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Entropy Production and Heat Flow in Liquid He II*

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ABSTRACT

An analysis of the effect of mutual friction between normal fluid and superfluid in the flow of liquid He II through narrow channels under large temperature gradients has been recently given by Craig, Keller and Hammel. This analysis is based on the Gorter-Mellink thermohydrodynamical equations. The present paper presents an alternative treatment of the problem in which the conservation of entropy is introduced naturally into the theory at an early stage. The role played by entropy production in the processes involved in the thermohydrodynamics is thereby considerably clarified and a simple relation is given for an estimate of the relative importance of normal fluid (Rayleigh) dissipation and mutual friction (Gorter-Mellink) dissipation.

I. INTRODUCTION

In a recent paper (hereafter referred to as CKH) Craig, Keller and Hammel (1) have given an analysis of experiments done at Los Alamos and reported in two previous papers (2,3) on the flow of liquid He II through narrow channels under large temperature and pressure gradients. In this analysis it is shown how the Gorter-Mellink (4) nonlinear thermohydrodynamic equations predict the saturation effects which are observed when very large heat current densities are present. The pointwise conservation of heat along the channel, through which the heat flow is observed, is assumed at the beginning of the derivation. Then after a solution to the Gorter-Mellink equations is obtained it is shown that the dissipation along the channel is equal to that required to produce the additional normal fluid flux which must flow out at the cold end of the channel.

In the derivation which follows we introduce the entropy directly into the hydrodynamic equations in the form of a source term added to the entropy continuity equation. From this it follows that heat is conserved pointwise in the channel. Thus, the role of entropy production in the two-fluid hydrodynamics becomes clearer and the derivations are

simpler than those given by CKH and suggest more generality. As a result of this, we are able to demonstrate clearly how the relative importance of the Rayleigh (5) and the Gorter-Mellink dissipations can be evaluated.

The physical problem treated below is exactly the same as the one treated by CKH. We define our channel geometry in a similar way. The z-direction is taken along the channel from low to high temperatures and the x-direction is taken across the small dimension of the channel. For details of the experimental setup the original papers (2,3) should be consulted.

II. INTEGRATION OF THE FLOW EQUATIONS

We begin with the two-fluid equations of motion in the form used by CKH.

$$\rho_s \frac{D\tilde{y}_s}{Dt} = -(\rho_s/\rho)\nabla P + \rho_s s\nabla T - \mathcal{F}_{sn} \quad (1)$$

$$\rho_n \frac{D\tilde{y}_n}{Dt} = -(\rho_n/\rho)\nabla P - \rho_s s\nabla T + \mathcal{F}_{sn} - \eta_n \nabla \times (\nabla \times \tilde{y}_n) + (2\eta_n + \eta') \nabla (\nabla \cdot \tilde{y}_n) \quad (2)$$

where

$$\frac{D\tilde{y}}{Dt} = \frac{\partial \tilde{y}}{\partial t} + (\tilde{y} \cdot \nabla) \tilde{y} \quad (3)$$

The subscripts s and n refer to the superfluid and the normal fluid, η_n is the normal fluid viscosity, and η' is the bulk or second viscosity. The form for F_{sn} will be given below. We ignore possible other forces acting separately on the normal fluid and the superfluid. In addition to the equations of motion we have the mass continuity equation

$$\nabla \cdot (\rho_s \vec{v}_s + \rho_n \vec{v}_n) + \frac{\partial \rho}{\partial t} = 0 \quad (4)$$

and the entropy continuity equation

$$\nabla \cdot (\rho s \vec{v}_n) + \frac{\partial (\rho s)}{\partial t} = \dot{\sigma} \quad (5)$$

where $\dot{\sigma}$ is the rate of increase of entropy due to dissipation per unit volume per unit time at the point under consideration. We neglect the possibility of heat transfer through the normal fluid by ordinary heat conduction.

Since we are dealing with steady state flows the local time derivatives drop out of the above equations. Adding (1) and (2)', we obtain

$$\rho_s (\vec{v}_s \cdot \nabla) \vec{v}_s + \rho_n (\vec{v}_n \cdot \nabla) \vec{v}_n = -\nabla P - \eta_n \nabla \times (\nabla \times \vec{v}_n) + (2\eta_n + \eta') \nabla (\nabla \cdot \vec{v}_n). \quad (6)$$

Using the vector identity

$$\nabla \times (\nabla \times \vec{v}) = \nabla (\nabla \cdot \vec{v}) - \nabla^2 \vec{v} \quad (7)$$

we reduce (6) to

$$\nabla P = (\eta_n + \eta') \nabla(\nabla \cdot \underline{y}_n) + \eta_n \nabla^2 \underline{y}_n - \rho_s (\underline{y}_s \cdot \nabla) \underline{y}_s - \rho_n (\underline{y}_n \cdot \nabla) \underline{y}_n. \quad (8)$$

The heat current density $\underline{q}$ is given by

$$\underline{q} = \rho s T \underline{y}_n = \beta^{-1} \underline{y}_n \quad (9)$$

where

$$\beta \equiv (\rho s T)^{-1} \quad (10)$$

At this point CKH introduce the pointwise conservation of heat in the form of the relation

$$\nabla \cdot \underline{q} = 0 \quad (11)$$

Actually it would seem that all that need be required for the conservation of heat is that the average over x of the heat flux be independent of z . The condition (11) implies not only this but also that the heat flux as a function of x has the same profile for all values of z . In our derivation (11) occurs as a result so that both of these conditions are met automatically.

Instead of introducing (11) as an assumption, we make use of the basic relation (5) for conservation of entropy. Since we do not assume (11), we must consider q as a function of both x and z .

$$\underline{q} = -\hat{k} q(x, z) \quad (12)$$

where $\hat{k}$ is the unit vector in the z -direction. In terms of (12) the requirement (11) would say that $q(x, z)$ becomes independent of z . We look for a solution $q(x, z)$ which is of the factored form

$$q(x, z) = f(x) g(z) \quad (13)$$

We assume that $T = T(z)$. Therefore we also have $\beta = \beta(z)$. The velocity $\underline{v}_n$ can then be written in accordance with (9), (12) and (13) as

$$\underline{v}_n = -\hat{k} f(x) g(z) \beta(z) = -\hat{k} f(x) \gamma(z) \quad (14)$$

where

$$\gamma(z) = g(z) \beta(z) \quad (15)$$

We further assume that the second order terms on the right of (8) can be separated into x and z components to give

$$\frac{\partial P}{\partial z} = -\eta_n \gamma \frac{d^2 f}{dx^2} - (2\eta_n + \eta') f \frac{d^2 \gamma}{dz^2} \quad (16)$$

$$\frac{\partial P}{\partial x} = -(\eta_n + \eta') \frac{df}{dx} \frac{d\gamma}{dz} \quad (17)$$

Thus, we see that formally we have the same equations as CKH except that now we have included the z variation of q in with the function $\gamma(z)$. Next, we drop the second order derivative with respect to z under the assumption that it is small in comparison with the one with respect to x , which is justified when the channel width is small compared to the channel length. Also, the right hand side of (17) involves a product of two small quantities. Therefore, we assume that the variation of P across the width of the channel is small and can be neglected. Accordingly we replace the partial derivative in (16) by the corresponding total derivative. Thus, finally we write

$$\frac{dP}{dz} = -\eta_n \gamma \frac{d^2 f}{dx^2} \quad (18)$$

in place of (16) and (17). The magnitudes of the terms dropped will be commented on at the conclusion of the derivation.

Subject to the condition that q vanishes at the channel boundaries $\pm d/2$, the solution for $f(x)$ is

$$f(x) = \frac{3}{2} \bar{f} \left(1 - \frac{4x^2}{d^2}\right) \quad (19)$$

where the constant $\bar{f}$ is the average value of $f(x)$ across the channel.

Then (18) becomes

$$\frac{dP}{dz} = \frac{12\eta_n g \bar{f}}{\rho_s T d^2} \quad (20)$$

We can now write (1) as

$$\frac{dP}{dz} = \rho_s \frac{dT}{dz} - (\rho/\rho_s) |\mathcal{F}_{sn}| \quad (21)$$

Eliminating dP/dz between (20) and (21) we get

$$\rho_s \frac{dT}{dz} - (\rho/\rho_s) |\mathcal{F}_{sn}| = \frac{12\eta_n g \bar{f}}{\rho_s T d^2} \quad (22)$$

This differs from the London-Zilsel (6) relation in the presence of the mutual friction term and in the variability of $g(z)$. For simplicity we let

$$\Lambda(z) = \frac{(\rho_s) T}{12\eta_n} \quad (23)$$

Then (22) becomes

$$\frac{dT}{dz} = \frac{1}{s\rho_s} |\mathcal{F}_{sn}| + \frac{\bar{f}g}{\Lambda d^2} \quad (24)$$

We take for $\mathcal{F}_{sn}$, the form

$$F_{sn}(\tilde{v}_s - \tilde{v}_n) = A \rho_s \rho_n (|\tilde{v}_s - \tilde{v}_n| - v_c)^{m-1} (\tilde{v}_s - \tilde{v}_n); \quad |\tilde{v}_s - \tilde{v}_n| > v_c$$

$$= 0 \quad ; \quad |\tilde{v}_s - \tilde{v}_n| < v_c \quad (25)$$

In the following derivations we will leave the second possibility in (25) understood. Now, (24) becomes

$$\frac{dT}{dz} = \frac{A \rho_n}{s} (|\tilde{v}_s - \tilde{v}_n| - v_c)^{m-1} |\tilde{v}_s - \tilde{v}_n| + \frac{\bar{f}g}{\Lambda d^2} \quad (26)$$

Next, we use the relation

$$\tilde{v}_s - \tilde{v}_n = - \frac{\rho}{\rho_s} \langle \tilde{v}_n \rangle \quad (27)$$

The argument for this is exactly the same as in CKH. From (14),

$$\langle \tilde{v}_n \rangle = -k \bar{f}g \quad (28)$$

Thus,

$$\tilde{v}_s - \tilde{v}_n = k \frac{\rho \bar{f}g}{\rho_s} \quad (29)$$

Therefore, with $m = 3$ (26) becomes

$$\frac{dT}{dz} = \frac{\bar{f}g}{\Lambda d^2} [1 + \alpha d^2 (\bar{f}g - q_c)^2] \quad (30)$$

where

$$q_c \equiv \rho_s s T v_c \quad (31)$$

and

$$\alpha \equiv \frac{A(\rho_n/\rho)}{12\eta_n(\rho_s/\rho)^3 s^2 T^2} \quad (32)$$

For constant g this would reduce to CKH - (25).

Next, we consider the conservation of entropy. For our steady state condition (5) becomes

$$\nabla \cdot (\rho s \underline{v}_n) = \dot{\sigma} = \dot{\sigma}_n + \dot{\sigma}_{sn} \quad (33)$$

where $\dot{\sigma}$ consists of contributions $\dot{\sigma}_n$ from Rayleigh dissipation and $\dot{\sigma}_{sn}$ from Gorter-Mellink dissipation. First, we calculate $\dot{\sigma}_n$.

$$\dot{\sigma}_n = \frac{\langle \Phi \rangle}{T} \quad (34)$$

where we use the form of Φ given by London (5).

$$\Phi = \frac{\eta_n}{2} \sum_{ik} \left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} \right)^2 + \eta' (\nabla \cdot \underline{v}_n)^2 \quad (35)$$

The evaluation of this quantity yields, for our geometry,

$$\langle \phi \rangle = \frac{\bar{f}^2 g^2}{\Lambda_d^2 T} \quad (36)$$

Thus,

$$\dot{\sigma}_n = \frac{\bar{f}^2 g^2}{\Lambda_d^2 T^2} \quad (37)$$

Next, we evaluate $\dot{\sigma}_{sn}$.

$$\dot{\sigma}_{sn} = \frac{\Phi_{GM}}{T} \quad (38)$$

where

$$\Phi_{GM} = A \rho_s \rho_n (|\underline{v}_s - \underline{v}_n| - v_c)^2 (\underline{v}_s - \underline{v}_n)^2 \quad (39)$$

Using (29), (31), and (32) we write (39) in the form

$$\Phi_{GM} = \frac{\bar{f}^2 g^2}{\Lambda_d^2 T} \alpha d^2 (\bar{f}g - q_c)^2 \quad (40)$$

Thus,

$$\dot{\sigma}_{sn} = \frac{\bar{f}^2 g^2}{\Lambda_d^2 T^2} \alpha d^2 (\bar{f}g - q_c)^2 \quad (41)$$

Therefore, $\dot{\sigma}$ is given by

$$\sigma = \frac{\bar{f}^2 g^2}{\Lambda_d^2 T^2} [1 + \alpha d^2 (\bar{f}g - q_c)^2] \quad (42)$$

The left hand side of (33) may be written

$$\nabla \cdot (\rho_{sv} \tilde{n}) = -\bar{f} \frac{d}{dT} \left(\frac{g}{T} \right) \frac{dT}{dz} \quad (43)$$

Using (30) in (43)

$$\nabla \cdot (\rho_{sv} \tilde{n}) = -\frac{\bar{f}^2 g}{\Lambda_d^2} [1 + \alpha d^2 (\bar{f}g - q_c)^2] \frac{d}{dT} \left(\frac{g}{T} \right) \quad (44)$$

Substituting (42) and (44) into (33) we obtain

$$\frac{d}{dT} \left(\frac{g}{T} \right) = -\frac{g}{T^2} \quad (45)$$

or

$$\frac{dg}{dT} = 0 \quad (46)$$

Since $T = T(z)$, this also implies that

$$\frac{dg}{dz} = 0 \quad (47)$$

Thus, we have shown that g is a constant when all relevant dissipation processes are taken into account. Therefore we can now go back and

replace $f(x)g(z)$ by $q(x)$ and $\bar{f}g(z)$ by $\bar{q}$ in all of our formulas. This gives us exactly the result of CKH for $T(z)$ and consequently also for the heat flow and fountain pressure. Therefore, this analysis predicts the same experimental results as CKH but places it on a more systematic foundation. Finally, we comment that the discussion of terms neglected in (16) and (17) becomes identical to that of CKH.

III. ENTROPY PRODUCTION AND HEAT FLOW

We begin this section with a review of the CKH treatment of entropy production along the channel. Equation (11) is postulated and the whole derivation is carried out without any specific reference to entropy production. The rate of energy dissipation along the channel is calculated on the basis of the solution to the flow equations. It is found that this is identically equal to the energy dissipation required to produce the increase in the normal fluid flux between the hot and the cold ends of the channel. We point out that this agreement is exact only when the Gorter-Mellink term is put back into CKH - (44) and also the effect of CKH - (46) is added in. This is implied by the next to last statement exception in CKH - Section II. However, we take^{to} the last statement. We assert that there is dissipation due to the Gorter-Mellink term, but that it is CKH - just the form of the result[^](44) which is not changed. The Gorter-Mellink term occurs in the denominator of the integrand because of the effect of the Gorter-Mellink dissipation on dT/dz . Then it comes into the numerator of the integrand because of the addition of the effect of CKH - (46). Thus, the Gorter-Mellink term divides out of the flux

integral leaving the same form as before, namely

$$\Delta \bar{N} = \frac{\bar{q}}{S_\lambda} \left(\frac{1}{T_0} - \frac{1}{T_1} \right) \quad (48)$$

where S_λ is the lambda point entropy. But as we have just commented, the invariance of this form depends specifically on putting the Gorter-Mellink dissipation term in consistently everywhere. Putting it another way, the Gorter-Mellink dissipation term changes dT/dz such that the flux integral considered by CKH keeps exactly the same form as it would have had if this Gorter-Mellink term were completely absent. It is of interest to note that these conclusions are all contained automatically in our derivation in Section II above and that we did not need to make any explicit mention of the production of normal fluid flux along the channel.

Now, we turn to another result which we have made clear by our method of derivation, namely, the relative rates of dissipation between the Rayleigh and the Gorter-Mellink processes.

The total rate $\dot{S}$ of entropy production throughout the channel is given by

$$\dot{S} = wd \int_0^z \dot{\sigma} dz \quad (49)$$

where $\dot{\sigma}$ is defined in Section II, w is the width of the slit in the y-direction and d is that in the x-direction.

$$\begin{aligned}\dot{S} &= \dot{S}_n + \dot{S}_{sn} \\ &= wd \int_0^z \dot{\sigma}_n dz + wd \int_0^z \dot{\sigma}_{sn} dz\end{aligned}\quad (50)$$

Therefore, we have

$$\dot{S}_n = wd\bar{q} \int_{T_0}^{T_1} \frac{dT}{T^2 [1 + \alpha d^2 (\bar{q} - q_c)^2]} \quad (51)$$

$$\dot{S}_{sn} = wd\bar{q} \int_{T_0}^{T_1} \frac{\alpha d^2 (\bar{q} - q_c)^2 dT}{T^2 [1 + \alpha d^2 (\bar{q} - q_c)^2]} \quad (52)$$

$$\begin{aligned}\dot{S} &= wd\bar{q} \int_{T_0}^{T_1} \frac{dT}{T^2} \\ &= wd\bar{q} \left(\frac{1}{T_0} - \frac{1}{T_1} \right)\end{aligned}\quad (53)$$

The relative rate of dissipation between the two types of friction is given by the ratio

$$R = \frac{\dot{S}_{sn}}{\dot{S}_n} \quad (54)$$

Meservey (8) defines a ratio which he calls the Gorter number to give the relative importance of these effects. However, instead of integrating entropy, he integrates energy over the volume. Although for almost isothermal conditions the energy ratio and the entropy ratio

are effectively identical, it is important to emphasize that it is the entropy which should be integrated since all the dissipation goes into producing entropy and none of it goes into the production of heat.

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