Argonne National Laboratory

A PROPOSED SOLUTION TO THE "MATCH" PROBLEM

by

Vera Pless

MASTER

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Applied Mathematics Division

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I. INTRODUCTION

In a bubble-chamber experiment, three cameras photograph the physical event simultaneously. These photographs consist of bubble-chamber tracks, which are sequences of bubbles, and there are approximately 20 tracks to a picture. The match problem in the analysis of these events is to identify the same track in the three views. The matching, even by a person, can be quite tricky; e.g., lines can cross in one view and not in another. People have many pattern-recognition processes, measuring abilities, and the flexibility to choose certain parameters only in certain cases. Computers do not have the flexibility people do, and when many parameters are given, the logic can get very involved and the program extremely time-consuming. The solution we present here to the match problem for use on computers uses only a few parameters and has only a few exceptional cases.

This solution first uses a simple, geometric criterion to determine which pair of tracks (from each of the three possible pairs of views) could represent the same track in space. Though this criterion is crude, it eliminates many of the possible pairings. From this, we construct possible triples of tracks (one from each view) such that each pair of tracks from the triple could represent the same track in space. We then associate an estimator with each triple and eliminate those whose estimators are too large. After this, we sort the triples into equivalence classes and choose the best triples depending in part on the estimator.)

II. PRELIMINARIES

In this section, we describe the geometric situation. We suppose the tracks are produced in a cube. Two of the cameras are situated at positions (1) and (2) at a distance h above two vertices of the cube. The two cameras are at a distance d_{12} apart, and the height of the cube is denoted by d . Camera (3) is at height h above the edge of the cube facing cameras (1) and (2). For greater ease of calculating on a computer, we will have three preferred coordinate systems, the 12 system, the 23 system, and the 31 system, depending on the respective pairs of cameras. These will be called the preferred ij systems. We will first describe the 12 system (see Fig. 1). The x -coordinate is taken to be along the edge of

the cube parallel to the line joining cameras (1) and (2). The y-coordinate is taken on an edge going out, and the z-coordinate is taken on an edge going down. If (x', y', z') is a bubble, $(x_1, y_1) = P_1$ is its conical projection on the x-y plane from position (1), which is where it would appear if photographed from camera (1), and $(x_2, y_2) = P_2$ is its conical projection on the x-y plane from position (2), its photograph from camera (2). Note that with this choice of coordinates, $y_1 = y_2$.

Fig. 1. The Preferred 12 System

Let d_{23} be the distance between cameras (2) and (3), let d_{31} be the distance between cameras (1) and (3), and let $P = d_{12} + d_{23} + d_{31}$. If we draw the back side of the cube and cameras (1) and (2), we get the picture shown in Fig. 2. We see that

$$\frac{z' + h}{x'} = \frac{h}{x_1},$$

and

$$\frac{z' + h}{d_{12} - x'} = \frac{h}{d_{12} - x_2}.$$

Fig. 2. Back Side of Cube and Cameras (1) and (2)

Hence,

$$z' + h = \frac{x'h}{x_1} = \frac{(d_{12} - x')h}{d_{12} - x_2},$$

so that

$$\frac{x'}{x_1} = \frac{d_{12} - x'}{d_{12} - x_2}.$$

From this we have

$$x'(d_{12} - x_2 + x_1) = d_{12}x_1$$

and

$$\frac{x'}{x_1} = \frac{d_{12}}{d_{12} - (x_2 - x_1)}.$$

Let

$$\Delta_{12} = \frac{P}{d_{12}}(x_2 - x_1).$$

Then

$$\frac{x'}{x_1} = \frac{P}{P - \Delta_{12}}.$$

Therefore,

$$z' + h = \frac{x'h}{x_1} = \frac{hP}{P - \Delta_{12}},$$

and solving for Δ_{12} in terms of z' , we obtain

$$\Delta_{12} = \frac{z'P}{z' + h}.$$

As (x', y', z') varies throughout the cube, Δ_{12} is a nonnegative increasing function of z' and so attains its maximum value when $z' = d$. This maximum value, which we denote by $\Delta_{12} \max$, equals $dP/(h + d)$.

Therefore, we have

$$0 \leq \Delta_{12}(z') \leq \frac{dP}{h + d} = \Delta_{12} \max.$$

Consider next the situation with the three cameras in place (see Fig. 3). Let P_3 be the intersection of the upper surface of the cube with the line through (3) and (x', y', z') . Note now that triangle $P_1P_2P_3$ is similar to triangle (1)(2)(3) since the corresponding sides are parallel. This is so for any triangle obtained in this fashion from any point (x', y', z') in the cube. If camera (3) is adjusted so that triangle (1)(2)(3) is equilateral, then triangle $P_1P_2P_3$ will be equilateral. In any event,

$$\frac{P_1P_2}{d_{12}} = \frac{P_2P_3}{d_{23}} = \frac{P_3P_1}{d_{31}}.$$

Fig. 3. Cube with Three Cameras in Place

Let Q_1 and Q_2 denote the vertices of the upper side of the cube beneath the points (1) and (2) (see Fig. 4). Let Q_3 denote the intersection of the edge of the cube facing Q_1Q_2 and the line perpendicular to this edge from camera (3). We let the x-coordinate axis in the preferred 23 system be Q_2Q_3 and the y-coordinate axis be a line on the upper face of the cube perpendicular to Q_2Q_3 . We use the same unit of measurement as in the preferred 12 system. The z-coordinate is as in the preferred 12 system. We let the x-coordinate axis in the preferred 31 system be Q_3Q_1 and the

y-coordinate axis be a line on the upper face of the cube perpendicular to Q_3Q_1 . The unit of measurement and the z-coordinate are the same for all three systems.

Fig. 4
Preferred 23 and 31 Systems

Let $(\bar{x}_1, \bar{y}_1)$ be the coordinates of P_1 in the preferred 31 system, and let $(\bar{x}_3, \bar{y}_3)$ be the coordinates of P_3 in this system. Since P_1P_3 is parallel to Q_1Q_3 , $\bar{y}_1 = \bar{y}_3$. Let $\Delta_{31} = (P/d_{31})(\bar{x}_1 - \bar{x}_3)$. Since

$$\frac{P_3P_1}{d_{31}} = \frac{P_1P_2}{d_{12}}, \Delta_{31} = \Delta_{12}.$$

Hence, the maximum value Δ_{31} attains, as our point varies inside the cube, is $dP/(h+d)$. Thus,

$$0 \leq \Delta_{31}(z') \leq \frac{dP}{h+d} = \Delta_{31} \max.$$

Similarly, for the preferred 23 system, we define Δ_{23} to be P/d_{23} times the x-coordinate of P_3 minus the x-coordinate of P_2 . The y-coordinates of P_2 and P_3 are equal, and $\Delta_{23} = \Delta_{31} = \Delta_{12}$. We have, as before,

$$0 \leq \Delta_{23}(z') \leq \frac{dP}{h+d} = \Delta_{23} \max.$$

Note that for a given point in the cube, all Δ_{ij} are equal.

We use the bound on $\Delta_{ij}(z')$ for the crude criterion to separate possible pairs of tracks. In addition to other things, this criterion separates tracks of different curvature. A finer criterion, called an estimator, based on the equality of the Δ_{ij} will be used on the possible triples.

III. THE METHOD

A. Possible Pairs

First we compare tracks from each pair of views to see which pairs could have come from the same track. Consider two tracks, one from view i and one from view j . Each track consists of a sequence of bubbles, which we join by straight-line segments. To find possible pairs of tracks, we start with the original bubble sequence on one of the tracks, called the master track; then we draw a line from a bubble of this track parallel to $(i)(j)$ and get a "corresponding" point on the other track, called the slave track. When the slave track is almost parallel to $(i)(j)$, it is quite difficult to find the "corresponding" points on it. To avoid this situation, we compute the angle θ_{ij}^{ik} between the tangent to the midpoint of track k from view i and $(i)(j)$. We also compute the angle θ_{ij}^{jl} between the tangent to the midpoint of track l from view j and $(i)(j)$. When θ_{ij}^{ik} or θ_{ij}^{jl} is too small, say less than 10° , we only use such a track for a master track.

We can start out by taking all tracks from view j as master tracks and comparing them with the tracks from view i . When we come to a track from view i whose θ angle is too small, we switch roles. If the θ angles of both tracks are too small, then they are almost parallel and we can define a distance between them. If the two tracks are farther apart than a value $d_{\max}$, which can be determined, they are not considered a pair; if they are equal to or less than $d_{\max}$, they are considered a possible pair, and we note that they were so classified because of a bad stereo view.

Fig. 5. The "Corresponding" Point

An upper limit for $d_{\max}$ is $\Delta_{\max} = dP/(h+d)$; experience with a particular bubble chamber can refine this upper limit. For each bubble on track l , we draw a line through it parallel to $(i)(j)$ (the x -axis for the preferred ij system) (see Fig. 5). This line intersects track k in a "corresponding" point. We compute Δ_{ij} for the points given by the bubbles on track l and the "corresponding"

points on track k . We then check if $0 \leq \Delta_{ij} \leq \Delta_{ij\max} = dP/(h+d)$. If this test fails for one pair of points, we stop. If it does not fail for all pairs of points obtained in this fashion, we say that tracks k and ℓ are a possible pair. They could be the same track in two views.

We then store all the possible pairs according to which pairs of views they come from, as follows: We suppose that all the tracks from each view are numbered and that this numbering is fixed. We write possible pairs of tracks from view (1)(2) as an ordered couple with the track from view 1 occurring first. The ordered couples of possible pairs from view (2)(3) has the track from view 2 first, and the ordered couples of possible pairs from view (3)(1) has the track from view 3 first. With each possible pair of tracks, we preserve their θ angles.

If we have a few tracks in each view; our three lists of possible pairs might look as follows:

View (1)(2)	View (2)(3)	View (3)(1)
1. (1,1)	1. (1,3)	1. (1,2)
2. (1,4)	2. (2,2)	2. (2,4)
3. (2,5)	3. (3,3)	3. (3,1)
4. (3,3)	4. (3,5)	4. (3,3)
5. (4,2)	5. (4,3)	
6. (4,3)	6. (5,1)	
7. (5,6)	7. (6,4)	

Our task is to find all possible triples from these three lists. Since (1,1) from view (1)(2), (1,3) from view (2)(3), and (3,1) from view (3)(1) occur as possible pairs, (1,1,3) is a possible triple. When we write a triple of tracks, it is understood that the first track mentioned is from view 1, the second from view 2, and the third from view 3. The other possible triples are (1,4,3), (2,5,1), and (4,2,2). Note that not all these triples could represent the three views of a track, as some of them (e.g., (1,1,3) and (1,4,3)) have a track in common. In addition to eliminating ambiguities like this, we will use an estimator to see if each triple is acceptable.

B. Triples from Pairs

The procedure we suggest to obtain the possible triples from the three lists of possible pairs starts at the top of the first two lists. We compare the second track of the first couple in the first list with the first track of the first couple in the second list. If these are unequal, we proceed down the second list until we either get equality or exhaust that list. If we exhaust the second list, we start this comparison again with the second couple in the first list and the first couple in the second list. If we do get equality in a comparison between the top couple of the first list and a couple of the second list, we proceed to compare the second track of the couple from the

second list with the first track of the couples in the third list starting from the top. Eventually, we either get equality or exhaust the third list. If we exhaust the third list, we continue down the second list. If we get an equality, we compare the second track of the couple from the third list with the first track of the couple from the first list. If we do not get equality, we continue down the third list. If we do get equality, we store the three tracks as a possible triple, keep the couple from the first list, and continue down the second list. In this procedure, when the third list is exhausted, we proceed down the second list, when the second list is exhausted, we proceed down the first list. We stop when the first list is exhausted.

To get a better idea of the logic involved, we consider the skeleton of a program to find the possible triples. We number the possible pairs from each pair of views and consider this number fixed.

Suppose the (1)(2) view has r pairs, the (2)(3) view has s pairs, and the (3)(1) view has t pairs. Let C_{pq}^{ij} stand for the q th track ($q = 1, 2$) in the p th couple of view ij . Possible track triples are stored in T . (When a triple is stored, the first track stored is from view (1), the second from view (2), and the third from view (3).)

With this notation, consider the following procedure:

1. Set $\ell = m = n = 1$. Go to 8.
2. Add 1 to n .
3. If $n \neq t + 1$, go to 9.
4. Put $n = 1$, and add 1 to m .
5. If $m \neq s + 1$, go to 8.
6. Put $m = 1$, and add 1 to ℓ .
7. Stop when $\ell = r + 1$.
8. If $C_{\ell 2}^{12} \neq C_{m 1}^{23}$, go to 4.
9. If $C_{m 2}^{23} \neq C_{n 1}^{31}$, go to 2.
10. If $C_{\ell 1}^{12} \neq C_{n 2}^{31}$, go to 2.
11. Store $C_{\ell 1}^{12}, C_{m 1}^{23}, C_{n 1}^{31}$, in T , and go to 4.

C. The Estimator

Now that we have all possible triples, we assign an estimator to each triple. If the value of its estimator is greater than a number, determined by experience, we will not consider that triple any further. The estimator associated with a triple measures how much the triangle formed

by a bubble on a master track from one view and its two corresponding points from the other two views deviates from a triangle similar to triangle (1)(2)(3). Given a triple, each track from the triple has two θ angles associated with it. Hence we have six θ angles associated with our triple of tracks. If θ_{ij}^{ik} is the smallest of these, we do not consider the (i)(j) view, and the track from the other view is chosen as the master track. Thus we consider two sides of a triangle, and these two sides are chosen to correspond to the best stereo views. For a correct matching, the Δ 's for these two sides are equal. Suppose that for a triple, θ_{23}^{3k} is the

Fig. 6. Sides for Estimator

is the smallest θ . We then choose the track from view 1 to be the master track and consider only views (1)(3) and (1)(2) (see Fig. 6). The estimator in this situation is obtained as follows: From the master track in view 1, we find the "corresponding" points on the tracks from views 2 and 3. Assume the track in view 1 has r points. Each of these points determines a Δ_{12} and a Δ_{13} . The estimator is

$$\frac{1}{r} \sum_{i=1}^r \left(\frac{2(\Delta_{12}^{(i)} - \Delta_{13}^{(i)})^2}{\Delta_{12}^{(i)} + \Delta_{13}^{(i)}} \right),$$

where $\Delta_{12}^{(i)}$ and $\Delta_{13}^{(i)}$ are computed for the i th bubble on the master track. The denominator in the estimator is there

to normalize the effects of the varying size of the triangle. Note that with this estimator we have no difficulty with those triangles that include a possible pair from a bad stereo view since that view would be automatically excluded. Note also that except for measuring errors and first-order optics, the estimator for a correct matching should be identically zero.

D. The Equivalence Classes

We consider only those triples whose estimators are small enough; all other triples are eliminated.

As we saw from the example cited before, the same track can occur in two possible triples. To handle this ambiguity, we divide the triples into classes. We call two triples related if they have at least one track in common. Thus, (1,3,6) and (2,3,7) are related. We call two triples, T_1 and T_n , equivalent if there are $(n-2)$ other triples T_i such that T_i is related to T_{i+1} for $i = 1, \dots, (n-1)$. So if our list of possible triples includes $T_1 = (1,3,6)$, $T_2 = (2,3,7)$, $T_3 = (2,4,3)$, and $T_4 = (6,7,3)$, then T_1 is said to be equivalent to T_4 (indeed any T_i is equivalent to any T_j , $i, j = 1, \dots, 4$). It can

easily be seen that this definition of equivalence is an equivalence relation (i.e., it is reflexive, symmetric, and transitive) and so divides the list of possible triples into equivalence classes.

In practice, we first divide the triples into classes that are easy to find, where each class is contained in an equivalence class. Then we join classes together until eventually we get the complete equivalence classes. Suppose that view 1 has n_1 tracks, view 2 has n_2 tracks, and view 3 has n_3 tracks. Without any loss of generality, we can assume that $n_1 \leq n_3$ and $n_2 \leq n_3$. We then divide the triples into classes according to their first track. All triples with the same first track will be in one class. With each class we associate three tuples of zeros and ones, one an n_1 -tuple denoted by x , one an n_2 -tuple denoted by y , and one an n_3 -tuple called z . The x -tuple associated with the class of triples containing the i th track from view 1 will be $(0, \dots, 0, 1, 0, \dots, 0)$. The y -tuple will have ones in those positions whose numbers are the same as the numbers of the tracks from view 2 occurring in its class, and zeros in other positions. The z -tuple is defined similarly with respect to the third view. If we form a new class by combining the triples of two classes, the new x -(or y - or z -)tuple is formed from the old one by putting ones in those positions in which either of the original ones has a one and zeros elsewhere. The x -, y -, and z -tuples then tell us which tracks from views 1, 2, or 3, respectively, occur in the class of triples. Two classes will be joined whenever either their x -, y -, or z -tuples have a one in common.

We will demonstrate by an example how to form the complete equivalence classes before describing the method in general. We suppose we have seven or fewer tracks in each view, and let the following be the list of possible triples: $(1,1,5)$, $(2,1,6)$, $(3,2,3)$, $(5,4,2)$, $(6,5,6)$, $(4,6,1)$, $(3,7,4)$, $(4,3,1)$. Then we get the first classes by considering the tracks in view 1. These are as follows:

<u>Class 1</u>	<u>Class 2</u>	<u>Class 3</u>
$(1,1,5)$	$(2,1,6)$	$(3,2,3)$ $(3,7,4)$
$x_1 = (1,0,0,0,0,0)$	$x_2 = (0,1,0,0,0,0)$	$x_3 = (0,0,1,0,0,0)$
$y_1 = (1,0,0,0,0,0,0)$	$y_2 = (1,0,0,0,0,0,0)$	$y_3 = (0,1,0,0,0,0,1)$
$z_1 = (0,0,0,0,1,0)$	$z_2 = (0,0,0,0,0,1)$	$z_3 = (0,0,1,1,0,0)$
<u>Class 4</u>	<u>Class 5</u>	<u>Class 6</u>
$(4,6,1)$ $(4,3,1)$	$(5,4,2)$	$(6,5,6)$
$x_4 = (0,0,0,1,0,0)$	$x_5 = (0,0,0,0,1,0)$	$x_6 = (0,0,0,0,0,1)$
$y_4 = (0,0,1,0,0,1,0)$	$y_5 = (0,0,0,1,0,0,0)$	$y_6 = (0,0,0,0,1,0,0)$
$z_4 = (1,0,0,0,0,0)$	$z_5 = (0,1,0,0,0,0)$	$z_6 = (0,0,0,0,0,1)$

We construct the following matrix consisting of the y-tuples from the various classes where y_i is the i th row of this matrix:

	1	0	0	0	0	0	0	
	1	0	0	0	0	0	0	
	0	1	0	0	0	0	1	
	0	0	1	0	0	1	0	
	0	0	0	1	0	0	0	
	0	0	0	1	0	0		

Classes j and k are combined if rows j and k have a one in the same column. Since rows 1 and 2 have a one in the first column we combine classes 1 and 2 and recompute the x -, y -, and z -tuples.

<u>Class 1</u>	<u>Class 2</u>	<u>Class 3</u>
(1,1,5)	(3,2,3)	(4,6,1)
(2,1,6)	(3,7,4)	(4,3,1)
$x_1 = (1,1,0,0,0,0)$	$x_2 = (0,0,1,0,0,0)$	$x_3 = (0,0,0,1,0,0)$
$y_1 = (1,0,0,0,0,0)$	$y_2 = (0,1,0,0,0,1)$	$y_3 = (0,0,1,0,0,1,0)$
$z_1 = (0,0,0,0,1,1)$	$z_2 = (0,0,1,1,0,0)$	$z_3 = (1,0,0,0,0,0)$

<u>Class 4</u>	<u>Class 5</u>
(5,4,2)	(6,5,6)
$x_5 = (0,0,0,0,1,0)$	$x_6 = (0,0,0,0,0,1)$
$y_5 = (0,0,0,1,0,0,0)$	$y_6 = (0,0,0,0,1,0,0)$
$z_5 = (0,1,0,0,0,0)$	$z_6 = (0,0,0,0,0,1)$

We now construct the following matrix whose rows are the z_i :

	0	0	0	0	1	1	
	0	0	1	1	0	0	
	1	0	0	0	0	0	
	0	1	0	0	0	0	
	0	0	0	0	1		

We again join those classes whose rows have a one in the same column; hence we combine classes 1 and 5. Our classes now are:

<u>Class 1</u>	<u>Classes 2, 3, and 4 remain the same</u>
(1,1,5)	
(2,1,6)	
(6,5,6)	
$x_1 = (1,1,0,0,0,1)$	
$y_1 = (1,0,0,0,1,0,0)$	
$z_1 = (0,0,0,0,1,1)$	

We now construct the following matrix whose rows are the x_i :

$$X = \begin{bmatrix} 110001 \\ 001000 \\ 000100 \\ 000010 \end{bmatrix}$$

There is no combining because of this. The matrices whose rows are, respectively, the y_i and z_i are as follows:

$$Y = \begin{bmatrix} 1000100 \\ 0100001 \\ 0010010 \\ 0001000 \end{bmatrix} \quad Z = \begin{bmatrix} 000011 \\ 001100 \\ 100000 \\ 010000 \end{bmatrix}$$

Since we join no classes because of matrices X, Y, or Z, we have the complete equivalence classes.

In the general situation, after we obtain the first classes according to the tracks in the view with the fewest tracks, say it is view 1, we compute the matrix Y whose rows are y_i , where y_i is the y-tuple of the i th class. We join those classes together when the corresponding rows of Y have a one in the same column. Then we compute the matrix Z whose rows are z_i , where z_i is the z-tuple of the current i th class. We join classes as before. The matrix X is obtained in a similar fashion from the x-tuples. We continue doing this until we reach three consecutive X, Y, and Z matrices whose rows do not have a one in a common column; i.e., we go through three matrices without joining any classes. At this point, the equivalence classes are complete.

E. The Maximal Independent Sets

We next describe the process of picking the best triples in an equivalence class. Recall class 1 of the example cited before. This equivalence class contained (1,1,5), (2,1,6), and (6,5,6). Hence this class could be viewed as either containing the three views of two tracks, namely, (1,1,5) and (6,5,6), or else containing the three views of one track (2,1,6). Notice that if we consider (2,1,6) as representing a track, then no other triple in class 1 could represent the views of a track since each of the other triples has a track in common with (2,1,6). When this situation occurs, we adopt the view that the larger number of tracks is the more likely situation and that the smaller number probably arose from some spurious occurrence. If experience proves that this view is wrong, we could choose the triple with smallest estimator in the equivalence class as the three views of one track, then eliminate all related triples from the class, redivide what is left of the class into new equivalence classes, and

repeat this process until nothing is left. From our point of view, (1,1,5) and (6,5,6), would be chosen as the correct views of two tracks.

To pick the best triples, we introduce a few simple notions from graph theory. An undirected graph, called simply a graph, is a set of points called vertices and a set of lines called edges joining some of the vertices. We also assume that any two vertices are joined by at most one edge. A set of vertices in a graph is called an independent set if no two of the vertices in the set are joined by edges. We call an independent set a maximal independent set if it is an independent set with the largest number of vertices. We obtain a graph from an equivalence class as follows: Consider each triple as a vertex, and join two vertices by an edge if the triples are related. Our equivalence class then becomes a connected graph, and the sets of candidates for the best tracks are the maximal independent sets in this graph. We will illustrate how to find these sets in the following example.

Suppose we have an equivalence class consisting of the following seven triples:

1. (1,3,6)
2. (1,4,6)
3. (2,3,7)
4. (1,6,7)
5. (2,4,7)
6. (3,5,8)
7. (3,4,9)

We will refer to these triples as either line triples or vertices. We determine which vertices each vertex is related to. For our example, this is given by the following table:

1	2	3	4	5	6	7
2	1	1	1	2	7	2
3	4	4	2	3		5
4	5	5	3	4		6
	7		5	7		

We rearrange the vertices and write them in the order of the number of vertices they are related to. Hence we have 6,1,3,7,2,4,5. We form a symmetric matrix of zeros and ones by writing the vertex numbers on top and on the side and putting a one whenever the column heading is related to the row heading and a zero otherwise. This gives us:

$$M = \begin{array}{c|cccccc} & 6 & 1 & 3 & 7 & 2 & 4 & 5 \\ \hline 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 7 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 4 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}$$

If we join related points, the resulting graph can be represented pictorially as follows:

This graph has seven vertices and eleven edges. The number of ones in M gives the number of edges emanating from each vertex of the graph, and since each edge joints two vertices this number is twice the number of edges in the graph. Note that M has 22 ones in it.

As an aid in finding the maximal independent sets in a graph, we state the following proposition:

Proposition: The number, N , of edges emanating from the vertices in any independent set must be less than or equal to the number of edges in the graph.

This is so since no two vertices in an independent set are joined by an edge of the graph.

To find the maximal independent sets, we work with the matrix M . The idea is to start with a vertex that has the smallest number of edges emanating from it, then choose an unrelated vertex with the smallest number of edges emanating from it, and continue this process as long as possible. If we have more than one choice at some stage, we go back and consider each of the different choices. We continue this process by starting with the vertex with the smallest number of edges which was not yet considered. At each stage we keep track of N , and we stop if the next possible vertex choice would result in a set with too large an N (that is, greater than the number of edges). We also keep track of the number of

vertices in the largest independent set found since we do not want the independent sets but only the maximal ones. We do this now for the specific M we started with.

$$M = \begin{array}{c|cccccc} & 6 & 1 & 3 & 7 & 2 & 4 & 5 \\ \hline 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 7 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 4 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}$$

Since there is only one vertex with one edge, we clearly start with 6. Here $N = 1$. As we proceed along the first row, we encounter a zero in the second column, which means that vertex 1 is unrelated to 6 and can now be chosen. Here $N = 4$. The last column is the only one with zeros in both its first and second positions; hence the independent set we have is 6,1,5 ($N = 8$). We list our choices and their associated N 's as follows:

	N		N		N		N		N		N		
6	1	6	1	6	1	1	3	1	3	3	3	7	3
1	4	3	4	2	5	7	6	5	7	7	6		
5	8	2	8										

We need not continue past this point since vertex 2 has four edges emanating from it, and we know that there are independent sets with three vertices, so that a set of three vertices not yet considered and containing vertex 2 must have an N of 12 and so cannot be independent. The maximal independent sets in this case then are 6,1,5 and 6,3,2.

For each independent set, we calculate the sum of the estimators of the triples in it. We choose, as representing the three views of the same track, those line triples in the maximal independent set whose estimator sum is the least.

After we have our list of accepted line triples, we want to go back and recapture those tracks that have appeared in only two views. To do this, we return to the listing of possible pairs. We remove each possible pair that has a track in common with a track in an accepted line triple. Of the list of possible pairs remaining, we consider only those whose tracks do not occur in any other pair in our list. We call these the two views of the same track. This does not guarantee that these are two views of the same track, but in reasonable situations they most likely will be.

The situation in which there are several possible "corresponding points" presents no logical difficulty if there are a sufficient number of points on the track so that a "sense" of direction can be kept. However, if the number of points on the track is small, there is a logical difficulty. This report does not treat this case. The matching of tracks that turn through more than 90° also presents a problem that can be treated in a number of simple ways. An example is to break such tracks into sections, each of which turns through less than 90° , and treat the sections separately.

We also have to consider the situation, which will be possible in automatic systems, where the same track is measured more than once but labeled differently. If there is one view in which the track is measured only once, then the above logic leads to no ambiguities. However, if a track is measured more than once in all views, this track will appear as two or more separately matched tracks. The resolution of this possible ambiguity is very straightforward. Note that the various measurements of the same track will always show up in the same equivalence class. Therefore, all we need do is to check whether the finally selected members of an equivalence class are indeed distinct tracks. There are several ways of doing this. One way might be to fit a circle to the various tracks and see if any have the same centers and radii. A more sensitive test would be to take the rms deviations of the points of one track from the curve represented by the other tracks. Tracks whose rms deviation is smaller than what would be expected from measuring errors would be considered the same track. Note that for two tracks to in fact be the same track, the information from all three views must be used. In any one view, two different tracks can be superimposed.