \dots + (-1)^M C(B_1 \cup B_2 \cup \dots \cup B_M) \leq 0 \quad (3.20)$$

which is Choquet's subadditive property of the capacity.

One simple generalization of this result is possible. Write

$$a[\psi] = \lim_{t \rightarrow \infty} \frac{1}{2\pi t} \int d\mathbf{r} \int d\mathbf{r}' \left\{ G_t^0(\mathbf{r}, \mathbf{r}') - G_t(\mathbf{r}, \mathbf{r}' | \psi) \right\} \quad (3.21)$$

When ψ is non-negative and approaches zero at infinity sufficiently rapidly, it is possible to show rigorously that this limit exists and $a[\psi]$ is equal to the quantum mechanical scattering length⁸ for the potential ψ . Combining (3.8) with (3.21) we have the following "subadditive" inequality for $a[\psi]$.

$$a[\underbrace{\psi_1 \psi_2 \dots \psi_M}_{\text{all together}}] - \sum_{i=1}^M a[\psi_i] + \sum_{|i| \geq 1}^M a[\underbrace{\psi_i \psi_j}_{\text{together}}] - \dots + (-1)^M a[\underbrace{\psi_1 \psi_2 \dots \psi_M}_{\text{all together}}] \leq 0$$

Using (3.6) and (3.7) instead of (3.8), other very similar inequalities for the scattering length may be written down; we shall, however, spare the reader.

The essential point of the preceding analysis was noticing that $E[\psi]$ (see (3.5)) was a completely monotonic functional of ψ . The Greens' function inequalities were then obtained by an "average over paths" of the inequalities for such functionals. It is therefore clear that any other way of averaging $E[\psi]$ will yield similar inequalities. The simplest type of generalization is when the "paths" are on a Riemannian manifold R_m rather than in the Euclidean space E_m . Let us

replace the Laplacian in (3.1) by the Laplace-Beltrami operator

$$H = -\frac{1}{2} \left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right) + \Psi(x_1, \dots, x_m) \quad (3.23)$$

where g^{ij} are the contravariant components of the Riemann tensor g_{ij} , and g is $\det(g_{ij})$. Defining $G_\pm(P, Q | \Psi)$ by (3.2) and (3.3) again, where the δ -function now means

$$\int \delta(P-Q) A(P) \sqrt{g} dx_1 \dots dx_m = A(Q) \quad (3.24)$$

we have at once that this Greens' function also satisfies the inequalities (3.6), (3.7) and (3.8). (It is well-known that this Greens' function has a path integral representation with non-negative weights.)

In general any replacement of the Laplacian in H by an operator which allows a path integral representation of the Greens' function with non-negative weight will yield the same inequalities for this Greens' function.

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M. Aissen, Proc. American Math. Soc. 4 360 (1953).
3. G. Choquet, Theory of Capacities, Ann. Istit. Fourier (Grenoble), 5 p. 131-295 (1953-1954).
4. As one sees from the proof, what we have actually shown is that if (2.1) is true for any fixed n , we have (2.7) for $M = n$, regardless of whether (2.1) is true for the smaller values of n .
5. These are the analogues of completely monotonic functions. (See, Widder, D. V., The Laplace Transform, Princeton Univ. Press, Princeton, N. J. (1941), p. 144 ff.) We do not exploit in this paper other consequences of this fact (Bernstein's theorem, Hankel determinants, and so forth).
6. Kac, M., Probability and Related Topics in Physical Sciences I, Interscience Publishers, New York (1959), p. 161 ff.
7. F. Spitzer, Zeit für Wahrschein. 3 110 (1964).
8. For the definition of the scattering length see, for example, M. L. Goldberger and K. M. Watson, Collision Theory, John Wiley and Sons, N.Y., 1964 p. 287.

The result that a $\left[\varphi \right]$ is the scattering length is due to M. Kac and one of us (JML), and will be published elsewhere.