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SYMMETRICALLY LOADED CYLINDRICAL SHELL WITH FIXED ENDS

L. E. Anderson

In this memo formulas are developed for the stresses and deflections in a circular cylindrical shell which is submitted to a uniform internal or external pressure. No axial forces are considered to be acting on the shell. The wall thickness of the shell is taken as constant and the ends of the shell are considered to be rigidly held.

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Deflections

The general equation for deflection of the shell is

$$(1) \quad \frac{d^4 w}{dx^4} + 4 \beta^4 x = \frac{p}{D}$$

where

w is the radial deformation, positive when directed inward

x is the axial position of the deformation

p is the acting pressure, positive when directed inward

$$\beta = \left[\frac{E h}{4 a^2 D} \right]^{1/4}$$

$$D = \frac{E h^3}{12 (1 - \nu^2)}$$

h is the wall thickness

a is the mean radius of the shell

ν is Poisson's ratio

E is Young's modulus

The general solution of this equation is

$$W = e^{-\beta x} \left[C_1 \cos \beta x + C_2 \sin \beta x \right] + e^{\beta x} \left[C_3 \cos \beta x + C_4 \sin \beta x \right] + \frac{p}{4 \beta^4 D}$$

or, in terms of hyperbolic functions,

$$(2) \quad W = C_1 \sin \beta x \sinh \beta x + C_2 \beta x \cosh \beta x + C_3 \cos \beta x \sinh \beta x \\ + C_4 \cos \beta x \cosh \beta x + \frac{p}{4 \beta^4 D}$$

The constants of integration may be evaluated by considering the boundary conditions. Since conditions at both ends of the shell are the same, the deflection curve must be an even function if the origin is taken at the longitudinal midpoint. Therefore, all odd terms on the right side of Eq. (2) must vanish. This is possible only if C_2 and C_3 are equal to zero. Then, Eq. (2) becomes

$$w = C_1 \sin \beta x \sinh \beta x + C_4 \cos \beta x \cosh \beta x + \frac{p}{4 \beta^4 D}$$

If both ends of the shell are fixed then the deflection and the slope are equal to zero at the ends. Thus,

$$w = \frac{dw}{dx} = 0 \quad \text{at} \quad x = \pm \frac{l}{2},$$

the total length of the shell being taken as l .

Substituting these relations into Eq. (2) yields the following:

$$C_1 = \frac{-p}{4 \beta^4 D} \left[\frac{\sin \alpha \cosh \alpha - \cos \alpha \sinh \alpha}{\sinh \alpha \cosh \alpha + \cos \alpha \sin \alpha} \right]$$

$$C_4 = \frac{-p}{4 \beta^4 D} \left[\frac{\sin \alpha \cosh \alpha + \cos \alpha \sinh \alpha}{\sinh \alpha \cosh \alpha + \cos \alpha \sin \alpha} \right]$$

where, for simplification, the term $\frac{\beta l}{2}$ has been replaced by α .

Substituting these values back into Eq. (2) gives the final expression for w :

$$(3) \quad w = \frac{p}{4 \beta^4 D} \left[1 - \left(\frac{\sin \alpha \cosh \alpha - \cos \alpha \sinh \alpha}{\sinh \alpha \cosh \alpha + \cos \alpha \sin \alpha} \right) \sin \beta x \sinh \beta x \right. \\ \left. - \left(\frac{\sin \alpha \cosh \alpha + \cos \alpha \sinh \alpha}{\sinh \alpha \cosh \alpha + \cos \alpha \sin \alpha} \right) \cos \beta x \cosh \beta x \right]$$

Forces and Moments

The following relations, which may be developed from equilibrium considerations in conjunction with Hook's Law, are necessary for finding stresses in the shell.

$$(4) \quad N_{\phi} = - \frac{E h w}{a}$$

$$(5) \quad N_x = 0 \quad (\text{original assumption})$$

$$(6) \quad M_{\phi} = \nu' M_x$$

$$(7) \quad M_x = - D \frac{d^2 w}{dx^2}$$

where

N is a force per unit length

M is a moment per unit length

Subscript x refers to the axial direction

Subscript ϕ refers to circumferential direction

By substituting values of the deflection w (or its derivatives) from Eq. (3) into Eqs. (4), (5), (6), and (7), the values of all moments and forces at any point may be found. (Shearing forces are neglected in this discussion.)

At the midpoint of the shell ($x=0$) the following relations are found for the forces and moments.

$$N_{\phi} = - p a \left[1 - \frac{\sin \alpha \cosh \alpha + \cos \alpha \sinh \alpha}{\sinh \alpha \cosh \alpha + \cos \alpha \sin \alpha} \right]$$

$$N_x = 0$$

$$M_x = \frac{p}{2 \beta} \left[\frac{\sin \alpha \cosh \alpha - \cos \alpha \sinh \alpha}{\sinh \alpha \cosh \alpha + \cos \alpha \sin \alpha} \right]$$

$$M_{\phi} = \nu M_x$$

At either end of the shell the expressions are found to be

$$N_{\phi} = 0$$

$$N_x = 0$$

$$M_x = \frac{p}{2\beta^2} \left[\frac{\sin 2\alpha - \sinh 2\alpha}{\sin 2\alpha + \sinh 2\alpha} \right]$$

$$M_{\phi} = \nu M_x$$

Effect of Longitudinal Forces

In the above derivations it is assumed that no axial force, N_x , is present. If such a force is acting on the shell it will have a constant value,

$$N_x = \frac{p' a}{2},$$

where p' is some uniform pressure acting on the ends of the shell, at all points in the shell. This force and its resultant deflection may be easily superimposed upon the forces and deflections caused by the lateral pressure.

For example, suppose the force, N_x , is caused by a pressure on the ends of the shell which has the same value as the lateral pressure. Due to this force an additional radial deflection is brought about.

$$w' = - \frac{\nu a^2 p}{2 E h} f(x)$$

where $f(x)$ is the same function as the one which appears within the brackets in Eq. (3).

Substituting for β in Eq. (3) gives

$$w = \frac{p a^2}{E h} f(x)$$

Adding these together algebraically gives

$$\text{Total deflection} = w' + w = \frac{p a^2}{E h} f(x) \left[1 - \frac{z'}{2} \right]$$

This expression differs from the equation for the deflection due to lateral pressure alone only by the term $\left[1 - \frac{z'}{2} \right]$. Therefore, to include the effect of N_x the previously developed equations for M_x , M_ϕ , and N_ϕ should be multiplied by $\left[1 - \frac{z'}{2} \right]$ since they depend upon w (or one of its derivatives) in a linear way.

Stresses

After moments and forces are obtained, stresses may be found simply by substituting in the following standard equations:

$$\sigma_x = \frac{N_x}{h} + \frac{6 M_x}{h^2}$$

$$\sigma_\phi = \frac{N_\phi}{h} + \frac{6 M_\phi}{h^2}$$