ARGONNE NATIONAL LABORATORY
P. O. Box 299
Lemont, Illinois

THE COUPLED ASPECTS OF A
FAST-THERMAL CRITICAL: ZPR-V

by
B. J. Toppel
Reactor Engineering Division

Assisted by:

C. E. Cohn
G. J. Fischer
W. Y. Kato

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ABSTRACT

ZPR-V is a zero-power, coupled fast-thermal critical assembly involving a dilute fast core and a light water-moderated, enriched uranium thermal annulus. The two regions are coupled through a natural uranium isolation blanket. This facility is useful in studying some of the properties of dilute fast reactor systems as well as those of coupled fast-thermal power breeders. Recent emphasis has been on the latter aspect of the problem and an investigation of the coupled aspects of the system has been made. An important parameter in the coupling concept is the division of reactivity. This can be determined by means of fast fuel substitutions and $1/v$ poisoning of the thermal annulus in conjunction with theoretical calculations. Auxiliary information obtained in such a procedure is the fast fuel worth throughout the core and the prompt neutron lifetime of the whole system. A second important parameter, S_1/S_2 , is the relative number of neutrons emitted per unit time due to fast and slow fissions. Information regarding this quantity can be obtained with the aid of U^{235} and U^{238} fission distributions. Measurements of the above parameters have been made for the case of a $U^{238}:U^{235}$ atomic ratio of 5:1 in the fast core and a 1-inch isolation blanket. The fractions of reactivity in the fast core, blanket, and annulus are, respectively, 24%, 14%, and 61%; the prompt lifetime is 39.0 ± 3.0 microseconds; the ratio S_1/S_2 is 0.43.

I. INTRODUCTION

R. Avery at the Argonne National Laboratory has recently made a study regarding the feasibility of a power breeder reactor utilizing the coupled reactor concept.¹ He has shown that the coupling of a fast and thermal assembly affords the possibility of obtaining a combined system having the breeding ratio characteristic of an all-fast system and at the same time exhibiting a prompt neutron lifetime characteristic of a slow

¹Paper 18-3 of the December, 1956 meeting of the American Nuclear Society; also Nuc. Sci. Eng. (in press).

system. By coupling, one means that in each of the assemblies some of the neutrons causing fissions were born in the other. The two "assemblies" may refer to two spatial assemblies or, as is perhaps more reasonable, one can divide the system into two parts as determined by the energy of the neutrons causing fissions. We shall consider the latter connotation here so that the slow assembly refers to thermal fissions while the fast assembly refers to epi-cadmium fissions.

A complete understanding of a coupled system involves knowledge of the coupling parameters as formulated by Avery. Owing to the coupled fast-slow aspect of ZPR-V, it was decided to determine the coupling parameters for that facility.

II. THEORETICAL BACKGROUND²

In the ordinary reactor, one can define the effective multiplication constant k_{eff} as the average number of fission neutrons in the next generation resulting from a single fission neutron. In the case of the coupled system, Avery¹ introduces four corresponding integral parameters. These are denoted by k_1 , k_2 , p_{12} , and p_{21} , and are defined as follows: k_1 is the average number of next-generation fission neutrons in assembly 1 resulting from a single fission neutron in assembly 1; k_2 is the average number of next-generation fission neutrons in assembly 2 resulting from a single fission neutron in assembly 2; p_{12} is the average number of next-generation fission neutrons in assembly 2 resulting from a single fission neutron in assembly 1; p_{21} is the average number of next-generation fission neutrons in assembly 1 resulting from a single fission neutron in assembly 2.

Let assembly 1 be the fast assembly for the purposes of the present discussion. Then $1 - k_1 = \Delta_1$ and $1 - k_2 = \Delta_2$ are measures of the subcriticality of assemblies 1 and 2 on their own, respectively; p_{12} and p_{21} are measures of the coupling between the two assemblies. The problem is to obtain the value of all four of these coupling parameters.

At criticality, the relative numbers of fission neutrons S_1 and S_2 in assemblies 1 and 2, respectively, obey the equations:

$$S_1 = k_1 S_1 + p_{21} S_2 \tag{1}$$

$$S_2 = k_2 S_2 + p_{12} S_1 \quad ;$$

therefore,

$$\frac{S_1}{S_2} = \frac{p_{21}}{\Delta_1} = \frac{\Delta_2}{p_{12}} \tag{2}$$

²The theoretical formalism is due to R. Avery and H. Hummel.

From Eq. (2) the ratio of p_{21} to Δ_1 and the ratio of Δ_2 to p_{12} is given by the ratio of the number of neutrons due to fast fissions to the number of neutrons due to thermal fissions. This ratio can be obtained experimentally by determining the thermal and epi-cadmium fission rate of U^{235} and the U^{238} fission rate throughout the system.

With the fast lifetime being obtained by calculation, one might hope that the results of a Rossi alpha-type experiment could be used to determine the sub-criticality of the fast assembly and hence the value of Δ_1 . Assuming, then, that this is done and that the value of S_1/S_2 has been obtained, one more relationship between the parameters is needed in order to determine the value of all four. This other relationship involves the division of reactivity between the two assemblies.

Define the fraction of reactivity in assembly 1 as

$$FR_1 = \frac{\int_1 \phi^\dagger \nu \Sigma_f \phi \, dV}{\int_{\text{total}} \phi^\dagger \nu \Sigma_f \phi \, dV}, \quad (3)$$

where the symbolic notation has the meaning

$$\int_1 \phi^\dagger \nu \Sigma_f \phi \, dV = \int_{\underline{r}} \int_{\underline{v}} \int_{\underline{v}'} \beta(v') \phi^\dagger(\underline{r}, v') [\nu \Sigma_f(\underline{r}, v)]_1 \phi(\underline{r}, v) \, d\underline{r} \, dv \, dv' \quad (4)$$

The bracketed $\nu \Sigma_f$ term is evaluated according to the meaning of assembly 1. FR_2 is defined in an analogous manner. The definition of FR_1 is reasonable, since it is essentially just the fissions in assembly 1 weighted according to their importance by means of the adjoint flux divided by the total fissions in the whole system again suitably weighted. Avery has shown that the fraction of reactivity is related to the coupling parameters by

$$FR_1 = \frac{\Delta_2}{\Delta_1 + \Delta_2} \quad (5)$$

$$FR_2 = \frac{\Delta_1}{\Delta_1 + \Delta_2}$$

If, then, FR_1 and FR_2 can be determined, assuming that Δ_1 has been obtained by the Rossi alpha experiment and that the ratio S_1/S_2 is known from the fission distributions, the value of all four of the coupling parameters will have been determined.

The procedure followed here has been to produce a perturbation in assembly 1 or assembly 2 and to determine the resulting δk experimentally. From these experimental values for δk and the computational techniques developed in the Appendix A, FR_1 and FR_2 can be computed. In the case

of the fast assembly, the perturbation consisted of fast fuel substitutions, while in the thermal system an effective perturbation in ν was produced by introducing a $1/\nu$ poison in the form of H_3BO_3 dissolved in the water moderator. In the latter case, one can in addition obtain the prompt neutron lifetime for the whole system, since, as is known from perturbation theory, if the reactor is uniformly poisoned by a $1/\nu$ absorber of magnitude C/ν , then ℓ is given by the resulting δk divided by C . The corrections made necessary by the fact that the poisoning was in fact not uniform are discussed in a later section.

III. DESCRIPTION OF THE FACILITY³

ZPR-V is a zero-power, coupled fast-thermal critical facility consisting of a dilute fast core surrounded by a light water-moderated, enriched uranium thermal annulus. The two regions are coupled through a natural uranium isolation blanket. This facility is useful in studying some of the properties of dilute fast reactor systems as well as those of coupled fast-thermal power breeders.

Figure 1 shows an elevation drawing and top view photograph of ZPR-V. The blanket is not visible in the photograph owing to the camera angle.

For the present experiments, the isolation blanket was 1 in. thick, while the fast core had a composition by volume of 35% uranium, 10% iron, 15% aluminum and 40% void. The fast fuel cans were loaded with 10 plates of enriched uranium, 5 plates of natural uranium, and 16 plates of aluminum. The uranium enrichment was such that this combination of 10 enriched and 5 natural plates resulted in a $U^{238}:U^{235}$ atomic ratio of 5:1. The physical constitution of the fast fuel cans permits fuel substitutions to be made in the sense that an enriched plate may be replaced by a natural plate.

IV. EXPERIMENTAL TECHNIQUES

A. Fast Core Perturbations

All reactivity determinations were made by using the inhour equation⁴ relating reactor period to reactivity. The procedure followed was to adjust the symmetrical thermal loading so that the reactor was on a convenient positive period with all safety and control rods fully withdrawn.

³ A more complete description of ZPR-V is contained in ANL-5547. Additional information appears in Engineering, 183, 404 (1957).

⁴ The delayed neutron data of G.R. Keepin was used in the inhour equation; see, e.g., "Progress in Nuclear Energy," Vol. 1, Series 1 (1956).

FIG. 1
ELEVATION DRAWING AND PLAN VIEW
PHOTOGRAPH OF ZPR-V

This configuration was chosen owing to its reproducibility. The perturbation was then performed and the new period determined. From the two periods, before and after the change, one can obtain the δk due to the perturbation. In all cases, the two conditions, unperturbed and perturbed, were run alternately a number of times so that a series of interlacing runs was obtained. In this way, any systematic drift in reactivity could be accounted for. Owing to the general warming up of the environment during the day, coupled with the negative temperature coefficient of reactivity, there is, in fact, a characteristic loss in reactivity for ZPR-V of the order of 0.5 ih/hour, depending, of course, upon the temperature of the day and recent past history of the system.

The periods were determined with the aid of two BF_3 proportional counters having good integral bias plateaus. An indication of the uncertainty of the determination was obtained from the agreement in periods found by the independent counter setups. Figure 2 shows the response of one of the counters to a positive reactor period. The statistics on the points are smaller than the sizes of the circles. The timing procedure involved the use of two Microflex electric timers interconnected so that the counters were "on" for an interval Δt_1 and "off" for a period Δt_2 , with the cycle repeating until manually stopped. During the "off" time the observer recorded the accumulated count and reset the scalars. A separate clock timed the duration of the measurement so that with a knowledge of the number of cycles the "time intervals" of Figure 2 could be translated into seconds. Although the agreement between the two counters was in general better than 0.5 ih, because of the occasional greater deviation plus the possibility of a systematic uncertainty, an individual reactivity measurement was assumed to be known to ± 1 ih.

Figure 3 shows the manner of interlacing runs in order to determine the δk . The procedure for the fuel substitutions was to maintain the total number of fuel plates at 15 but vary the number of enriched relative to natural plates. In the case of Figure 3, the δk produced by replacing 3 natural plates with 3 enriched plates is obtained. The procedure used to obtain the lines through the points was to determine the least squares fit to the points using two lines constrained to have the same slope (see Appendix B). The error on the individual points is shown as ± 1 ih. On the basis of this error, the typical standard deviation of the difference of the two lines is 0.8 ih.

Figure 4 shows the worth of enriched plates relative to natural plates in the central fast fuel can. The total worth of the 10 enriched plates was determined in five small steps, since the full exchange would have involved too great a δk to be handled by means of a period determination. Having now determined the worth of 10 enriched plates in the central can position, it was possible to obtain the values shown in Figure 5 by determining the relative worth of a can containing 15 natural plates as a function

FIG. 2
TYPICAL RESPONSE OF BF_3 PROPORTIONAL
COUNTER TO POSITIVE REACTOR PERIOD

FIG. 3
INTERLACING REACTIVITY DETERMINATIONS FOR
TWO CONFIGURATIONS IN THE CENTRAL FAST
FUEL CAN. THE LINES ARE "DOUBLE STRAIGHT
LINE" LEAST SQUARE FITS TO THE DATA.

FIG. 4
RELATIVE WORTH OF ENRICHED FUEL
IN CENTER OF FAST CORE

of position in the core. The method of interlacing runs was followed for these runs as before. It can be seen from Figure 5 that, proceeding from the central can toward the blanket, the worth of enriched fuel at first decreases but then increases again in the can nearest the blanket. This effect reflects the fact that the spectrum becomes softer as the blanket is approached.

Since one now has δk as a function of position in the core, it is possible to calculate FR_1 as indicated in Appendix A. The results are shown later in Table I.

B. Thermal Annulus Perturbations

For the thermal annulus perturbations, boric acid solution was added to the water moderator. The double line, least squares fitting technique was applied as before, the only difference being that now the "no poison" and "poisoned" reactivity values were temporally separated rather than being interlaced as in the case of the fast core experiments. The reason for this is, of course, the impracticality of extracting the poison from the water in a short time. Owing to this necessary separation of configurations, seven determinations of reactivity were obtained before adding

poison so that the slope of the reactivity vs. time curve was well established. In the case of the poison runs, the typical standard deviation in the difference δk is 1.4 ih. This higher value as compared with the core result is a reflection of the fact that the runs could not be temporally interlaced.

THE PROBABLE ERROR ON ALL POINTS IS ± 1.6 ih.

FIGURE 5
 REACTIVITY LOSS IN INHOURS PRODUCED BY REPLACING A NORMAL 5:1 FUEL CAN WITH 35% NATURAL URANIUM AS A FUNCTION OF CAN POSITION IN THE FAST CORE. THE VALUES FOR THE UNMARKED CANS CAN BE OBTAINED BY SYMMETRY ARGUMENTS.

Three poison samples in all were added to the moderator. Two of the three were closely equal in size, while the third sample was smaller. This procedure furnished a check on the consistency of the method. The boron concentration was determined by the Chemistry Division by means of quantitative chemical analysis of water samples taken at the end of each poisoning sequence. A typical poisoning introduced 1.56 micrograms of boron per cc of water and resulted in a δk of -17.54 ih.

Since δk has been determined as a function of boron concentrations, FR_2 may be computed using the method in Appendix A.

C. Prompt Lifetime

As mentioned above, the prompt lifetime may be obtained if the reactor is uniformly poisoned with a $1/v$ absorber. The present experiment deviated from the theoretical assumption of uniform poisoning since the boric acid was added only to the thermal annulus water. Therefore, the fast core and blanket were not poisoned. The inaccuracy caused by not poisoning the fast core and blanket is, however, negligible. The fast core contains essentially no thermal flux, so that the effect of the small amount of $1/v$ poison would be unimportant. Calculations indicate that less than 1% of the spatially integrated thermal flux is in the blanket, the remainder being in the annulus. In particular, UNIVAC calculations involving uniform and non-uniform $1/v$ poisoning indicate that the discrepancy in lifetime thereby produced is less than 2%.

A second reason for non-uniform poisoning is due to the fact that the thermal annulus is heterogeneous in nature. The reflector and thermal fuel were therefore poisoned to different degrees. In order to correct for the heterogeneous annulus effect, observations were next made on the effect of poisoning only the reflector in one quadrant. This was accomplished by constructing water-tight aluminum cans which could be filled with a boric acid solution and placed beside (radially outward) and above the thermal fuel. The large can placed beside the annulus was approximately 22 in. x 37 in. x 7 in., while the two small cans placed above each thermal basket were approximately $2\frac{1}{2}$ in. x 21 in. x 8 in. The can wall thickness was $1/16$ in. so as not to perturb the flux appreciably. That the reactor was indeed not perturbed by the cans was shown experimentally by observing no difference in reactivity with or without the cans when filled with plain water.

The cans were filled with a boric acid solution of concentration approximately four times that resulting from each poison addition in the whole reactor experiment. Since the earlier results indicated that the effects were small enough to be treated as first-order perturbations, it was felt that this somewhat higher concentration localized in one quadrant would still lead to a valid result.

The effect of the "poison cans" on reactivity was noted as before; that is, the reactor periods were observed with and without cans, always running with all safety and control rods withdrawn, and the usual series of interlacing runs was made. Although the reflector below the fuel could not be poisoned, it was found that the top poison cans had an unobservable effect and so it is reasonable to assume a similar result for the bottom of the fuel. The region at the corners of the thermal sections was also inaccessible. It was assumed, however, that the reflector in that region would be worth no more than the large region beside the fuel. The effect of this uncertainty on the lifetime fortunately lies inside the experimental uncertainty. In the following, the above experiment will be referred to as the "reflector experiment."

The correction to the lifetime was made as follows. Assuming

$$\ell = \frac{\delta k}{C} ,$$

we must use:

$$\delta k = (\delta k_0 - \frac{4}{r} \delta k_r - \delta k_c) + \frac{4f}{r} \delta k_r + f \delta k_c$$

and

$$C = f C_0 ,$$

where:

δk_0 = the δk observed in the whole reactor runs

δk_r = the δk observed in the reflector runs

δk_c = the unobserved δk of the corner reflector region, which is assumed to be $\leq \delta k_r$

f = the water fraction in the vicinity of the thermal fuel

r = the ratio of boron concentration in the poison cans to the concentration added in the whole reactor runs

C_0 = the $v\Sigma_a$ used in the whole reactor runs.

D. Determination of the Ratio of S_1 to S_2

As indicated above, S_1 and S_2 are, respectively, the relative number of neutrons emitted per unit time due to fast fissions and thermal fissions. In order to obtain these quantities, Cd-covered and bare U^{235} foils as well as U^{238} foils highly depleted in U^{235} were exposed throughout

the reactor, and the resulting gamma activity was counted using a NaI(Tl) scintillation spectrometer. The discrimination level was set to count only gamma rays greater than 400 kev in energy. The relative activations of the U^{235} and U^{238} foils are shown in Figure 6. From results obtained using a double ionization chamber,⁵ the U^{235} to U^{238} fission ratio was determined to an accuracy of about 10% as 26 at the center of the fast core. Using this result to normalize the foil data, the relative number of fissions per cc was calculated. This result is seen in Figure 7. The coincidence of epi-cadmium and thermal fissions per cc at the annulus-blanket interface is accidental. The ratio S_1/S_2 can be obtained by a spatial integration of the results in Figure 7.

V. RESULTS AND DISCUSSION

The division of reactivity in Table I has been computed using the δk values determined from the fuel substitution and poisoning experiments in conjunction with UNIVAC calculations (see Appendix A). The division of the annulus and core values for fractions of reactivity into three energy groups as well as the values quoted for the blanket were obtained by means of calculated values of spatially integrated fissions. It can be seen from Table I that the distinction between a spatial assembly division or an energy assembly division is not very important for ZPR-V. Also listed in Table I are the corrected prompt neutron lifetime and the ratio S_1/S_2 .

Table I

DIVISION OF REACTIVITY BY REGIONS AND BY ENERGY

<u>Region</u>	<u>Thermal (0-0.7 ev)</u>	<u>Resonance (0.7-10⁴ ev)</u>	<u>Fast (10⁴ - ∞)</u>
Annulus	59%	2%	-
Blanket	8	2	4%
Core	-	3	21
Total:	67%	7%	25%

Prompt Neutron Lifetime = $39.0 \pm 3.0 \mu\text{sec}$

$$S_1/S_2 = 0.43$$

⁵ The double ion chamber results are due to R. Nobles.

FIG. 6
 U^{235} AND U^{238} FISSION FOIL ACTIVATION AS A
 FUNCTION OF POSITION

FIG. 7
RELATIVE FISSIONS/cc AS A FUNCTION OF POSITION

If we consider an energy division so that the fraction of thermal reactivity is taken to be 0.67, then Δ_1 is about twice Δ_2 . Also $p_{21} = 0.43 \Delta_1$, while $p_{12} = 2.3 \Delta_2$. It is hoped that in the near future a Rossi alpha experiment will yield a value for Δ_1 so that the four parameters can be determined explicitly.

Future investigations will involve a determination of the coupling parameters for differing core compositions as well as varying blanket thicknesses. Preliminary data for the case of a 5.1 core but with a 2-inch isolation blanket indicate that the lifetime and fraction of reactivity in the annulus are increased by about 10% while the fraction of reactivity in the fast core is decreased by about the same amount. The direction of the effect is certainly correct and the magnitude seems reasonable.

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APPENDIX A

METHOD OF CALCULATING THE FRACTION OF REACTIVITY

For the case of a perturbation in the fast core, one can write the familiar perturbation expression δk symbolically as⁶

$$\delta k_1 = \frac{\int_1 \phi^\dagger \delta [fJ - K] \phi dV}{\int_{\text{total}} \phi^\dagger \nu \Sigma_f \phi dV} \quad , \quad (\text{A-1})$$

where fJ and K represent the usual expressions for production and destruction of neutrons, respectively.⁷ Comparing Eqs. (A-1) and (3) and assuming that the adjoint fluxes are independent of energy and position, one can write FR_1 as

$$\begin{aligned} FR_1 &= \delta k_1 \frac{\int_1 \phi^\dagger \nu \Sigma_f \phi dV}{\int_1 \phi^\dagger \delta [fJ - K] \phi dV} \\ &= \delta k_1 \frac{\int_1 \nu \Sigma_f \phi dV}{\int_1 \delta [(\nu-1) \Sigma_f - \Sigma_c] \phi dV} \end{aligned} \quad (\text{A-2})$$

By making $U^{235} - U^{238}$ replacements in the fast core, one can produce a δk_1 which is determinable experimentally. The corresponding ratio of integrals in Eq. (A-2) is calculable with the aid of fluxes obtained from 19-group, one-dimensional, diffusion approximation UNIVAC calculations. Since the δk resulting from a fuel change varies as a function of position in the core, the FR_1 calculation involves a spatial integration of a series of determinations over the core.

In the case of the water-moderated annulus, an effective change in ν can be produced by inserting poison in the water. If then any change in thermal leakage caused by introduction of the poison is neglected, the general expression for δk can be specialized so that the resulting δk_2 can be written

$$\delta k_2 = \frac{\int_2 \phi^\dagger \delta (\nu \Sigma_f) \phi dV}{\int_{\text{total}} \phi^\dagger \nu \Sigma_f \phi dV} \quad . \quad (\text{A-3})$$

⁶ For the present discussion we use the sub 1 and sub 2 designation to refer to a spatial division rather than an energy division. Actually, owing to the spectral-regional correspondence in ZPR-V the distinction is not too important.

⁷ The Reactor Handbook, Volume 1, AECD-3645, Chapter 1.6.

By comparison with the expression for FR_2 which is just Eq. (3) with 2 replacing the 1, we see that

$$FR_2 = \delta k_2 \frac{\int_2 \phi^\dagger \nu \Sigma_f \phi dV}{\int_2 \phi^\dagger \delta(\nu \Sigma_f) \phi dV} \quad (A-4)$$

Now in the case of the thermal annulus, one expects $\nu \Sigma_f$ to be relatively independent of position and if one assumes the spectrum to be essentially a Maxwellian thermal distribution, one can write

$$\frac{\int_2 \phi^\dagger \nu \Sigma_f \phi dV}{\int_2 \phi^\dagger \delta(\nu \Sigma_f) \phi dV} = \left(\frac{\nu \Sigma_f}{\delta(\nu \Sigma_f)} \right)_2 \quad \frac{\int_2 \phi^\dagger \phi dV}{\int_2 \phi^\dagger \phi dV} = \left(\frac{k_\infty}{\delta k_\infty} \right)_2, \quad (A-5)$$

where k_∞ and δk_∞ are evaluated in an assumed Maxwellian thermal spectrum. In particular, since

$$(k_\infty)_2 \cong (\eta f)_2 = \left(\frac{\nu \Sigma_f}{\sum_a U_a^{235} + \sum_a H_2O + \sum_a Al} \right)_2 \quad (A-6)$$

and after the poisoning with boron (in the form of H_3BO_3 dissolved in the water)

$$(k'_\infty)_2 = \left(\frac{\nu \Sigma_f}{\sum_a U_a^{235} + \sum_a H_2O + \sum_a Al + \sum_a B} \right)_2, \quad (A-7)$$

then

$$\left(\frac{k_\infty}{\delta k_\infty} \right)_2 = \left(\frac{k_\infty}{k'_\infty - k_\infty} \right)_2 \cong \left(\frac{\sum_a U_a^{235} + \sum_a H_2O + \sum_a Al}{\sum_a B} \right)_2. \quad (A-8)$$

Thus finally

$$FR_2 \approx \delta k_2 \left(\frac{k_\infty}{\delta k_\infty} \right)_2, \quad (A-9)$$

where δk_2 is measured experimentally and $(k_\infty/\delta k_\infty)_2$ is calculated as in Eq. (A-8).

APPENDIX B

DOUBLE STRAIGHT LINE, LEAST SQUARES FITTING TECHNIQUE

Having two sets of data to which it is desired to fit two straight lines of equal slope

$$\begin{aligned} y_i &= a + bx_i & n \text{ observations} \\ y'_j &= a' + bx'_j & n' \text{ observations} \end{aligned} \quad , \quad (B-1)$$

a , a' , and b are determined by forming

$$m = \sum_i w_i [y_i - (a + bx_i)]^2 + \sum_j w'_j [y'_j - (a' + bx'_j)]^2 \quad (B-2)$$

and setting

$$\frac{\partial m}{\partial a} = \frac{\partial m}{\partial b} = \frac{\partial m}{\partial a'} = 0$$

The weights w_i and w'_j are assumed inversely proportional to the squared standard deviations of the points:

$$w_i = \frac{c}{\sigma^2(y_i)} ;$$

$$w'_j = \frac{c'}{\sigma^2(y'_j)} .$$

For the present purpose, assume that w_i and w'_j are independent of i and j . In particular, set $\sigma^2(y_i) = \sigma^2(y'_j)$ and choose the arbitrary constants c and c' such that $w_i = w'_j = 1$ for all i and j .

Since the formulas for a , a' , b , and $\Delta = a - a'$ are functions of the y_i and y'_j , errors in the latter will result in non-zero standard deviations in the former. In particular, $\sigma^2(a)$, $\sigma^2(a')$, $\sigma^2(b)$, and $\sigma^2(\Delta)$ can be computed by remembering that if, for example,

$$e = \sum_k r \mu_k$$

where the μ_k have squared standard deviations $\sigma^2(\mu_k)$, then

$$\sigma^2(e) = \sum_k r^2 \sigma^2(\mu_k) .$$

The results obtained in this manner can be shown to be independent of the normalization of the w_i and w'_j .