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RATIONAL CHEBYSHEV APPROXIMATIONS
FOR FERMI-DIRAC INTEGRALS
OF ORDERS $-1/2$, $1/2$, AND $3/2$

by

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ABSTRACT

Rational Chebyshev approximations are given for the complete Fermi-Dirac integrals of orders $-1/2$, $1/2$, and $3/2$. Maximal relative errors vary with the function and interval considered, but generally range down to 10^{-9} or less.

I. INTRODUCTION

The complete Fermi-Dirac integrals are usually defined by

$$F_k(x) = \int_0^\infty \frac{t^k dt}{e^{t-x} + 1}, \quad k > -1, \quad (1)$$

although Dingle⁴ prefers the definition,

$$F_k(x) = (k!)^{-1} \int_0^\infty \frac{t^k dt}{e^{t-x} + 1}, \quad (2)$$

which places no restriction on the order k . We will use definition (1) but will employ some formulas, suitably modified, derived by Dingle.

These integrals appear in a variety of applications subject to Fermi-Dirac statistics, for example, in the theory of semiconductors. The most frequently used functions are those for which k is either an integer or a half-integer. Function values are quite difficult to compute for k a half-integer and x positive. Consequently a number of useful tables have been published over the last 30 years.^{1,2,4,9} Recently Werner and Raymann¹² used interpolation in the McDougall and Stoner⁹ table to generate a compatible pair of Chebyshev approximations for $k = 1/2$. Their work allows easy computation of $F_{1/2}(x)$ with a maximal relative error less than 5×10^{-4} . The present work presents portions of the arrays, termed by Rice¹¹ the "L_∞ Walsh arrays," of rational Chebyshev approximations for $k = -1/2$, $1/2$, and $3/2$. Maximal errors range down to 10^{-9} or less.

II. FUNCTIONAL DISCUSSION

The well-known expansion,^{4,9}

$$F_k(x) = k! \sum_{r=1}^{\infty} (-1)^{r-1} \frac{e^{rx}}{r^{k+1}}, \quad (3)$$

is convergent for $k > -1$ and $x < 0$. The Taylor series,

$$F_k(x) = k! \sum_{r=0}^{\infty} \frac{x^r (1 - 2^{r-k}) \zeta(k+1-r)}{r!}, \quad (4)$$

where ζ is the Riemann zeta-function, is convergent for $k > -1$ and $|x| < \pi$.

For positive x , Dingle⁴ has shown

$$\begin{aligned} \mathcal{F}_k(x) = \cos \pi k \mathcal{F}_k(-x) + 2 \sum_{r=0}^{[1/2(k+1)]} \frac{t_{2r} x^{k+1-2r}}{(k+1-2r)!} \\ + \frac{2 \sin k\pi}{\pi} \sum_{r=[1/2(k+3)]}^{\infty} \frac{t_{2r} (2r-k-2)!}{x^{2r-k-1}}, \end{aligned} \quad (5)$$

where $[x]$ denotes the integer part of x , and

$$t_{2r} = \frac{1}{2} (2\pi)^{2r} (1 - 2^{1-2r}) |B_{2r}| / (2r)!,$$

where the B_r are the Bernoulli numbers. This expansion is finite, hence exact, for k an integer. However, for k half an odd integer, the expansion is only asymptotic, equivalent to the well-known Sommerfeld representation,⁹

$$F_k(x) = \frac{x^{k+1}}{k+1} \left\{ 1 + \sum_{r=1}^n a_{2r} x^{-2r} \right\} + R_{2n}, \quad (6)$$

where

$$a_{2r} = \frac{(1 - 2^{1-2r}) (k+1)! (2\pi)^{2r}}{(k+1-2r)! (2r)!} |B_{2r}|, \quad (7)$$

and R_{2n} is a remainder term. Dingle^{4,5,6} has transformed (5) into a convergent representation by replacing the final sum with

$$\sum_{r=[1/2(k+3)]}^n \frac{t_{2r} (2r-k-2)!}{x^{2r-k-1}} + \frac{(2n-k)!}{x^{2n-k+1}} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^{2n+2}} A_{2n-k}(jx), \quad (8)$$

where

$$A_s(x) = \frac{-\pi x^{s+1}}{2(s!) \sin \pi s} (e^x - e^{-x} \cos \pi s) - \sum_{m=1}^{\infty} \frac{(s-2m)!}{s!} x^{2m}. \quad (9)$$

In the Sommerfeld form,

$$R_{2n} = \frac{x^{k+1}}{k+1} \frac{2}{\pi} \frac{\sin(k\pi)(k+1)!(2n-k)!}{x^{2n+2}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^{2n+2}} A_{2n-k}(jx). \quad (10)$$

The reader is referred to Dingle's works for the derivations of (8) and (9) and for some useful asymptotic expressions for the $A_s(x)$.

III. APPROXIMATION FORMS

Three different approximation forms and associated intervals were chosen for each function, reflecting the basically different functional behaviors displayed in Eqs. (3), (4), and (6). The forms and intervals are

$$(F_k)_{\ell,m}^{1*}(x) = e^x [\Gamma(k+1) + e^x R_{k,\ell,m}^{1*}(e^x)], \quad -\infty < x \leq 1; \quad (11)$$

$$(F_k)_{\ell,m}^{2*}(x) = R_{k,\ell,m}^{2*}(x), \quad 1 \leq x \leq 4; \quad (12)$$

and

$$(F_k)_{\ell,m}^{3*}(x) = x^{k+1} \left[\frac{1}{k+1} + \frac{1}{x^2} R_{k,\ell,m}^{3*}(1/x^2) \right], \quad 4 \leq x < \infty; \quad (13)$$

where the $R_{k,\ell,m}^{i*}$ are rational Chebyshev approximations of degree ℓ in the numerator and m in the denominator. The first and third forms were also used by Werner and Raymann,¹² although our choice of interval for the third form is different. The choice of intervals used here is the result of experimentation. Reasonable choices of ℓ and m give reasonable accuracy on each interval and, although not optimal in this sense, a given choice of k, ℓ , and m results in about the same accuracy for each interval.

IV. COMPUTATIONS

All computations to be described were performed on a CDC-3600 computer in 25-decimal, floating-point arithmetic.

The basic tools for obtaining the approximations were two versions of the second algorithm of Remes.^{3,7} Functional values were computed as needed in a number of ways. For $x < -1$, Eq. (3) gave at least 20S results. The series in Eq. (4) was transformed by the QD algorithm⁸ into a continued fraction, the first 40 terms of which gave about 11S for $|x| < 4$ (higher accuracy for smaller x and less accuracy for larger x). Finally, the Sommerfeld-Dingle

expansion, Eqs. (6)-(10), was the basis for a computation that gave maximal relative errors of 3×10^{-9} for $k = -1/2$, 3×10^{-11} for $k = 1/2$, and 5×10^{-13} for $k = 3/2$ and $x \geq 4$. Because of large subtraction errors in the Dingle method, these last accuracies appear nearly maximal when 25-decimal arithmetic is used. All three methods of computation were cross-checked in regions where they overlapped and were checked for gross errors against existing tables in the literature, although none of the tables contained as many significant figures as the computations. Additional detailed numerical checking was performed for the computations based on Dingle's work because of the large subtraction error involved in Eq. (9) for certain values of s and x . As a final check, the relative error functions

$$\delta_{k,\ell,m}^{i*}(x) = \frac{F_k(x) - (F_k)_{\ell,m}^{i*}(x)}{F_k(x)} \quad (14)$$

were plotted on a cathode-ray tube, photographed and examined for smoothness.

FORTTRAN listings of the subroutines used to compute the function values are presented in the appendix.

Because the approximation forms (11) and (13) correctly emulate the asymptotic behavior of $F_k(x)$ as $x \rightarrow \pm\infty$, the errors (14) vanish asymptotically. Thus computations in the Remes algorithm could be restricted to large finite intervals, generally $[-10,1]$ in the first case and $[4,60]$ in the second.

In the original computations, all error curves were leveled to at least $3S$. The rounded coefficients presented in this report were separately tested for 2000 random arguments against the original function routines. In each case the maximal error agreed within $2S$ in magnitude and position with one of the extremal points found in the Remes algorithm.

Out of about 200 different approximations generated for the intervals and approximation forms described above, almost a dozen gave nonstandard error curves on the interval considered, or a slightly larger interval, or gave computational difficulty because of near-degeneracy. The nonstandard error curves were typified by an extra extremal point of magnitude different from the others, while the near-degeneracy was frequently typified by a pole just outside the approximation interval, and a near-common factor in the numerator and denominator. As expected, if $(F_k)_{\ell,m}^{i*}$ had a nonstandard error curve, $(F_k)_{\ell+1,m+1}^{i*}$ was nearly degenerate. Behaviors of this type have previously been noted and commented upon, particularly by Ralston¹⁰ and Rice.¹¹

The behavior of the two versions of the Remes algorithm used in these difficult cases points up a basic difference in the numerical stability of the two approaches. For example, the program based on the Fraser-Hart technique⁷ failed to converge to $(F_{1/2})_{3,3}^{3*}(x)$ even when $10S$ initial guesses at the critical points and a $5S$ initial guess at the maximal error, based on the approximation obtained by the Cody-Stoer technique,⁷ were used. The difficulty in this case is that the denominator of $R_{1/2,3,3}^{3*}$ vanishes for $x^2 \approx 15.93994663$, while the numerator vanishes for $x^2 \approx 15.93994749$, and the interval of approximation is $16 \leq x^2 < \infty$. This approximation is not very stable numerically.

The techniques devised to handle such nearly-degenerate cases are still being revised and will be the subject of a future report.

V. RESULTS

Tables I-III list the values of

$$E_{k,l,m}^{i*} = -100 \log \max |\delta_{k,l,m}^{i*}(x)|,$$

where the maximum is taken over the appropriate interval, for the initial segments of the L_∞ Walsh arrays. Tables IV-XII present the coefficients for all approximations listed in Tables I-III. All coefficients are given to an accuracy greater than that justified by the maximum errors. Reasonable rounding of the coefficients generally should not greatly affect the overall accuracy. However, some care is necessary for nearly degenerate approximations such as $(F_{1/2})_{3,3}^{3*}(x)$. In a number of other isolated cases, notably $(F_{-1/2})_{7,0}^{1*}(x)$, $(F_{-1/2})_{8,0}^{1*}(x)$, and $(F_{3/2})_{0,3}^{2*}(x)$, the maximum relative error is abnormally increased by further rounding of the coefficients.

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TABLE I. Relative Errors $E_{-1/2, \ell, m}^{i*} = -100 \log_{10} \left\| \frac{F_{-1/2}(x) - (F_{-1/2})_{\ell, m}^{i*}(x)}{F_{-1/2}(x)} \right\|_{\infty}$

$i = 1, -\infty < x \leq 1$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8
0	65	129	187	243	298	351	404	457	510
1	265	345	418	486	552				
2		492	580	661	738				
3		580	714	807	893				
4			806	934	1030				

$i = 2, 1 \leq x \leq 4$

0	44	173	285 [†]	316	397	534 [†]	558	621	727
1		266	312	362	484	557		701	
2			413	561	752	795	863		
3				744	794	822			
4					879				

$i = 3, 4 \leq x < \infty$

0	317 [†]	348 [†]	454 [†]	465	503	566	607	617	644
1		381	465		560	600			
2			524	616 [†]	633				
3				630					
4					762				

[†]Nonstandard error curve.

TABLE II. Relative Errors, $E_{1/2, \ell, m}^{i*} = -100 \log_{10} \left\| \frac{F_{1/2}(x) - (F_{1/2})_{\ell, m}^{i*}(x)}{F_{1/2}(x)} \right\|_{\infty}$

$i = 1, -\infty < x \leq 1$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8
0	109	181	247	309	369	428	485	542	598
1	271	359	439	513	584				
2		495	588	674	755				
3		586	715	812	902				
4			811	935	1033				
5		746							

$i = 2, 1 \leq x \leq 4$

0	21	118	265	393 [†]	431	519	666 [†]	692	760
1		207	374	427		614	691		
2		289	436	531	667	802	920		
3		371	516	648	846	904			
4		456			905				
5		544							

$i = 3, 4 \leq x < \infty$

0	312	407	480	516	586	640	652	687	737
1		464							
2			647 [†]						
3				654					
4					817				

[†]Nonstandard error curve.

TABLE III. Relative Errors, $E_{3/2, \ell, m}^{i*} = -100 \log_{10} \left\| \frac{F_{3/2}(x) - (F_{3/2})_{\ell, m}^{i*}(x)}{F_{3/2}(x)} \right\|_{\infty}$

$i = 1, -\infty < x \leq 1$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8
0	150	232	305	374	439	503	565	625	685
1	301	397	483	563	639				
2		528	626	716	801				
3		622	750	850	944				
4			848	971	1072				

$i = 2, 1 \leq x \leq 4$

0	13	79	199	358	499	542	635	789	821
1	59	171	304	479	538		737	819	
2	122		397	546	645	781	908		
3	195			630	766	942			
4					858				

$i = 3, 4 \leq x < \infty$

0	291	485 [†]	488	567	643	657	699	761	811 [†]
1		488							
2			637						
3				784					
4					944				

[†] Nonstandard error curve.

TABLE IV. Coefficients for $(F_{-1/2})_{l,m}^{1*} = e^x \left[\Gamma(1/2) + e^x \sum_{s=0}^l p_s e^{sx} / \sum_{s=0}^m q_s e^{sx} \right]$,
 $-\infty < x \leq 1$

l	m	s	p_s	q_s

0	0	0	-4.61	(-01) 1.00 (00)

0	1	0	-1.2198	(00) 1.0000 (00)
		1		7.3975 (-01)

1	0	0	-8.414	(-01) 1.000 (00)
		1	1.649	(-01)

1	1	0	-1.24470	(00) 1.00000 (00)
		1	-1.52654	(-02) 7.98207 (-01)

1	2	0	-1.25280 0	(00) 1.00000 0 (00)
		1	-4.67039 3	(-01) 1.18610 7 (00)
		2		2.69776 1 (-01)

1	3	0	-1.25322 05	(00) 1.00000 00 (00)
		1	-5.97991 17	(-01) 1.29286 92 (00)
		2		3.51302 56 (-01)
		3		-7.06264 55 (-04)

2	0	0	-1.063	(00) 1.000 (00)
		1	4.172	(-01)
		2	-6.497	(-02)

2	1	0	-1.25098 1	(00) 1.00000 0 (00)
		1	-3.73341 3	(-02) 8.34512 9 (-01)
		2	3.59541 7	(-03)

2	2	0	-1.25322 15	(00) 1.00000 00 (00)
		1	-6.01723 59	(-01) 1.29585 46 (00)
		2	-1.22715 51	(-03) 3.54694 31 (-01)

2	3	0	-1.25330 8005	(00) 1.00000 0000 (00)
		1	-1.00231 7312	(00) 1.61615 9896 (00)
		2	-1.35492 2115	(-01) 7.20948 0969 (-01)
		3		7.73159 3686 (-02)

2	4	0	-1.25331 32175	(00) 1.00000 00000 (00)
		1	-1.17038 54694	(00) 1.75031 56729 (00)
		2	-2.10444 17181	(-01) 8.90016 57132 (-01)
		3		1.21194 98495 (-01)
		4		-7.23010 70715 (-05)

TABLE IV. (Contd.)

ℓ	m	s	p_s	q_s
4	3	0	-1.25331 39847	(00) 1.00000 00000 (00)
		1	-1.30813 72929	(00) 1.86023 63897 (00)
		2	-2.84288 96320	(-01) 1.03862 13128 (00)
		3	-4.51835 15684	(-04) 1.65371 21843 (-01)
		4	1.69497 15445	(-05)
4	4	0	-1.25331 41288 20	(00) 1.00000 00000 00 (00)
		1	-1.72366 35577 01	(00) 2.19178 09259 80 (00)
		2	-6.55904 57292 58	(-01) 1.60581 29554 06 (00)
		3	-6.34228 31976 82	(-02) 4.44366 95274 81 (-01)
		4	-1.48838 31061 16	(-05) 3.62423 22881 12 (-02)
5	0	0	-1.24021	(00) 1.00000 (00)
		1	9.14609	(-01)
		2	-5.51063	(-01)
		3	2.22592	(-01)
		4	-5.07552	(-02)
		5	4.85492	(-03)
6	0	0	-1.24829 7	(00) 1.00000 0 (00)
		1	9.70148 3	(-01)
		2	-6.79477 4	(-01)
		3	3.60057 8	(-01)
		4	-1.25503 8	(-01)
		5	2.49157 0	(-02)
		6	-2.11027 8	(-03)
7	0	0	-1.25143 39	(00) 1.00000 00 (00)
		1	9.98499 44	(-01)
		2	-7.66993 94	(-01)
		3	4.88938 27	(-01)
		4	-2.27446 02	(-01)
		5	6.93674 38	(-02)
		6	-1.21791 29	(-02)
		7	9.25405 74	(-04)
8	0	0	-1.25262 124	(00) 1.00000 000 (00)
		1	1.01216 553	(00)
		2	-8.21205 789	(-01)
		3	5.93515 846	(-01)
		4	-3.39488 875	(-01)
		5	1.39521 113	(-01)
		6	-3.77148 044	(-02)
		7	5.93123 886	(-03)
		8	-4.08601 489	(-04)

TABLE V. Coefficients for $(F_{1/2})_{\ell,m}^{1*} = e^x \left[\Gamma(3/2) + e^x \sum_{s=0}^{\ell} p_s e^{sx} / \sum_{s=0}^m q_s e^{sx} \right]$, $-\infty < x \leq 1$

ℓ	m	s	p_s	q_s

0	0	0	-1.525	(-01) 1.000 (00)

0	1	0	-3.0017	(-01) 1.0000 (00)
		1		4.4005 (-01)

1	0	0	-2.455	(-01) 1.000 (00)
		1	4.097	(-02)

1	1	0	-3.10391	(-01) 1.00000 (00)
		1	-1.00423	(-02) 5.38275 (-01)

1	2	0	-3.13104 6	(-01) 1.00000 0 (00)
		1	-1.02287 9	(-01) 8.65935 9 (-01)
		2		1.29887 2 (-01)

1	3	0	-3.13290 15	(-01) 1.00000 00 (00)
		1	-1.39206 04	(-01) 9.87450 43 (-01)
		2		1.87895 54 (-01)
		3		-1.26432 39 (-03)

1	5	0	-3.13326 9338	(-01) 1.00000 0000 (00)
		1	-1.85787 4534	(-01) 1.13719 8219 (00)
		2		2.65906 7030 (-01)
		3		-5.64719 9331 (-03)
		4		6.37389 0600 (-04)
		5		-4.07029 6708 (-05)

2	0	0	-2.8718	(-01) 1.0000 (00)
		1	9.0129	(-02)
		2	-1.2925	(-02)

2	1	0	-3.12629 4	(-01) 1.00000 0 (00)
		1	-2.34283 2	(-02) 6.05737 3 (-01)
		2	1.84516 8	(-03)

2	2	0	-3.13291 80	(-01) 1.00000 00 (00)
		1	-1.42756 95	(-01) 9.98828 53 (-01)
		2	-1.00908 90	(-03) 1.97169 67 (-01)

TABLE V. (Contd.)

ℓ	m	s	P_s	q_s
2	3	0	-3.13325 7406	(-01) 1.00000 0000 (00)
		1	-2.33341 3133	(-01) 1.28892 8282 (00)
		2	-2.72819 5186	(-02) 4.35695 1559 (-01)
		3		3.29828 7723 (-02)
2	4	0	-3.13328 13573	(-01) 1.00000 00000 (00)
		1	-2.78977 04596	(-01) 1.43467 48241 (00)
		2	-4.60619 48001	(-02) 5.74527 98957 (-01)
		3		5.80505 94853 (-02)
		4		-1.42372 59572 (-04)
3	0	0	-3.03733	(-01) 1.00000 (00)
		1	1.26890	(-01)
		2	-3.61760	(-02)
		3	4.44225	(-03)
3	1	0	-3.13154 50	(-01) 1.00000 00 (00)
		1	-3.62052 61	(-02) 6.55302 08 (-01)
		2	5.26949 02	(-03)
		3	-4.24923 34	(-04)
3	2	0	-3.13321 838	(-01) 1.00000 000 (00)
		1	-1.74702 263	(-01) 1.10163 079 (00)
		2	-2.99276 299	(-03) 2.56849 951 (-01)
		3	1.39102 692	(-04)
3	3	0	-3.13328 14419	(-01) 1.00000 00000 (00)
		1	-2.80582 65535	(-01) 1.43979 96246 (00)
		2	-4.71780 05580	(-02) 5.80877 04412 (-01)
		3	-1.18443 08954	(-04) 6.00800 57319 (-02)
3	4	0	-3.13328 50447 6	(-01) 1.00000 00000 0 (00)
		1	-3.65440 77502 6	(-01) 1.71064 71034 6 (00)
		2	-1.06722 78464 1	(-01) 9.18231 83574 7 (-01)
		3	-6.52474 17608 5	(-03) 1.68753 53052 2 (-01)
		4		7.70686 19606 8 (-03)
4	0	0	-3.09912	(-01) 1.00000 (00)
		1	1.48926	(-01)
		2	-6.06170	(-02)
		3	1.51297	(-02)
		4	-1.61145	(-03)

TABLE V. (Contd.)

TABLE V. (Contd.)

ℓ	m	s	P_s	q_s
7	0	0	-3.13189 12	(-01) 1.00000 00 (00)
		1	1.68674 01	(-01)
		2	-1.01706 97	(-01)
		3	5.64746 46	(-02)
		4	-2.44159 47	(-02)
		5	7.16957 74	(-03)
		6	-1.23440 92	(-03)
		7	9.28638 14	(-05)
8	0	0	-3.13281 50	(-01) 1.00000 00 (00)
		1	1.69779 56	(-01)
		2	-1.06223 96	(-01)
		3	6.53935 82	(-02)
		4	-3.41539 70	(-02)
		5	1.33632 26	(-02)
		6	-3.51889 08	(-03)
		7	5.45793 86	(-04)
		8	-3.73344 14	(-05)

TABLE VI. Coefficients for $(F_{3/2})_{\ell,m}^{1*} = e^x \left[\Gamma(5/2) + e^x \sum_{s=0}^{\ell} p_s e^{sx} / \sum_{s=0}^m q_s e^{sx} \right]$,
 $-\infty < x \leq 1$

ℓ	m	s	p_s	q_s

0	0	0	-1.402	(-01) 1.000 (00)

0	1	0	-2.26015	(-01) 1.00000 (00)
		1		2.79353 (-01)

1	0	0	-2.0257	(-01) 1.0000 (00)
		1	2.7773	(-02)

1	1	0	-2.33268	(-01) 1.00000 (00)
		1	-1.06386	(-02) 3.80655 (-01)

1	2	0	-2.34851 35	(-01) 1.00000 00 (00)
		1	-6.75255 59	(-02) 6.46378 25 (-01)
		2		6.64570 69 (-02)

1	3	0	-2.34972 957	(-01) 1.00000 000 (00)
		1	-9.44874 442	(-02) 7.64083 900 (-01)
		2		1.03270 297 (-01)
		3		-1.10216 837 (-03)

2	0	0	-2.24498	(-01) 1.00000 (00)
		1	5.41190	(-02)
		2	-7.00389	(-03)

2	1	0	-2.34636 0	(-01) 1.00000 0 (00)
		1	-2.39138 2	(-02) 4.55977 3 (-01)
		2	1.51156 1	(-03)

2	2	0	-2.34974 804	(-01) 1.00000 000 (00)
		1	-9.90504 038	(-02) 7.83564 382 (-01)
		2	-1.14559 597	(-03) 1.15033 976 (-01)

2	3	0	-2.34994 6434	(-01) 1.00000 0000 (00)
		1	-1.58962 7447	(-01) 1.03924 0157 (00)
		2	-1.64723 8393	(-02) 2.70866 7647 (-01)
		3		1.47926 8388 (-02)

2	4	0	-2.34996 16134	(-01) 1.00000 00000 (00)
		1	-1.93018 6 609	(-01) 1.18423 90314 (00)
		2	-2.91563 03772	(-02) 3.77136 13107 (-01)
		3		2.84183 83369 (-02)
		4		-1.25408 75607 (-04)

TABLE VI. . (Contd.)

ℓ	m	s	p_s	q_s
3	0	0	-2.31672	(-01) 1.00000 (00)
		1	7.04044	(-02)
		2	-1.74462	(-02)
		3	2.01404	(-03)
3	1	0	-2.34916 55	(-01) 1.00000 00 (00)
		1	-3.62065 37	(-02) 5.14273 17 (-01)
		2	4.09037 23	(-03)
		3	-2.88081 88	(-04)
3	2	0	-2.34992 8140	(-01) 1.00000 0000 (00)
		1	-1.24818 5148	(-01) 8.93855 4914 (-01)
		2	-3.37885 4298	(-03) 1.62736 8527 (-01)
		3	1.26476 0763	(-04)
3	3	0	-2.34996 17182	(-01) 1.00000 00000 (00)
		1	-1.95392 64014	(-01) 1.19434 20572 (00)
		2	-3.06557 11516	(-02) 3.87179 30021 (-01)
		3	-1.41222 30260	(-04) 3.08879 90780 (-02)
3	4	0	-2.34996 38233 1	(-01) 1.00000 00000 0 (00)
		1	-2.53765 84108 6	(-01) 1.44275 68663 5 (00)
		2	-6.78523 16384 9	(-02) 6.35531 80238 0 (-01)
		3	-3.68952 96423 8	(-03) 9.24256 24033 3 (-02)
		4		3.15574 24108 7 (-03)
4	0	0	-2.33955 1	(-01) 1.00000 0 (00)
		1	7.87280 1	(-02)
		2	-2.68168 7	(-02)
		3	6.15592 1	(-03)
		4	-6.29609 2	(-04)
4	1	0	-2.34977 878	(-01) 1.00000 000 (00)
		1	-4.67124 740	(-02) 5.60840 342 (-01)
		2	7.00409 477	(-03)
		3	-9.23800 805	(-04)
		4	6.51095 874	(-05)
4	2	0	-2.34995 75343	(-01) 1.00000 00000 (00)
		1	-1.46108 33321	(-01) 9.84592 22315 (-01)
		2	-6.30011 99126	(-03) 2.07555 31821 (-01)
		3	4.29361 72986	(-04)
		4	-1.93454 54597	(-05)

TABLE VII. Coefficients for $(F_{-1/2})_{\ell,m}^{2*} = \sum_{s=0}^{\ell} p_s x^s / \sum_{s=0}^m q_s x^s$, $1 \leq x \leq 4$

ℓ	m	s	p_s	q_s

0	0	0	2.48	(00) 1.00 (00)

1	0	0	1.157	(00) 1.000 (00)
		1	6.974	(-01)

1	1	0	9.2012	(-01) 1.0000 (00)
		1	1.0331	(00) 7.5323 (-02)

2	0	0	9.5182	(-01) 1.0000 (00)
		1	9.1065	(-01)
		2	-4.4659	(-02)

2	1	0	9.78850	(-01) 1.00000 (00)
		1	7.99584	(-01) -7.01494 (-02)
		2	-8.70049	(-02)

2	2	0	1.17909 1	(00) 1.00000 0 (00)
		1	1.33436 7	(00) 8.97500 7 (-01)
		2	1.15108 8	(00) 1.15382 4 (-01)

3	0	0	9.85318	(-01) 1.00000 (00)
		1	8.55422	(-01)
		2	-1.76135	(-02)
		3	-3.96073	(-03)

3	1	0	8.55663	(-01) 1.00000 (00)
		1	-6.45133	(-01) -1.81040 (00)
		2	-1.78045	(00)
		3	9.43084	(-02)

3	2	0	1.07688 85	(00) 1.00000 00 (00)
		1	7.66121 17	(-01) 1.00214 78 (-01)
		2	2.71062 35	(-01) 7.46077 27 (-02)
		3	2.45934 98	(-02)

3	3	0	1.07161 9310	(00) 1.00000 0000 (00)
		1	7.59564 5943	(-01) 7.79454 5888 (-02)
		2	2.52371 0602	(-01) 9.21173 5007 (-02)
		3	5.09743 3764	(-02) 2.49051 0221 (-03)

TABLE VII. (Contd.)

ℓ	m	s	p_s	q_s
4	0	0	1.04864	(00) 1.00000 (00)
		1	7.22814	(-01)
		2	7.82344	(-02)
		3	-3.24449	(-02)
		4	2.96718	(-03)
4	1	0	1.10603 7	(00) 1.00000 0 (00)
		1	9.98433 7	(-01) 4.04948 1 (-01)
		2	5.16390 2	(-01)
		3	-6.75920 5	(-02)
		4	4.35166 1	(-03)
4	2	0	1.07198 0228	(00) 1.00000 0000 (00)
		1	7.26622 8684	(-01) 4.84796 2678 (-02)
		2	2.29829 8375	(-01) 8.76964 9288 (-02)
		3	4.06344 9460	(-02)
		4	-7.59453 8767	(-04)
4	3	0	1.07181 6279	(00) 1.00000 0000 (00)
		1	7.39403 4230	(-01) 5.98390 7395 (-02)
		2	2.38509 0240	(-01) 8.94886 4578 (-02)
		3	4.47670 3287	(-02) 9.79176 7837 (-04)
		4	-4.64044 6743	(-04)
4	4	0	1.07381 27694	(00) 1.00000 00000 (00)
		1	5.60033 03660	(00) 4.60318 40667 (00)
		2	3.68822 11270	(00) 4.30759 10674 (-01)
		3	1.17433 92816	(00) 4.21511 32145 (-01)
		4	2.36419 35527	(-01) 1.18326 01601 (-02)
5	0	0	1.08026 79	(00) 1.00000 00 (00)
		1	6.41448 71	(-01)
		2	1.56861 77	(-01)
		3	-6.82099 07	(-02)
		4	1.06660 51	(-02)
		5	-6.31436 00	(-04)
5	1	0	1.08337 02	(00) 1.00000 00 (00)
		1	6.70588 64	(-01) 3.54434 55 (-02)
		2	1.90302 78	(-01)
		3	-6.90299 79	(-02)
		4	1.02974 84	(-02)
		5	-5.90999 68	(-04)

TABLE VII. (Contd.)

ℓ	m	s	P_s	q_s

5	2	0	1.07181 9778	(00) 1.00000 0000 (00)
		1	7.27410 9124	(-01) 4.86640 5987 (-02)
		2	2.30378 2202	(-01) 8.89181 0299 (-02)
		3	4.21642 5666	(-02)
		4	-9.13975 8985	(-04)
		5	8.27404 6194	(-06)

5	3	0	1.07188 88751	(00) 1.00000 00000 (00)
		1	5.03265 79663	(-01) -1.60188 65290 (-01)
		2	7.84074 42020	(-02) 7.82236 73744 (-02)
		3	-6.51696 79405	(-03) -1.82990 31947 (-02)
		4	-9.31821 65739	(-03)
		5	1.62673 42406	(-04)

6	0	0	1.08318 53	(00) 1.00000 00 (00)
		1	6.32859 09	(-01)
		2	1.66897 05	(-01)
		3	-7.41802 91	(-02)
		4	1.25805 51	(-02)
		5	-9.46403 90	(-04)
		6	2.08504 77	(-05)

6	2	0	1.07199 90840	(00) 1.00000 00000 (00)
		1	7.17208 85162	(-01) 3.98630 45189 (-02)
		2	2.23906 21201	(-01) 8.71325 39369 (-02)
		3	3.87101 09429	(-02)
		4	-1.45381 72252	(-04)
		5	-8.82658 30124	(-05)
		6	4.94724 92164	(-06)

7	0	0	1.07684 100	(00) 1.00000 000 (00)
		1	6.55745 112	(-01)
		2	1.32949 058	(-01)
		3	-4.73283 302	(-02)
		4	3.32269 933	(-04)
		5	2.28232 349	(-03)
		6	-4.35743 850	(-04)
		7	2.67949 680	(-05)

TABLE VIII. Coefficients for $(F_{1/2})_{\ell,m}^{2*} = \sum_{s=0}^{\ell} p_s x^s / \sum_{s=0}^m q_s x^s$, $1 \leq x \leq 4$

ℓ	m	s	p_s	q_s

0	0	0	2.25	(00)
1	0	0	-5.768	(-02)
		1	1.362	(00)
1	1	0	4.7314	(-01)
		1	7.7863	(-01)
1	2	0	6.1007	(-01)
		1	5.2596	(-01)
		2		1.7706
1	3	0	6.55032	(-01)
		1	3.79108	(-01)
		2		5.25739
		3		-3.62667
1	4	0	6.70525 3	(-01)
		1	2.81923 4	(-01)
		2		9.87557 5
		3		-1.31311 6
		4		7.38087 6
1	5	0	6.75788 07	(-01)
		1	2.11156 09	(-01)
		2		1.50039 27
		3		-2.83494 53
		4		3.05667 38
		5		-1.44147 54
2	0	0	6.3377	(-01)
		1	5.8352	(-01)
		2	1.7598	(-01)
2	1	0	7.11750	(-01)
		1	4.88466	(-01)
		2	2.65133	(-01)
2	2	0	6.94327 4	(-01)
		1	4.91885 5	(-01)
		2	2.14556 1	(-01)

TABLE VIII. (Contd.)

l	m	s	P_s	q_s
2	3	0	6.83793 84	(-01) 1.00000 00 (00)
		1	4.55851 30	(-01) -9.25984 29 (-02)
		2	1.51936 65	(-01) 1.86829 17 (-02)
		3		-1.13763 36 (-03)
3	0	0	7.04404	(-01) 1.00000 (00)
		1	4.69599	(-01)
		2	2.30287	(-01)
		3	-7.74968	(-03)
3	1	0	6.97061 6	(-01) 1.00000 0 (00)
		1	4.47976 9	(-01) -5.34822 3 (-02)
		2	1.92989 4	(-01)
		3	-1.62631 4	(-02)
3	2	0	6.60285 87	(-01) 1.00000 00 (00)
		1	1.12971 85	(00) 7.86010 96 (-01)
		2	5.44822 92	(-01) 5.56166 69 (-02)
		3	2.36763 29	(-01)
3	3	0	6.76208 535	(-01) 1.00000 000 (00)
		1	6.51664 310	(-01) 1.59803 695 (-01)
		2	2.63424 203	(-01) 3.05417 676 (-02)
		3	6.96443 154	(-02) -8.78750 815 (-04)
4	0	0	6.95843 7	(-01) 1.00000 0 (00)
		1	4.88374 6	(-01)
		2	2.15982 6	(-01)
		3	-3.26704 0	(-03)
		4	-4.90344 6	(-04)
4	2	0	6.76853 838	(-01) 1.00000 000 (00)
		1	6.42001 951	(-01) 1.49222 401 (-01)
		2	2.69861 983	(-01) 4.25490 855 (-02)
		3	7.15207 988	(-02)
		4	3.92131 914	(-03)
4	3	0	6.78110 83388	(-01) 1.00000 00000 (00)
		1	6.24037 03618	(-01) 1.29918 59343 (-01)
		2	2.84010 76536	(-01) 6.71283 90678 (-02)
		3	7.59568 01178	(-02) 1.79512 91391 (-03)
		4	1.19180 49933	(-02)

TABLE VIII. (Contd.)

ℓ	m	s	p_s	q_s
4	4	0	6.78176 62666 0	(-01) 1.00000 00000 0 (00)
		1	6.33124 01791 0	(-01) 1.43740 40039 7 (-01)
		2	2.94479 65177 2	(-01) 7.08662 14845 0 (-02)
		3	8.01320 71141 9	(-02) 2.34579 49473 5 (-03)
		4	1.33918 21294 0	(-02) -1.29449 92883 5 (-05)
5	0	0	6.82612 13	(-01) 1.00000 00 (00)
		1	5.23388 99	(-01)
		2	1.81099 79	(-01)
		3	1.30949 41	(-02)
		4	-4.11706 11	(-03)
		5	3.05557 33	(-04)
5	1	0	6.73763 552	(-01) 1.00000 000 (00)
		1	7.55181 055	(-01) 2.98751 505 (-01)
		2	3.02649 687	(-01)
		3	8.92997 704	(-02)
		4	-7.73701 652	(-03)
		5	3.86129 735	(-04)
5	2	0	6.78002 84280	(-01) 1.00000 00000 (00)
		1	5.99576 35111	(-01) 9.31494 46381 (-02)
		2	2.57973 40249	(-01) 5.90873 68499 (-02)
		3	6.51217 69768	(-02)
		4	8.43035 94767	(-03)
		5	-1.49735 28933	(-04)
5	3	0	6.78177 60416 1	(-01) 1.00000 00000 0 (00)
		1	6.38250 12281 3	(-01) 1.51306 14398 8 (-01)
		2	2.99229 34508 6	(-01) 7.18791 24202 8 (-02)
		3	8.22840 72325 8	(-02) 2.84704 36521 7 (-03)
		4	1.39726 87071 8	(-02)
		5	8.73836 46369 9	(-05)
6	0	0	6.77125 535	(-01) 1.00000 000 (00)
		1	5.40552 553	(-01)
		2	1.59801 137	(-01)
		3	2.65204 122	(-02)
		4	-8.66050 609	(-03)
		5	1.09080 605	(-03)
		6	-5.43532 456	(-05)

TABLE IX. Coefficients for $(F_{3/2})_{\ell, m}^{2*} = \sum_{s=0}^{\ell} p_s x^s / \sum_{s=0}^m q_s x^s$, $1 \leq x \leq 4$

ℓ	m	s	p_s	q_s
0	0	0	4.63	(00) 1.00 (00)
0	1	0	2.61	(00) 1.00 (00)
		1		-2.21 (-01)
0	2	0	1.721	(00) 1.000 (00)
		1		-4.453 (-01)
		2		5.531 (-02)
0	3	0	1.3797	(00) 1.0000 (00)
		1		-6.2697 (-01)
		2		1.5304 (-01)
		3		-1.3490 (-02)
1	0	0	-1.95	(00) 1.00 (00)
		1	4.19	(00)
1	1	0	4.986	(-01) 1.000 (00)
		1	1.729	(00) -1.468 (-01)
2	0	0	1.917	(00) 1.000 (00)
		1	-2.650	(-01)
		2	1.037	(00)
2	1	0	1.31361	(00) 1.00000 (00)
		1	6.01871	(-01) -6.95719 (-02)
		2	5.63293	(-01)
2	2	0	1.19607	(00) 1.00000 (00)
		1	7.33852	(-01) -1.53064 (-01)
		2	3.52295	(-01) 1.04035 (-02)
3	0	0	1.19468	(00) 1.00000 (00)
		1	9.39850	(-01)
		2	4.39685	(-01)
		3	8.81772	(-02)

TABLE IX. (Contd.)

ℓ	m	s	p_s	q_s
7	0	0	1.15310 2291	(00) 1.00000 0000 (00)
		1	1.01555 9575	(00)
		2	4.05630 5952	(-01)
		3	7.97099 3415	(-02)
		4	1.00407 6968	(-02)
		5	-2.62552 9962	(-03)
		6	2.76862 5861	(-04)
		7	-1.19071 7378	(-05)
7	1	0	1.15329 35625	(00) 1.00000 00000 (00)
		1	1.04777 96293	(00) 2.86168 66388 (-02)
		2	4.36030 26590	(-01)
		3	9.00484 39570	(-02)
		4	1.30499 77962	(-02)
		5	-2.59708 88046	(-03)
		6	2.57537 33284	(-04)
		7	-1.06655 27384	(-05)
8	0	0	1.15328 38888	(00) 1.00000 00000 (00)
		1	1.01482 14756	(00)
		2	4.06898 15134	(-01)
		3	7.85081 28560	(-02)
		4	1.07295 10914	(-02)
		5	-2.87017 78452	(-03)
		6	3.29552 28679	(-04)
		7	-1.82094 36427	(-05)
		8	3.21144 25144	(-07)

TABLE X. Coefficients for $(F_{-1/2})_{l,m}^{3*} = \sqrt{x} \left[2 + x^{-2} \frac{\sum_{s=0}^l p_s x^{-2s}}{\sum_{s=0}^m q_s x^{-2s}} \right]$, $4 \leq x < \infty$

l	m	s	p_s	q_s

0	0	0	-9.84535	(-01) 1.00000 (00)

1	0	0	-9.13416	(-01) 1.00000 (00)
		1	-1.63268	(00)

1	1	0	-5.86246	(-01) 1.00000 (00)
		1	-1.58903	(02) 1.50627 (02)

2	0	0	-8.02227 9	(-01) 1.00000 0 (00)
		1	-7.48362 9	(00)
		2	6.80125 6	(01)

2	1	0	-7.95655 2	(-01) 1.00000 0 (00)
		1	-9.45646 7	(00) 1.74603 8 (00)
		2	6.97840 6	(01)

2	2	0	-8.14958 47	(-01) 1.00000 00 (00)
		1	4.05212 66	(00) -1.08676 28 (01)
		2	-3.25435 65	(02) 3.84615 01 (02)

3	0	0	-7.95844 0	(-01) 1.00000 0 (00)
		1	-8.03586 8	(00)
		2	8.22691 9	(01)
		3	-1.13698 2	(02)

3	2	0	-8.25012 894	(-01) 1.00000 000 (00)
		1	-1.12755 836	(01) 1.10714 048 (01)
		2	-9.08601 950	(02) 7.32006 445 (02)
		3	2.06417 674	(03)

3	3	0	-8.24391 144	(-01) 1.00000 000 (00)
		1	-2.04495 807	(00) -4.88152 379 (-01)
		2	-8.96893 377	(02) 8.05727 048 (02)
		3	4.88655 638	(03) -3.56730 597 (03)

4	0	0	-8.10832 95	(-01) 1.00000 00 (00)
		1	-5.43271 21	(00)
		2	-5.40522 97	(01)
		3	2.64299 50	(03)
		4	-1.89535 53	(04)

TABLE X. (Contd.)

l	m	s	P_s	q_s
4	1	0	-8.31127 51	(-01) 1.00000 00 (00)
		1	-5.49534 68	(01) 6.74773 69 (01)
		2	-8.40753 47	(02)
		3	1.35278 82	(04)
		4	-6.53744 54	(04)
4	2	0	-8.24257 595	(-01) 1.00000 000 (00)
		1	-4.76805 483	(00) 2.72740 320 (00)
		2	-9.68394 313	(02) 8.98120 468 (02)
		3	1.90769 269	(01)
		4	1.29436 252	(04)
4	4	0	-8.22255 9330	(-01) 1.00000 0000 (00)
		1	-3.62036 9345	(01) 3.93568 9841 (01)
		2	-3.01538 5410	(03) 3.56875 6266 (03)
		3	-7.04987 1579	(04) 4.18189 3625 (04)
		4	-5.69814 5924	(04) 3.38513 8907 (05)
5	0	0	-8.21950 59	(-01) 1.00000 00 (00)
		1	-2.81509 44	(00)
		2	-2.53481 70	(02)
		3	9.23757 98	(03)
		4	-1.17463 07	(05)
		5	5.45286 55	(05)
5	1	0	-8.26076 882	(-01) 1.00000 000 (00)
		1	-1.31956 104	(01) 1.41015 626 (01)
		2	-4.15796 591	(02)
		3	1.14592 915	(04)
		4	-1.26624 512	(05)
		5	5.43624 554	(05)
6	0	0	-8.25631 271	(-01) 1.00000 000 (00)
		1	-1.77016 944	(00)
		2	-3.56500 509	(02)
		3	1.39963 592	(04)
		4	-2.29698 400	(05)
		5	1.85480 331	(06)
		6	-5.99517 045	(06)

TABLE X. (Contd.)

ℓ	m	s	p_s	q_s
7	0	0	-8.25056 485	(-01) 1.00000 000 (00)
		1	-2.05733 265	(00)
		2	-3.09685 666	(02)
		3	1.04830 212	(04)
		4	-9.19037 085	(04)
		5	-1.06211 675	(06)
		6	2.55534 312	(07)
		7	-1.36641 261	(08)
8	0	0	-8.23861 027	(-01) 1.00000 000 (00)
		1	-2.74845 432	(00)
		2	-1.75529 446	(02)
		3	-1.91443 226	(03)
		4	5.33947 278	(05)
		5	-1.92904 148	(07)
		6	3.31093 215	(08)
		7	-2.87275 145	(09)
		8	1.01363 470	(10)

TABLE XI. Coefficients for $(F_{1/2})_{l,m}^{3*} = x\sqrt{x} \left[\frac{2}{3} + x^{-2} \sum_{s=0}^l p_s x^{-2s} / \sum_{s=0}^m q_s x^{-2s} \right]$, $4 \leq x < \infty$

l	m	s	p_s	q_s

0	0	0	8.66045	(-01) 1.00000 (00)

1	0	0	8.27563 5	(-01) 1.00000 0 (00)
		1	7.71260 8	(-01)

1	1	0	8.16118 1	(-01) 1.00000 0 (00)
		1	8.76882 9	(00) 8.94339 7 (00)

2	0	0	8.17952 1	(-01) 1.00000 0 (00)
		1	1.25752 1	(00)
		2	-5.52433 6	(00)

2	2	0	8.22713 535	(-01) 1.00000 000 (00)
		1	5.27498 049	(00) 5.69335 697 (00)
		2	2.90433 403	(02) 3.22149 800 (02)

3	0	0	8.20220 92	(-01) 1.00000 00 (00)
		1	1.00406 08	(00)
		2	2.17390 50	(00)
		3	-6.86511 96	(01)

3	3	0	8.22752 754	(-01) 1.00000 000 (00)
		1	-7.55890 283	(00) -9.89594 310 (00)
		2	2.07024 852	(02) 2.31227 330 (02)
		3	-4.71158 007	(03) -5.22142 719 (03)

4	0	0	8.22233 18	(-01) 1.00000 00 (00)
		1	6.74158 01	(-01)
		2	1.87884 69	(01)
		3	-3.95379 81	(02)
		4	2.20003 88	(03)

4	4	0	8.22449 97626	(-01) 1.00000 00000 (00)
		1	2.00463 03393	(01) 2.34862 07659 (01)
		2	1.82680 93446	(03) 2.20134 83743 (03)
		3	1.22265 30374	(04) 1.14426 73596 (04)
		4	1.40407 50092	(05) 1.65847 15900 (05)

TABLE XI. (Contd.)

ℓ	m	s	p_s	q_s
5	0	0	8.22847 690	(-01) 1.00000 000 (00)
		1	5.42973 691	(-01)
		2	2.81626 642	(01)
		3	-6.91504 148	(02)
		4	6.47339 934	(03)
		5	-2.30214 680	(04)
6	0	0	8.22766 554	(-01) 1.00000 000 (00)
		1	5.73872 521	(-01)
		2	2.43957 869	(01)
		3	-4.86789 764	(02)
		4	9.79722 285	(02)
		5	4.81988 969	(04)
		6	-3.55961 480	(05)
7	0	0	8.22607 910	(-01) 1.00000 000 (00)
		1	6.45811 775	(-01)
		2	1.35686 010	(01)
		3	2.73705 461	(02)
		4	-2.72387 751	(04)
		5	6.18151 015	(05)
		6	-6.27757 443	(06)
		7	2.47713 340	(07)
8	0	0	8.22498 5547	(-01) 1.00000 0000 (00)
		1	7.05693 4573	(-01)
		2	2.41085 2430	(00)
		3	1.27282 3178	(03)
		4	-7.64398 5240	(04)
		5	2.02280 8196	(06)
		6	-2.94393 1153	(07)
		7	2.29368 1289	(08)
		8	-7.49246 4681	(08)

TABLE XII. Coefficients for $(F_{3/2})_{\ell,m}^{3*} = x^2 \sqrt{x} \left[\frac{2}{5} + x^{-2} \sum_{s=0}^{\ell} p_s x^{-2s} / \sum_{s=0}^m q_s x^{-2s} \right]$, $4 \leq x < \infty$

ℓ	m	s	p_s	q_s

0	0	0	2.4247	(00) 1.0000 (00)

1	0	0	2.46912 7	(00) 1.00000 0 (00)
		1	-8.86420 6	(-01)

1	1	0	2.46929 3	(00) 1.00000 0 (00)
		1	-6.56036 0	(-01) 9.64792 7 (-02)

2	0	0	2.46929 3	(00) 1.00000 0 (00)
		1	-8.94237 6	(-01)
		2	8.53079 7	(-02)

2	2	0	2.46721 347	(00) 1.00000 000 (00)
		1	1.99900 983	(01) 8.36525 114 (00)
		2	1.56338 125	(02) 6.90842 636 (01)

3	0	0	2.46768 69	(00) 1.00000 00 (00)
		1	-7.24123 33	(-01)
		2	-4.89536 43	(00)
		3	4.32686 53	(01)

3	3	0	2.46741 6637	(00) 1.00000 0000 (00)
		1	9.77546 3043	(01) 3.99113 6128 (01)
		2	2.29398 3407	(03) 9.41059 1778 (02)
		3	8.00686 7097	(03) 3.70648 4478 (03)

4	0	0	2.46723 529	(00) 1.00000 000 (00)
		1	-6.54201 170	(-01)
		2	-8.29176 315	(00)
		3	1.08396 848	(02)
		4	-4.30414 210	(02)

4	4	0	2.46740 02368 4	(00) 1.00000 00000 0 (00)
		1	2.19167 58236 8	(02) 8.91125 14061 9 (01)
		2	1.23829 37907 5	(04) 5.04575 66966 7 (03)
		3	2.20667 72496 8	(05) 9.09075 94630 4 (04)
		4	8.49442 92003 4	(05) 3.89960 91564 1 (05)

TABLE XII. (Contd.)

ℓ	m	s	p_s	q_s
5	0	0	2.46727 333	(00) 1.00000 000 (00)
		1	-6.65016 991	(-01)
		2	-7.34016 183	(00)
		3	7.31554 453	(01)
		4	1.46300 618	(02)
		5	-3.44035 357	(03)
6	0	0	2.46735 012	(00) 1.00000 000 (00)
		1	-6.91874 085	(-01)
		2	-4.28705 298	(00)
		3	-8.34706 678	(01)
		4	4.15218 036	(03)
		5	-5.33094 923	(04)
		6	2.40781 348	(05)
7	0	0	2.46739 4366	(00) 1.00000 0000 (00)
		1	-7.11154 6044	(-01)
		2	-1.46718 6920	(00)
		3	-2.77370 3219	(02)
		4	1.12308 9911	(04)
		5	-1.94470 7749	(05)
		6	1.69243 9397	(06)
		7	-6.02181 0767	(06)
8	0	0	2.46741 05262	(00) 1.00000 00000 (00)
		1	-7.19062 08391	(-01)
		2	-9.79874 41439	(-02)
		3	-3.93717 79544	(02)
		4	1.67372 54465	(04)
		5	-3.46808 75216	(05)
		6	4.14068 12729	(06)
		7	-2.71878 72694	(07)
		8	7.61006 09106	(07)

APPENDIX

FORTRAN Listings of Subroutines Used to Compute Fermi-Dirac Functions

```

      SUBROUTINE FERDIREX(P,X,Y)
C
C   THE FERMI-DIRAC FUNCTION F(P,X) IS EVALUATED
C   FOR X L.T. OR = -1 BY THE EQUATION
C    $F(P,X) = P\text{-FACTORIAL} * \text{SUM (FROM R=1 TO INF.)}$ 
C    $(-1)^{R-1} * \text{EXP}(R*X)/(R^{K+1})$ .
C
      TYPE DOUBLE P,X,Y,A,TP,TEMP,FACT,T1,TX,EXPON,SUM
      TYPE DOUBLE SIGN
      DIMENSION A(50)
      DATA (TP=-10.000)
      IF(TP-P)100,150,100
100  TP=P
      TEMP=TP+1.00
      CALL DGAMMA(TEMP,FACT)
      T1 = 2.00
      A(1)=1.00
      SIGN=-1.00
      DO 125 I=2,50
      A(I)=SIGN*T1**(-TEMP)
      T1=T1+1.00
125  SIGN=-SIGN
      IN=-1
150  TX=X
      EXPON=DEXP(TX)
      SUM=0.00
      N=-48.000/TX
      N=N+1
      DO 175 I=N,1,IN
175  SUM= (SUM+A(I))*EXPON
      Y=SUM*FACT
      RETURN
      END

```

```

      SUBROUTINE FERDIRQD(P,X,Y,NUM)
C
C   THE FERMI-DIRAC FUNCTION F(P,X) IS COMPUTED
C   BY APPLYING THE QD-ALGORITHM TO THE TAYLOR
C   SERIES EXPANSION AND EVALUATING THE NUM-TH
C   CONVERGENT OF THE RESULTING CONT. FRACTION.
C
      TYPE DOUBLE P,PFAC,FAC,TWOPWR,ARG,T,FNU,C
      X,QDA,X,Y,TP
      DIMENSION C(40),QDA(40,40)
      DATA(TP=-1.3400)
      IF(TP-P)10,15,10
10  TP=P
      FAC=1.00
      TWOPWR= 2.000**(-P)
      ARG= P+1.00
      CALLDGAMMA(ARG,PFAC)
      DO 20 NU=1,40
      CALLDZETA(ARG,T,25.000)
      T=T*PFAC*(1.000-TWOPWR)/FAC
      FNU= NU-1
      C(NU)=T
      FAC=FAC+FAC*FNU
      TWOPWR=TWOPWR*2.000
20  ARG=ARG-1.00
      CALL DQD(C,QDA,40)
      DO 12 I=1,40
12  C(I)=QDA(I,I)
15  CALL DCONF1(X,C,Y,NUM)
      RETURN
      END

```

```

SUBROUTINE FERDIRAS(P,ETA,Y)
C
C   THE FERMI-DIRAC FUNCTION F(P,X) IS EVALUATED
C   IN THE ASYMPTOTIC REGION BY THE METHOD OF
C   DINGLE, APPL. SCI. RES. B-6, P230.
C   P MUST BE OF THE FORM 2*N+.5
C
  TYPE DOUBLE P,TP,Y,ETA,A,X,SUM1,TEMP1,PI,
  XSUM2,SIGN,EXPON,FACT2,TERM,DENOM,XJ,DX,TEM
  X,TEMP,FACTOR,TETA,EPS
  X,SINKPI,S,ESS
  X,DABSF
  DIMENSION A(25)
  DATA (TP=1.34D0)
  IF(TP-P)100,125,100
100 PI=1.570796326794896619231322D
  TP=P
  CALL DFERMID(P,A)
125 DX=ETA
  X=DX*DX
  TEMP1=DX*DX**P/(P+1.0D0)
  TETA = X
  SUM1=1.0D
  SUM2=1.0D
  DO 150 I=1,24
  N=I-1
  TERM = A(I+1)/TETA
  IF (TERM/SUM2-1.0D0) 140,151,151
140 SUM2=TERM
  SUM1=SUM2+SUM1
150 TETA=TETA*X
151 IF(N)140,140,152
152 SUM1=SUM1*TEMP1
  IF(DX-25.0D0)158,154,154
154 Y=SUM1
  RETURN
C
C   COMPUTE FACTOR FOR CURLY-A SERIES
C
158 M=TP-.5D
  M=XMODF(M,2)
  IF(M)165,160,165
160 SINKPI=1.0D
  GO TO 167
165 SINKPI=-1.0D
167 EPS=SUM1*1.0D-12
  NN=N+N
  TEMP=NN
  S=TEMP-P
  FACTOR=S+1.0D
  CALL DGAMMA(FACTOR,TEM)
  TERM=P+2.0D
  CALL DGAMMA(TERM,FACTOR)
  FACTOR=FACTOR*TEM*SINKPI/PI/DX**(NN+2)
C
C   COMPUTE CURLY-A SERIES
C
180 TETA=DX
  SUM2=0.0D
  SIGN=1.0D
  IGNU=1
  DO 250 K=1,11
  IF(IGNU)190,210,190
190 CONTINUE

```

```

EXPON=DX+(S+1.0D0)*DLOG(DX)
FACT2=PI*DEXP(EXPON)/TEM/SINKPI
ESS=S
TERM=1.0D0
C
C
C   EXIT CURLY-A LOOP BECAUSE EXP TERM TOO LARGE
C
C   IF(DABSF(FACT2)-1.0D20)200,210,210
200 DENOM=ESS*(ESS-1.0D0)
    TERM=TERM*X/DENOM
    ESS=ESS-2.0D
    FACT2=FACT2-TERM
C
C   EXIT CURLY-A LOOP ON EPS TEST
C
C   IF(DABSF(TERM/SUM1 )- EPS )220,200,200
C
C   ASYMPTOTIC FORM OF CURLY-A
C
210 ESS=S
    IGNU=0
    FACT2=TERM=1.0D0
    XJ=FACT2
211 TERM=(ESS+1.0D0)*(ESS+2.0D0)*TERM/X
    IF(ESS-1.5D0)213,2110,2110
2110 IF(TERM/XJ-.300)212,220,220
212 IF(DABSF(TERM)-1.0D-10)220,213,213
213 FACT2=FACT2+TERM
    ESS=ESS+2.0D
    XJ=TERM
    GO TO 211
220 XJ=K** (N+1)
    XJ=XJ*XJ
    XJ= SIGN *FACT2/XJ
    SUM2 = SUM2 + XJ
C
C   EXIT SUM CURLY-A ON EPS TEST
C
C   IF(DABSF(XJ/SUM2)-1.0D-12)300,230,230
230 DX= DX + TETA
240 X=DX*DX
    SIGN = -SIGN
250 CONTINUE
300 SUM2=SUM2*TEMP1*FACTOR
    SUM2= SUM1 + SUM2
    Y=SUM2
    RETURN
    END

```

SUBROUTINE DFERMID(YAK,B)

C
C
C
C
C

THIS SUBROUTINE COMPUTES THE COEFFICIENTS
FOR THE ASYMPTOTIC FORM OF THE FERMI-DIRAC
FUNCTION F(P,X).

```

      TYPE DOUBLE A,B,PISQ,DK,YAK,DX,DY,DPI,TWOPOW
      X,FACTK,FACTOR,PI2,DN
      DIMENSION A(25),B(25)
      DATA (ISET=314159 )
      IF(ISET-314159)150,100,150
100  ISET=0
      A(1)=1.000/6.00
      A(2)=1.000/30.00
      A(3)=1.000/42.00
      A(4)=1.000/30.00
      A(5)=5.000/66.00
      A(6)=691.000/2730.00
      A(7)=7.000/6.000
      A(8)=3617.000/510.00
      A(9)=43867.000/798.000
      A(10)=174611.000 / 330.00
      A(11)= 854513.000 / 138.00
      A(12)= 236364091.000 / 2730.00
      A(13)= 8553103.000 / 6.00
      A(14)= 23749461029.000/ 870.00
      A(15)= 8615841276005.000 / 14322.00
      A(16)= 7709321041217.000 / 510.00
      A(17)= 2577687858367.000 / 6.00
      A(18)= 26315271553053477373.000 / 1919190.00
      A(19)= 2929993913841559.000 / 6.00
      A(20)= 261082718496449122051.000 / 13530.00
      A(21)=1520097643918070802691.000 / 1806.00
      A(22)= 27833269579301024235023.000 / 690.00
      A(23)= 5964511111593912163277961.00 / 282.00
      A(24)= 5.609403368997817686249128027 / 46410.00
      A(25)= 4.950572052410796482124775026 / 66.00
      PI2=3.14159265358979323846264
      PI2=PI2+PI2
      PISQ=PI2*PI2
150  DK=YAK+1.00
      TWOPOW=.50
      FACTK=DK
      DN=2.00
      B(1)=1.00
      DX=1.00
      FACTOR=2.00
      DO 300 I=2,25
      DX= DX*FACTK*PISQ/FACTOR
      DY=1.000-TWOPOW
      FACTK=FACTK-1.00
      DX=DX*FACTK
      B(I)=DX*DY*A(I-1)
      FACTK=FACTK-1.00
      TWOPOW=TWOPOW*.250
      DN=DN+2.00
      FACTOR=DN*DN-DN
300  CONTINUE
      RETURN
      END

```

```

SUBROUTINE DGAMMA(X,Y)
C
C   COMPUTES GAMMA(X) FOR X=M/2, ABS(M) L.E. 15.
C
  TYPE DOUBLE X,Y,GAM,PI,Z,T1
  DATA (PI=3.141592653589793238462643D)
  X,(IGNAL=-1)
  DIMENSION GAM(80)
  IF(IGNAL)10,10,100
10  IGNAL=1
  GAM(1)=DSQRT(PI)
  Z=.5D
  DO 12 I=3,79,2
  GAM(I)=Z*GAM(I-2)
12  Z=Z+1.0D
  GAM(2)=1.0D
  Z=1.0D
  DO 15 I=4,80,2
  GAM(I)=Z*GAM(I-2)
15  Z=Z+1.0D
100 Z=X
  IF(Z)110,110,200
110 Z=-Z
  M=Z+Z+.01D
  T1=DSIN(PI*Z)
  Y=-PI/(Z*T1*GAM(M))
  RETURN
200 M=Z+Z+.01D
  Y=GAM(M)
  RETURN
  END

```

```

SUBROUTINE DQD(S,A,N)
C
C   THIS SUBROUTINE PRODUCES THE QD ARRAY FOR THE
C   SEQUENCE S. A VARIETY OF APPLICATIONS ARE
C   GIVEN BY HENRICI, SOME APPLICATIONS OF THE
C   QUOTIENT-DIFFERENCE ALGORITHM (PROC. SYMP.
C   APPL. MATH. VOL. 15(1963) PP.159-183). IN
C   PARTICULAR, IF S CONTAINS THE COEFFICIENTS
C   OF A FORMAL POWER SERIES  $S(1) + S(2)*X + \dots$ ,
C   THE COEFFICIENTS OF THE CORRESPONDING CON-
C   TINUED FRACTION,  $F = A(1)/(1-A(2)*X/(1-A(3)*X$ 
C    $/(\dots$  WILL BE THE DIAGONAL ELEMENTS OF THE ARRAY A.
C
  TYPE DOUBLE S,A
  DIMENSION S(40), A(40,40)
  DO 1 I = 1,N
  1 A(I, 1) = S(I)
  DO 2 I = 2, N
  2 A(I, 2) = A(I, 1)/A(I - 1, 1)
  DO 3 I = 3, N
  3 A(I, 3) = A(I,2) - A(I - 1, 2)
  DO 10 J = 4, N, 2
  DO 12 I = J, N
  12 A(I,J) = A(I-1, J- 2)*A(I, J- 1)/A(I - 1,J - 1)
  IF(N - J - 1)5, 6, 6
  6 JJ = J + 1
  DO 10 I = JJ, N
  10 A(I, J + 1) = A(I-1, J- 1) + A(I, J) - A(I - 1, J)
  5 CONTINUE
  END

```


SUBROUTINE DCONF1(X,A,RES,N)

C
C THIS SUBROUTINE EVALUATES THE NTH CONVERGENT OF
C THE CONTINUED FRACTION $RES = A(1)/(1-A(2)*X/(1- \dots))$.
C THIS IS THE FORM PRODUCED BY THE QD-ALG, BUT IS NOT
C FROM A POWER SERIES

C
C TYPE DOUBLE X,A,RES,SUM,XX
C TYPE DOUBLE DABSF
C DIMENSION A(25)
C MIN1=-1
C XX=X
C SUM=0.0D
C DO 1 I=N,1,MIN1
C IF(DABSF(1.0D0-XX*SUM)-1.0D-300)2,2,3
2 SUM= 1.0D300
C GO TO 1
3 SUM= A(I)/(1.0D0-XX*SUM)
1 CONTINUE
C RES=SUM
C RETURN
C END

SUBROUTINE DZETA(ES,EA,ZETAF,D)

C
C THIS SUBROUTINE COMPUTES $(S-1)*ZETA(S,A)$ FOR
C S GREATER THAN 0. THE ABSOLUTE ERROR IS
C LESS THAN $10**D$. THE COMPUTATION USES THE
C RECURRENCE $ZETA(S,A) = ZETA(S,A+1)+A**(-S)$
C TO INCREASE A UNTIL FOUR TERMS OF THE ASYMPTOTIC
C SERIES GIVE THE REQUIRED ACCURACY. ALGORITHM
C DUE TO THACHER.

C
C TYPE DOUBLE S,A,ZETAF,D,EM,ALPHA,SUM,EA,ES,T
C TYPE DOUBLE ALPHA2
C S=ES
C A=EA
C IF(S) 1,2,2
1 PRINT 1000,S
1000 FORMAT(* S=* E16.8,*(NEGATIVE)*)
C RETURN
2 M=((S+3.0D0)**9*10.0D0**D*1.3091275D-7)**(1.0D0/
X (S+8.0D0))+1.0D0-A
C EM=M
C ALPHA=EM+A
C ALPHA2=ALPHA * ALPHA
C SUM = ALPHA + (S-1.0D0)*.5D
C T= (S-1.0D0)*S/ALPHA
C SUM= SUM +T /12.0D
C T=(S+1.0D0)*(S+2.0D0)*T/ALPHA2
C SUM=SUM-T/720.0D
C T=(S+3.0D0)*(S+4.0D0)*T/ALPHA2
C SUM= SUM + T/ 30240.0D
C T= (S+5.0D0)*(S+6.0D0)*T/ALPHA2
C SUM= SUM-T/1209600.0D
C SUM= SUM*ALPHA**(-S)
C T=0.0D0
C EM= ALPHA
C DO 5 K=1,M
C EM = EM-1.0D
5 T=T +EM**(-S)
C ZETAF= SUM +(S-1.0D0)*T
C RETURN
C END

```

SUBROUTINE DRZETA(S,RES,D)
C
C   THIS SUBROUTINE USES THE SUBROUTINES DZETA
C   FOR THE HURWITZ ZETA FUNCTION, AND DGAMMA
C   FOR THE GAMMA FUNCTION TO COMPUTE THE
C   RIEMANN ZETA FUNCTION Z(S).  FOR S=1, THE
C   VALUE 1.0D300 IS RETURNED.
C
  TYPE DOUBLE S,RES,D,DL,SL,T,F
  DL=D
  IF(S-1.0D0)2,1,3
1 RES=1.0D300
  RETURN
3 SL=S
  CALL DZETA(SL,1.0D0,T,DL)
  RES=T/(SL-1.0D0)
  RETURN
2 SL=1.0D0-S
  CALL DGAMMA(SL,T)
  F=.5D0*.3183098861837906715377675D
  F=2.0D0*F**SL*DCOS (.25D0*SL/F)*T
  CALL DZETA(SL,1.0D0,T,DL)
  RES=-T*F/S
  RETURN
END

```

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