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THE EFFECT OF FLUCTUATIONS IN THE WIDTHS ON
NEUTRON REACTION CROSS SECTIONS

Abstract

The effect of fluctuations in the widths on the reaction cross sections when the widths are distributed in χ^2 -distributions is expressed in terms of a single infinite integral. The integral is evaluated explicitly in terms of tabulated functions in some simple cases, and the results discussed.

Introduction

The importance of fluctuations in the widths to the calculation of average reaction cross sections was first pointed out by Wigner in a calculation of the ratio of capture to fission in U^{235} .¹ Since that time two other cross section calculations have been done including careful account of fluctuation phenomena. One is a refinement of Wigner's work done by Oleksa;² the other is a calculation of the average radiative cross section of the fission products done by Greebler, Hurwitz, and Storm.³

In the earlier papers (Ref. 1 and 2) the statistical distributions of the widths had to be guessed. Oleksa, in particular, tried several distributions. Lately, however, evidence has been found which supports a particular choice of distribution, the so-called Porter-Thomas distribution.⁴ Briefly, Porter and Thomas argued that the reduced width, $\gamma_{\lambda S}$,⁵ should be distributed normally. The neutron width, $\Gamma_{\lambda S}$, which is proportional to the square of the reduced width should then be distributed in a χ^2 -distribution with one degree of freedom,⁶ which in this context is known by their names. Porter and Thomas found this distribution in excellent agreement with the experimental data. We shall assume it in this paper for neutron widths.

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Further, Porter and Thomas concluded that if the radiative width were distributed in a χ^2 -distribution the number of degrees of freedom would be very large (~ 100). For simplicity in the ensuing calculations we shall take the radiative width constant, which corresponds to infinitely many degrees of freedom.

In this paper we shall confine ourselves to the study of the effect of fluctuations on neutron reaction cross sections in non-fissionable nuclei.

Expressions for the Cross Section

From the Breit-Wigner one-level formula it follows that the average cross section for reaction from channel S to S' through states of a given total angular momentum, J, and parity, π , is⁷

$$\sigma_{J\pi}(S, S') = \frac{2J+1}{2j+1} \frac{2\pi^2}{k_S^2 D_{J\pi}} \left\langle \frac{\Gamma_{\lambda S} \Gamma_{\lambda S'}}{\Gamma_{\lambda}} \right\rangle \quad (1)$$

where j is the channel spin in channel S, k_S is the incident neutron wave number in channel S, and $D_{J\pi}$ is the average spacing of levels with the given J and π . The bracket denotes an average with respect to the levels λ . The total width, Γ_{λ} , is the sum of the partial widths, $\Gamma_{\lambda S}$, over channels, S. This cross section is not what is experimentally observed; it must be summed over all J, π , all ingoing and outgoing orbital angular momenta, l, l' , and the outgoing channel spins, j' , and averaged over the incoming channel spins, j. If we let σ denote all that is specified by S except l and j then

$$\sigma(\sigma, \sigma') = \frac{2\pi^2/k_{\sigma}^2}{2(2I+1)} \sum_{J\pi l l' j j'} \frac{2J+1}{D_{J\pi}} \left\langle \frac{\Gamma_{\lambda \sigma l j} \Gamma_{\lambda \sigma' l' j'}}{\sum_{\sigma l j} \Gamma_{\lambda \sigma l j} + \Gamma_{\lambda \gamma}} \right\rangle \quad (2)$$

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where I is the spin of the target nucleus.

In schematic theories of neutron reaction cross sections $\Gamma_{\lambda\sigma\ell j}$ is often taken to be independent of j .⁸ In such a case the sums over j and j' can be carried out, yielding

$$\sigma(\sigma, \sigma') = \frac{2\pi^2/k^2}{2(2I+1)} \sum_{J\pi\ell\ell'} \frac{2J+1}{D_{J\pi}} \left\langle \frac{\epsilon(\lambda\sigma\ell) \Gamma_{\lambda\sigma\ell} \cdot \epsilon(\lambda\sigma'\ell') \Gamma_{\lambda\sigma'\ell'}}{\sum_{\sigma\ell} \epsilon(\lambda\sigma\ell) \Gamma_{\lambda\sigma\ell} + \Gamma_{\lambda\gamma}} \right\rangle \quad (3)$$

where $\epsilon(\lambda\sigma\ell)$ is essentially the symbol introduced by Hauser and Feshbach and is equal to the number of channel spins which can form state λ of angular momentum J and parity π in channel σ with orbital angular momentum ℓ . For neutrons $\epsilon(\lambda\sigma\ell)$ can be 0, 1, or 2 only. The quantity $\epsilon(\lambda\sigma\ell) \Gamma_{\lambda\sigma\ell}$ is to be interpreted, for the purpose of averaging, as a random variable which is the sum of $\epsilon(\lambda\sigma\ell)$ random variables, $\Gamma_{\lambda\sigma\ell}$, all of which have the same distribution. If the latter is a χ^2 -distribution with one degree of freedom then $\epsilon(\lambda\sigma\ell) \Gamma_{\lambda\sigma\ell}$ is distributed in a χ^2 -distribution with $\epsilon(\lambda\sigma\ell)$ degrees of freedom and mean $\epsilon(\lambda\sigma\ell) \langle \Gamma_{\lambda\sigma\ell} \rangle$.

We shall therefore be interested in the averages of quantities like $\Gamma_s \Gamma_t / (\Gamma_s + \Gamma_t + \Gamma_r + \dots)$ where $\Gamma_s, \Gamma_t, \Gamma_r, \dots$ are independently distributed in χ^2 -distributions with 1, 2, or ∞ degrees of freedom. In this paper we confine ourselves to evaluating explicitly in terms of tabulated functions averages in which there are no more than three widths. The following table summarizes the cases done:

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Γ_s	Γ_t	Γ_r
1	1	absent, 1, 2 ∞
1	2	absent, 1, 2 ∞
2	2	absent, 1, 2 ∞
1	∞	absent, 1, 2
2	∞	absent, 1, 2

The entries in the table represent the number of degrees of freedom in the distribution of the corresponding width. We can find the averages of expressions like $\Gamma_s^2 / (\Gamma_s + \Gamma_t + \Gamma_r)$ from the identity:

$$\langle \Gamma_s \rangle = \left\langle \frac{\Gamma_s \Gamma_t}{\Gamma_s + \Gamma_t + \Gamma_r} \right\rangle + \left\langle \frac{\Gamma_s \Gamma_r}{\Gamma_s + \Gamma_t + \Gamma_r} \right\rangle + \left\langle \frac{\Gamma_s^2}{\Gamma_s + \Gamma_t + \Gamma_r} \right\rangle \quad (4)$$

Reduction of the Multiple Integrals

In general, the averages will be represented by multiple integrals, one integration for each random variable (width) present. For widths distributed in χ^2 -distributions an artifice exists for reducing these multiple integrals to a single integral. The result is general, but we prove it only in detail for the case of three widths.

A normalized χ^2 -distribution with 2ρ degrees of freedom is

$$P(x, \rho) = \frac{\rho}{\Gamma(\rho)} (\rho x)^{\rho-1} e^{-\rho x} \quad (5)$$

where $\Gamma(\rho)$ is a gamma function. If we let $a_s = \langle \Gamma_s \rangle$ then

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$$\left\langle \frac{\Gamma_S \Gamma_t}{\Gamma_S + \Gamma_t + \Gamma_r} \right\rangle = \int_0^\infty dx_S \int_0^\infty dx_t \int_0^\infty dx_r \times P(x_S, \rho_S) P(x_t, \rho_t) P(x_r, \rho_r) \frac{a_S^{x_S} \cdot a_t^{x_t}}{a_S^{x_S} + a_t^{x_t} + a_r^{x_r}} \quad (6)$$

If we now set

$$\frac{1}{a_S^{x_S} + a_t^{x_t} + a_r^{x_r}} = \int_0^\infty d\lambda e^{-\lambda(a_S^{x_S} + a_t^{x_t} + a_r^{x_r})} \quad (7)$$

in the right hand side of Eq. (6) and interchange the order of integration so that the λ -integration is last, the x -integrations are independent, and easily carried out. The result, after some simple rearrangement, is

$$\left\langle \frac{\Gamma_S \Gamma_t}{\Gamma_S + \Gamma_t + \Gamma_r} \right\rangle = a_S^{a_t} \int_0^\infty \left(1 + \frac{\lambda a_S}{\rho_S}\right)^{-(\rho_S+1)} \times \left(1 + \frac{\lambda a_t}{\rho_t}\right)^{-(\rho_t+1)} \left(1 + \frac{\lambda a_r}{\rho_r}\right)^{-\rho_r} d\lambda \quad (8)$$

When a given ρ_k approaches ∞ , the corresponding bracket in Eq. (8) approaches $e^{-\lambda a_k}$. We define

$$R = \left\langle \frac{\Gamma_S \Gamma_t}{\Gamma_S + \Gamma_t + \Gamma_r} \right\rangle \frac{a_S + a_t + a_r}{a_S^{a_t}}$$

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Results

We first consider the two width cases, i.e., those in which a_r is absent and $\rho_S = 1/2$ or 1 and $\rho_t = 1/2, 1$, or ∞ . The resulting integrals are all tabulated.⁹

$$\rho_S = \rho_t = 1/2 \quad R = \frac{a_S + a_t}{(\sqrt{a_S} + \sqrt{a_t})^2} \quad 10 \quad (9a)$$

$$\rho_S = 1/2, \rho_t = 1 \quad R = \frac{3 + r^2}{2} \left(\frac{3 + 2r^2}{r^4} - 3 \frac{1 + r^2}{r^5} \tan^{-1} r \right) \text{ where } r^2 = \frac{2a_S}{a_t} - 1 \quad (9b)$$

$$\rho_S = \rho_t = 1 \quad R = \left(\frac{a_S + a_t}{a_S - a_t} \right)^2 \left[1 - \frac{2a_S a_t}{a_S^2 - a_t^2} \log \left(\frac{a_S}{a_t} \right) \right] \quad (9c)$$

$$\rho_S = 1/2, \rho_t = \infty \quad R = \frac{a_S + a_t}{a_S} \left[1 - \sqrt{\frac{\pi a_t}{2a_S}} e^{\frac{a_t}{2a_S}} \operatorname{Erfc} \left(\sqrt{\frac{a_t}{2a_S}} \right) \right] \quad 11 \quad (9d)$$

$$\rho_S = 1, \rho_t = \infty \quad R = \frac{a_S + a_t}{a_S} e^{\frac{a_t}{a_S}} E_2 \left(\frac{a_t}{a_S} \right) \quad 11 \quad (9e)$$

The formulae for the remaining (three width) cases are quite complicated and are noted for reference in Appendix I, along with some brief description of their manner of derivation.

Discussion

In Figs. 1 and 2 the ratio R for the two width cases is plotted for ratios of the average widths between .03 and 30. All of the curves have minima in the neighborhood of $a_S/a_t = 1$; the symmetric curves (those for

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Fig. 1. R as a Function of a_s/a_t .

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Fig. 2. R as a Function of a_t/a_s .

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which $\rho_S = \rho_t$) have minima exactly as $a_S/a_t = 1$. For the cases plotted in Figs. 1 and 2 R is always less than unity.

R cannot be less than one for all reactions, however, since the sum of all reaction cross sections equals the cross section for compound nucleus formation, for which $R = 1$ irrespective of the distribution of widths. Thus

if we define $R_{km} = \frac{\sum a_i}{a_k a_m} \left\langle \frac{\Gamma_k \Gamma_m}{\sum \Gamma_i} \right\rangle$, it can easily be shown that

$$\sum_k a_k (R_{km} - 1) = 0 \quad (10)$$

Hence, if some of the R_{km} are less than unity, others must be greater. For two width reactions we have seen in the special cases given by Eq. (9) that $R_{St} \leq 1$ (and consequently $R_{SS}, R_{tt} \geq 1$). For two widths this is a general conclusion, and we can establish it as follows:

Let us consider the inequality $\langle (\Gamma_S + \lambda \Gamma_t)^2 / (\Gamma_S + \Gamma_t) \rangle \geq 0$. By expanding the square it can be shown easily that

$$\lambda^2 \frac{a_t}{a_S} R_{tt} + 2\lambda R_{St} + \frac{a_S}{a_t} R_{SS} \geq 0 \quad (11)$$

Since Eq. (11) must be true for all real λ the discriminant of the quadratic form must not be positive. Hence

$$R_{St}^2 \leq R_{SS} R_{tt} \quad (12)$$

If we define $R_{St} = 1 + D_{St}$ we can rewrite Eq. (12) as

$$D_{St}^2 + 2D_{St} \leq D_{SS} D_{tt} + D_{SS} + D_{tt} \quad (13)$$

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It follows from Eq. (10) that

$$-D_{St} = \frac{a_S}{a_t} D_{SS} = \frac{a_t}{a_S} D_{tt} \quad (14)$$

Combining Eqs. (13) and (14) yields

$$D_{St} \leq \frac{1}{2} (D_{SS} + D_{tt}) \quad (15)$$

Since D_{SS} and D_{tt} must have the same sign, and opposite sign to D_{St} , it follows that D_{St} is never positive, and if it vanishes so do D_{SS} and D_{tt} .

It is a general theorem that D_{SS} is never negative, even when there are more than two widths. For the purpose of deriving this inequality we can replace the sum of all the widths which appear in the total width except those for the entrance channels by a single random variable, Γ_r . Then from the results of the two width case

$$\left\langle \frac{\Gamma_S(\Gamma_t + \Gamma_r)}{\Gamma_S + (\Gamma_t + \Gamma_r)} \right\rangle \leq \frac{a_S + (a_t + a_r)}{a_S + (a_t + a_r)} \quad (16)$$

From Eq. (16) we can prove that

$$a_t D_{St} + a_r D_{Sr} \leq 0 \quad (17)$$

Since, however, Eq. (10) in this case is written

$$a_S D_{SS} + a_t D_{St} + a_r D_{Sr} = 0 \quad (18)$$

we must have $D_{SS} \geq 0$ always.

One may inquire, then, whether D_{St} is always ≤ 0 ? The answer is no as we can see as follows: If $\Gamma_r \geq \Gamma_S + \Gamma_t$ with very small probability D_{St}

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will be very close to its value for $\Gamma_r = 0$ which is less than one. On the other hand, if $\Gamma_r \leq \Gamma_s + \Gamma_t$ with very small probability D_{st} will be close to the value $\langle \Gamma_r^{-1} \rangle a_r$, which is greater than one.* These considerations indicate that there may be considerable enhancement of a cross section due to fluctuations. The appropriate circumstances may arise, for example, in the radiative capture cross section when there is strong inelastic competition by several excited states. The inelastic width may then greatly exceed the elastic (entrance) and radiative (exit) widths.

*We prove this with Schwarz' inequality. If $f(x)$ is normalized probability distribution,

$$1 = \left(\int \sqrt{\frac{f(x)}{x}} \sqrt{xf(x)} dx \right)^2 \leq \left(\int \frac{f(x)}{x} dx \right) \left(\int f(x) x dx \right) = \langle x^{-1} \rangle \langle x \rangle$$

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Appendix I

Let us consider first the cases in which none of the widths have $\rho = \infty$. The resulting integrals are then all tabulated and we merely note the results.

Case 1

$$\rho_S = \rho_t = \rho_r = 1/2$$

$$\text{Let } t = \text{larger} \left(\sqrt{\frac{a_S}{a_t}}, \sqrt{\frac{a_t}{a_S}} \right) \text{ and } q = \frac{a_r}{\sqrt{a_S a_t}}$$

$$R = \frac{t + \frac{1}{t} + q}{2\sqrt{q}} I(t, q)$$

Case (i): $t > \frac{1}{t} > q$

$$I = 2 \left(\frac{qt}{t - q} \right)^{5/2} \frac{1}{k^2 k'^4} \left[k'^2 F(\phi, k) - (1 + k^2) E(\phi, k) + k^2 \sin \phi \left(\frac{\cos \phi}{\Delta(\phi, k)} + \frac{\Delta(\phi, k)}{\cos \phi} \right) \right]$$

$$\text{where } k = \sqrt{\frac{1 - qt}{1 - \frac{q}{t}}} \quad \text{and } \phi = \arccos \left(\frac{q}{t} \right)^{1/2}$$

Case (ii): $t > q > \frac{1}{t}$

$$I = 2 \left(\frac{t}{t^2 - 1} \right)^{5/2} \frac{1}{k^2 k'^2} \left[(1 - 2k^2) E(\phi, k) - k'^2 F(\phi, k) + k^2 \Delta(\phi, k) \tan \phi \right]$$

$$\text{where } k = \sqrt{\frac{t^2 - \frac{1}{t}}{t^2 - 1}} \quad \text{and } \phi = \arccos \left(\frac{1}{t} \right)$$

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Case (iii): $q > t > \frac{1}{t}$

$$I = 2 \left(\frac{q}{qt - 1} \right)^{5/2} \frac{1}{k'^2 k^4} \left[(1 + k'^2) E(\phi, k) - 2k'^2 F(\phi, k) - \frac{k^2 \sin \phi \cos \phi}{\Delta(\phi, k)} \right]$$

$$\text{where } k = \sqrt{\frac{t^2 - 1}{t^2 - \frac{1}{q}}} \quad \text{and } \phi = \arccos \left(\frac{1}{qt} \right)^{1/2}$$

Case 2

$$\rho_S = \rho_t = \frac{1}{2}, \rho_r = 1$$

$$\text{Let } t = \sqrt{\frac{a_S}{a_t}}, \quad q = \frac{a_r}{2\sqrt{a_S a_t}}$$

$$R = \left[\frac{q + \frac{1}{2} \left(t + \frac{1}{t} \right)}{(q - t) \left(q - \frac{1}{t} \right)} \right] \left[\frac{q^2}{\sqrt{(q - t) \left(q - \frac{1}{t} \right)}} \log \left(\frac{\frac{q}{2} \left(t + \frac{1}{t} \right) - 1 - \sqrt{(q - t) \left(q - \frac{1}{t} \right)}}{q \left[q - \frac{1}{2} \left(t + \frac{1}{t} \right) - \sqrt{(q - t) \left(q - \frac{1}{t} \right)} \right]} \right) \right. \\ \left. - \frac{q \left(t + \frac{1}{t} + 1 \right) - 1}{1 + \frac{1}{2} \left(t + \frac{1}{t} \right)} \right]$$

Case 3

$$\rho_S = \rho_r = 1/2, \rho_t = 1 \quad \text{Let } t = \sqrt{\frac{a_S}{a_r}}, \quad q = \frac{a_t}{2\sqrt{a_S a_r}}$$

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$$R = \left[\frac{q + \frac{1}{2} \left(t + \frac{1}{t} \right)}{(q - t)^2} \right] \left[\frac{t^2(q - 1)}{\frac{q}{2} \left(t + \frac{1}{t} \right) - 1} + \frac{q(q - 1)}{q - \frac{1}{t}} \right. \\ \left. + \frac{q}{\sqrt{(q - t)(q - \frac{1}{t})}} \left\{ t + \frac{\frac{q}{2} \left(t + \frac{1}{t} \right) - 1}{q - \frac{1}{t}} \right\} \right. \\ \left. \times \log \left(\frac{q \left[q - \frac{1}{2} \left(t + \frac{1}{t} \right) - \sqrt{(q - t)(q - \frac{1}{t})} \right]}{\frac{q}{2} \left(t + \frac{1}{t} \right) - 1 - \sqrt{(q - t)(q - \frac{1}{t})}} \right) \right]$$

Case 4

$$\rho_S = 1/2, \rho_t = \rho_r = 1$$

$$\text{Let } t = \sqrt{\frac{a_t}{a_r}}, \quad q = \frac{2a_S}{\sqrt{a_t a_r}}$$

$$R = \left[\frac{t + \frac{1}{t} + \frac{1}{2} q}{t q^{3/2}} \right] \left[\frac{2 \sqrt{q}}{(t - \frac{1}{q})(\frac{1}{t} - \frac{1}{q})^2} + \frac{\log \left(\frac{1 + \sqrt{1 - qt}}{1 - \sqrt{1 - qt}} \right)}{(\frac{1}{q} - t)^{3/2} (t - \frac{1}{t})^2} \right. \\ \left. + \frac{\frac{t}{\sqrt{q}} - \frac{1}{2} \left(\frac{1}{q} - \frac{1}{t} \right)^{-1/2} \log \left(\frac{1 + \sqrt{1 - q/t}}{1 - \sqrt{1 - q/t}} \right)}{(\frac{1}{q} - \frac{1}{t})^2 (t - \frac{1}{t})} \right. \\ \left. + \frac{(\frac{2}{t} - t - \frac{1}{q}) \log \left(\frac{1 + \sqrt{1 - q/t}}{1 - \sqrt{1 - q/t}} \right)}{(\frac{1}{q} - \frac{1}{t})^{5/2} (t - \frac{1}{t})^2} \right]$$

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Case 5

$$\rho_s = \rho_t = 1, \quad \rho_r = 1/2$$

$$\text{Let } t = \sqrt{\frac{a_s}{a_t}}, \quad q = \frac{2a_r}{\sqrt{a_s a_t}}$$

$$R = \left[\frac{t + \frac{1}{t} + \frac{1}{2}q}{\sqrt{q} \left(t - \frac{1}{t}\right)^2} \right] \left[\frac{\frac{t}{\sqrt{q}} - \frac{1}{2} \left(\frac{1}{q} - \frac{1}{t}\right)^{-1/2} \log \left(\frac{1 + \sqrt{1 - q/t}}{1 - \sqrt{1 - q/t}} \right)}{\left(\frac{1}{q} - \frac{1}{t}\right)} \right. \\ + \frac{\frac{1}{t\sqrt{q}} - \frac{1}{2} \left(\frac{1}{q} - t\right)^{-1/2} \log \left(\frac{1 + \sqrt{1 - qt}}{1 - \sqrt{1 - qt}} \right)}{\left(\frac{1}{q} - t\right)} - 2 \frac{\log \left(\frac{1 + \sqrt{1 - q/t}}{1 - \sqrt{1 - q/t}} \right)}{\left(t - \frac{1}{t}\right) \sqrt{\frac{1}{q} - \frac{1}{t}}} \\ \left. + 2 \frac{\log \left(\frac{1 + \sqrt{1 - qt}}{1 - \sqrt{1 - qt}} \right)}{\left(t - \frac{1}{t}\right) \sqrt{\frac{1}{q} - t}} \right]$$

Case 6

$$\rho_s = \rho_t = \rho_r = 1$$

$$\text{Let } t = \sqrt{\frac{a_s}{a_t}}, \quad q = \frac{a_r}{\sqrt{a_s a_t}}$$

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$$R = \left[\frac{t + \frac{1}{t} + q}{q(t - \frac{1}{t})^2} \right] \left[\frac{\log \left(\frac{q}{t} \right) - 1 + \frac{t}{q}}{\left(\frac{1}{t} - \frac{1}{q} \right)^2} + \frac{\log (qt) - 1 + \frac{1}{qt}}{\left(t - \frac{1}{q} \right)^2} \right. \\ \left. - 2 \frac{\log \left(\frac{q}{t} \right)}{\left(t - \frac{1}{t} \right) \left(\frac{1}{t} - \frac{1}{q} \right)} + 2 \frac{\log (qt)}{\left(t - \frac{1}{t} \right) \left(t - \frac{1}{q} \right)} \right]$$

Let us next turn to cases in which one $\rho = \infty$. There are seven such cases in all; in only two can the ratio R be expressed in terms of tabulated functions. In the remaining cases we must be content with an asymptotic expansion which will, in general, be a good representation when the radiative width ($\rho = \infty$) is small compared to the geometric mean of the two neutron widths. This latter requirement is very frequently fulfilled in practice.

Case 7

$$\rho_S = \rho_t = 1/2, \rho_r = \infty$$

$$\text{Let } p = \frac{1}{2} \left(\sqrt{\frac{a_S}{a_t}} + \sqrt{\frac{a_t}{a_S}} \right), \quad q = \frac{a_r}{\sqrt{a_S a_t}}$$

$$R = \left[p + \frac{q}{2} \right] \left[\frac{1}{p+1} - \frac{\frac{q}{2}}{p+1} + \left(\frac{q}{2} \right)^2 e^{pq/2} \right. \\ \left. \times \sum_{n=1}^{\infty} \frac{(|2n-3|)!!}{(2n)!!} E_{2n-1} \left(\frac{1}{2} pq \right) \left(\frac{p^2-1}{p^2} \right)^{n-1} \right]$$

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where $x!! = x(x-2)(x-4) \dots 2$ or 1 for positive integers and unity otherwise, and E_n are the exponential integrals mentioned in Ref. 9. The manner of derivation of this series is as follows:

From Eq. (8) it follows that

$$R = \left(p + \frac{q}{2}\right) \int_0^{\infty} d\mu (\mu^2 + 2p\mu + 1)^{-3/2} e^{-1/2 q\mu}$$

If we integrate twice by parts and then add and subtract $\mu + p$ to the integrand of the resulting integral we obtain

$$\begin{aligned} R\left(p + \frac{q}{2}\right)^{-1} &= \frac{p + \frac{1}{2}q}{p^2 - 1} - \frac{\left(\frac{1}{2}q\right)^2}{p^2 - 1} \int_0^{\infty} (\mu + p) e^{-1/2 q\mu} d\mu \\ &\quad + \frac{\left(\frac{1}{2}q\right)^2}{p^2 - 1} \int_0^{\infty} (\mu + p) \left(1 - \sqrt{\frac{\mu^2 + 2p\mu + 1}{(\mu + p)^2}}\right) e^{-\frac{1}{2}q\mu} d\mu \\ &= \frac{1}{p+1} \left(1 - \frac{q}{2}\right) + \frac{\left(\frac{1}{2}q\right)^2}{p^2 - 1} \int_0^{\infty} (\mu + p) \left(1 - \sqrt{1 - \frac{p^2 - 1}{(p + \mu)^2}}\right) \\ &\quad \times e^{-\frac{1}{2}q\mu} d\mu \end{aligned}$$

If we expand the square root in the last line and integrate term-by-term we obtain the desired series.

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Case 8

$$\rho_s = 1/2, \rho_t = 1, \rho_r = \infty$$

$$\text{Let } t = \sqrt{\frac{2a_s}{a_t}}, \quad q = \frac{a_r}{\sqrt{2a_s a_t}}, \quad r = \sqrt{t^2 - 1}$$

$$R = t^2(t^2 + 2 + 2qt) \left[\sum_{n=0}^{\infty} (-)^n \left(\frac{q}{t}\right)^n \sum_{k=\max(0, n-2)}^{\infty} (-)^k \frac{(2k+3-2n)!!}{(2k+5)!!} (k+1)r^{2k} \right. \\ \left. - I_0\left(\frac{q}{t}\right) \sum_{k=0}^{\infty} \frac{(k+1)}{(2k+5)!!} r^{2k} \left(\frac{q}{t}\right)^{k+3} \right]$$

$$\text{where } I_0(\alpha) = e^{\alpha} \int_1^{\infty} e^{-\alpha y^2} dy = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} e^{\alpha} \text{Erfc}(\sqrt{\alpha})$$

The manner of deriving this series is as follows. From Eq. (8) it follows that

$$R = t^2(t^2 + 2 + 2qt) L(r, \frac{q}{t}) \quad \text{where}$$

$$L(r, \alpha) = e^{\alpha} \int_1^{\infty} \frac{dx}{x^2(x^2 + r^2)^2} e^{-\alpha x^2}$$

$$\text{Now } x^{-2}(x^2 + r^2)^{-2} = \sum_{k=0}^{\infty} (-)^k (k+1) \frac{r^{2k}}{x^{2(k+3)}}$$

$$\text{Therefore } L(r, \alpha) = \sum_{k=0}^{\infty} (-)^k (k+1) r^{2k} I_{k+3}(\alpha)$$

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$$\text{where } I_k(\alpha) = e^\alpha \int_1^\infty \frac{e^{-\alpha y^2}}{y^{2k}} dy.$$

It can be shown by partial integration that

$$(2k + 1) I_{k+1}(\alpha) = 1 - 2\alpha I_k(\alpha)$$

from which it follows that

$$I_{k+1}(\alpha) = \sum_{m=0}^k (-)^m \frac{(2k - 2m - 1)!!}{(2k + 1)!!} (2\alpha)^m + \frac{(-)^{k+1} (2\alpha)^{k+1}}{(2k + 1)!!} I_0(\alpha)$$

We now insert this equation in the sum giving $L(r, \alpha)$ and interchange the order of summation, which gives the desired result. The k -series of the $n=0$ in the double sum can be evaluated explicitly, of course, since

$$\sum_{k=0}^{\infty} (-)^k (k+1) \frac{r^{2k}}{2k+5} = \int_1^{\infty} \frac{dx}{x^2(x^2 + r^2)} = \frac{1}{r^5} \left(r - \frac{3}{2} \tan^{-1} r + \frac{1}{2} \frac{r}{1+r^2} \right)$$

Any of the other k -series may be generated from this relation by multiplying by an appropriate power of r and integrating. E.g., the $n = 1$ k -series can be obtained as follows:

$$\begin{aligned} \sum_{k=0}^{\infty} (-)^k (k+1) \frac{r^{2k}}{(2k+5)(2k+3)} &= r^{-3} \int \frac{dr}{r^3} \left(r - \frac{3}{2} \tan^{-1} r + \frac{1}{2} \frac{r}{1+r^2} \right) \\ &= \frac{1}{2r^3} - \frac{1}{4r^4} + \frac{3+r^2}{4r^5} \tan^{-1} r, \text{ etc.} \end{aligned}$$

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Such a procedure is particularly useful when q is small but r is not, so that only a few terms need be taken in the n -series, while many must be taken in the k -series.

Case 9

$$\rho_S = \rho_t = 1, \rho_r = \infty$$

In this case the integrals are tabulated. Let $t = \sqrt{\frac{a_S}{a_t}}$ and $q = \frac{a_r}{\sqrt{a_S a_t}}$.

$$R = \left[\frac{t + \frac{1}{t} + q}{(t - \frac{1}{t})^2} \right] \left[t e^{\frac{q}{t}} E_2\left(\frac{q}{t}\right) + \frac{1}{t} e^{qt} E_2(qt) - \frac{2e^{\frac{q}{t}} E_1\left(\frac{q}{t}\right)}{(t - \frac{1}{t})} + \frac{2e^{qt} E_1(qt)}{(t - \frac{1}{t})} \right]$$

Case 10

$$\rho_S = \frac{1}{2}, \rho_t = \infty, \rho_r = \frac{1}{2}$$

This integral is handled by the method of Case 7. Let

$$t = \sqrt{\frac{a_r}{a_S}} \quad \text{and} \quad q = \frac{a_t}{\sqrt{a_r a_S}}$$

$$R = \frac{1}{2} (t^2 + 1 + qt) \left[\frac{2}{1+t} - q e^{\frac{1}{2} qt} \sum_{k=1}^{\infty} \frac{(|2k-3|)!!}{(2k)!!} \left(\frac{t^2-1}{t^2} \right)^{k-1} E_k\left(\frac{1}{2} qt\right) \right]$$

Case 11

$$\rho_S = \frac{1}{2}, \rho_t = \infty, \rho_r = 1$$

This integral can be handled by the method of Case 8. Let

$$t = \sqrt{\frac{a_S}{a_r}}, \quad q = \frac{a_t}{\sqrt{a_S a_r}}, \quad r = \sqrt{2t^2 - 1}$$

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$$R = 2(t^2 + 1 + qt) \left[\sum_{n=0}^{\infty} (-)^n \left(\frac{q}{t} \right)^n \sum_{k=\max(0, n-1)}^{\infty} (-)^k r^{2k} \frac{(2k+1-2n)!!}{(2k+3)!!} \right. \\ \left. + I_0 \left(\frac{q}{2t} \right) \sum_{k=0}^{\infty} \frac{r^{2k} \left(\frac{q}{t} \right)^{k+2}}{(2k+3)!!} \right]$$

The k-series can be summed explicitly in a manner similar to that employed in Case 8 starting from the $n = 0$ k-series:

$$\sum_{k=0}^{\infty} (-)^k \frac{r^{2k}}{2k+3} = \int_1^{\infty} \frac{dx}{x^2(x^2 + r^2)} = \frac{1}{r^2} \left(1 - \frac{\tan^{-1} r}{r} \right)$$

Case 12

$$\rho_S = 1, \rho_t = \infty, \rho_r = \frac{1}{2}$$

This integral can be handled by the method of Case 8. Let

$$t = \sqrt{\frac{a_r}{a_S}}, \quad q = \frac{a_t}{\sqrt{a_r a_S}}, \quad r = \sqrt{2t^2 - 1}$$

$$R = 4t^2(t^2 + 1 + qt) \left[\sum_{n=0}^{\infty} (-)^n \left(2 \frac{q}{t} \right)^n \sum_{k=\max(0, n-1)}^{\infty} (-)^k (k+1) r^{2k} \frac{(2k+1-2n)!!}{(2k+3)!!} \right. \\ \left. + I_0 \left(\frac{q}{2t} \right) \sum_{k=0}^{\infty} \frac{(k+1) r^{2k} \left(\frac{q}{t} \right)^{k+2}}{(2k+3)!!} \right]$$

The k-series can be summed explicitly in a manner similar to that employed in Case 8 starting from the $n = 0$ k-series:

$$\sum_{k=0}^{\infty} (-)^k \frac{(k+1) r^{2k}}{2k+3} = \int_1^{\infty} \frac{dx}{(x^2 + r^2)^2} = \frac{1}{2r^2} \left(\frac{\tan^{-1} r}{r} - \frac{1}{1+r^2} \right)$$

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Case 13

$$\rho_s = \rho_r = 1, \rho_t = \infty$$

The integrals are again tabulated. Let $t = \sqrt{\frac{a_s}{a_r}}$, $q = \frac{a_t}{\sqrt{a_s a_r}}$

$$R = \left[\frac{t^2 + qt + 1}{t^2 - 1} \right] \left[e^{q/t} E_2\left(\frac{q}{t}\right) - \frac{e^{q/t} E_1\left(\frac{q}{t}\right)}{t^2 - 1} + \frac{e^{qt} E_1(qt)}{t^2 - 1} \right]$$

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1. E. P. Wigner, "On the Variation of η with Energy in the 100-1000 ev Region," BNL-25 (Nov. 1955).
2. S. Oleksa, Nuclear Science and Technology 1, 275 (1955)
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3. P. Greebler, H. Hurwitz, Jr., and M. L. Storm, "Statistical Evaluation of Fission Product Absorption Cross Sections at Intermediate and High Energies," Nuclear Science and Engineering (to be published).
4. C. E. Porter and R. G. Thomas, "Fluctuations of Nuclear Reaction Widths," Phys. Rev. 104, 483 (1956).
5. Our notation will be that of R. G. Sachs, "Nuclear Theory," Addison-Wesley Publishing Company, Ch. 10 (1953). This notation conforms largely with that of Teichmann and Wigner.
6. E. Cramer, "Mathematical Methods of Statistics," Princeton University Press, Ch. 18 (1956).
7. The restriction to the one-level formula is not essential. R. G. Thomas ["Collision Matrices for the Compound Nucleus," Phys. Rev. 97, 224 (1955)] discusses in detail the conditions under which expressions like Eq. (1) are valid.
8. See e.g., W. Hauser and H. Feshbach, "Inelastic Scattering of Neutrons," Phys. Rev. 87, 366 (1952).
9. If there are no radiative widths (i.e., if none of the ρ_k 's equals ∞) then the integrand of (8) is of the form $R \left[\lambda, P(\lambda) \right]$ where R is a rational function and $P(\lambda)$ is a polynomial. When P is of the order

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two or less the integral can be expressed in terms of elementary functions. The relevant formulae may be found in "A Short Table of Integrals," B. O. Pierce, Ginn and Co., 1929, for example. When P is of the order three or four the integral is elliptic. An exhaustive compilation of the relevant formulae may be found in "Handbook of Elliptic Integrals for Engineers and Physicists," P. F. Byrd and M. D. Friedman, Springer Verlag, 1954. We use the latter's notation in referring to elliptic integrals. In cases where one of the ρ_k 's equals ∞ an exponential occurs in the integrand. Many of the integrals occurring in the cases may be found in "Tables of Integral Transforms," A. Erdely et al., McGraw Hill Co., 1954, Ch. IV. When in the text, an integral is said to be tabulated, it or some simply related form can be found in one of the foregoing references. No attempt will be made, in general, to specify which one.

10. This first result has been given by Weinberg and Wigner (unpublished). The last two results have been given in slightly different form by Greebler, et al. (Ref. 3). These authors chose to do their calculations entirely with the case $\rho_S = \frac{1}{2}$ which is correct for S-wave neutrons. Their treatment of fluctuation phenomena at higher energies is inadequate, however. In particular, the use of the S-wave R will produce an underestimate of the cross section, though generally of the order of 10%, which for their case is insignificant.
11. For the sake of clarity the reader is reminded of the definitions of the even function and exponential integrals, viz,

$$\text{Erf} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{Erfc}(x) = 1 - \text{Erf}(x) \quad E_n(x) = \int_1^\infty \frac{e^{-\mu x}}{\mu^n} d\mu$$

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