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THE INTERACTION OF BARE SYSTEMS OF CONTAINERS. PART I

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- 2 -

THE INTERACTION OF BARE SYSTEMS OF CONTAINERS: PART ONE

INTRODUCTION

The problem of interaction between containers filled with fissile material is a constantly recurring one in the K-25 Plant. The rules employed at the present time for regulating this interaction are mainly rules of thumb, with little experimental or theoretical justification, and appear in many cases to be more conservative than necessary. In this report a theoretical solution for the interaction problem is given, based on a macroscopic, multigroup diffusion model in which a dual coupling matrix is the fundamental interaction parameter, and which is essentially a modified first order perturbation.

The first section of this report is devoted to the general theory. Here the fundamental equations are set up for a system of containers of arbitrary shape and content. A discussion of the errors inherent in the physical assumptions made precedes the derivation. In the second part the general formulae are particularized, and a number of special cases which are pertinent to the problems arising at the K-25 Plant are considered.

In table I experimental separations for criticality, obtained in part IV of the Critical Mass Studies, are compared with those predicted on the basis of the present theory. The close agreement obtained would seem to indicate that the theory, if properly applied, might be useful in determining plant specifications.

A. GENERAL THEORY

I. ASSUMPTIONS MADE IN THE DEDUCTION OF THE CRITICAL FORMULA

- 1) In the usual diffusion approximation to the Boltzmann equation, it is

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2572-2

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- 3 -

assumed that the neutron flux (for any given energy) in a critical reactor is a wave function, i.e., a solution of the equation

$$\Delta^2 \phi + B^2 \phi = 0 \quad (1)$$

with B^2 independent of energy. Since this approximation is established without reference to the geometrical construction of the reactor, it is reasonable to assume that also in the case of a reactor made up of separate containers, the flux is a wave function in each container. Thus, it is specifically assumed in this report that the neutron fluxes in a container are solutions of an equation of type (1), and furthermore, that they are solutions of such an equation in which the constant B^2 has the same value as it would have if no other containers existed. This assumption and the approximation to which it leads, are, like the diffusion approximation itself, extremely difficult to gauge quantitatively, and only good checks with experiment can at present substantiate it.

2) In any energy group (assuming a multigroup treatment of order n) of a fixed container it is assumed that the neutrons leaking into that group from another container, leak in at the highest energy which that group contains. This appears, in general, to be a non-conservative assumption, since it probably underestimates the rate of slowing down in the system as a whole. This may be (over)-corrected for by setting up the equations so that the neutrons leaking into any energy group of a fixed container do so at the lowest energy in that group. But this considerably complicates the formulae, especially when n , the number of energy groups is large. However, in the important special case when all containers are filled with the same material, it turns out that the criticality criterion developed is unaffected by this assumption.

3) Finally, it is assumed that the interaction between pairs of containers is a function of a suitably defined coupling coefficient depending only upon the

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~~CONFIDENTIAL~~ 3

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geometrical data. A thorough discussion on the choice for such a coupling coefficient will be given later. For the purpose of setting up the general criticality equation, its definition will suffice here. If $F_A(E)$ is the number of neutrons of energy E leaking from the container A per second and $F_{A \rightarrow B}(E)$ is the number of these neutrons which get into the container B , then the coupling coefficient from A to B , written $c_{A \rightarrow B}$, is defined as their ratio:

$$c_{A \rightarrow B} = \frac{F_{A \rightarrow B}(E)}{F_A(E)}.$$

The assumption made in this paragraph is that $c_{A \rightarrow B}$ is independent of E .

II. DEDUCTION OF THE GENERAL CRITICAL FORMULA

Since a sufficient condition for non-criticality is the purpose of this report, the usual method will be followed of first setting up a necessary condition for criticality. To this end, let S be a system consisting of m separate containers, $F_1, F_2, \dots, F_p, \dots, F_m$, which, as a whole, is just critical. For each container, F_p , the following are defined:

1) The flux $\phi_{j,p}(\underline{r})$ at the point $\underline{r}$. The index j denotes the flux of the j th energy group in an n -group diffusion picture; $\phi_{1,p}$ is the thermal flux, $\phi_{n,p}$ the fastest flux in F_p .

2) $\Sigma_{j,p}, D_{j,p}, L_{j,p}, P_{j,p}$, the (fictitious) absorption cross section, diffusion coefficient, diffusion length, and resonance capture escape probability respectively, of the j th group of neutrons in F_p . Here, by $P_{1,p}$ is meant $\nu U_p f_p$, the thermal production rate. As usual $L_{j,p}^2 = D_{j,p} / \Sigma_{j,p}$.

3) B_p^2 the geometrical buckling of F_p . It is defined as the minimum value of B^2 such that the equation (1) has a solution vanishing on the extrapolated boundary of F_p . (The effect of varying the extrapolation length in the buckling

2572-4

on the critical separation can be guessed at from physical considerations but is brought out more clearly by the discussion of formula (7) at the end of I,1, part B.)

4) $c_{q \rightarrow p}$ the coupling coefficient between F_q and F_p , measuring the effect of the container F_q on F_p . In general $c_{q \rightarrow p} \neq c_{p \rightarrow q}$. The symbol $c_{p \rightarrow p}$ is the self-coupling of the p th container. For a container of convex shape it is evidently zero. However, for a "container" having a re-entrant surface it will differ from zero, as is the case, for example, when the container has the shape of a "T".

5) $G_{j,p} = \sum_{j,p} \int_{F_p} \phi_{j,p}(\mathbf{r}) dv$ = total number of neutrons which have energies in the j th group and which are absorbed per second in F_p . For the sake of uniformity of notation, the index j will be allowed to take on the value $n+1$, and in this case, it will mean the same as 1, i.e. $\phi_{n+1,p}$ means $\phi_{1,p}$, etc.

In order for the system S to be critical it is necessary that the total number of neutrons of any energy remain constant. Hence, considering all possibilities, the following balance equation for total neutrons of the j th energy group in F_p may be written:

$$\begin{aligned} & (\text{Number which leak out per second}) + (\text{number which are absorbed per second}) = \\ & (\text{number which slow down from the } (j+1)\text{st energy group in } F_p) + \\ & (\text{number leaking in from the other containers}). \end{aligned}$$

An analytical form for each of the expressions in parenthesis may be obtained by using the assumptions in section I.

a) Number which leak out per second is

$$- D_{j,p} \int_{\text{surface } F_p} \nabla \phi_{j,p} \cdot d\mathbf{A}.$$

Using Green's theorem and assumption 1), this is

$$D_{j,p} \int_{F_p} \nabla^2 \phi_{j,p} dV = B_p^2 D_{j,p} \int_{F_p} \phi_{j,p} dV = B_p^2 L_{j,p}^2 G_{j,p}.$$

b) Number which are absorbed per second is

$$\int_{F_p} \sum_{j,p} \phi_{j,p} dV = G_{j,p}.$$

c) Number which slow down from the $(j+1)$ th group in F_p per second is

$$\int_{F_p} P_{j+1,p} \sum_{j+1,p} \phi_{j+1,p} dV = P_{j+1,p} G_{j+1,p}.$$

d) Number leaking into F_p from the other containers is

$$-\sum_{q=1}^m c_{q \rightarrow p} D_{j,q} \int_{\text{surface } F_q} \nabla \phi_{j,q} \cdot dA = \sum_{q=1}^m c_{q \rightarrow p} L_{j,q}^2 B_q^2 G_{j,q}.$$

Using a), b), c), and d) the balance equations are

$$B_p^2 L_{j,p}^2 G_{j,p} + G_{j,p} = P_{j+1,p} G_{j+1,p} + \sum_{q=1}^m c_{q \rightarrow p} L_{j,q}^2 B_q^2 G_{j,q}. \quad (2)$$

$$1 \leq j \leq n: \quad 1 \leq p \leq m.$$

This system of equations may be written in a more useful form by employing matrix notation. Matrices and vectors will, in what follows, be symbolized by underscored capital letters such as C, I, etc. The component form of the matrix or vector is given in the usual manner; e.g., A = $(a_{p,q})$ is the matrix whose p th row and q th column element is $a_{p,q}$. The determinant of the matrix A is denoted by $\det \underline{A}$.

Let C be the m -by- m matrix with $c_{q \rightarrow p}$ as the element in the p th row and q th column; and let

$$\underline{L}_j = (L_{j,p} \delta_{p,q}), \quad \underline{B} = (B_p \delta_{p,q}), \quad \underline{P}_j = (P_{j,p} \delta_{p,q})$$

2572-6

$$\underline{I} = (\delta_{p,q}), \quad \underline{I} - \underline{C} = \underline{Z}, \quad \underline{G}_j = (G_{j,p}),$$

where $\delta_{p,q}$ is the Kronecker delta.

The matrices $\underline{Z}$ $\underline{B}^2$ and $\underline{C}$ will be called the leakage and coupling matrices, respectively, for the system of containers under consideration. Within the scope of the present method, the former completely determines the interaction of the system.

In matrix notation, equations (2) become

$$(\underline{I} + \underline{Z} \underline{B}^2 \underline{L}_j^2) \underline{G}_j = \underline{P}_{j+1} \underline{G}_{j+1}, \quad 1 \leq j \leq n.$$

Eliminating $G_n, G_{n-1}, \dots, G_2$ in turn, leads to

$$(\underline{I} + \underline{Z} \underline{B}^2 \underline{L}_n^2) \underline{P}_n^{-1} (\underline{I} + \underline{Z} \underline{B}^2 \underline{L}_{n-1}^2) \underline{P}_{n-1}^{-1} \dots (\underline{I} + \underline{Z} \underline{B}^2 \underline{L}_1^2) \underline{P}_1^{-1} \underline{G}_1 = \underline{G}_1 \quad (3)$$

and this in turn leads to

$$\det \left[\underline{I} - (\underline{I} + \underline{Z} \underline{B}^2 \underline{L}_n^2) \underline{P}_n^{-1} (\underline{I} + \underline{Z} \underline{B}^2 \underline{L}_{n-1}^2) \underline{P}_{n-1}^{-1} \dots (\underline{I} + \underline{Z} \underline{B}^2 \underline{L}_1^2) \underline{P}_1^{-1} \right] = 0. \quad (4)$$

Equation (4) is the most general of the various criticality criteria to be derived and hence is basic in the sequel. It will be called the General Criticality Formula. (If there is no interaction, that is, if all $c_{q \rightarrow p} = 0$, then $\underline{Z} = \underline{I}$ and equation (4) may easily be seen to reduce to

$$\sum_{p=1}^n E_p = 0,$$

where

$$E_p = 1 - \sum_{j=1}^n P_{j,p} (1 + L_j^2 B_p^2)^{-1}.$$

Since $\sum_{j=1}^n P_{j,p} (1 + L_j^2 B_p^2)^{-1}$ is just the multiplication constant for the container F_p , it follows that, in this case, at least one of the containers must be critical

2572-7

individually, as would be expected.)

III. EVALUATION OF THE COUPLING COEFFICIENT

a) Approximation to the Coupling Coefficient for a Fixed Energy

In the case when the true flux in the container A as well as its surface angular distribution is known, it is possible to write down the exact value for the coupling coefficient, $c_{A \rightarrow B}$, governing the leakage of neutrons from the container A to the container B. For, if $W(\underline{r}_A, \underline{\Omega})$ is the leakage flux density in the direction $\underline{\Omega}$ at the point $\underline{r}_A$ of the boundary of A, then it follows from the definition of the coupling coefficient that

$$c_{A \rightarrow B} = \frac{\int_{A_1} dS_A \int_{B_1} W(\underline{r}_A, \underline{\Omega}) d\underline{\Omega}}{\int_{A_2} dS_A \int_{\text{all } \underline{\Omega}} W(\underline{r}_A, \underline{\Omega}) d\underline{\Omega}} \quad (5)$$

Here A_2 is the surface of container A, A_1 is that part of A_2 from which neutrons may get to B in a straight line flight, and $B_1 = B_1(\underline{r}_A)$ is that portion of the surface of B which may be reached by neutrons in a straight line flight from the point $\underline{r}_A$.

However, in most cases $W(\underline{r}, \underline{\Omega})$ will not be known due to limited experimental information and the prohibitively huge amount of labor necessary to solve the exact equation which it satisfies. Even if it were known, the above expression for $c_{A \rightarrow B}$ would be very complicated, and hence from the viewpoint of practical computation nearly useless. Furthermore, its use would, in general, lead to a value for $c_{A \rightarrow B}$ dependent upon core materials, which in turn, would make the interaction equations extremely complex. It is thus necessary to consider certain approximate expressions for $c_{A \rightarrow B}$ which are more suitable for numerical evaluation but are still close enough to the true value to give reasonably good appraisals of the interaction.

These approximations to $c_{A \rightarrow B}$ arise from physical and mathematical simplifications of three kinds, namely:

- (a) Simplification of the form of the surface leakage flux from A.
- (b) Simplification of the form of the angular distribution with which neutrons leave a given point of the boundary of A.
- (c) Use of an average or majorant for one of the quantities appearing in formula (5).

Several examples illustrating these simplifications will now be given. (All considerations in this section are for a fixed energy. In the following section the effect of energy variation will be discussed.)

Example 1. The most obvious simplification is the assumption that the angular distribution is isotropic, and that the surface leakage flux is constant over the surface of A. In this case, the approximate expression for $c_{A \rightarrow B}$, is

$$c_{A \rightarrow B} \text{ uniform} = \frac{\int_{A_1} dS_A \int_{B_1} d\Omega}{2 \pi \int_{A_2} dS_A}.$$

This may be recognized as a multiple of the average fractional solid angle subtended by B at A, which has played a major part in practical interaction considerations at K-25. From the nature of the assumptions leading to it, however, it would seem that no great reliance, as far as safety considerations go, may be placed in an interaction theory using it without the simultaneous use of large safety factors. In fact, in a number of cases where it has been used it has led to non-conservative results as compared with experiment. (See table I, page 28.)

Example 2. For many configurations and especially for those of interest at K-25 a more conservative coupling coefficient is obtained if the angular distribution is not required to be isotropic but is still assumed to have the same form at

all points of A_1 , while the leakage from A_2 is still assumed to be constant.

When A is a cylinder, a natural choice for the angular distribution is the one which is isotropic in any plane parallel to a base and which in any plane thru a cylinder element has the same shape as the angular distribution of neutrons emerging from an infinite half-space bounded by a plane, and filled with a scattering, non-absorbing material. The latter distribution is very closely given by the Fermi distribution, $1 + \sqrt{3} \mu$, μ being the cosine of the angle between the direction considered and the normal to the plane bounding the surface. With this distribution, the expression for $c_{A \rightarrow B}$ is

$$c_{A \rightarrow B, \text{ Fermi}} = \frac{\int_{A_1} dS_A \int_{B_1} \frac{1 + \sqrt{3} \mu_A}{1 + \sqrt{3} \frac{\pi}{4}} d\Omega}{2\pi \int_{A_2} dS_A},$$

where now μ_A is the cosine of the angle between the normal to A at $\underline{r}_A$ and the projection of $\underline{r}_A - \underline{r}_B$ on the base of A.

Example 3. A better approximation of the true leakage flux may be obtained by assuming that the surface leakage flux is that given by diffusion theory, i.e., $W(\underline{r}, \underline{\Omega}) = -D_{n_A} \cdot \nabla \phi(\underline{r}) \cdot h(\underline{\Omega})$, where $\phi(\underline{r})$, the flux in A considered as a reactor, is obtained by diffusion theory, and $h(\underline{\Omega})$ governs the angular distribution of neutrons at $\underline{r}$. If A is a cylinder, the angular distribution used in example 2 is still appropriate, so that now $c_{A \rightarrow B}$ is

$$c_{A \rightarrow B, \text{ diff}} = \frac{\int_{A_1} dS_A \underline{n}_A \cdot \nabla \phi \int_{B_1} \frac{1 + \sqrt{3} \mu_A}{1 + \sqrt{3} \frac{\pi}{4}} d\Omega}{2\pi \int_{A_2} \underline{n}_A \cdot \nabla \phi dS_A}.$$

The expressions given in the preceding examples assume a comparatively simple form when A and B are equal right cylinders having the same base plane. Let R, H,

Re, He, and S be their common radius, height, extrapolated radius, extrapolated height, and edge-to-edge separation respectively. These expressions then become:

$$c_{A \rightarrow B, \text{ uniform}} = \frac{1}{1 + \frac{R}{H}} \cdot \frac{1}{\pi^2} \int_0^{\pi/2} d\phi \int_0^{\psi_2} \psi_1^2 d\psi \int_0^{\alpha} \frac{(1 - \frac{y}{\alpha}) dy}{(1 + y^2)^{3/2}}, \quad (5a)$$

$$c_{A \rightarrow B, \text{ Fermi}} = \frac{1}{1 + \frac{R}{H}} \cdot \frac{1}{\pi^2} \int_0^{\pi/2} d\phi \int_0^{\psi_2} \psi_1^2 d\psi \int_0^{\alpha} \frac{(1 - \frac{y}{\alpha})}{(1 + y^2)^{3/2}} \left(\frac{1 + \frac{\sqrt{3}}{\sqrt{1 + y^2}}}{1 + \sqrt{3} \frac{\pi}{4}} \right) dy \quad (5b)$$

$$c_{A \rightarrow B, \text{ diff}} = \frac{1}{1 + 1.7 \left(\frac{Re}{He} \right)^2} \cdot \frac{1}{\pi^2} \int_0^{\pi/2} d\phi \int_0^{\psi_2} \psi_1^2 d\psi \int_0^{\alpha} \frac{\cos^2 \frac{\pi y}{2\alpha}}{(1 + y^2)^{3/2}} \left(\frac{1 + \frac{\sqrt{3}}{\sqrt{1 + y^2}}}{1 + \sqrt{3} \frac{\pi}{4}} \right) dy. \quad (5c)$$

In these formulas, $\alpha = \frac{H}{P}$, where P is the projection of $\underline{r}_A - \underline{r}_B$ on the common base plane, and ψ is the angle between P and the normal to $\underline{r}_A$. It may be shown from these formulas that all the approximate $c_{A \rightarrow B}$'s are equal when H is infinite, as would be expected from symmetry considerations. Thus the uniform coupling is adequate in this case. Further consequences of these formulas will now be derived.

By comparing formulas (5b) and (5c), the inequalities

$$0.93 \frac{1 + 1.71 \left(\frac{Re}{He} \right)^2}{1 + \frac{R}{H}} < \frac{c_{A \rightarrow B, \text{ Fermi}}}{c_{A \rightarrow B, \text{ diff}}} < \frac{1 + 1.71 \left(\frac{Re}{He} \right)^2}{1 + \frac{R}{H}}$$

may be obtained. The factor 0.93 is the maximum over all α of the ratio of the innermost integrals. This inequality shows that $c_{A \rightarrow B, \text{ Fermi}}$ greatly overestimates the interaction as compared with $c_{A \rightarrow B, \text{ diff}}$ when R is roughly greater than H/2, and that it slightly underestimates the interaction when R is roughly less than H/2.

2572-11

The former fact is of comparatively little importance since not many squat cylinders will be found in the plant, and in those cases where they have to be considered, the interaction as determined by $c_{A \rightarrow B, \text{Fermi}}$ will be very conservatively estimated. The latter fact, however, is significant from the safety viewpoint. The worst underestimation occurs when the factor

$$\frac{1 + 1.71 \left(\frac{Re}{He} \right)^2}{1 + \frac{R}{H}}$$

has its smallest value. It is shown in appendix 2 that this minimum value is 0.885. Hence

$$0.823 < \frac{c_{A \rightarrow B, \text{Fermi}}}{c_{A \rightarrow B, \text{diff}}} < 1.$$

This shows that if $c_{A \rightarrow B, \text{diff}}$ be regarded as the exact coupling coefficient, then the much simpler expression $c_{A \rightarrow B, \text{Fermi}}$ may be used with assurance for cylinders only if multiplied by the factor $1.22 = 1/0.823$.

In a similar manner it may be shown that

$$0.864 < \frac{c_{A \rightarrow B, \text{uniform}}}{c_{A \rightarrow B, \text{Fermi}}} < 1,$$

so that $c_{A \rightarrow B, \text{uniform}}$ slightly underestimates the interaction as compared with $c_{A \rightarrow B, \text{Fermi}}$. If then, the simplest coupling coefficient $c_{A \rightarrow B, \text{uniform}}$ is to be used, it should be multiplied by the factor $1.16 = 1/0.864$, in order to be sure that under all conditions it is at least as safe as $c_{A \rightarrow B, \text{Fermi}}$. Since the latter has the factor 1.22 accompanying it in an analogous manner with regard to $c_{A \rightarrow B, \text{diff}}$, it follows that use of the factor $1.41 = (1.16)(1.22)$, is sufficient to guarantee that the uniform coupling is at least as safe as the coupling which takes the diffusion flux distribution into account.

2572-12

For greater clarity, these results will now be restated:

For two equal cylinders of any shape or size:

- 1) $1.22 c_{A \rightarrow B, \text{Fermi}}$ when used as a coupling coefficient never gives less conservative estimates for the interaction than $c_{A \rightarrow B, \text{diff.}}$
- 2) $1.16 c_{A \rightarrow B, \text{uniform}}$ when used as a coupling coefficient never gives less conservative estimates for the interaction than $c_{A \rightarrow B, \text{Fermi}}$.
- 3) $1.41 c_{A \rightarrow B, \text{uniform}}$ when used as a coupling coefficient never gives less conservative estimates for the interaction than $c_{A \rightarrow B, \text{diff.}}$

Another set of useful expressions for the coupling coefficient, (and probably the most useful ones if a one foot spacing convention is followed) is derived from the so-called maximum fractional solid angle. These are obtained from formula (5) by replacing the inner integral $\int_{B_1} W(\underline{r}_A, \underline{\Omega}) d\underline{\Omega}$, by

$$\max_{\underline{\Omega}} \left(W(\underline{r}_A, \underline{\Omega}) \right) \left(w_{A \rightarrow B} \right),$$

where

$$w_{A \rightarrow B} = \frac{r_A}{r_A} \left(\int_{B_1} d\underline{\Omega} \right)$$

is the maximum solid angle subtended by B at any point of A. The result is

$$c_{A \rightarrow B, \text{max.}} = w_{A \rightarrow B} \cdot \frac{\int_{A_1} ds_A \max_{\underline{\Omega}} W(\underline{r}_A, \underline{\Omega})}{\int_{A_2} ds_A \int d\underline{\Omega} W(\underline{r}_A, \underline{\Omega})}$$

If $W(\underline{r}, \underline{\Omega})$ is assumed uniform in $\underline{r}$ and $\underline{\Omega}$, this becomes

$$c_{A \rightarrow B, \text{max., uniform}} = w_{A \rightarrow B} \cdot \frac{\int_{A_1} ds_A}{2\pi \int_{A_2} ds_A};$$

if it is assumed uniform in $\underline{r}$ but Fermi in $\underline{\Omega}$ it is

$$c_{A \rightarrow B, \text{ max. }, \text{ Fermi}} = 1.157 c_{A \rightarrow B, \text{ max. }, \text{ uniform}},$$

while if A is a cylinder and the diffusion form for $\underline{r}$ and the Fermi form for $\underline{\Omega}$ are assumed it becomes

$$c_{A \rightarrow B, \text{ max. }, \text{ diff}} = 1.157 \frac{1 + \frac{R}{H}}{1 + 1.71 \left(\frac{Re}{He} \right)^2} \cdot c_{A \rightarrow B, \text{ max. }, \text{ uniform}}.$$

From the last equation it follows, for example, that in order to use

$c_{A \rightarrow B, \text{ max. }, \text{ uniform}}$ with assurance it must be multiplied by the factor
 $1.34 = (1.157)(1.16)$.

In deriving the above results it has been assumed that A and B were equal cylinders having their bases in the same plane. However, the comparisons between the various coupling coefficients remain valid when B is arbitrary and A is a cylinder.

Table I gives the results obtained for the critical separation of 2 equal cylinders based upon a number of the coupling coefficients introduced above. Comparison with experiment is also made.

b) Effect of Core and Energy Group on the Coupling Coefficient

Since the exact coupling coefficient, $c_{A \rightarrow B}$, depends upon the form of the flux in container A, it will have different values if this form changes from one energy group to another. For the same reason, it may have different values even for a fixed energy group when the contents of container A are changed. When A is not too near individual criticality, or far from B, this variation in the value of c will be slight and usually negligible. On the other hand, significant discrepancies may arise when A is small and close to B. This is due mainly to the fact that a greater portion of the total leakage flux flows in the direction of container B than is accounted for in the derivation of the preceding section, with a resultant underestimation of the true coupling.

2572-14

This may be overcome by assuming that the surface leakage flux from A in the direction of container B is the sum of the leakage flux in that direction from the container A considered as an isolated reactor and the flux arriving at A from container B also considered as an isolated reactor. Since B is supposed to lie close to A, this leakage flux may be taken as being equal to the sum of the surface leakage flux from A and the surface leakage flux from B. The expression (5) for $c_{A \rightarrow B}$ now leads to the more conservative approximation

$$\begin{aligned}
 c_{A \rightarrow B} &= \frac{\int_{A_1} dS_A \int_{B_1} d\Omega \left[W_{\text{diff},A}(\underline{r}_A, \underline{\Omega}) + W_{\text{diff},B}(\underline{r}_A, \underline{\Omega}) \right]}{\int_{A_2} dS_A \int W_{\text{diff},A}(\underline{r}_A, \underline{\Omega}) d\Omega + \int_{A_1} dS_A \int W_{\text{diff},B}(\underline{r}_A, \underline{\Omega}) d\Omega} \\
 &\approx \frac{\int_{A_1} dS_A \int_{B_1} d\Omega \left[W_{\text{diff},A}(\underline{r}_A, \underline{\Omega}) + W_{\text{diff},B}(\underline{r}_B, \underline{\Omega}) \right]}{\int_{A_2} dS_A \int W_{\text{diff},A}(\underline{r}_A, \underline{\Omega}) d\Omega} \\
 &\approx c_{A \rightarrow B, \text{ diff}} + c_{B \rightarrow A, \text{ diff}} \frac{\int_{B_2} dS_B W_{\text{diff},B}(\underline{r}_B)}{\int_{A_2} dS_A W_{\text{diff},A}(\underline{r}_A)} \\
 &\approx c_{A \rightarrow B, \text{ diff}} + c_{B \rightarrow A, \text{ diff}} \cdot \frac{(\text{Buckling of B})(\text{total flux in B})}{(\text{Buckling of A})(\text{total flux in A})}
 \end{aligned}$$

In the particular case of a critical system each container of which contains the same core material, the total flux in each container will be nearly proportional to the container volume. Using this fact, it follows that for such a system a conservative coupling is $c_{A \rightarrow B, \text{ sum}}$ given by

$$c_{A \rightarrow B, \text{ sum}} = c_{A \rightarrow B, \text{ diff}} + c_{B \rightarrow A, \text{ diff}} \frac{(\text{Buckling of B})(\text{Volume of B})}{(\text{Buckling of A})(\text{Volume of A})}$$

c) On the Approximations Involved in Using W_{diff}

In all the derivations involving W_{diff} , it has been tacitly assumed that W_{diff} is represented sufficiently accurately by the flux which would exist in the reactor

if it were individually critical. This imposes restrictions upon the general validity of the results thereby obtained. In particular, it means that the theory as here developed is inapplicable for systems which include containers of widely divergent sizes or containers which are not nearly directly opposite each other. For example, the problem of an 8" sphere interacting with an 8" infinite cylinder would definitely be outside the range of problems for which reasonably valid results can be assumed. Extension of the method to include such systems would not be very difficult, and amounts in effect, only to recalculating a suitable coupling coefficient. In the example just quoted, the flux in the cylinder is considerably different from that in an individually critical infinite cylinder, since it decreases exponentially in both directions along its axis, but if this exponential decrease is taken into account in computing the coupling coefficients, then a generally valid result should be expected.

B. APPLICATIONS

I. CONTAINERS FILLED WITH THE SAME MATERIAL

1. Criterion A

The general critical equation (4) may be considerably simplified when all the containers F_p are filled with material of the same composition. For then $L_{j,p}^2 = L_j^2$ and $P_{j,p} = P_j$ are independent of the container F_p and the corresponding matrices are simply

$$\underline{L}_j = L_j \underline{I}, \quad \underline{P}_j = P_j \underline{I}.$$

Using these equations, and observing that $\sum_{j=1}^n P_j = k_\infty$ is just the multiplication constant for an infinite medium composed of the common container contents, the general equation becomes

$$\det \left[k_\infty \underline{I} - \sum_{j=1}^n (\underline{I} + L_j^2 \underline{Z} \underline{B}^2) \right] = 0.$$

2572-16

This will now be transformed into a more tractable form by assuming that the matrix $\underline{Z} \underline{B}^2$ may be diagonalized, i.e., by assuming that there exists a non-singular matrix $\underline{U}$ and a diagonal matrix $\underline{D} = (d_p \delta_{p,q})$ such that

$$\underline{Z} \underline{B}^2 = \underline{U} \underline{D} \underline{U}^{-1}.$$

(From the physical point of view this is no restriction at all, for if $\underline{Z} \underline{B}^2$ cannot be diagonalized for a given system, it is always possible to change the system very slightly so that in the new system this matrix may be diagonalized. It is well known that the numbers d_p , (which may be complex) are the eigenvalues of the matrix $\underline{Z} \underline{B}^2$, so that if necessary they may be determined by the usual methods.)

Then

$$\det \left[k_{\infty} \underline{I} - \frac{1}{\pi} \sum_{j=n} (\underline{I} + \underline{L}_j^2 \underline{Z} \underline{B}^2) \right] = \det \left[k_{\infty} \underline{I} - \frac{1}{\pi} \sum_{j=n} (\underline{I} + \underline{L}_j^2 \underline{U} \underline{D} \underline{U}^{-1}) \right] =$$

$$(\det \underline{U}) \det \left[k_{\infty} \underline{I} - \frac{1}{\pi} \sum_{j=n} (\underline{I} + \underline{L}_j^2 \underline{D}) \right] (\det \underline{U}^{-1}) = \det \left[k_{\infty} \underline{I} - \frac{1}{\pi} \sum_{j=n} (\underline{I} + \underline{L}_j^2 \underline{D}) \right].$$

Therefore, a necessary condition for criticality is

$$\det \left[k_{\infty} \underline{I} - \frac{1}{\pi} \sum_{j=n} (\underline{I} + \underline{L}_j^2 \underline{D}) \right] = 0. \quad (6)$$

Since each of the matrices appearing in (6) is diagonal, the determinant may be simply evaluated:

$$\det \left[k_{\infty} \underline{I} - \frac{1}{\pi} \sum_{j=n} (\underline{I} + \underline{L}_j^2 \underline{D}) \right] = \prod_{p=1}^m t_p,$$

where

$$t_p = k_{\infty} - \frac{1}{\pi} \sum_{j=1}^n (1 + \underline{L}_j^2 d_p).$$

Consequently for equation (6) to hold, at least one of the t_p must be zero. The

formal statement of this result will be called criterion A and may be stated as follows:

A necessary condition that a group of non-individually critical containers filled with the same fissile material form a just critical system is the equality

$$k_{\infty} = \prod_{j=1}^n (1 + L_j^2 d_p), \quad (7)$$

where k_{∞} is the (infinite) multiplication constant of the fissile material and d_p is an appropriate eigenvalue of the system's leakage matrix $\underline{Z} \underline{B}^2$.

This criterion may be used as stated in particular cases where it is necessary to know the interaction as accurately as is possible. However, for most purposes the interaction is not needed exactly; furthermore, under most conditions it will be extremely difficult to calculate the eigenvalues d_p . It is therefore essential to put this criterion in a more usable form.

Before doing so however, formula (7) may be employed to determine the effect of varying the extrapolation length, E , in the geometrical bucklings B_p^2 upon the critical data. For simplicity it will be assumed that all containers are identical so that $\underline{B}^2 = B^2 \underline{I}$, B^2 being the common buckling. Formula (7) then becomes

$$k_{\infty} = \prod_{j=1}^n \left[1 + L_j^2 B^2 (1 - \lambda_p) \right]$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the coupling matrix $\underline{C}$. In most cases of physical interest, the λ_p are less than 1. Hence, if the extrapolation length is lowered with a resultant increase in the buckling it will be necessary to increase λ_p in order to maintain the above equality. This, in turn, means that

each of the coupling coefficients would also be increased. Physically, this means that if more neutrons are allowed to leak from each container (corresponding to a decrease in extrapolation length) it is necessary to move the containers closer together (corresponding to increasing their couplings) in order to maintain criticality. From the safety viewpoint, this means that in applying formula (7) or deductions from it, the larger the extrapolation length used in the bucklings the more conservative the critical estimate. Although the above arrangement assumes identical containers, a system of unequal containers would behave similarly, and hence a similar conclusion will be assumed for the general interacting system.

2) Simplification of Criterion A: Criterion B

If equation (7) is to hold for some d_p , it follows that for this d_p

$$k_{\infty} = \prod_{j=1}^n \left| 1 + L_j^2 d_p \right|.$$

But

$$\left| 1 + L_j^2 d_p \right| \geq 1 + L_j^2 \operatorname{Re} \{d_p\},$$

where $\operatorname{Re} \{d_p\}$ is the real part of d_p .*

Consequently, (if all $\operatorname{Re} \{d_p\} \geq 0$, as they are in most cases of physical interest)

$$k_{\infty} \geq \prod_{j=1}^n \left(1 + L_j^2 \operatorname{Re} \{d_p\} \right).$$

If now ξ denotes any positive number such that

$$\xi \leq \operatorname{Re} \{d_p\}, \quad (p = 1, 2, \dots, m), \quad (8)$$

* It is very likely that for the particular forms of $\underline{Z} \underline{B}^2$ considered, all d_p are real. However, since the resulting criticality criterion would not be changed even if this were true, it was not deemed worthwhile to try to prove this fact.

2572-19

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then a fortiori there must hold

$$k_{\infty} \geq \frac{n}{\pi} (1 + L_j^2 \xi). \quad (9)$$

This relation may be given a simple physical interpretation which will be used as the basis for a practical interaction criterion. Let G be a reactor whose geometric buckling is a number ξ , satisfying (8) and whose core is composed of the material common to the container system $F_1, F_2, \dots, F_m$. Then the effective multiplication constant k_G for G , as determined by n -group diffusion theory, is just

$$k_G = \frac{k_{\infty}}{\frac{n}{\pi} (1 + L_j^2 \xi)}.$$

Thus, if the system $F_1, F_2, \dots, F_m$ is critical, (9) is satisfied and every reactor G of this type must be at least critical; conversely, if any reactor, G , of this type, is sub-critical, (9) will not be satisfied, and the system of containers will also be sub-critical.

It is shown in appendix 1, that one value of ξ satisfying (8) is $\xi = B_S^2$, where

$$B_S^2 = \min_{1 \leq p \leq m} \left(B_p^2 - \frac{1}{2} \sum_{q=1}^m B_p^2 c_{p \rightarrow q} - \frac{1}{2} \sum_{q=1}^m B_p^2 c_{q \rightarrow p} \right), \quad (10)$$

whenever the expression on the right is positive.

Using the value B_S^2 (which will be called the system buckling) for ξ , the working criterion for a safe distribution of containers, which is defined as criterion B , may be stated as follows:

Let the system, S , of containers $F_1, F_2, \dots, F_m$ be filled with a common fissile material whose (infinite) multiplication constant is k_{∞} , and let the system have buckling B_S^2 as defined by formula (10). Let G be any reactor whose core is also composed of the common fissile material and whose geometrical buckling

2572-20

is B_S^2 . Then, if G is sub-critical so is the system of containers.

A possible scheme for applying this criterion is the following:

- a) Decide upon a shape for an individual reactor.
- b) Determine from critical mass experiments the actual linear dimensions of a single reactor G having this shape and having a core of the common fissile material.
- c) Determine the geometrical buckling B_G^2 of G.
- d) Restrict the coupling coefficients (and hence the relative position of the various containers) by the condition

$$\frac{B_G^2}{B_S^2} \leq 1. \quad (11)$$

where B_S^2 is given by equation (10). This scheme will be more fully treated along with some examples in a second half of this report to be printed separately.

3) Comments on the Preceding Criterion

a) The value B_S^2 is, of course, only one of a possible infinity of values for ξ . It is however a good one in the sense that under certain conditions the equality sign may hold in equation (9) with $\xi = B_S^2$. It is therefore of some importance to notice that it differs in a marked way from what might usually be expected. Ordinarily, it is assumed that the total number of neutrons reaching F_p from the other containers is proportional to $\sum_{q=1}^m c_{p \rightarrow q} = w_p$. But from the definition of $c_{p \rightarrow q}$ it follows that w_p really measures the number of neutrons leaving F_p and getting to the remaining containers, so that this assumption, in the general case, is quite unfounded. The expression for d on the other hand is seen to involve the mean value of the quantity $B_p^2 \sum_{q=1}^m c_{p \rightarrow q}$, measuring the leakage from F_p to all other containers

of the system, and the quantity $\sum_{q=1}^m B_q^2 c_{q \rightarrow p}$, measuring the leakage into F_p from all of the remaining containers.

b) Criterion B is simple to apply (especially in the case of equal containers when it contains essentially only one parameter, namely $w_0 = \max_p \sum_{q=1}^m c_{p \rightarrow q}$), yet it is not too crude since it allows for changes in container material as well as container shape.

c) It should prove to be very useful, and, from the nature of its derivation should give safe results. There are a number of reasons for this. In the first place, no microscopic parameters (with the possible exception of the extrapolation length) enter into it, all the need for such parameters being obviated by the experimental determination of the individually critical reactor G. In the second place, the shape of the container G is disposable. By its choice, it is to be expected that since only a ratio of bucklings of containers appear in (11), any shape error inherent in the group-diffusion theory, would tend to be minimized or conservatively displaced. Finally, as another point in its favor, it should be observed that although in the derivation the n-group diffusion method was used, the formula (11) contains no reference to any group picture.

4) Identical Containers

When all the containers of the system are identical the formula for B_S^2 becomes particularly simple:

$$B_S^2 = B^2 (1 - w), \quad (10a)$$

where B^2 is the geometrical buckling of a single container and w is the maximum total coupling between any one container and all the others.

In this simple case it is also possible to check on how good an approximation B_S^2 (as given by (10a)), is to the actual least eigenvalue of the matrix $\underline{Z} \underline{B}^2$. Since

2572-22

now $\underline{B}^2 = B^2 \underline{I}$, the eigenvalues of $\underline{Z} \underline{B}^2 = B^2 (\underline{I} - \underline{C})$ are B^2 times the eigenvalues of $\underline{I} - \underline{C}$; and since the latter are merely obtained from the eigenvalues of $\underline{C}$ by subtraction from 1, it is only necessary to calculate the eigenvalues of $\underline{C}$, and in particular (since the minimum eigenvalue of $\underline{Z} \underline{B}^2$ is being sought) the maximum eigenvalue of $\underline{C}$. This will be done for a number of special cases.

a) Example 1. For 2 convex containers

Figure 1

$$\underline{C} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, \text{ with } c = c_{1 \rightarrow 2} = c_{2 \rightarrow 1}.$$

The eigenvalues of $\underline{C}$ are easily seen to be

$$\lambda_1 = c, \lambda_2 = -c$$

so that the eigenvalues of $\underline{Z} \underline{B}^2$ are merely

$$d_1 = B^2 (1 - c), d_2 = B^2 (1 + c).$$

Consequently the minimum eigenvalue is $B^2 (1 - c)$ and it is given by (10a) exactly.

b) Example 2. For 3 convex containers

Figure 2

$$\underline{C} = \begin{pmatrix} 0 & c_{1 \rightarrow 2} & c_{3 \rightarrow 1} \\ c_{1 \rightarrow 2} & 0 & c_{2 \rightarrow 3} \\ c_{3 \rightarrow 1} & c_{2 \rightarrow 3} & 0 \end{pmatrix}$$

so that the eigenvalues λ_1, λ_2 , and λ_3 are the roots of the characteristic equation

$$\lambda^3 - (c_{1 \rightarrow 2}^2 + c_{2 \rightarrow 3}^2 + c_{3 \rightarrow 1}^2) \lambda - 2 c_{1 \rightarrow 2} c_{2 \rightarrow 3} c_{3 \rightarrow 1} = 0.$$

If the containers are symmetrically arranged, so that $c_{1 \rightarrow 2} = c_{2 \rightarrow 3} = c_{3 \rightarrow 1} = c$,

2572-23

the equation is

$$\lambda^3 - 3c^2\lambda - 2c^3 = 0,$$

the roots of which are

$$\lambda_1 = 2c, \lambda_2 = -c, \lambda_3 = -c.$$

Therefore the eigenvalues of $\underline{Z} \underline{B}^2$ in the symmetrical case are

$$d_1 = B^2 (1 - 2c), d_2 = B^2 (1 + c), d_3 = B^2 (1 + c).$$

But $w = 2c$, so that also in this case formula (10a) is exact. From this fact, it may be shown that for 3 containers the symmetrical arrangement is at least as reactive as any other arrangement having the same value for w .

c) Example 3. If the containers in the preceding example are not symmetrically arranged, but, rather, are placed as shown in figure 3 so that $c_{1 \rightarrow 2} = c_{2 \rightarrow 3} = c$,

$c_{3 \rightarrow 1} = 0$, then the characteristic equation becomes

$$\lambda^3 - 2c^2\lambda = 0.$$

The eigenvalues of the coupling matrix are thus

$$\lambda_1 = 0, \lambda_2 = +c\sqrt{2}, \lambda_3 = -c\sqrt{2},$$

and hence, the minimum eigenvalue of $\underline{Z} \underline{B}^2$ is $d_2 = B^2 (1 - c\sqrt{2})$. For this arrangement, w is still obviously given by $w = 2c$. Therefore, in the approximation (10a), when w is used instead of the true least eigenvalue of $\underline{Z} \underline{B}^2$, there is an over-estimation of the coupling amounting to

$$\frac{w}{\lambda_2} = \frac{2c}{c\sqrt{2}} = 1.41,$$

i.e. about 41%. This also indicates that this system is about 71% as reactive as the symmetrical system considered in example 2.

2572-24

d) Example 4. The situation is no longer so simple when 4 or more containers

are considered, since no completely symmetrical arrangement exists. One arrangement for 4 cylinders is shown in figure 4. For this arrangement the characteristic equation is

$$X^4 - (3 + \beta) X^2 - 2(\beta + 1) X - \beta = 0,$$

with $\beta = 3 \left(\frac{c_1}{c_2} \right)^2$, $X = \frac{\lambda}{c_2}$.

This may be solved for X to obtain

$$X_1 = -1, X_2 = -1, X_3 = 1 + \sqrt{1 + \beta}, X_4 = 1 - \sqrt{1 + \beta}.$$

Since β is non-negative, X_3 is always the maximum root, and so the maximum eigenvalue of $\underline{C}$ is

$$\lambda_3 = c_2 X_3 = c_2 \sqrt{1 + 3 \frac{c_1^2}{c_2^2}} + c_2 = \sqrt{c_2^2 + 3 c_1^2} + c_2.$$

Thus the exact value to use for ξ in equation (8) is

$$\xi_{\text{exact}} = B^2 (1 - c_2 - \sqrt{c_2^2 + 3 c_1^2}).$$

The value for w is evidently $3 c_1$ so that B_S^2 as given by formula (10a) is

$$B_S^2 = B^2 (1 - 3 c_1).$$

An idea of how much the coupling is overestimated using B_S^2 as compared with the exact coupling may be obtained by comparing $3 c_1$ and $c_2 + \sqrt{c_2^2 + 3 c_1^2}$. For large

separations $c_2 \approx \frac{c_1}{\sqrt{3}}$, and so

$$\frac{3 c_1}{c_2 + \sqrt{c_2^2 + 3 c_1^2}} \approx \frac{3 c_1}{\frac{c_1}{\sqrt{3}} + \sqrt{3 c_1^2 + \frac{c_1^2}{3}}} = \frac{3 \sqrt{3}}{1 + \sqrt{10}} = 1.2484.$$

Consequently the true coupling is overestimated by only 25%.

5) Containers Filled with Different Materials

The preceding sections may be used as a basis for a simple treatment of systems in which it is not required that all containers be filled with a common fissile material. In order to do this, however, an additional assumption has to be made.

Assumption 4.) Let $F_1, \dots, F_m$ be a system of interacting containers, filled with any materials whatsoever. If the contents of any one of these be replaced by a material whose composition lies closer to that giving minimum geometry in the case of a single reactor, then the system as a whole is more reactive.

If the validity of this assumption is granted, then any system may be replaced by one in which each container is filled with the "most" reactive core material, and then if this latter system is safe so is the original one. Criterion B may be used to test the safety of the substitute system.

II. COMPARISON OF RESULTS WITH EXPERIMENT IN SOME SIMPLE CASES

a) A number of simple interaction problems were checked on the basis of formula 11 using various types of coupling coefficients. The results, given in terms of edge-to-edge separation at criticality are listed in table I. The experimental data were taken from Critical Mass Studies, Part IV. The problems

considered were the critical separations for a pair of equal highly enriched

10" cylinders

1572-25A
26

at an H/X of 169 as a function of the common height of either of the interacting cylinders. In this particular case

$$B_1^2 = B_2^2 = B^2 = \frac{23.13}{(d + E)} + \frac{9.87}{(h + E)^2}$$

$$c_{2 \rightarrow 1} = c_{1 \rightarrow 2} = c; \quad c_{1 \rightarrow 1} = c_{2 \rightarrow 2} = 0. \quad (12)$$

It has been shown that under these conditions the equality sign should hold in (11), so that these examples should furnish a good idea of the errors inherent in the various coupling coefficients. The notation is as follows:

h, d = height and diameter of an interacting cylinder.

$\gamma = \frac{d}{h}$ = shape factor of an interacting cylinder.

H, D = height and diameter of the individually critical cylinder G as determined by experiment.

s = edge-to-edge separation between the pair of interacting cylinders.

Schematic Procedure

Step a.) Since the reactors of the system are identical there is no question of choosing a shape. G is, of course, chosen to have the same shape as either of the interacting cylinders.

Step b.) H and D are thus uniquely determined from experiment once it is known that they must satisfy $\frac{D}{H} = \gamma$. This is done by using a graph of H versus D for a critical cylinder at the desired core composition. Such a graph may be obtained from the data of Critical Mass Studies, Part III.

Step c.) Once H and D are known, B_G^2 is determined by

$$B_G^2 = \frac{23.13}{(D + E)^2} + \frac{9.87}{(H + E)^2}.$$

As usual E is the total extrapolation distance, and in this paper is numerically

2572-28

4.8 cm. According to diffusion theory, B_G^2 should be independent of γ ; actually, since diffusion theory is not exact, it varies slightly with γ .

Step d.) From the conditions (12) it follows that

$$B_S^2 = B^2 (1 - c),$$

and hence by (11) and the remark that equality obtains in this formula

$$\frac{B_G^2}{B^2 (1 - c)} = 1;$$

$$c = 1 - \frac{B_G^2}{B^2}. \quad (11a)$$

This determines c , and hence the separation s . The value of s will of course depend upon the form of the coupling coefficient c that is used. The table gives values of s for $c_u = c_{A \rightarrow B}$, uniform, $1.41 c_u$, $c_m = c_{A \rightarrow B}$, max. u , and $1.34 c_m$. (See III, a, for the definitions of these couplings and the derivations of numerical factors.)

TABLE 1 - ALL DIMENSIONS IN CM. $d = 25.4$ CM.

h	γ	H	D	B^2	B_G^2	$c = 1 - \frac{B_G^2}{B^2}$	s = Separation predicted using different couplings				
							c_u	$1.41 c_u$	c_m	$1.34 c_m$	Exp.
28.7	0.8850	31.33	27.73	0.03415	0.02942	0.1385	<0	1.64	6.2	9.7	0.3
30.7	0.8273	32.9	27.22	0.03319	0.02950	0.1118	0.92	4.30	9.4	13.0	2.3
32.8	0.7744	34.5	26.72	0.03234	0.02967	0.0826	4.33	9.28	14.0	18.3	6.1
34.3	0.7405	35.75	26.47	0.03181	0.02966	0.0676	7.58	13.1	17.3	22.2	9.9
35.8	0.7095	36.88	26.17	0.03134	0.02980	0.0491	13.3	19.3	23.5	28.9	15.9
37.2	0.6827	38.0	25.95	0.03095	0.02985	0.0355	19.6	26.9	30.5	37.2	24.9
38.2	0.6649	38.76	25.77	0.03069	0.02995	0.0241	28.0	37.5	39.7	47.6	34.2

These results are plotted in figure 1.

APPENDIX I

THE DERIVATION OF THE SYSTEM BUCKLING

Theorem: Let $\underline{T} = (t_{p,q})$ be a real diagonalizable matrix, with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. Then

$$\min_{1 \leq p \leq m} \left(\operatorname{Re} \left\{ \lambda_p \right\} \right) \geq \left(\min_{1 \leq p \leq m} g_p \right),$$

where

$$g_p = t_{p,p} - \frac{1}{2} \sum_{\substack{q=1 \\ q \neq p}}^m |t_{p,q}| - \frac{1}{2} \sum_{\substack{q=1 \\ q \neq p}}^m |t_{q,p}|.$$

Proof: Let $\underline{X}_p$ be the column eigen-vector corresponding to the eigenvalue λ_p .

Then

$$\underline{T} \underline{X}_p = \lambda_p \underline{X}_p. \quad (1)$$

If (1) is premultiplied by the transposed conjugate $\overline{\underline{X}}_p^t$, of $\underline{X}_p$, and both sides of the result are divided by $\overline{\underline{X}}_p^t \underline{X}_p$ there results

$$\frac{\overline{\underline{X}}_p^t \underline{T} \underline{X}_p}{\overline{\underline{X}}_p^t \underline{X}_p} = \lambda_p, \text{ or } \operatorname{Re} \left\{ \frac{\overline{\underline{X}}_p^t \underline{T} \underline{X}_p}{\overline{\underline{X}}_p^t \underline{X}_p} \right\} = \operatorname{Re} \left\{ \lambda_p \right\}. \quad (2)$$

Now let $\underline{X} = (x_p)$ be an arbitrary vector. Then

$$\begin{aligned} \overline{\underline{X}}^t \underline{T} \underline{X} &= \sum_{p=1}^m \sum_{q=1}^m \overline{x}_p t_{p,q} x_q = \sum_{p=1}^m |x_p|^2 t_{p,p} + \sum_{p=1}^m \sum_{\substack{q=1 \\ q \neq p}}^m \overline{x}_p x_q t_{p,q} \\ \operatorname{Re} \left\{ \overline{\underline{X}}^t \underline{T} \underline{X} \right\} &= \sum_{p=1}^m |x_p|^2 t_{p,p} + \sum_{p=1}^m \sum_{\substack{q=1 \\ q \neq p}}^m t_{p,q} \operatorname{Re} \left\{ \overline{x}_p x_q \right\} \end{aligned}$$

Using Schwarz's inequality

$$\left| \operatorname{Re} \left\{ \overline{x}_p x_q \right\} \right| \leq \left| \overline{x}_p x_q \right| \leq \frac{1}{2} \left\{ |x_p|^2 + |x_q|^2 \right\};$$

whence

$$\begin{aligned} \operatorname{Re} \left\{ \bar{\underline{X}}^t \underline{T} \underline{X} \right\} &= \sum_{p=1}^m |x_p|^2 t_{q,p} - \frac{1}{2} \sum_{p=1}^m \sum_{\substack{q=1 \\ p \neq q}}^m |t_{p,q}| \left\{ |x_p|^2 + |x_q|^2 \right\}, \\ &\geq \sum_{p=1}^m |x_p|^2 t_{p,q} - \frac{1}{2} \sum_{p=1}^m |x_p|^2 \sum_{\substack{q=1 \\ q \neq p}}^m |t_{p,q}| - \frac{1}{2} \sum_{q=1}^m |x_q|^2 \sum_{\substack{p=1 \\ p \neq q}}^m |t_{p,q}|, \\ &\geq \sum_{p=1}^m |x_p|^2 g_p \geq \left(\min_{1 \leq p \leq m} g_p \right) \left(\sum_{p=1}^m |x_p|^2 \right) = \left(\min_{1 \leq p \leq m} g_p \right) (\bar{\underline{X}}^t \underline{X}). \end{aligned}$$

Therefore,

$$\operatorname{Re} \frac{\left\{ \bar{\underline{X}}^t \underline{T} \underline{X} \right\}}{\bar{\underline{X}}^t \underline{X}} \geq \left(\min_{1 \leq p \leq m} g_p \right) g_p.$$

From this result and equation (2) the theorem follows immediately.

The results stated on page 20 follows from this theorem by taking $\underline{T} = \underline{Z} \underline{B}^2$.

APPENDIX II

DETERMINATION OF THE FACTOR FOR COMPARING $c_{A \rightarrow B}$, diff and $c_{A \rightarrow B}$, Fermi
The minimum of

$$f(R, H, E) = \frac{1 + 1.71 \left(\frac{R + E}{H + 2E} \right)^2}{1 + \frac{R}{H}}$$

is desired when R, H, and E are allowed to vary. R and H will be allowed to vary freely, while E (for the sake of convenience in the argument) will be restricted so that

$$E \leq 0.93 H. \quad (1)$$

This restriction on E does not in any way affect the general applicability of the result. There are two cases to be considered.

a.) $2R \leq H$. In this case $f(R, H, E)$ increases as E increases so that

$$f(R, H, E) \geq f(R, H, 0) = \frac{1 + 1.71 \left(\frac{R}{H} \right)^2}{1 + R/H}$$

The minimum of the last expression as a function of R/H is 0.8854, occurring when $R/H = 0.2588$. Hence for $2R \leq H$,

$$f(R, H, E) \geq 0.8854.$$

b.) $2R > H$. In this case $f(R, H, E)$ increases as E increases so that with the restriction (1)

$$\min_E f(R, H, E) = f(R, H, 0.93H) = \frac{1 + 1.71 \left(\frac{\frac{R}{H} + 0.93}{2.86} \right)^2}{1 + \frac{R}{H}}$$

The minimum of the last expression as a function of R/H is greater than 0.8854 and occurs near $R/H = 1.19$. Hence in either case, i.e. for all R, H, and E satisfying (1),

$$f(R, H, E) \geq 0.885$$

and the result desired is proved.

2572-32