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Controlled Thermonuclear Processes

DIFFUSION OF PLASMA PARTICLES ACROSS A MAGNETIC FIELD

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ABSTRACT

A previous calculation of the rate of diffusion of like charged particles across a magnetic field is generalized. No "a priori" assumption as to the relative magnitude of certain terms need be made and spatial density gradients are permitted in both directions perpendicular to the field. The final result agrees with that given earlier.

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I. INTRODUCTION

We are concerned with a gas of charged particles in a magnetic field. It is well known[1] that if a gas is composed of electrons and ions the diffusion of these particles across a magnetic field is approximately represented by Fick's law and the effective diffusion coefficient D is inversely proportional to the square of the magnetic field H:

$$D = \frac{cnkT}{\sigma H^2}$$

where c is the light velocity, σ the conductivity of the plasma gas and kT is the thermal energy.

A quite different type of diffusion exists if there is only one species of charged particles. One of us[2] has previously shown that to the first order in the density gradient there is no diffusion across a magnetic field. The lowest order term which contributes to the diffusion is proportional to the third derivative of the density. The effective diffusion constant in this case is found to be proportional to H^{-4}.

This diffusion rate was calculated in I by taking into account the off-diagonal terms of the stress tensor which appear in the dynamical equation of motion. In the derivation, it was assumed that all space derivatives were in a single direction (say the x direction). In addition, two inequalities were assumed which were justified "a posteriori."

More recently, a different approach to the same problem was followed by Longmire and Rosenbluth.[3] They also assumed the one-dimensional variation of density gradients and arrived at essentially the same result as that in I.

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1. L. Spitzer, Physics of Fully Ionized Gases, Interscience Publishers, New York (1956), p. 38.
2. A. Simon, Phys. Rev. 100, 1557 (1955). This paper will be referred to as paper I hereafter.
3. C. L. Longmire and M. N. Rosenbluth, Phys. Rev. 103, 507 (1956).

It is the purpose of this paper to generalize the treatment in I by allowing density gradients to exist both in the x and y directions. The magnetic field is in the z-direction. This two-dimensional variation of density may be more realistic than the one-dimensional case considered before. As a result, it will be seen that no "a priori" assumption concerning the magnitudes of the components of the stress tensor is necessary and the calculation is straightforward.

II. DIFFUSION EQUATION

We assume that there is no electric field and that the magnetic field is homogeneous and in z-direction. The dynamical equation for a simple gas (in the terminology of Chapman and Cowling[4]) takes the following form in a steady state:

$$nav_y = \frac{\partial}{\partial x} T_{xx} + \frac{\partial}{\partial y} T_{yx}, \tag{1}$$

$$- nav_x = \frac{\partial}{\partial x} T_{xy} + \frac{\partial}{\partial y} T_{yy}, \tag{2}$$

$$0 = \frac{\partial}{\partial x} T_{xz} + \frac{\partial}{\partial y} T_{yz} \tag{3}$$

where

$$a = \frac{eH}{c}, \tag{4}$$

and n is the particle density. It is assumed in the above equations that there are no gradients in the z-direction.

The components of the stress tensor (T_{ij}) are given in reference 3 and take the following forms.*

4. Chapman and Cowling, Mathematical Theory of Non-Uniform Gases, Cambridge University Press, Cambridge, 1952.

*The sign of the last term under the brackets in Eqs. (11) and (13) of I should be negative. This change does not enter into any of the remaining equations.

$$T_{xx} = p - \frac{2\mu}{1 + \frac{16}{9}\omega^2\tau^2} \left\{ e_{xx} + \frac{1}{2}(e_{xx} + e_{yy}) \frac{16}{9}\omega^2\tau^2 - e_{xy}\frac{4}{3}\omega\tau \right\}, \quad (5)$$

$$T_{yy} = p - \frac{2\mu}{1 + \frac{16}{9}\omega^2\tau^2} \left\{ e_{yy} + \frac{1}{2}(e_{xx} + e_{yy}) \frac{16}{9}\omega^2\tau^2 + e_{xy}\frac{4}{3}\omega\tau \right\}, \quad (6)$$

$$T_{xy} = - \frac{2\mu}{1 + \frac{16}{9}\omega^2\tau^2} \left\{ e_{xy} + \frac{1}{2}(e_{yy} - e_{xx}) \frac{4}{3}\omega\tau \right\},$$

$$T_{zx} = - \frac{2\mu}{1 + \frac{4}{9}\omega^2\tau^2} \left\{ e_{zx} - \frac{2}{3}\omega\tau\, e_{zy} \right\}, \quad (7)$$

$$T_{zy} = - \frac{2\mu}{1 + \frac{4}{9}\omega^2\tau^2} \left\{ e_{zy} + \frac{2}{3}\omega\tau\, e_{zx} \right\}. \quad (8)$$

Here τ denotes the average time between collisions and ω is the cyclotron frequency. The quantity μ is the coefficient of viscosity in the absence of a magnetic field and is related to the pressure p as the following:

$$\mu \cong \frac{2}{3} p\tau. \quad (9)$$

The matrix (e_{ij}) has the following components:

$$e_{xx} = \frac{1}{3}\left(2\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right), \quad (10)$$

$$e_{yy} = \frac{1}{3}\left(2\frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial x}\right), \quad (11)$$

$$e_{xy} = \frac{1}{2} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) . \tag{12}$$

Therefore, we have

$$e_{xx} - e_{yy} = \frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \tag{13}$$

We note here that Eq. (3) is identically satisfied because of the forms of T_{zx} and T_{zy} of Eqs. (7) and (8) and because of our assumption that there are no gradients in the z-direction.

Let us now use the equation of continuity

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0, \tag{14}$$

Substituting for $\vec{v}$ from Eqs. (1) and (2), we obtain:

$$\frac{\partial n}{\partial t} + \frac{1}{a} \left[\frac{\partial^2}{\partial x \partial y} (T_{xx} - T_{yy}) + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right) T_{xy} \right] = 0. \tag{15}$$

From Eqs. (5), (6), and (7) we see that

$$T_{xx} - T_{yy} = - \frac{2\mu}{1 + \frac{16}{9} (\omega\tau)^2} \left\{ (e_{xx} - e_{yy}) - \frac{8}{3} \omega\tau \, e_{xy} \right\} \tag{16}$$

$$T_{xy} = - \frac{2\mu}{1 + \frac{16}{9} (\omega\tau)^2} \left\{ e_{xy} + \frac{2}{3} \omega\tau \, (e_{xx} - e_{yy}) \right\} \tag{17}$$

We are interested in the case where the magnetic field is strong and particles make many Larmor gyrations between collisions. This is expressed by the condition $\omega\tau \gg 1$. We are not concerned with the opposite case because in the limit of weak magnetic field usual diffusion must be the case.

We now substitute Eqs. (12), (13), (16), and (17) in Eq. (15) and carry out differentiations. The calculation is straightforward but the following points must be remembered. First, the mean free time τ $(= \lambda/v)$ is inversely proportional to the density n and thus μ is independent of n. Second, the terms which are independent of τ correspond to Hall drift currents rather than diffusion and should be ignored.

We obtain:

$$\frac{\partial n}{\partial t} = -\frac{9}{8}\frac{\mu}{(\omega\tau)^2 a}\left\{\frac{2}{n}\frac{dn}{dx}\Delta v_y - \frac{2}{n}\frac{dn}{dy}\Delta v_x + \frac{1}{2}\left[\frac{\partial}{\partial x}(\Delta v_y) - \frac{\partial}{\partial y}(\Delta v_x)\right] + \left[\frac{1}{n}\left(\frac{d^2n}{dx^2} - \frac{d^2n}{dy^2}\right) + \frac{1}{n^2}\left\{\left(\frac{dn}{dx}\right)^2 - \left(\frac{dn}{dy}\right)^2\right\}\right]\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \left[\frac{2}{n}\frac{d^2n}{dxdy} + \frac{2}{n^2}\frac{dn}{dx}\frac{dn}{dy}\right]\left(\frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial x}\right)\right\} \tag{18}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The terms which have been ignored in Eq. (18) are all independent of τ.

Now to lowest order in τ, we have from Eqs. (1), (2), (5), and (6):

$$v_y = \frac{1}{na}\frac{\partial p}{\partial x} = \frac{kT}{a}\frac{1}{n}\frac{\partial n}{\partial x} \tag{19}$$

$$v_x = -\frac{1}{na}\frac{\partial p}{\partial y} = -\frac{kT}{a}\frac{1}{n}\frac{\partial n}{\partial y} \tag{20}$$

Substituting these in Eq. (18) we obtain:

$$\frac{\partial n}{\partial t} = - D \Bigg[\Delta^2 \log n + 4 \frac{\partial \log n}{\partial x} \left(\frac{\partial}{\partial x} \Delta \log n \right) + 4 \frac{\partial \log n}{\partial y} \left(\frac{\partial}{\partial y} \Delta \log n \right)$$

$$+ 2 \left\{ \frac{\partial^2 \log n}{\partial x^2} - \frac{\partial^2 \log n}{\partial y^2} + 2 \left(\frac{\partial \log n}{\partial x} \right)^2 - 2 \left(\frac{\partial \log n}{\partial y} \right)^2 \right\} \left(\frac{\partial^2 \log n}{\partial x^2} - \frac{\partial^2 \log n}{\partial y^2} \right)$$

$$+ 8 \left(\frac{\partial^2 \log n}{\partial x \partial y} + 2 \frac{\partial \log n}{\partial x} \frac{\partial \log n}{\partial y} \right) \left(\frac{\partial^2 \log n}{\partial x \partial y} \right) \Bigg] \qquad (21)$$

where

$$D = \frac{9 \mu k T}{16 \, \omega^2 \tau^2 a^2} \qquad (22)$$

This is rather a complicated result. However, if we assume the density variation in x-direction only, we are led to

$$\frac{\partial n}{\partial t} = - \frac{9 c^2 k T}{16 \, c^2 H^2} \frac{d^2}{dx^2} \left(\frac{\mu}{(\omega \tau)^2} \frac{d}{dx} \left\{ \frac{d \log n}{dx} \right\} \right) \qquad (23)$$

which is nothing but Eq. (17) of I.

It is noted that both Eqs. (21) and (23) have the same effective diffusion constant which is proportional to H^{-4}.

It should be noted that one can iterate further by use of Eqs. (1), (2), (5), and (6) and obtain higher terms in Eqs. (19) and (20). These in turn can produce other terms of order τ^{-1} upon substitution in Eq. (18). A $1/\tau$ contribution also results in this fashion upon substitution of these higher terms in the terms which have been omitted from Eq. (18). All of these additional τ^{-1} terms, however, are at least two powers higher in the space derivatives and may be neglected.

The result in Eq. (23), since it is identical to that derived in I, still differs from the result in Longmire and Rosenbluth by the numerical factor 4/3. In this regard it should be noted that there is some ambiguity in the choice of the velocity distribution function in their paper. Longmire and Rosenbluth determined the velocity distribution by choosing it to be a pure Maxwellian, undistorted by the presence of spatial density gradients. This seems to us to be entirely analagous to the methods used in kinetic theory derivations of transport coefficients. One always obtains the proper dimensional form of the coefficients but the numerical value is usually slightly in error since the calculation is not completely self-consistent.

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