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THEORY OF DRIFT-CONE MODES OF COLLISIONLESS PLASMA IN CYLINDRICAL GEOMETRY

by

R. L. McCrory, Jr., and J. E. McCune

ABSTRACT

In this report, we consider the problem of the collisionless electrostatic flute-like drift-cone instability of a hot, fully ionized plasma in a magnetic field in cylindrical geometry. Our cylindrical plasma model includes those sources of free energy available to drive the instability inherent in many mirror confinement devices, i.e., radially varying density gradients and velocity space "loss-cone" like distribution functions. Our zeroth order density distribution is a Gaussian in the radial direction. In our theoretical treatment, we derive a singular integral equation for the Hankel transform for each azimuthal component of the perturbed electrostatic potential in the plasma. Our treatment includes the finite Larmor radius corrections to all orders of a_i/R (a_i is the ion Larmor radius, R the plasma radius) for those cases where the quantity is less than unity. We find the radial eigenmodes of the system without resorting to the "local approximation" and our results give the cylindrical geometry marginal stability boundaries for the nonlocal flute-like drift-cone modes which indicates that the critical values of R/a_i are substantially lower than have previously been reported. We apply our cylindrical results to the Livermore 2X machine and find that we predict stability for values of the azimuthal mode number less than 5. Our results show that the frequency at threshold for the drift-cone modes is sensitive to the ion-electron temperature ratio but relatively insensitive to a_i/R .

I. INTRODUCTION

Plasma confinement in mirror machines has been the subject of numerous studies¹⁻⁵ concerning the instabilities (both electrostatic and electromagnetic) of the plasma in the presence of both a loss-cone distribution for the ions as well as the negative radial density gradient inherent to such devices.

In this report, we formulate the problem of the linear stability theory of the drift cyclotron (drift-cone) modes of a slightly inhomogeneous, collisionless plasma in cylindrical geometry. In particular, we treat the case of the flute-like drift-cone mode. The flute-like drift-cone mode has, as its source of free energy to drive the in-

stability, a combination of the free energy available from the loss-cone distribution in velocity space and the free energy associated with the spatial inhomogeneity of the plasma. Although we analyze the problem in the electrostatic limit, Tang,⁶ among others, has shown that finite-length effects, as well as electromagnetic effects, do not radically alter the marginal stability parameters for local modes from those obtained in the electrostatic limit.

Our theoretical derivation yields a singular integral equation for the Hankel transform for each azimuthal component of the perturbed electrostatic potential in the plasma by a method similar to that of Shima-Fowler.² The treatment includes finite

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Larmor radius effects for which the ratio $\frac{a_1}{R}$ (a_1 is the ion Larmor radius, R the plasma radius) is less than unity. In addition, our theoretical formulation allows us to examine the nonlocal radial eigenmodes in cylindrical geometry. We calculate the marginal stability boundaries analytically for parameters suited to mirror machine plasmas by the use of a WKB analysis in the Hankel transform space (k -space).

The flute-like drift-cone modes, first predicted by Mikhailovskii,¹ and later analyzed by Rosenbluth and Post,³ Berk, Pearlstein, and Cordey,⁷ Baldwin et. al.,⁴ Tang,⁶ and others have previously been treated by using an eikonal approximation that does not include the radial eigenmode structure in the plasma and leaves open the question of the effects of nonlocal modes in the radial direction concerning the linear evolution of the drift-cone modes.

Our analysis here is thus the opposite limit of previous work in that we do not consider finite-length effects, but determine only the radial structure of the drift-cone modes of a loss-cone inhomogeneous plasma in cylindrical geometry which is infinite in the direction parallel to the magnetic field.

Our results give the cylindrical geometry marginal stability boundaries for the drift-cone modes for various values of the azimuthal mode number ℓ . Our cylindrical results indicate that the critical values of $\frac{R}{a_1}$ necessary to stabilize the drift-cone modes are substantially lower than those previously estimated.³ In particular, we apply our cylindrical results to the parameter regime suitable to the Livermore 2X machine and predict stability for many values of the azimuthal mode number ℓ , for which others^{3,4,6} have predicted instability. The results for the drift-cone modes indicate that the threshold frequency for instability is relatively insensitive to $\frac{a_1}{R}$ but is quite sensitive to the ratio of ion to electron temperature, T_i/T_e .

In particular, for our assumed Gaussian profile for the plasma density in the radial direction we find that no positive frequency ω , in the range $0 < \omega < \Omega_i$ (Ω_i is the ion gyro frequency) can occur for the assumed form for the ion loss-cone distribution function. This result has previously been

derived by Callen⁵ and others¹⁻⁶ for the slab model in the limit for which local approximation is valid. Specifically, we demonstrate that a necessary condition for occurrence of the drift-cone modes is that the quantity β^* be less than zero, where

$$\begin{aligned} \beta^* = & \frac{\sqrt{\pi}}{4(1-\epsilon_1)} \frac{1}{\sqrt{\epsilon_1(1-\epsilon_1)}} \left(\frac{\omega_{*1}}{\Omega_1} \left(-\gamma \frac{\pi}{\Omega_1} \csc^2 \left(\pi \frac{\omega_r}{\Omega_1} \right) + \right. \right. \\ & \left. \left. - \frac{\gamma}{\Omega_1} (1-2\epsilon_1) \left[\cot \left(\pi \frac{\omega_r}{\Omega_1} \right) - \pi \frac{\omega_r}{\Omega_1} \csc^2 \left(\pi \frac{\omega_r}{\Omega_1} \right) \right] \right) \right. \\ & \left. + \frac{\sqrt{\pi}}{2\sqrt{\epsilon_e(1-\epsilon_e)}} \frac{T_i}{T_e} \left[\left(-\frac{\pi\gamma}{\Omega_e} \frac{\omega_{*e}}{\Omega_e} - \frac{\pi\omega_r}{\Omega_e} \frac{\gamma}{\Omega_e} \csc^2 \left(\pi \frac{\omega_r}{\Omega_e} \right) \right) \right. \right. \\ & \left. \left. + \frac{\gamma}{\Omega_e} \cot \left(\pi \frac{\omega_r}{\Omega_e} \right) \right] \right) \end{aligned} \quad (1)$$

with

$$\omega = \omega_r + i\gamma \quad \gamma > 0 \quad (2)$$

$$\epsilon_{1,e} = a_{1,e}^2/R^2 \quad (3)$$

$$\omega_{*1,e} = \ell \epsilon_{1,e} \Omega_{1,e} \quad (4)$$

in conjunction with the requirement that

$$\lambda_{D1}^2 (\hat{g}_1(x) + \hat{g}_e(x)) > 0 \quad (5)$$

where the quantities $\hat{g}_1(x)$, $\hat{g}_e(x)$ have the dimensions of $(\text{length})^{-2}$ and are defined in the text and λ_{D1}^2 is the square of the ion Debye length.

These results can be shown to indicate, as one would expect, that the larger the plasma cross section in terms of the ion Larmor radius, the more stable is the system. In addition we find that the drift-cone modes, if any, which should occur in a deuterium (D^+) plasma for which $T_i/T_e = 40$ in a plasma for which $\frac{R}{a_1} > 15$ will require azimuthal mode numbers, ℓ , greater than 100, for example. These results disagree with the previous estimates of Post-Rosenbluth³ who require $\frac{R}{a_1} > 115$ for stability for the same plasma parameters.

Even Tang's⁶ finite-length electromagnetic theory indicates $\frac{R}{a_1}$ should be about 70. We attribute our much smaller estimate of the critical value of $\frac{R}{a_1}$ to the effects of cylindrical geometry and the nonlocal nature of the radial eigenmodes. By comparing Tang's⁶ stability parameters with those of Post-Rosenbluth³ we conjecture that the electromagnetic correction necessary in the analysis of the 2X machine ($\beta \sim .05$) should not affect our results by more than a factor of two.

In the following sections, we summarize our theoretical analysis and present the results applicable to a large class of mirror machine plasmas.

II. DISPERSION EQUATIONS FOR THE DRIFT-CONE MODES

From Shima - Fowler,² the integrodifferential equation for the ℓ^{th} component of the perturbed potential can be written in the form (for the case of cylindrical symmetry for perturbations of the form $\psi_\ell(r) = \psi_\ell(r) \exp[i(\ell\theta + k_z z)]$):

$$\left[\frac{\ell^2}{r^2} + k_z^2 - \frac{1}{r} \left(\frac{d}{dr} r \frac{d}{dr} \right) \right] \psi_\ell(r) = \hat{\rho}_\ell(r) \quad (6)$$

$$\hat{\rho}_\ell(r) = \sum_j \frac{e_j^2}{m_j} \psi_\ell(r) \int d^3v \frac{\partial f_{oj}}{\partial E_\perp} \bigg|_{p_\theta} + \int d^3v \left[-i \right.$$

$$\left. \left(k_z v_z - \omega \right) \frac{\partial f_{oj}}{\partial E_\perp} \bigg|_{p_\theta} + i\ell \frac{\partial f_{oj}}{\partial p_\theta} \bigg|_{E_\perp} + i k_z v_z \frac{\partial f_{oj}}{\partial E_\parallel} \right] \int_{-\infty}^{\infty} d\tau \left[e^{i k_z v_z \tau - i \omega \tau} e^{i \ell (\theta - \theta')} \psi_\ell(r') \right]$$

where we have assumed that the distribution functions f_{oj} depend on the constants of motion along the unperturbed orbits given by:

$E_\perp$ - The energy of the particle due to motion perpendicular to the magnetic field, B .

$E_\parallel$ - The energy of the particle due to motion parallel to the magnetic field.

p_θ - The canonical angular momentum about the z -axis. (The z -axis is parallel to $\underline{B}$.)

$$p_\theta = r v_\perp \sin(\phi - \theta) + \frac{1}{2} r^2 \Omega_j$$

where, as in Shima-Fowler² z , r , θ and v_z , $v_\perp$, ϕ are the usual cylindrical coordinates in $\underline{r}$ and $\underline{v}$ space, respectively. In (6), above, the primed coordinates designate the particle coordinates at time t' . These orbits are given explicitly by Shima-Fowler.²

Taking the Hankel transform⁸ of (6), of order ℓ , there results the following singular integral equation for the ℓ^{th} harmonic component of the electrostatic potential

$$(k_z^2 + k^2) H_\ell(k) = 4\pi \int_0^\infty \hat{\rho}_\ell(r) J_\ell(kr) r dr \quad (7a)$$

$$\text{where} \quad \psi_\ell(r) = \int_0^\infty H_\ell(k) J_\ell(kr) k dk \quad (7b)$$

In order to model a mirror plasma, we use the following distribution function for the ions:

$$f_{oi} = \frac{\partial}{\partial \alpha_{1i}^2} \left(\frac{n_{io}}{\pi^{3/2} \alpha_{zj}} \left[e^{-v_\perp^2 / \alpha_{1j}^2} e^{-v_z^2 / \alpha_{zj}^2} e^{-p_{\theta j} / P_{\alpha j}} \right] \right) \quad (8)$$

where

$$\alpha_{1,zj}^2 = \frac{2 T_{1,zj}}{m_j} \quad (9)$$

$$P_{\alpha j} = \frac{\alpha_{1j}^2}{2 \Omega_j \epsilon_j} \quad (10)$$

where $T_{1,zj}$ is the temperature of the j th species in the direction perpendicular and parallel to B , respectively; m_j = mass of the j th species; ϵ_j is dimensionless; and n_{io} is the ion density at $r=0$. The distribution function (8) gives

$$n_i(r) = n_{io} e^{-r^2 / R_1^2} \left(1 + \frac{r^2 \epsilon_j^2}{a_1^2} \right) \quad (11)$$

where

$$R_1^2 = \frac{4p_{o1}^2}{2p_{o1}\Omega_1 - \alpha_1^2} \quad (12)$$

$$\alpha_1^2 = \alpha_{1f}^2 / \Omega_1^2 \quad (13)$$

for the ion density as a function of radius. For the electrons, we use

$$f_{oe} = (1 + bp_{oe} \frac{\partial}{\partial p_{oe}})$$

$$\left(\frac{n_{eo}}{\pi^{3/2} \alpha_{1e}^2 \alpha_{ze}^2} e^{-v_z^2 / \alpha_{ze}^2} e^{-v_1^2 / \alpha_{1e}^2} \cdot p_o / p_{oe} \right)_{\alpha_{1e}^2, \alpha_{ze}^2 \text{ constant}} \quad (14)$$

$$n_e(r) = n_{eo} e^{-r^2/R_e^2} (1 + \frac{b \Omega_e r^2}{2p_{oe}} (1 - 2\epsilon_e)) \quad (15)$$

where setting

$$b = \frac{\epsilon_1^2}{\alpha_1^2} \frac{2p_{oe}}{\Omega_e (1 - 2\epsilon_e)} \quad (16)$$

gives us charge neutrality to lowest order provided we also set $R_e = R_1$.

Note from (8) that the ions have a distribution which is like that used previously by others⁹ to simulate the loss-cone like distribution function of a mirror plasma. For such a distribution, Dory and Guest⁹ have shown that the distribution is stable to loss-cone instabilities. Thus, we are assured that any instabilities we find will be due to the combination of both the loss-cone and density gradient sources of free energy in the plasma.

Using (6), (8), and (14) in (7) we find that there results the following integral equation in k-space for the Hankel transform of the electrostatic potential

$$H_L(k) = \int_0^\infty k' dk' \frac{\hat{G}(k, k')}{k^2 + k_z^2} H_L(k') \quad (17)$$

where

$$\hat{G}(k, k') = \hat{G}_1(k, k') + \hat{G}_e(k, k') \quad (18)$$

$$\hat{G}_1(k, k') = \frac{\partial}{\partial \alpha_1^2} (a_1^2 G_1(k, k')) \quad (19)$$

$$\hat{G}_e(k, k') = (1 + bp_{oe} \frac{\partial}{\partial p_{oe}}) \hat{G}_e(k, k') \quad (20)$$

where

$$G_j(k, k')$$

$$= \frac{4\pi n_{jo} e^2}{T_1} \left(\frac{R_1^2}{2} \right) e^{-\frac{1}{4} (k' - k)^2 R_j^2} e^{-\frac{1}{2} k k' R_j^2} \quad (21)$$

$$\left\{ -I_L \left(\frac{1}{2} k k' R_j^2 \right) + (-1\omega - 1\ell \frac{\alpha_{1j}^2}{2p_{oj}}) \int_{-\infty}^0 d\tau \tau \Xi_j(k, k', \tau) \right.$$

$$\left. - \frac{k^2}{2} (\alpha_{zj}^2 - \alpha_{1j}^2) \int_{-\infty}^0 d\tau \tau \Xi_j(k, k', \tau) \right\}$$

$$\Xi_j(k, k', \tau)$$

$$= \left(\frac{s_1}{s_j} \right)^{1/2} \cdot I_L \left(\frac{1}{2} k k' R_1^2 \sqrt{s_j s_1^*} \right) e^{-1\omega \tau - k_z^2 \alpha_{zj}^2 \frac{\tau^2}{4}} \quad (22)$$

$$s_j = 1 - \epsilon_j + \epsilon_j e^{-1\Omega_j \tau}$$

We now restrict ourselves to the flute-like modes ($k_z=0$) primarily for two reasons: (1) The flute-like modes are "absolute" in nature, and do not propagate out the ends of the confining device. (2) Callen⁵ has shown for the slab model that the flute-like drift-cone modes occur at lower values of the threshold density and " $k_1 a_1$ " (where k_1 is the wave number of the perturbation perpendicular to B) than do the drift-cyclotron instabilities which occur in the regime where no flute-like loss-cone instabilities occur.

For the flute-like modes, we obtain

$$\hat{G}_1(k, k') \Big|_{k_z=0} = -\frac{R_1^2}{2\lambda_{D1}^2} e^{-\frac{1}{2}(k'-k)^2 R_1^2} e^{-\frac{1}{2}kk'R^2} \quad (23)$$

$$\left\{ \frac{\epsilon_1}{1-\epsilon_1} \left[(1-\frac{1}{2}(k^2+k'^2) R^2 + \frac{1}{2}kk'R^2) \frac{I_{\ell}'}{I_{\ell}} \right] (\frac{1}{2}kk'R^2) \right. \\ \left. I_{\ell}(\frac{1}{2}kk'R^2) \right\} +$$

$$\sum_{n=-\infty}^{\infty} \frac{\omega + \omega_*}{\omega + n\Omega_1} I_{\ell-n}(\frac{1}{2}kk'R^2(1-\epsilon_1)) \frac{I_n'}{I_n}(\frac{1}{2}kk'R^2\epsilon_1)$$

$$(1-\frac{1}{2}(k^2+k'^2)R^2 + kk'R^2) \frac{I_n'}{I_n}$$

$$\sum_{n=-\infty}^{\infty} \frac{\omega_*}{\omega + n\Omega_1} I_{\ell-n}(\frac{1}{2}kk'R^2(1-\epsilon_1)) \frac{I_n}{I_n}(\frac{1}{2}kk'R^2\epsilon_1) \Big\}$$

where

$$\omega_* = \ell \epsilon_1 \Omega_1$$

$$\lambda_{D1}^2 = (k_B T_1 / 8\pi M_{10} e^2) k_B = 1.3804 \times 10^{-16} \text{ ergs/}^\circ\text{K}$$

where the " , " on $I_{\ell,n}$ means differentiation with respect to argument and the argument of I_{ℓ} is $\frac{1}{2}kk'R_1^2$, and the argument of I_n is $\frac{1}{2}kk'R\epsilon_1$.

Our resulting integral equation for the drift-cone modes has the important property that the kernel is sharply peaked about $k' = k$ for large R .

In fact, in the infinite medium homogeneous limit, viz.:

$$\lim_{R_j \rightarrow \infty} R_j + \infty \quad \text{but } a_j = \text{constant, } j = 1 \text{ or } e \quad (24)$$

$$\epsilon_j \rightarrow 0$$

the kernel is diagonalized. We now consider the delta-convergent sequence defined by¹⁰

$$\delta(\beta, \kappa) = \frac{2}{\sqrt{\pi}\beta} e^{-\kappa^2/\beta} \quad (25)$$

with its associated relations

$$\frac{\partial \delta(\beta, \kappa)}{\partial \kappa} = \frac{1}{2} \frac{\partial^2 \delta(\beta, \kappa)}{\partial \kappa^2} \quad (26)$$

and expand $\delta(\beta, \kappa)$ in a Taylor series about β_0 in the formal sense of Gel'fand and Shilov.¹⁰ We use the addition relationships

$$\lim_{\beta_0 \rightarrow 0} \frac{\partial^n \delta(\beta, \kappa)}{\partial \kappa^n} \Big|_{\beta_0} = \delta^{(n)}(\kappa) \quad (27)$$

where the superscript n on the right denotes differentiation n times and is formally defined by:

$$\int_0^\infty \delta^{(n)}(x-x_0) h(x) dx = (-)^n \frac{d^n h(x)}{dx^n} \Big|_{x=x_0} \quad (28)$$

If we now make the identifications

$$k = k' - k \quad \beta = 4/R^2$$

together with the substitution

$$v_{\ell}(x) = \sqrt{g(x, x)} \sqrt{x} H_{\ell}(x); \quad x = kR \quad (29)$$

where

$$g(k, k') \\ k_z = 0 \\ = -\frac{\sqrt{kk'R}}{2\lambda_{D1}^2} e^{-\frac{1}{2}kk'R^2} \left[\frac{\epsilon_1}{1-\epsilon_1} I(\frac{1}{2}kk'R^2) f_1(k, k') + \right. \quad (30)$$

$$- \frac{\epsilon_1}{1-\epsilon_1} \sum_n \frac{\omega+\omega_n}{\omega+n\Omega} I_{\ell-n} (\frac{1}{2}kk'R^2(1-\epsilon_1))$$

$$\times I_n (\frac{1}{2}kk'R^2\epsilon_1) f_2(k, k')$$

$$\begin{aligned} & - \sum_n \frac{\omega_n}{\omega+n\Omega} I_{\ell-n} (\frac{1}{2}kk'R^2(1-\epsilon_1)) I_n (\frac{1}{2}kk'R^2\epsilon_1) + \\ & + \frac{\lambda_{D_1}^2}{\lambda_{D_e}^2} [I_{\ell} (\frac{1}{2}kk'R^2) - (\omega+\omega_n)e \sum_n \frac{1}{\omega+n\Omega} I_{\ell-n} \\ & (\frac{1}{2}kk'R^2(1-\epsilon_e)) I_n (\frac{1}{2}kk'R^2\epsilon_e)] \end{aligned}$$

and

$$f_1(k, k') = 1 - \frac{1}{2}(k^2 + k'^2) R^2 + \frac{1}{2}kk'R^2 \frac{I_{\ell}'(\frac{1}{2}kk'R^2)}{I_{\ell}(\frac{1}{2}kk'R^2)} \quad (31)$$

$$f_2(k, k') = 1 - \frac{1}{2}(k^2 + k'^2) R^2 + \frac{1}{2}kk'R^2 \frac{I_n'(\frac{1}{2}kk'R^2\epsilon_1)}{I_n(\frac{1}{2}kk'R^2\epsilon_1)}$$

we obtain a differential equation approximation to (17) in the form:

$$\frac{d^2 V_{\ell}(x)}{dx^2} + Q(x) V_{\ell}(x) = 0 \quad (32)$$

with

$$\begin{aligned} Q(x) = & \frac{1}{2} \left(\frac{g'}{g} \right)^2 - \frac{1}{2} \left(\frac{g''}{g} \right) + 1 + \frac{1}{g} \frac{d^2}{dx^2} g(x, x') \bigg|_{x'=x} + \\ & - \frac{(x^2 + x'^2) \lambda_{D_1}^2}{\lambda_{D_1}^2 g(x, x)} \end{aligned} \quad (33)$$

where

$$g = g(x, x); \quad g' \equiv \frac{dg(x, x)}{dx}; \quad g'' \equiv \frac{d^2 g(x, x)}{dx^2} \quad (34)$$

where $\hat{g}(x, x)$ is defined by equation (30) and can also be written as:

$$\hat{g}(x, x') = \hat{g}_1(x, x') + g_e(x, x') \quad (35)$$

The following identities are necessary to complete the definition of $Q(x)$ of (33);

$$\frac{dg(x, x')}{dx'} \bigg|_{x'=x} = \frac{1}{2} g' \quad (36)$$

$$\frac{d^2 g_1(x, x')}{dx'^2} \bigg|_{x'=x} = \frac{1}{4} \left(\frac{d^2 g(x, x)}{dx^2} - \frac{1}{x} \frac{dg(x, x)}{dx} \right) + \quad (37)$$

$$- \frac{\sqrt{xx'}\pi}{2\lambda_{D_1}^2} e^{-\frac{1}{2}xx'} \frac{\epsilon_1}{1-\epsilon_1} I_{\ell}(\frac{1}{2}xx') (1 + \frac{1}{2}xx' \frac{I_{\ell}'(\frac{1}{2}xx')}{I_{\ell}(\frac{1}{2}xx')})$$

$$\begin{aligned} & - \frac{\epsilon_1}{1-\epsilon_1} \sum_n \frac{\omega+\omega_n}{\omega+n\Omega} I_{\ell-n} (\frac{1}{2}xx'(1-\epsilon_1)) I_n (\frac{1}{2}xx'\epsilon_1) \\ & \left(1 + \frac{1}{2}xx' \frac{I_n'(\frac{1}{2}xx')}{I_n(\frac{1}{2}xx'\epsilon)} \right) \end{aligned}$$

$$\frac{d^2 g_e(x, x')}{dx'^2} \bigg|_{x'=x} = \left(\frac{1}{4} \frac{d^2 g_e(x, x')}{dx^2} - \frac{1}{x} \frac{dg_e(x, x)}{dx} \right)$$

In general, the Hankel transform $H_{\ell}(k)$, (7b), does not always exist for an arbitrary $\psi_{\ell}(r)$. By a theorem referred to by Watson¹¹ as the analog of the Riemann-Lebesgue lemma, it can be demonstrated that

$$\int_0^{\infty} \psi_{\ell}(r) J_{\ell}(kr) \ell dr = O\left(\frac{1}{\sqrt{k}}\right) \text{ as } k \rightarrow \infty \quad (38)$$

provided $\int_0^{\infty} \psi_{\ell}(k) \sqrt{r} dr$ exists and is absolutely convergent. Thus, we require that the solution to the differential equation (32) gives $H_{\ell}(k)$ from (17) so that $H_{\ell}(k)$ is of order $k^{-\frac{1}{2}}$ as $k \rightarrow \infty$.

Similarly, we establish the k-space scaling condition for small k by additionally requiring that (1) $\psi_k(r)$ be bounded, and (2) $\psi_k(r)$ join the vacuum solution for large r ($r \gg R$). The result so obtained requires $H_k(k) < O(k^\ell)$ for $k \ll \ell/R_0$. (Where $R_0 = \alpha R_{1,e}$; $\alpha \gg 1$.) Note, additionally, that the requirement for $kR > 1$ (the requirement that established the validity of the differential equation) does not allow us to use the small k scaling condition as a boundary condition for the $\ell = 1$ differential equation. In those cases where the solution to the $\ell=1$ differential equation is small to the right of $k=\ell$ ($kR=1$) we can join the solution of the differential equation to the (vanishing) solution of the integral equation as specified by the small-k scaling condition.

In the next section, we analyze the equation (32) in conjunction with our k-space conditions above to determine the necessary critical conditions for the onset of the flute-like drift-cone instability. In particular we discuss the behavior of $Q(x)$ over the entire range of x before proceeding with our WKB analysis of (30).

III ANALYSIS OF THE DRIFT-CONE DIFFERENTIAL EQUATION

In this section, we analyze $Q(x)$ of (33) to determine the conditions under which one can expect the flute-like drift-cone modes to propagate. Let us first consider the case of small x . The dominant behavior of $Q(x)$ in (33) is given by

$$Q(x) \sim - \frac{\lambda_{D1}^2 (x^2 + x_2^2) - \frac{1}{R^2}}{\lambda_{D1}^2 g(x, x)}; \quad \frac{1}{2} x^2 \ll 1 \quad (39)$$

and

$$\lim_{x \rightarrow 0} g(x, x) \sim x^{2\ell+1}$$

$$\frac{1}{2} x^2 \ll 1.$$

In order to satisfy the small-k scaling condition for the $g(x, x)$ scaling for small x , it can easily be demonstrated that $Q(x)$ must be negative for small x . That $Q(x)$ must be negative for small x gives us the following necessary condition:

$$\alpha_0 - \beta_0 \frac{\omega_*}{\omega} - \frac{\omega(1+\epsilon_1) + 2\omega_*}{\omega + \Omega_1} \ll 0 \quad (40)$$

where

$$\alpha_0 = \frac{\epsilon_1}{1-\epsilon_1} (\ell+1 - (1-\epsilon_1)^\ell)$$

$$\beta_0 = (1-\epsilon_1)^{\ell-1} - (1-\epsilon_1)$$

$$\gamma_0 = \ell \epsilon_1 (1-\epsilon_1)^{\ell-2} \quad \omega_* = \omega_* > 0.$$

From (40) it follows that no positive frequencies ω , in the range $0 < \omega < \Omega_1$, can satisfy the relation. For the chosen sign of ω_* , positive frequencies correspond to modes which propagate azimuthally in the direction opposite to that of the ion diamagnetic current. On the other hand, for many interesting values of the parameters in (40) it is found that negative frequencies easily satisfy the relation. The negative frequency modes propagate azimuthally in the direction of the ion diamagnetic current. This result has been demonstrated by Callen⁵ and others in a slab model analysis of the flute-like drift-cone modes.

In addition to the above result, we can also determine, for a frequency, ω , in the range $-\Omega_1 < \omega < 0$, the minimum value of $\ell\sqrt{\epsilon_1}$ for which drift-cone modes of the given frequency will propagate. If we assume $\ell\epsilon_1 < 1$ (but not $\ell\sqrt{\epsilon_1} < 1$), we find

$$\ell\sqrt{\epsilon_1} > \frac{1}{2} |\omega(1+\epsilon_1)|.$$

The above result gives the minimum value of $\ell\sqrt{\epsilon_1} \sim "k_\perp" a_i$ where $"k_\perp"$ is the wave number (perpendicular to B) used in the slab model. In the slab model, results specifying a minimum $"k_\perp" a_i$ have been found for mode propagation.

For larger values of x , when the above relation holds and $-\Omega_1 < \omega < 0$, we can determine additional properties about $Q(x)$.

For

$$\frac{1}{2} x^2 \epsilon_e \ll 1; \quad \frac{1}{2} x^2 > \ell \quad (41)$$

we can have

$$\frac{1}{2} x^2 \epsilon_i < 1 \quad (42)$$

or

$$\frac{1}{2} x^2 \epsilon_i > 1 \quad (43)$$

When (41) holds, we write

$$\lambda_{D_1}^2 \hat{g}_e(x, x) \sim \frac{\pi \sqrt{\pi}}{2} e^{-\frac{1}{2} x^2} I_{\frac{1}{2}}(\frac{1}{2} x^2) \frac{T_1}{T_e} \quad (44)$$

$$x \left(1 - \frac{\omega + \omega_*}{\omega} (1 - \ell \epsilon_e) (1 - \frac{1}{2} x^2 \epsilon_e) \right).$$

From (44) we see that $\lambda_{D_1}^2 \hat{g}_e(x, x)$ is zero when

$$x = x_{oe} \sim \frac{2}{\epsilon_e} \left(1 - \frac{1}{1 + \frac{\omega_*}{\omega}} \right) \quad (45)$$

and

$$\lambda_{D_1}^2 \hat{g}_e(x, x) > 0 \text{ for } 0 < x < x_{oe}. \quad (46)$$

Similarly, for $\lambda_{D_1}^2 \hat{g}_e(x_{o1}, x_{o1}) = 0$, x must be larger than that value allowed by (41) and in fact, we find

$$x_{o1} \sim \frac{\sqrt{\pi}}{\epsilon_1 \sqrt{\epsilon_1 (1 - \epsilon_1)}} \cot \left(\pi \frac{\omega}{\Omega_1} \right) \left[\frac{\omega_*}{\Omega_1} - (1 - 2\epsilon) \frac{\omega}{\Omega_1} \right] \quad (47)$$

where

$$\lambda_{D_1}^2 \hat{g}_1(x, x) > 0 \text{ for } 1 < x < x_{o1}. \quad (48a)$$

From the foregoing, we find $\lambda_{D_1}^2 \hat{g}(x, x) = 0$ for some x_p to the right of x_{o1} , whichever is smaller, which is given approximately by the solution of

$$\begin{aligned} & -\frac{1}{4} \frac{\epsilon_1}{1 - \epsilon_1} x_p + \frac{\sqrt{\pi}}{4 \sqrt{1 - \epsilon_1}} \frac{\omega_*}{\Omega_1} \frac{\cot \pi \frac{\omega}{\Omega_1}}{\sqrt{\epsilon_1 (1 - \epsilon_1)}} \\ & - \frac{1 - 2\epsilon_1}{4(1 - \epsilon_1)} \frac{\omega}{\Omega_1} \frac{\sqrt{\pi} \cot \left(\pi \frac{\omega}{\Omega_1} \right)}{\sqrt{\epsilon_1 (1 - \epsilon_1)}} - \frac{1}{2} \frac{T_1}{T_e} x_p + \\ & + \frac{1}{2} \frac{T_1}{T_e} x_p (1 - \frac{1}{2} x_p^2 \epsilon_e) (1 - \ell \epsilon_e) \left(1 + \frac{\omega_*}{\omega} \right) = 0. \end{aligned} \quad (48b)$$

For $x > x_p$, $Q(x)$ is positive, and, in fact for

$$\frac{1}{2} x^2 \gg \ell \quad (49)$$

and

$$\frac{1}{2} x^2 \epsilon_{1,e} \gg n \quad (n \sim 10) \quad (50)$$

$$\text{then } Q(x) \sim \frac{\lambda_{D_1}^2}{\mu R^2} (x^2 + i \frac{\beta_*}{\mu} x) \quad (51)$$

where

$$\mu = \frac{1}{4} \frac{\epsilon_1}{1 - \epsilon_1} + \frac{1}{2} \frac{T_1}{T_e}$$

and β_* is given by (1).

For x such that $\frac{1}{2} x^2 \sim \ell^2$ but $\frac{1}{2} x^2 \epsilon_1 \sim 0(1)$

$$\frac{1}{2} x^2 \epsilon_e \ll 1,$$

$$Q(x) \sim 1 - \frac{(x^2 + x^2) \frac{\lambda_{D_1}^2}{R^2}}{\lambda_{D_1}^2 \hat{g}(x, x)} \quad (52)$$

where $\lambda_{D_1}^2 \hat{g}(x, x) = \lambda_{D_1}^2 (\hat{g}_1(x, x) + \hat{g}_e(x, x))$ and $\hat{g}_1(x, x)$ is given by (30) and near $x = \ell$:

$$Q(x) \sim 1 - \frac{(x^2 + x^2) \frac{\lambda_{D_1}^2}{R^2}}{x_e^{-\frac{1}{2}} x^2 I_{\frac{1}{2}}(\frac{1}{2} x^2) V(x)} \quad (53)$$

where $V(x)$ is a weak function of x of order unity. Thus, we can qualitatively construct $Q(x)$ over the entire range of x . We illustrate the form of $Q(x)$ in Figure 1. Note that there is a minimum threshold density for $Q(x)$ to be positive near its relative maximum about $x = \ell$.

In the next section, we show that we must be above threshold density in order to have a drift-cone solution, and, in addition, by a WKB analysis of (32) for $Q(x)$ like that shown in Figure 1(3), the large- k scaling conditions impose severe restrictions on the plasma parameters which allow the drift-cone modes to propagate.

Fig. 1. The quantity $Q(x)$ various values of the plasma density at $r=0$. 1-Low Density, 2-Low Density, 3-High Density, ($Q(x)$ is positive between the two zeros about $\frac{x}{l} = 1$).

IV. NECESSARY CONDITIONS FOR THE OCCURRENCE OF DRIFT-CONE MODES

In the previous section we derived one necessary condition, (40), for the occurrence of the flute-like drift-cone modes. Here, we supplement that condition with the additional requirements on the solution to (32) required by the large- k scaling condition of Section II. The requirement (38) on $H_2(k)$ requires that

$$v_{\ell}(x) = \sqrt{x} g(x, x) < O'(1) \text{ as } x \rightarrow \infty. \quad (54)$$

In the previous section we found that the large- k behavior of $Q(x)$ under the conditions stated was given by (51). The WKB solutions to (32) using (51) give

$$v_{\ell}^{WKB\pm} = Q^{-1/4} e^{\pm i \lambda_{D1} (\frac{1}{2} x^2 + i \frac{\beta^*}{2\mu} x)} / \sqrt{\mu} R \quad (55)$$

In order to choose the decaying solution of (55) as required by (54) we need to determine the sign of

β^* . Our k -space selection rule is:

- (i) choose v_{ℓ}^{WKB+} when $\beta^* > 0$
- (ii) choose v_{ℓ}^{WKB-} when $\beta^* < 0$.

In order to construct the Stokes diagram¹² for (32) we need to determine the sign of imaginary part of x_p . From (48b) we find

$$\begin{aligned} Y_p = \text{Im}(x_p) = & \quad (56) \\ = -\gamma & \left\{ \frac{-\sqrt{\pi} \frac{\pi}{\Omega_1} \csc^2(\pi \frac{\omega}{\Omega_1}) [\frac{\omega^*}{\Omega_1} - (1-2\epsilon_1) \frac{\omega}{\Omega_1}]}{4(1-\epsilon_1) x \sqrt{\epsilon_1(1-\epsilon_1)}} \right. \\ & - \frac{1-2\epsilon_1}{\Omega_1} \sqrt{\pi} \cot^2(\pi \frac{\omega}{\Omega_1}) \\ & \left. - \frac{1}{2} \frac{T_1}{T_e} \frac{\omega^*}{\omega^2} e^{-\frac{1}{2} x^2 \epsilon} I_0(\frac{1}{2} x^2 \epsilon_1) \right\} \end{aligned}$$

$$\begin{aligned} & \left\{ -\frac{\sqrt{\pi} \cot(\pi \frac{\omega}{\Omega_1})}{4(1-\epsilon_1) x^2 \sqrt{\epsilon_1(1-\epsilon_1)}} \left(\frac{\omega^*}{\Omega_1} - (1-2\epsilon_1) \frac{\omega}{\Omega_1} \right) + \right. \\ & \left. + \frac{1}{2} \frac{T_1}{T_e} \left(1 + \frac{\omega^*}{\omega} \right) (-x\epsilon_e) \left(1 - \frac{3}{4} x^2 \epsilon_e \right) \right\}^{-1} \end{aligned}$$

where

$$x = x_p$$

$$\omega = \omega_r$$

If $-1 < \frac{\omega}{\Omega_1} < -\frac{1}{2}$, (56) gives the result $Y_p < 0$ for $\gamma > 0$ if $|\omega| > |\omega^*|$. For this case, we illustrate the Stokes diagram for this case in Figure 2. The anti-Stokes line emanating from the singular region about x_p crosses the real x -axis from below if $\beta^* < 0$ and does not if $\beta^* > 0$.

Fig. 2. Stokes Diagram for the Drift-Cone differential equation (equation 32 of the text) for the two cases $\beta^* > 1$, $\beta^* < 1$. β^* is defined by equation (1) of the text.

By tracing the solution throughout the entire x -plane, by extending the method of Furry¹³ (see Heading¹² and Berk, Pearlstein, Cordey⁷ for more details about the extension of Furry's method) we may determine the nature of the solutions consistent with small- k and large- k requirements of Section II. Using the notation of Heading¹², we denote the WKB solutions near a point $x=a$ by

$$(a, x) = Q^{-1/2} \exp i \int_a^x \sqrt{Q(x)} dx \quad (57)$$

$$(x, a) = Q^{-1/2} \exp -i \int_a^x \sqrt{Q(x)} dx \quad (58)$$

and define the auxiliary quantities $[a, b]$ and $[b, a]$ by

$$[a, b] = \exp i \int_a^b \sqrt{Q(x)} dx \quad (59)$$

$$[b, a] = \exp -i \int_a^b \sqrt{Q(x)} dx \quad (60)$$

The Stokes and anti-Stokes lines in Figure 2 near the turning points and singular region are defined by

$$\text{Im} \int_a^x \sqrt{Q(x)} dx = 0 \quad (\text{Stokes lines}) \quad (61)$$

$$\text{Re} \int_a^x \sqrt{Q(x)} dx = 0 \quad (\text{Anti-Stokes lines}).$$

In addition, we affix a subscript "s" for subdominant (or "d") for dominant to the solutions (57), (58) if the solution is exponentially decreasing (increasing) away from the point $x=a$. For our two cases ($\beta^* > 0$, $\beta^* < 0$) we trace the subdominant solution required by the small- k scaling condition from region 1 in Figure 2 along the indicated contours for the two cases and obtain, in the notation above:

- (1) $(z_1, z)_s$
- (2) $(z_1, z)_s$
- (3) $(z_1, z)_d$
- (4) $(z_1, z)_d + (z, z_1)_s \equiv (z_2, z)_d [z_1, z_2] + i (z, z_2)_s [z_2, z_1]$
- (5) $(z_2, z)_s [z_2, z_1] + i (z, z_2)_d [z_2, z_1]$
- (6) $i (z, z_2) [z_2, z_1] + \left\{ [z_1, z_2] + [z_2, z_1] \right\} (z_2, z)_s$
- (7) $i (z, z_2) [z_2, z_1] + \left\{ [z_1, z_2] + [z_2, z_1] \right\} (z_2, z)_d$

where $(z_2, z) = (z_p, z) [z_2, z_p] = [z_2, z_p] v_L^{\text{WKB}+}$

and $(z, z_2) = v_L^{\text{WKB}-}$

where $v_L^{\text{WKB}\pm}$ is defined by (55).

The requirement that the coefficient of $v_L^{\text{WKB}+}$ vanish in order to satisfy our large- k scaling condition for the case $\beta^* < 0$ yields the following k -space local dispersion relation for the flute-like drift-cone modes as

$$\int_{z_1}^{z_2} \sqrt{Q(x_0, \ell, \dots) dz} = (n + \frac{1}{2}) \pi; \quad x_0 = \ell \quad (62)$$

$$n = 0, 1, \dots$$

For the case $\beta^* > 0$, we see that the coefficient of $V_{\ell}^{\omega_{KB}}$ must vanish in accordance with selection rule (i). It is not possible to satisfy the requirement unless $V_{\ell}(x) = 0$ everywhere. Hence, we have, in addition to (40), the additional necessary conditions that $\beta^* < 0$ when $\gamma > 0$. In the next section, we discuss these necessary conditions to predict properties of the drift-cone modes. In addition, we present numerical data similar to that of the Livermore 2X machine⁴ in order to compare our theory with an available data base.

V. RESULTS CONCERNING THE DRIFT-CONE MODES IN CYLINDRICAL GEOMETRY

Before considering the properties of the k-space local dispersion relation (62), we note that a good deal of information about the drift-cone modes can be derived from the necessary conditions (40) and (1). A useful approximation to the condition (1) is given by

$$\left(-\frac{W}{\pi} + \frac{\omega^*}{\Omega_i} \right) W^2 \csc^2 W > 2 \frac{a_i}{a_e} \left(\frac{\omega^*}{\Omega_i} \right) \quad (63)$$

where $W = \pi \frac{\omega_r}{\Omega_i}$.

From (65) note that as T_i/T_e increases, ω_r must approach Ω_i . Also note that increasing the plasma radius, R, for fixed a_i, a_e , the inequality (65) becomes increasingly easier to satisfy for values of ω_r greater than $-\Omega_i$.

Using (40) with (63) for a fixed azimuthal mode number and temperature T_i/T_e allows us to predict the critical value of ϵ_i , below which the drift-cone modes are stable. This corresponds to the result established previously by Rosenbluth and Post,³ and states that the larger the plasma radius, in terms of the ion Larmor radius, the more stable is the system.

In Table I we present these results for a hydrogen plasma with $T_i/T_e = 100$ for various values of the azimuthal mode number ℓ .

Table I

ℓ	$T_i/T_e = 100$ ϵ_i critical	$k_i a_i$	$(R/a_i)_{critical}$
5	.113	1.6	3.16
10	.053	2.2	4.46
15	.035	2.7	5.47
20	.026	3.2	6.33
50	.010	5.0	10.00
99	.005	7.0	14.08
1000	.00050	22.36	44.73
10000	.00005	70.71	141.42

ℓ	$T_i/T_e = 40$ ϵ_i critical	$k_i a_i$	$(R/a_i)_{critical}$
10	.053	2.2	4.47
15	.035	2.7	5.48
20	.026	3.2	6.32
50	.010	5.0	10.00
99	.005	7.2	14.17

$$k_i a_i = 1/\sqrt{\epsilon_i(1-\epsilon_i)}$$

In addition, we include the results for a deuterium plasma for which $T_i/T_e = 40$, which is like that of the Livermore 2X machine.⁴ We list values of $\ell \sqrt{\epsilon_i}$ in the table for comparison with the slab model $k_i a_i$ behavior.⁵ Note that values of $\ell \sqrt{\epsilon_i}$ are of order unity or larger. This result is similar to that noted by Collen⁵ and others in their analysis of the flute-like drift-cone modes of a slab model plasma.

Associated with each critical value of ϵ_i , there is a minimum frequency in the range $-1 < \frac{\omega}{\Omega_i} < 0$ above which the drift-cone modes will not propagate. For the parameters of Table I we find that the critical frequency for the hydrogen plasma to be $\omega/\Omega_i \text{ crit} = -.94$ and for the deuterium plasma $\omega/\Omega_i \text{ crit} = -.95$ over the entire range of ℓ . It turns out that the critical frequency for a given machine is relatively insensitive to the azimuthal mode number, but is more sensitive to T_i/T_e , as illustrated in Figure 3.

Fig. 3 Critical value of ω/Ω_i vs electron-ion temperature ratio. The critical value of ω in the range $-1 < \frac{\omega}{\Omega_i} < 0$ is defined as that value of ω above which the drift-cone modes will not propagate.

By an analysis of the k-space local dispersion relation, we can also determine the threshold frequency for the drift cone modes. This local dispersion relation is given by

$$k^2 \lambda_{D1}^2 = -\frac{1}{4} \left(\frac{\epsilon_1}{1-\epsilon_1} \right) + \frac{1}{2} \left(\frac{\epsilon_1}{1-\epsilon_1} \right) \left(1 - \frac{1}{2\epsilon_1} \right) \left(\frac{\omega}{\Omega_i} + \frac{\omega^*}{\Omega_i} \right) + \frac{\sqrt{\pi} \cot \left(\frac{\pi \omega}{\Omega_i} \right)}{ka_1} + \frac{1}{2} \frac{\omega^*}{\Omega_i} \frac{\sqrt{\pi}}{ka_1} \cot \left(\frac{\pi \omega}{\Omega_i} \right) + \frac{1}{2} \frac{T_i}{T_e} \left(\frac{\omega^* e}{\omega} \right) e^{-\frac{1}{2} \epsilon_1^2} e^{-\frac{1}{2} \epsilon_1^2} I_0 \left(\frac{1}{2} \epsilon_1^2 \right). \quad (64)$$

In writing (64) we have used the large argument limit of the Bessel functions terms involving the wave number and the ion Larmor radii. An analysis of (64) for $\epsilon_1 > \epsilon_{1,crit}$ and $-1 < \frac{\omega}{\Omega_i} < 0$ shows that ω/Ω_i approaches -1 more closely as ϵ_1 is increased above its critical value. The net result is that increasing ϵ_1 above its critical value decreases the threshold density. From (64), it also follows that $k^2 \lambda_{D1}^2$ increases with decreasing frequency ($\omega/\Omega_i \rightarrow -1$). Thus, for a fixed ϵ_1 , we find that the critical frequency is also the threshold frequency. Larger values of ϵ_1 above $\epsilon_{1,crit}$

decreases the threshold density and makes a given machine more unstable as the density is increased.

Fig. 4. Threshold properties and stability boundaries for the drift-cone modes in a D^+ plasma for which $T_i/T_e = 40$. The vertical lines are due to the existence of a critical value of ϵ_1 , below which a given mode is stable.

In Figure 4 we have plotted the value of ϵ_1 vs the threshold density as measured by $k^2 \lambda_{D1}^2$ for a D^+ plasma with $T_i/T_e = 40$, for $k \sim \ell/R$ as noted in the discussion following equation (64). [For the 2X machine, $R/a_1 \sim 4$ ($\epsilon_1 \sim 6.25 \times 10^{-3}$) and our results indicate that we would not expect to see any drift cone modes for $\ell < 5$.] If we estimate the effect of the electromagnetic corrections (the 2X machine has $\beta \sim .05$ so our electrostatic theory is not strictly applicable to the machine), from Tang's⁶ work, we predict that the corrections should not affect our results for the critical value of R/a_1 by more than a factor of 2. For parameters like those of 2X, $\omega_{p1}^2/\Omega_i^2 \sim 200$ or $k^2 \lambda_{D1}^2 = .03$ for an $\ell = 10$ mode and 8×10^{-3} for an $\ell = 5$. The $\ell = 5$ mode gives $\epsilon_{1,crit} \sim .1$ ($R/a_1 \sim 3$) and we predict stability for ℓ modes less than 5. The drift-cone modes that appear in the machine during the first millisecond of operation for which the density is typically a few times $10^{12}/cc$ should have ℓ values greater than 5. These results are in disagreement with those of Post-Rosenbluth³ whose estimate of the critical ϵ_1

gives $R/a_1 > 115$ and Tang's⁶ electromagnetic results which require $R/a_1 > 70$. Our results indicate that $R/a_1 > 15$ should stabilize the instability for $k < 99$. We attribute our smaller estimate of R/a_1 to the effects of cylindrical geometry and nonlocal nature of the radial eigenmodes.

VI. SUMMARY AND CONCLUSIONS

Historically, the determination of the potentially unstable drift-cone microinstabilities¹⁻⁵ has been simulated and best understood from the relatively simple plane-geometry or "slab model" of a magnetized plasma. In this report we have formulated a cylindrical plasma model and exploited that model to determine the nature of the drift-cone modes. We did not resort to the "local" approximation in the radial direction and found that the stability boundaries for the drift-cone modes differ appreciably from previous estimates. Our model treated the finite Larmor radius effects to all orders while examining the threshold properties of potentially dangerous low-frequency ($|\omega| \leq \Omega_i$) flute-like drift-cone modes. In our analysis we demonstrated that the eigenmodes have a sharp spectrum in the Hankel transform space about $k \sim (R/r)$ and are therefore highly nonlocal in real space. Taking advantage of the local k -space solution, we derived the dispersion relation for the drift-cone modes. Our results demonstrate the importance of the nonlocal nature of the modes when we contrast our results with previous theoretical results for the 2X machine. We have shown that the drift-cone modes, if any, that appear in the 2X machine will have azimuthal mode numbers greater than 5 for typical values of the plasma parameters. We feel that a major contribution to our present theory could be made by extending the theory to include a steady-state radial electric field,¹⁴ electromagnetic effects, and finite-length effects. We have shown that current theoretical work,^{6,7} which typically employs the local approximation for the radial eigenmode while considering the finite-length and electromagnetic effects should be carefully examined in the light of our nonlocal radial results to determine the effects nonlocal radial eigenmodes might have on the theoretical results.

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