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UNITED STATES ATOMIC ENERGY COMMISSION

THE HARMODOTRON—A BEAM HARMONIC,
HIGHER ORDER MODE DEVICE FOR
PRODUCING MILLIMETER AND
SUBMILLIMETER WAVES

Technical Report No. 1

By
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September 10, 1956

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MILLIMETER AND SUBMILLIMETER WAVES

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by

M.D. Sirkis

Contract No. AT(11-1)-392

U.S. ATOMIC ENERGY COMMISSION

10 September 1956

Approved by

A handwritten signature in cursive script, reading "Paul D. Coleman". The signature is written in dark ink and is positioned over the printed name and title.

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ABSTRACT

A new method for the generation of relatively high powers at submillimeter wavelengths through the excitation of a single selected TM_{0mn} mode in a cylindrical cavity is theoretically investigated by means of a field theory approach. Experiments at a wavelength of 8.310 millimeters which successfully demonstrate the method are described.

The oscillation in the cavity is excited by a high energy (1 mev range), bunched electron beam, produced by a single cavity electron accelerator operating at 2775 megacycles per second. It is shown that satisfactory coupling between the electron beam and the field within the resonant cavity is possible only for beam velocities approaching the velocity of light. The bunched electron beam contains current frequency components at high harmonics of the 2775 megacycles per second fundamental frequency. The cylindrical cavity is tuned to resonate at one of these harmonics.

The theoretical relationship between the generated power, P , and the harmonic current component, I_s , is typically of the order of

$$P(\text{watts}) \approx 3,000 [I_s (\text{amps})]^2 \sqrt{\lambda(\text{mm})}$$

for a 1 mev electron beam. If a harmonic current amplitude of 100 milliamperes (a reasonable value) is assumed, the theory predicts a pulsed generated power of approximately 30 watts at a wavelength of one millimeter.

1. INTRODUCTION

The study of the interaction of electromagnetic radiation with matter has brought rich rewards to the physicist in his quest for a better understanding of the material world in which he lives. From investigations ranging over the entire electromagnetic spectrum, he has gathered a large part of his knowledge of matter and energy. Indeed, the entire structure of modern physics has its foundations in experiments concerning the interaction of electromagnetic radiation and matter. Planck's treatment of black body radiation¹, Einstein's discussion of the photoelectric effect², and Bohr's introduction of the quantized atom³--all efforts to explain radiation experiments--played important roles in the transformation from classical physics to modern physics which occurred in the late nineteenth century and early twentieth century.

The importance of electromagnetic radiation in physical research is undiminished today. The nuclear physicist gathers knowledge of the nucleus with the aid of the very short wavelength γ -rays. The atomic physicist and the solid state physicist find answers to some of their questions with the aid of radiation in the region from the infrared to the ultraviolet portion of the spectrum. The microwave spectroscopist investigates matter with the aid of radiation in the range of centimeter and decimeter wavelengths, and the infrared spectroscopist concentrates his efforts at wavelengths less than 100 microns.

There is, however, a range of wavelengths between 0.1 millimeter and 1.0 millimeter which has hardly been used as a diagnostic tool. At these

wavelengths the continuous spectrum sources familiar from infrared work are unsatisfactory for work at wavelengths longer than about 0.05 millimeter, whereas electronic oscillators have not yet been built for work below wavelengths of about 2.6 millimeters.

The difficulties associated with the continuous spectrum sources can be illustrated by a consideration of the power radiated by a black body. The power radiated per unit area by a black surface of area A per unit increment of wavelength λ in a direction normal to the surface is given by

$$\epsilon_{\lambda T}/A = \{1.177 \times 10^{-12} \lambda^{-5} / e^{1.432/\lambda T} - 1\} \text{ watts/cm}^2 \text{ steradian, (1.1)}$$

where λ is in centimeters and T is the temperature in degrees Kelvin. Such a source is capable of appreciable power in the 1 to 50 micron range of wavelengths. However, at longer wavelengths of the order of 0.5 millimeter, these sources are unsatisfactory. For a wavelength of 0.5 millimeter and a temperature of 10^4 degrees,

$$\lambda T = 5 \times 10^2 \text{ cm degrees.}$$

For this value of λT , Eq. 1.1 may be approximated by the Rayleigh-Jeans law:

$$\epsilon_{\lambda T}/A \approx 8.22 \times 10^{-13} T/\lambda^4 \text{ watts/cm}^2 \text{ steradian.}$$

For a one percent bandwidth

$$d\lambda = 0.01 \times 0.05 = 5 \times 10^{-4} \text{ cm.}$$

and

$$(\epsilon_{\lambda T}/A)d\lambda = 6.6 \times 10^{-7} \text{ watts/cm}^2 \text{ steradian.}$$

The total power radiated by a black surface is given by the Stefan-Boltzmann Law.

$$\mathcal{E} = \sigma AT^4 = 5.710 \times 10^{-12} AT^4, \text{ watts}$$

where as before A is the area of the surface in square centimeters and T is the temperature in degrees Kelvin. For the case considered above

$$\mathcal{E} \approx 5.7 \times 10^4 \text{ watts.}$$

Thus it is seen that extremely large input powers are required to produce very small power in the range of wavelengths near 0.5 millimeter.

By frequency multiplication techniques, using crystal harmonic generators (See Chapter 2), Gordy⁴ and his co-workers at Duke University have been able to generate radiation at a wavelength of 0.77 millimeter. The power level thus obtained is quite low, probably less than 10^{-12} watt. In practice it is found that even this small amount of power is obtained only with considerable patience, because crystal harmonic generators are extremely tedious to adjust. Even though the Duke group have been able to make spectroscopic and superconductivity measurements at wavelengths in the 1 to 5 millimeter range using crystal harmonic generators as sources, these devices can at best be considered only as a stop-gap measure. Nevertheless, at the present time there is no better source available at millimeter wavelengths.

If this range of the spectrum is to be fully exploited, better sources of radiation at these wavelengths must be developed. It is not difficult to point out several uses for radiation in this so-called submillimeter range of the spectrum.

Perhaps the most obvious use for this radiation is to extend the

range of investigation of microwave spectroscopy. Gordy and his co-workers have already extended this range to the millimeter wave region with their crystal harmonic generators. Further extension will require more powerful sources at shorter wavelengths.

A second field in which radiation at submillimeter wavelengths would be useful as a diagnostic tool is the study of superconductivity.⁵ The postulate of a superconducting quantum state implies an energy gap of the order of kT_c between the superconducting state and the normal state, where T_c is the temperature at which DC superconductivity occurs. For tin, T_c is 3.72 degrees Kelvin, and

$$kT_c/h = 77 \text{ kmc/s.}$$

The onset of superconductivity occurs at progressively smaller temperatures (below T_c) as the frequency is increased. This effect enables one to measure the energy gap between the superconducting state and the ordinary state as a function of temperature. To pursue this work to temperatures of about two degrees Kelvin, submillimeter radiation will be required.

A third use for submillimeter wavelength radiation is in the study of gaseous discharges by microwave methods.⁶ These studies can be extended to higher electron densities only with the aid of higher frequency sources. One method of investigating a gaseous discharge is to establish the discharge in a resonant cavity. The interaction of the free electrons in the discharge with the oscillating electric field in the cavity perturbs the resonant frequency. The change in the resonant frequency is given by

$$\frac{\omega_a' - \omega_a}{\omega_a} = - \frac{1}{2\epsilon_0\omega_a} \int_V \sigma_i E^2 dV / \int_V E^2 dV,$$

where ω_a is the unperturbed resonant angular frequency, ω'_a is the perturbed resonant angular frequency, V is the volume of the cavity, and ϵ_0 is the permittivity of free space. One of the conditions for the validity of this formula is that

$$-\sigma_i = \frac{Ne^2}{m\omega} \ll \omega_a \epsilon_0$$

where σ_i is the imaginary component of the complex conductivity arising from the interaction of the free electrons with the oscillating field. N is the number of free electrons per cubic meter, e is the electronic charge, and m is the electronic mass. Typical experimental values are

$$\begin{aligned} N &= 10^{17} \text{ electrons/m}^3, \\ \omega_a &= 18.8 \times 10^{10} \text{ sec}^{-1} \\ (\lambda_a &= 1 \text{ cm}). \end{aligned}$$

With these numbers

$$\frac{Ne^2}{m\omega_a^2 \epsilon_0} \approx 0.009.$$

Because the frequency appears as a square in the denominator, radiation at a wavelength of 0.5 millimeter would make it possible to study discharges with electron densities of the order of four hundred times those which are admissible to study with centimeter wavelengths.

From the three examples presented here, it should be clear that a real need exists for a satisfactory source of power at submillimeter wavelengths. The present effort to develop a source of submillimeter waves has been undertaken to satisfy that need.

A new and fundamentally different approach to the generation of radiation at millimeter and submillimeter wavelengths is presented here.

This method has been theoretically investigated with the aid of field theory techniques and has been successfully demonstrated at a wavelength of 8.310 millimeters. The extension of this technique to wavelengths less than one millimeter appears entirely reasonable.

A tightly bunched high energy (1 mev range) electron beam obtained from a pulsed single cavity electron accelerator operating at 2775 mc/s is used to excite a higher order TM_{0mn} mode in a cylindrical cavity. The dimensions of the cavity are chosen so that the desired mode resonates at a harmonic of the fundamental frequency of the accelerator. The resonant cavities are of the order of several wavelengths in diameter and length, as opposed to the usual resonant structures in low voltage devices which have dimensions less than a wavelength. The high velocity of the electron beam makes satisfactory coupling between the electron beam and the resonant cavity possible. Power measurements at a wavelength of 8.310 millimeters using the TM_{018} and TM_{028} modes in cylindrical cavities excited by the thirteenth harmonic of the accelerator frequency are in reasonable agreement with the theory.

The relationship between the generated power, P , the wavelength, λ , and the harmonic current amplitude, I_s , is typically of the order of

$$P(\text{watts}) \approx 3,000 [I_s(\text{amps})]^2 \sqrt{\lambda(\text{mm})},$$

for a 1 mev electron beam. For a harmonic current amplitude of 100 milliamperes (a reasonable value), the generated pulsed power at a wavelength of 1 millimeter is of the order of 30 watts. This power is of the order of 10^8 (ten raised to the eighth power) greater than the power available in a one percent bandwidth from a black body at 10^4 degrees Kelvin with a surface area of one square centimeter. It is of the order of 10^{13} (ten

raised to the thirteenth power) greater than the power produced by crystal harmonic generators.

If an electron beam can be produced which has high harmonic content into the submillimeter range of wavelengths, it appears that pulsed power at the watt level at wavelengths below one millimeter can be generated through the harmonic excitation of higher order modes in cavity resonators. At the present time work in progress in this laboratory to produce an electron beam suitable for the generation of submillimeter wavelength radiation by this method is proceeding at an encouraging pace.

2. HISTORICAL BACKGROUND

2.1 Incoherent Sources

The sources of radiation may be classified as either coherent or incoherent. A coherent source produces an electromagnetic field which is characterized by a sinusoidal oscillation at a single frequency. The klystron oscillator is an example of a coherent source. An incoherent source, on the other hand, produces radiation which consists of a series of damped wave trains of random phase relative to each other. Radiation of this type is found to cover a band of frequencies. An incandescent body is an example of an incoherent source.

The earliest attempts to generate radiation at millimeter wavelengths were with incoherent sources. In 1895 P. Lebedew was able to generate radiation at a wavelength of 0.6 millimeter with a spark oscillator.⁷ In 1923 Nichols and Tear worked in the region from 0.22 millimeter to 50 millimeters, also using a spark gap as a source.⁸ During the period from 1924 to 1929 Mme. Glagoleva-Arkadieva experimented with a mass radiator in which metal particles suspended in oil were carried into a spark gap by a rotating wheel which extended into the oil and to which the metal particles adhered. With this device she was able to obtain radiation at wavelengths from 0.129 millimeter to 50 millimeters.⁹ Similar investigations were made by Lewitsky¹⁰ during the period from 1924 to 1927 and more recently by Cooley and Rohrbaugh in 1945.¹¹ An improved form of the mass radiator was built by Daunt and his associates at Oxford University. This device emitted approximately 30 milliwatts of power in the range of wavelengths between 3 and 9 millimeters, with the peak

intensity occurring at 5.7 millimeters.¹² Unfortunately there are large variations in the output power due to fluctuations in the number of metal balls in the spark gap.

Plasma oscillations have also been investigated as a source of millimeter wavelength radiation.¹³ Yunker has succeeded in generating wavelengths from 30 millimeters to 8 millimeters by exciting plasma oscillations in a gassy magnetron.¹⁴ Lampert and his co-workers have studied the possibility of plasma excitation in semi-conductors, but his analysis indicates that this method is not promising.¹⁵

Even if it were possible to develop incoherent sources so that, for example, the range of infrared spectroscopy could be extended into the region around one millimeter, they would still be less desirable than good coherent sources. In spectroscopy the investigator measures absorption or reflection as a function of frequency. If his source is incoherent, the spectroscopist inevitably decreases the power with which he may work as he increases the frequency resolution of his apparatus. This complication puts the infrared spectroscopist at a considerable disadvantage with respect to the microwave spectroscopist who works with a coherent, essentially monochromatic source. Consequently more recent efforts have been directed toward the achievement of a good coherent source.

2.2 Coherent Sources

Since the end of World War II, a considerable effort has been made to extend the useful range of conventional microwave oscillators, in particular the klystron, the magnetron, and the traveling wave tube, into

the millimeter wave region of the spectrum. Four general problems harass this development:¹⁶

1. The fabrication of the devices is difficult because of the small size of the component parts.
2. The required beam current densities become extremely high as the cathode area is decreased.
3. The circuit losses increase in inverse proportion to the square root of the wavelength.
4. Adequate cooling of the small internal structures becomes increasingly difficult as the wavelength is decreased.

These problems will be illustrated by a consideration of some of the devices which have been built to generate power at millimeter wavelengths.

In 1946 Lafferty described a klystron which operated in the 5.80 to 4.15 millimeter range. Data for some of the tubes built by Lafferty appear in Table 1.¹⁷ The dimensions of the klystron structure are shown in Fig. 1. Lafferty concluded in his paper that it is doubtful if oscillations appreciably shorter than four millimeters can be achieved

Tube	Wavelength mm	Beam Potential volts	Reflector Potential volts	Starting Current ma	Peak Current ma	Power mw
70	5.80	1750	-40	32	70	12.7
77	5.60	1600	-50	31	48	5.5
78	4.51	1700	-45	38	49	2.6
87	4.15	1700	-350	16	25	0.2

Table 1. Power Output Measurements (After Lafferty)

by the velocity modulation reflex oscillator principle until new cathodes, capable of emission with a current density considerably higher than that

presently attainable, have been developed.

Figure 1. Reflex Klystron for a Wavelength of 5.5 Millimeters (After Lafferty).

Attempts to scale the magnetron for millimeter wave generation have been made at Columbia University. Data for some 22 vane rising sun magnetrons built at Columbia are given in Table 2.¹⁸ At short wavelengths the high values of magnetic field required becomes a problem. However, Willshaw and his co-workers have shown that magnetrons with very small cathode diameters can be made to oscillate, although at reduced efficiency, in magnetic fields with only about 25% of the usual intensity.¹⁹ Using this technique the Columbia group has made a pulsed magnetron which pro-

Wavelength (millimeters)	6.3	6.3	4.5	3.3	2.5
Anode Diameter (inches)	0.096	0.123	0.0645	0.050	0.038
Small Resonator Diameter (inches)	0.165	0.181	0.1105	0.086	0.064
Large Resonator Diameter (inches)	0.217	0.241	0.148	0.113	0.084
Vane Thickness (inches)	0.0085	0.0105	0.005	0.004	0.003
Anode Height (inches)	0.120	0.200	0.075	0.065	0.048
Cathode Diameter (inches)	0.062	0.077	0.041	0.032	0.025
Anode Potential (volts)	3150	5100	3050	3000	3100
Magnetic Field (gauss)	5200	5200	7700	10,000	13,000

Table 2. 22-Vane Columbia Magnetrons(after Bernstein)

duced 1-2 kilowatts peak power at a wavelength of 2.6 millimeters with about one percent efficiency. The tube operates with an anode potential of about 20 kilovolts and with a magnetic field strength of about 15,000 gauss. The lifetime of this tube is of the order of several hours.

Pulsed operation is used to circumvent the heat problem.

Traveling wave tubes²⁰ using different types of slow wave structures have been made for the range of wavelengths from 4.5 to 6 millimeters. Christensen and Watkins described a backward wave tube covering the

range from 4.5 to 6 millimeters having the following characteristics:²¹

0.025 inch internal diameter of helix

0.002 inch x 0.005 inch tungsten tape for helix

4.0 milliampere electron beam

850-2400 volt beam potential

2000 gauss focusing field

1 milliwatt output power

1300°C. helix temperature due to electron bombardment.

The problems of mechanical construction and of heat dissipation are obvious. In addition the efficiency is quite low. Concerning this tube the authors have written that the power output was limited by the heat dissipation capabilities of the helix. It can be shown that for a given beam voltage and current density there is a minimum wavelength at which oscillations can be made to occur in a traveling wave tube no matter how long the helix is made. For Christensen and Watkin's tube this wavelength is about 2.4 millimeters.

Millman²² and Karp²³ have worked on traveling wave tubes in which the electron beam interacts only part of the time with an electromagnetic wave which propagates along a guiding structure. With a structure of the type shown in Fig. 2 power has been produced in the

Figure 2. Space Harmonic Circuit (After Karp)

range of wavelengths from 2.7 to 2.9 millimeters. In this structure there is an interchange of energy between the electron beam and the electromagnetic field as the electrons pass across the slots. There is no appreciable interaction in the regions between the slots. Only the electrons close to the slotted surface are effective in transferring energy to the electromagnetic field. It appears that the problem of maintaining a sufficiently large number of electrons near the slotted surface will thwart attempts to extend this technique to much shorter wavelengths.

These examples illustrate some of the problems encountered if conventional means are used to extend electronic generation of power into the low millimeter and submillimeter range of the spectrum.

Coherent generation of power at wavelengths below one millimeter has been achieved at very low power levels with the aid of crystal harmonic generators.²⁴ The non-linear behavior of a silicon-tungsten point contact is used to generate harmonics of an incident signal in these devices. In this way a wavelength of 0.77 millimeter has been obtained through the excitation of the 16th harmonic of the 1.2 centimeters signal from a 2K33 klystron oscillator.²⁵ Figure 3 is a simplified sketch of a crossed waveguide crystal harmonic generator. Energy traveling down the larger waveguide excites currents in the tungsten-crystal circuit. The non-linear property of the contact causes harmonic current components to be established in the tungsten wire, which acts like an antenna in the smaller waveguide. The size of the small waveguide is chosen so that all wavelengths longer than the desired wavelength cannot propagate.

Figure 3. Simplified Drawing of a Crossed-Waveguide Crystal Harmonic Generator

Because the harmonic power falls off rapidly with harmonic number, the presence of any higher harmonics causes no complications in practice.

Table 3 gives the results of some measurements made with several common klystrons operating as primary oscillators for crystal harmonic generators.²⁶

Klystron	Power Output	Fundamental Wavelength	Harmonic Number	Harmonic Wavelength	Conversion Loss in db	Signal to Noise
2K33A	35 mw	12.5 mm	2	6.25	18	350
			3	4.17	35	
			4	3.12	45	
			5	2.50	60	
QK290	35 mw	9.5	2	4.75	20	30
			3	3.17	40	
			4	2.38	55	
			5	1.90	70	
QK291	12 mw	8.5	2	4.25	21	100
			3	2.83	40	
			4	2.12	60	
QK294	3 mw	7.5	2	3.75	33	900
			3	2.50	45	

Table 3. Harmonic Signals Generated Using Silicon Crystal Multiplier
(After C.M. Johnson, *et al.*, *Rev. Sci. Instr.* Vol. 25, 214,
March 1954)

These crystal multipliers can absorb several hundred milliwatts of incident power. If a primary oscillator capable of producing approximately 0.5 watt at a wavelength of about two millimeters were available, it would appear that the harmonic multiplication work could be carried readily into the range of submillimeter wavelengths.

Magnetrons have also been used as harmonic generators. Harmonics up to the tenth, corresponding to a wavelength of 1.25 millimeters, have been obtained with a 3J31 magnetron at Columbia University.²⁷ According to Nethercot, the only advantage of the magnetron as a harmonic generator is the relative ease with which it can be operated as compared to the crystal harmonic generator. He states that it is possible to generate radiation at 1.25 millimeters wavelength within several hours with a magnetron, whereas it may take several weeks to do so with a crystal harmonic generator because of the more critical adjustments required.

The experimental apparatus for harmonic generation with a magnetron is shown in Fig. 4.²⁸ The tapered section is used to attenuate all harmonics with wavelengths longer than the one desired. The position of the tapered section is varied to maximize the power generated at the desired harmonic.

Nethercot has found that the signal-to-noise ratio for a $1/3$ megacycle per second bandwidth is approximately given by

$$S/N = 10^{\lambda(\text{mm.})} + 1$$

where $\lambda(\text{mm})$ is the wavelength of the harmonic in millimeters. A signal-to-noise ratio of unity occurs when the peak to peak noise between magnetron pulses is raised by its own height during a pulse. At a

Figure 4. Apparatus for Harmonic Generation with a Magnetron²⁷

wavelength of 1.5 millimeters a signal-to-noise ratio of 1 corresponds to approximately 0.5 microwatt.

Another class of sources, the so-called radiation oscillators, has been investigated by a number of workers.^{29, 30, 31, 32, 33} In these devices a deflecting system of some sort is used to impose a transverse simple harmonic oscillation on a bunched beam of high energy electrons. These electrons then behave as translated oscillators which radiate directly. The magnetic deflecting system of Coleman and Lewis for a radiation oscillator is shown in Fig. 5.³⁴ The wavelength radiated at the angle θ with respect to the x-axis is given by

$$\lambda_p = \frac{\lambda_e}{\beta} (1 - \beta \cos \theta),$$

where β is the ratio of the electron velocity to the velocity of light. Radiation at wavelengths less than 1.9 millimeters has been observed by Motz with a 3 mev electron beam traversing a deflecting system of the type shown in Fig. 5.³⁵

Figure 5 Schematic Diagram of Alternating Gradient Magnetic Deflecting System (After Coleman and Lewis)

Cerenkov found that electrons traveling through a dielectric medium would, under certain conditions, emit radiation.³⁶ The work of Frank and Tamm shows that it is necessary for the electrons to travel faster than the phase velocity of the radiation in the medium for this to occur.³⁷ Danos and Lashinsky have succeeded in generating 0.1 microwatt at K-band wavelengths by using the Cerenkov effect.³⁸

A more recent development is the molecular microwave oscillator which uses a quadripolar cylindrical electrostatic field to separate a beam of

ammonia molecules into two components. The excited component is allowed to enter a cavity which is resonant at the wavelength of the transition radiation emitted when the ammonia molecules return to the ground state. If the power transfer from the beam to the cavity is large enough, self-sustained oscillations will result as subsequent transitions are influenced by the fields in the cavity. Approximately 10^{-8} watts has been obtained with a device of this type at 23.870 kmc/s.³⁹

Although all attempts at generation of millimeter and submillimeter waves have not been discussed here, it should be clear that the problem is difficult and has steadfastly defied solution. Even today there is no device comparable in effectiveness and convenience of operation to the oscillators available at centimeter wavelengths.

In the next chapter a new approach to the problem will be presented.

3. A NEW APPROACH TO THE GENERATION OF MILLIMETER WAVES

3.1 General Considerations

Some of the obstacles which have prevented the generation of appreciable power with a line spectrum at millimeter wavelengths were cited in the preceding chapter. In particular it was pointed out that the scaling of the conventional centimeter wavelength oscillators for use at wavelengths near one millimeter is difficult partly because of the small physical size of the resulting scaled structure. Generally, any resonant structure operating in its fundamental mode must have dimensions which are of the order of one-half of the resonant wavelength. In the usual klystron oscillator with a beam potential of the order of a kilovolt, a reentrant cavity is used because of transit time effects. A cavity of this type has dimensions which are of the order of one-third of the resonant wavelength. An obvious way to circumvent the problems associated with small physical size is to use a higher order oscillation in a non-reentrant structure. In this way it is possible to obtain a practical structure with dimensions which are of the order of several wavelengths.

The TM_{0mn} modes of a right, circularly cylindrical cavity are suitable modes for excitation by an axial electron beam, because they are characterized by a non-zero axial electric field against which the electron beam can do work. The problem of electromagnetic oscillations in a cylindrical cavity with perfectly conducting walls can be solved exactly. The effect of a finite wall conductivity can then be included

by a perturbation technique. For TM_{0mn} modes it is found that the resonant wavelength λ is related to the cavity diameter D and the length L by the expression

$$(D/\lambda)^2 = (r_{0m}/\pi)^2 + (nD/2L)^2$$

where r_{0m} is the m^{th} root of the first order Bessel function $J_0(x)$.

It is interesting to compare the dimensions of a cavity of this type with those of the reentrant cavity which Lafferty used in his 5.5 millimeter klystron. For a TM_{018} mode with $D/L = 0.6$ (a value which the theoretical work of Chapter 5 will prove to be reasonable), it is found that

$$D \approx 0.545 \text{ inch,}$$

and

$$L \approx 0.906 \text{ inch.}$$

This structure is shown in Fig. 6 with Lafferty's cavity drawn to the same scale for comparison. Indeed, a similar TM_{018} cavity, resonant at a wavelength of one millimeter, is larger than the reentrant structure resonant at a wavelength of 5.5 millimeters.

It will be shown that the efficiency of the coupling between the electron beam and the electromagnetic field for the TM_{0mn} mode is measured in terms of the longitudinal coupling coefficient, k_L , defined by

$$k_L^2 = k_+^2 + 2(-1)^n k_+ k_- + k_-^2,$$

Figure 6. A Comparison of a Reentrant Cavity (Lafferty's) and a TM_{018} Cylindrical Cavity, Both Resonant at 5.5 Millimeters. Each Cavity Is Shown Four Times its Actual Size.

where

$$k_{\pm} = \frac{1}{2} \frac{\sin \left(\frac{n\pi}{2} \left(\frac{v_{omn}}{v} \mp 1 \right) \right)}{\left(\frac{n\pi}{2} \left(\frac{v_{omn}}{v} \mp 1 \right) \right)}$$

In this last expression v_{omn} , the phase velocity of the TM_{omn} radiation in the cavity, is greater than c ; v , the axial velocity of the electron beam is, of course, less than c . For low electron velocities both k_{+} and k_{-} are small. Although k_{-} remains small even for high energy beams, k_{+} approaches one-half as the beam velocity approaches the velocity of light. For this reason high energy electrons are required to excite higher order mode structures.

A bunched electron beam contains current components at harmonics of the bunching frequency. The amplitude of these harmonics depends

on the degree of bunching. A higher order mode resonant cavity can be tuned to resonate at a frequency corresponding to a particular harmonic current component of the bunched electron beam. In this way harmonic excitation of higher order modes in a resonant cavity can be achieved. By using harmonic excitation, it is possible to form the electron beam at lower frequencies where high power pulsed equipment is available. In the ten to three centimeter region of wavelengths, a machine capable of producing a tightly bunched electron beam in the mev range of energies, with pulsed output currents of the order of 100 milliamperes, can be built. By Fourier analysis of the theoretical output current waveform of this machine, it can be shown that there are present in appreciable amounts frequency components corresponding to wavelengths in the submillimeter range.

For a TM_{018} cavity with $D/L = 0.6$ and excited by a bunched electron beam with $\beta = 0.95$, it will be shown that the power available at a matched waveguide output is given by

$$P(\text{watts}) \approx 4 \times 10^8 [I_s (\text{amps})]^2 \sqrt{\lambda(\text{mm})}$$

where I_s is the amplitude of the harmonic current component, corresponding to the wavelength, λ . For a bunched beam having a harmonic component of only 10 milliamperes at a wavelength of one millimeter, the generated power is of the order of one-half watt.

3.2 Program

In the chapters which follow, this new approach to the generation of millimeter waves is considered in detail. The development begins in Chapter 4 with a presentation of Slater's treatment of oscillations in a hollow cavity. His approach to the electromagnetic field problem in a waveguide is also used to relate the power delivered to the cavity by the electron beam to the power delivered to the load through the coupling iris which joins the cavity and its waveguide output. Chapter 5 deals with the application of these general methods to the special case of a right circularly cylindrical cavity with a waveguide output. In Chapter 6 a pulsed single cavity electron accelerator suitable for exciting the oscillations is described. Chapter 7 treats the problems which arise from the pulsed excitation of the higher order mode cavities at harmonics of the accelerator frequency.

The remaining part of the discussion concerns the apparatus built for experimental testing of the theory and the results of the experimental investigations. The design of the higher order mode resonators is considered in Chapter 8. In Chapter 9 a magnetic deflection electron spectroscope which was used to measure the electron velocity is described. Chapter 10 deals with the experimental measurement of the power generated as a function of the electron beam velocity. The discussion is concluded in Chapter 11 with the conclusions which were formed as a result of this investigation. A bibliography follows Chapter 11.

4. THE SOLUTION OF MAXWELL'S EQUATIONS IN A HOLLOW CAVITY* 41

4.1 Introduction

In this section the general problem of the solution of Maxwell's equations in a hollow cavity of arbitrary shape will be considered. Following Slater's work, the solution will be obtained as a summation over the normal mode fields of the cavity which will be shown to have certain orthogonality properties. The quantities which appear in Maxwell's equations will be expanded in a series of the appropriate normal modes, and substitution of these expansions into Maxwell's equations will lead to the differential equations which the expansion coefficients must obey. The problem will then be solved when these expansion coefficients have been determined. The excitation of oscillations in a hollow cavity by electronic currents will also be considered.

4.2 Orthogonal Functions in a Hollow Cavity

Consider a hollow cavity of volume V bounded in part by a perfectly conducting surface, S , and closed by a perfectly insulating surface, S' . It is possible to set up a set of orthogonal vector functions which may be used to expand an arbitrary vector field within the cavity. It is necessary to include a set of irrotational vectors and a set of solenoidal vectors. It is desirable to construct two sets of solenoidal vectors, however, either of which may be used with the set of irrotational vectors to expand an arbitrary vector function. One of these sets of solenoidal functions will satisfy boundary conditions similar to

*The material of this chapter is taken from Chapters 1 and 4 of Slater, J.C. *Microwave Electronics*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950. It is included for the purpose of completeness and with the permission of the author and publisher.

those which the electric field satisfies within the cavity on S and the other set will satisfy boundary conditions similar to those which the magnetic field satisfies within the cavity on S' . These functions will be called the $\bar{E}_a$'s and $\bar{H}_a$'s, respectively. The $\bar{E}_a$'s will be normal to the surface S and the $\bar{H}_a$'s will be normal to the surface S' . Because the magnetic field has no divergence (the magnetic permeability will be assumed to be constant throughout the entire volume being considered), a set of irrotational functions which are tangential on S will not be required. The set of irrotational functions which are normal on S will be called the $\bar{F}_a$'s. The development of the equations which determine these functions will now be given, following Slater.

The functions $\bar{E}_a$ and $\bar{H}_a$ are to be solenoidal; consequently, they must be the curls of other vectors. In particular, it may be required that

$$k_a \bar{E}_a = \text{curl } \bar{H}_a, \quad k_a \bar{H}_a = \text{curl } \bar{E}_a, \quad (4.2.1)$$

where k_a is a constant simply related to the resonant frequency of the a^{th} mode of the cavity. In addition, $\bar{E}_a$ and $\bar{H}_a$ are to satisfy the boundary conditions

$$\hat{n} \times \bar{E}_a = 0 \text{ on } S, \quad \hat{n} \times \bar{H}_a = 0 \text{ on } S'. \quad (4.2.2)$$

It can be shown readily that Eqs. 4.2.1 and 4.2.2 lead through the use of Stokes' theorem to the additional properties of $\bar{E}_a$ and $\bar{H}_a$:

$$\hat{n} \cdot \bar{H}_a = 0 \text{ on } S, \quad \hat{n} \cdot \bar{E}_a = 0 \text{ on } S'. \quad (4.2.3)$$

The differential equations satisfied separately by $\bar{E}_a$ and $\bar{H}_a$ are easily found by taking the curl of Eqs. 4.2.1. The result is

$$\text{curl curl } \bar{H}_a = k_a^2 \bar{H}_a, \quad (4.2.4a)$$

and

$$\text{curl curl } \bar{E}_a = k_a^2 \bar{E}_a. \quad (4.2.4b)$$

With the aid of the vector identity

$$\text{curl curl } \bar{A} = \text{grad}(\text{div } \bar{A}) - \nabla^2 \bar{A},$$

and noting that

$$\text{div } \bar{E}_a = \text{div } \bar{H}_a = 0,$$

Eqs. 4.2.4 may be written

$$\nabla^2 \bar{H}_a + k_a^2 \bar{H}_a = 0 \quad (4.2.5a)$$

and

$$\nabla^2 \bar{E}_a + k_a^2 \bar{E}_a = 0. \quad (4.2.5b)$$

Thus $\bar{E}_a$ and $\bar{H}_a$ are both solutions of the vector Helmholtz equation.

There will be an infinite set of solutions of Eqs. 4.2.5 corresponding to different values of k_a and satisfying the boundary conditions of Eq. 4.2.2. For each k_a there will be both an $\bar{E}_a$ and an $\bar{H}_a$, related to each other by Eq. 4.2.1.

It can be shown that the functions $\bar{E}_a$ and $\bar{H}_a$ have orthogonality properties of the form

$$\begin{aligned} \int_V \bar{E}_a \cdot \bar{E}_b \, dV &= 0, \quad \text{if } a \neq b, \\ \int_V \bar{H}_a \cdot \bar{H}_b \, dV &= 0, \quad \text{if } a \neq b, \end{aligned} \quad (4.2.6)$$

The proof for the first follows:

$$\begin{aligned} \text{div}(\bar{E}_b \times \text{curl } \bar{E}_a) - \text{div}(\bar{E}_a \times \text{curl } \bar{E}_b) &= \text{curl } \bar{E}_a \cdot \text{curl } \bar{E}_b - \bar{E}_b \cdot \text{curl curl } \bar{E}_a \\ &\quad - \text{curl } \bar{E}_b \cdot \text{curl } \bar{E}_a + \bar{E}_a \cdot \text{curl curl } \bar{E}_b. \end{aligned}$$

With the aid of Eq. 4.2.4b, this equation becomes

$$\operatorname{div}(\bar{\mathbf{E}}_b \times \operatorname{curl} \bar{\mathbf{E}}_a) - \operatorname{div}(\bar{\mathbf{E}}_a \times \operatorname{curl} \bar{\mathbf{E}}_b) = (k_b^2 - k_a^2) \bar{\mathbf{E}}_a \cdot \bar{\mathbf{E}}_b.$$

Integration over the volume V leads, with the aid of Green's theorem, to

$$\int_{S, S'} \hat{\mathbf{n}} \cdot \{k_a \bar{\mathbf{E}}_b \times \bar{\mathbf{H}}_a - k_b \bar{\mathbf{E}}_a \times \bar{\mathbf{H}}_b\} dA = (k_b^2 - k_a^2) \int_V \bar{\mathbf{E}}_a \cdot \bar{\mathbf{E}}_b dV.$$

Because of the boundary conditions on $\bar{\mathbf{E}}_a$ and $\bar{\mathbf{H}}_a$, the surface integral is zero. Consequently,

$$\int_V \bar{\mathbf{E}}_a \cdot \bar{\mathbf{E}}_b dV = 0 \quad \text{if } k_a \neq k_b. \quad (4.2.7)$$

If several functions occur for the same value of k_a , it is still possible to satisfy Eq. 4.2.7 by suitable linear combinations of these functions. Such functions will, of course, satisfy Eqs. 4.2.1 and 4.2.5.

Using the same vector identity, it may be shown in a similar way that

$$\int_V \bar{\mathbf{H}}_a \cdot \bar{\mathbf{H}}_b dV = 0 \quad \text{if } a \neq b. \quad (4.2.8)$$

It is convenient to normalize the $\bar{\mathbf{E}}_a$'s and the $\bar{\mathbf{H}}_a$'s so that

$$\int_V \bar{\mathbf{E}}_a^2 dV = \int_V \bar{\mathbf{H}}_a^2 dV = 1. \quad (4.2.9)$$

To show that this is indeed possible, use is made of the vector identity

$$\operatorname{div}(\bar{\mathbf{E}}_a \times \operatorname{curl} \bar{\mathbf{E}}_a) = \operatorname{curl} \bar{\mathbf{E}}_a \cdot \operatorname{curl} \bar{\mathbf{E}}_a = \operatorname{curl} \operatorname{curl} \bar{\mathbf{E}}_a.$$

Integration over V with the aid of Green's theorem and Eqs. 4.2.1 and

4.2.4 leads to

$$k_a \int_{S, S'} \hat{n} \cdot (\bar{E}_a \times \bar{H}_a) dA = k_a^2 \int_V H_a^2 dV = k_a^2 \int_V E_a^2 dV.$$

From the boundary conditions on $\bar{E}_a$ and $\bar{H}_a$, the surface integral can be shown to vanish. Therefore

$$\int_V E_a^2 dV = \int_V H_a^2 dV,$$

and consequently the normalization may be chosen as indicated in

Eq. 4.2.9. The orthonormality relations may be conveniently expressed in the form

$$\int_V \bar{E}_a \cdot \bar{E}_b dV = \int_V \bar{H}_a \cdot \bar{H}_b dV = \delta_{ab}, \quad (4.2.10)$$

where δ_{ab} is equal to one if $a = b$ but is zero otherwise.

The next step in the development is the construction of the irrotational functions, which will be denoted as the $\bar{F}_a$'s. Because $\bar{F}_a$ has zero curl, it is possible to write

$$k_a \bar{F}_a = \text{grad } \psi_a. \quad (4.2.11)$$

The eigenvalues k_a associated with the set of vectors $\bar{F}_a$ are, in general, different from the eigenvalues associated with the $\bar{E}_a$'s and $\bar{H}_a$'s. In addition, the requirement that

$$\nabla^2 \psi_a + k_a^2 \psi_a = 0, \quad (4.2.12)$$

is imposed upon the scalar functions ψ_a . From this equation it may be shown that $\bar{F}_a$ satisfies the vector Helmholtz equation as did $\bar{E}_a$ and $\bar{H}_a$:

$$\nabla^2 \bar{F}_a = \text{grad}(\text{div } \bar{F}_a) - \text{curl curl } \bar{F}_a = -k_a^2 \bar{F}_a,$$

which may be written

$$\text{grad} \left\{ \frac{1}{k_a} \nabla^2 \psi_a \right\} + k_a \text{grad } \psi_a = 0.$$

Finally

$$\frac{1}{k_a} \text{grad}\{\nabla^2 \psi_a + k_a \psi_a\} = 0,$$

which by Eq. 4.2.12 is indeed satisfied. The boundary conditions which are imposed on the functions ψ_a are

$$\psi_a = 0 \text{ on } S \text{ and } S', \quad (4.2.13)$$

from which it immediately follows that

$$\hat{n} \times \bar{F}_a = 0 \text{ on } S \text{ and } S'. \quad (4.2.14)$$

The functions ψ_a and $\bar{F}_a$ possess orthogonality properties in the volume V of the form

$$\begin{aligned} \int_V \psi_a \psi_b \, dV &= 0 \quad \text{if } a \neq b \\ \int_V \bar{F}_a \cdot \bar{F}_b \, dV &= 0 \quad \text{if } a \neq b. \end{aligned} \quad (4.2.15)$$

These equations are readily proved by a consideration of the identity

$$\begin{aligned} \text{div}(\psi_a \text{grad } \psi_b) &= \psi_a \nabla^2 \psi_b + \text{grad } \psi_a \cdot \text{grad } \psi_b \\ &= -k_b^2 \psi_a \psi_b + \text{grad } \psi_a \cdot \text{grad } \psi_b. \end{aligned} \quad (4.2.16)$$

If the order of a and b are interchanged and if the equation thus obtained is subtracted from Eq. 4.2.16, the result is

$$\text{div}(\psi_a \text{grad } \psi_b) - \text{div}(\psi_b \text{grad } \psi_a) = (k_a^2 - k_b^2) \psi_a \psi_b.$$

With the aid of Green's theorem, integration over the volume V leads to

$$\int_{S, S'} \hat{n} \cdot [\psi_a \text{grad } \psi_b - \psi_b \text{grad } \psi_a] dA = (k_a^2 - k_b^2) \int_V \psi_a \psi_b \, dV.$$

Because of the boundary conditions on ψ_a and ψ_b the surface integral vanishes, leaving

$$\int_V \psi_a \psi_b \, dV = 0 \quad \text{if } a \neq b. \quad (4.2.17)$$

Integration of Eq. 4.2.16 now leads easily to the result that

$$\int_V \bar{F}_a \cdot \bar{F}_b \, dV = 0 \quad \text{if } a \neq b. \quad (4.2.18)$$

It is convenient to normalize the functions ψ_a and $\bar{F}_a$ by requiring

$$\int_V \psi_a^2 \, dV = \int_V \bar{F}_a^2 \, dV = 1.$$

That this may be done follows from Eq. 4.2.16, by letting $a = b$ and integrating over V . It is therefore possible to write

$$\int_V \psi_a \psi_b \, dV = \int_V \bar{F}_a \cdot \bar{F}_b \, dV = \delta_{ab}. \quad (4.2.19)$$

It is also possible to show that

$$\int_V \bar{F}_a \cdot \bar{E}_a \, dV = 0, \quad (4.2.20a)$$

and

$$\int_V \bar{F}_a \cdot \bar{H}_a \, dV = 0. \quad (4.2.20b)$$

Integration of the equation

$$\text{div}(\psi_a \bar{E}_a) = \psi_a \text{div} \bar{E}_a + \text{grad} \psi_a \cdot \bar{E}_a$$

throughout V and the use of Green's theorem to transform the left side to a surface integral leads to

$$\int_{S, S'} \psi_a (n \cdot \bar{E}_a) \, dA = k_a \int_V \bar{F}_a \cdot \bar{E}_a \, dV,$$

because $\text{div} \bar{E}_a = 0$. Now ψ_a is zero on S and S' ; therefore, the surface integral vanishes, completing the proof for the first of Eqs. 4.2.20.

The proof for the second is analogous.

Two families of solenoidal vector functions, the $\bar{E}_a$'s and the $\bar{H}_a$'s, and a family of irrotational vectors, the $\bar{F}_a$'s have been constructed. It has been shown that the $\bar{E}_a$'s are a mutually orthogonal family of functions as are the $\bar{H}_a$'s. In addition, it has been shown that the irrotational functions, the $\bar{F}_a$'s, are mutually orthogonal and are also orthogonal with respect to the $\bar{E}_a$'s and the $\bar{H}_a$'s. It will now be assumed that an arbitrary vector function within the volume V may be expanded in terms of either the $\bar{H}_a$'s and the $\bar{F}_a$'s or the $\bar{E}_a$'s and the $\bar{F}_a$'s. No attempt will be made to prove this statement. The expansion in functions which satisfy the same type of boundary conditions as the function being expanded should be expected to converge better than the other expansion. The expansion in $\bar{H}_a$'s and $\bar{F}_a$'s is normal to the surface S_0 , whereas the expansion in $\bar{E}_a$'s and $\bar{F}_a$'s is normal to the surface S . Consequently, at the surface the expansion will generally not represent an arbitrary function which may be neither tangential or normal there. In general, it will not be possible to differentiate and integrate the expansions for this reason.* The expansion of an arbitrary vector A will have the form

$$\bar{A} = \sum_a \left\{ \bar{E}_a \int_V \bar{A} \cdot \bar{E}_a \, dV + \bar{F}_a \int_V \bar{A} \cdot \bar{F}_a \, dV \right\}, \quad (4.2.21a)$$

or

$$\bar{A} = \sum_a \left\{ \bar{H}_a \int_V \bar{A} \cdot \bar{H}_a \, dV + \bar{F}_a \int_V \bar{A} \cdot \bar{F}_a \, dV \right\}. \quad (4.2.21b)$$

*A similar situation arises in the case of Fourier series. See, for example, Whittaker, E.T. and Watson, G.N. *A Course of Modern Analysis*, Cambridge University Press (Cambridge: 1952) p. 168.

4.3 The Relationships of $\bar{E}_a$ and $\bar{H}_a$ to the Electric Field, $\bar{E}$, and the Magnetic Field, $\bar{H}$, in a Cavity with Perfectly Conducting Walls

Consider a cavity completely enclosed by a surface S . On S ,

$$\hat{n} \times \bar{E}_a = \hat{n} \cdot \bar{H}_a = 0, \quad (4.3.1)$$

where $\bar{E}_a$ and $\bar{H}_a$ are the functions defined in Section 4.2. It has also been shown that

$$\text{curl curl } \bar{E}_a = k_a^2 \bar{E}_a \quad (4.3.2)$$

and

$$\text{curl curl } \bar{H}_a = k_a^2 \bar{H}_a. \quad (4.3.3)$$

If there are no sources within the cavity, the electric field, $\bar{E}$, and the magnetic field, $\bar{H}$, satisfy the equations

$$\text{curl curl } \bar{E} = k^2 \bar{E}, \quad (4.3.4)$$

and

$$\text{curl curl } \bar{H} = k^2 \bar{H}, \quad (4.3.5)$$

where it has been assumed that the time dependence of these fields is of the form $\exp(i\omega t)$. At the boundary surface S

$$\hat{n} \times \bar{E} = \hat{n} \cdot \bar{H} = 0. \quad (4.3.6)$$

As a particular solution of Eq. 4.3.4 let

$$\bar{E} = \bar{E}_a e^{j\omega_a t}$$

and assume the same time dependence for $\bar{H}$. Then, from Maxwell's equations,

$$\text{curl } \bar{E} = -j\omega_a \mu \bar{H} = \text{curl } \bar{E}_a e^{j\omega_a t} = k_a^2 \bar{H}_a e^{j\omega_a t}.$$

Therefore

$$\bar{H} = \frac{jk_a}{\omega_a \mu} \bar{H}_a e^{j\omega_a t} = j\sqrt{\epsilon/\mu} \bar{H}_a e^{j\omega_a t}. \quad (4.3.7)$$

For the case under consideration a possible solution of Maxwell's equations is

$$\bar{E} = \bar{E}_a e^{j\omega_a t} \quad (4.3.8a)$$

and

$$\bar{H} = \frac{J}{\eta_k} \bar{H}_a e^{j\omega_a t} \quad (4.3.8b)$$

where

$$\eta_k = \sqrt{\mu/\epsilon}. \quad (4.3.9)$$

It is now apparent that the expansion in terms of the $\bar{E}_a$ functions or the $\bar{H}_a$ functions is actually an expansion in terms of the normal mode electric or magnetic field distributions.

4.4 Normal Mode Expansion of Maxwell's Equations in a Hollow Cavity

The orthogonal vector functions which were set up in Section 4.2 will now be used to expand the electric and magnetic fields and the other quantities appearing in Maxwell's equations. By means of Maxwell's equations it will be possible to develop the equations which the expansion coefficients must satisfy. From the results of Section 4.2 it is possible to write down immediately:

$$\begin{aligned} \bar{E} &= \sum_a (\bar{E}_a \int_V \bar{E} \cdot \bar{E}_a dV + \bar{F}_a \int_V \bar{E} \cdot \bar{F}_a dV), \\ \bar{H} &= \sum_a \bar{H}_a \int_V \bar{H} \cdot \bar{H}_a dV, \\ \bar{J} &= \sum_a (\bar{E}_a \int_V \bar{J} \cdot \bar{E}_a dV + \bar{F}_a \int_V \bar{J} \cdot \bar{F}_a dV), \\ \rho &= \sum_a \psi_a \int_V \rho \psi_a dV. \end{aligned} \quad (4.4.1)$$

The expansion of the other quantities which appear in Maxwell's equa-

tions requires some manipulation. For example,

$$\text{curl } \bar{E} = \sum_a \bar{H}_a \int_V (\text{curl } \bar{E}) \cdot \bar{H}_a dV.$$

To evaluate the integral, consider

$$\begin{aligned} \text{div}(\bar{E} \times \text{curl } \bar{E}_a) &= \text{curl } \bar{E} \cdot \text{curl } \bar{E}_a - \bar{E} \cdot \text{curl curl } \bar{E}_a \\ &= k_a \bar{H}_a \text{curl } \bar{E} - k_a \bar{E} \cdot \bar{E}_a. \end{aligned}$$

With the use of the divergence theorem, integration over the volume V leads to

$$\int_{S, S'} \hat{n} \cdot (\bar{E} \times \bar{H}_a) dA = \int_V \text{curl } \bar{E} \cdot \bar{H}_a dV - k_a \int_V \bar{E} \cdot \bar{E}_a dV.$$

Because $\bar{H}_a$ is normal to the surface S' there is no contribution to the surface integral from integration over S' . Consequently

$$\text{curl } \bar{E} = \sum_a \bar{H}_a \left\{ k_a \int_V \bar{E} \cdot \bar{E}_a dV + \int_S (\hat{n} \times \bar{E}) \cdot \bar{H}_a dA \right\}. \quad (4.4.2)$$

The surface term in Eq. 4.4.2 will be zero for the case of perfectly conducting walls, for in this case $\bar{E}$ will be normal to the surface S .

In the general case, however, there will be a small tangential component of $\bar{E}$ because of the finite conductivity of the walls. This small tangential component of $\bar{E}$ will then make a contribution to the expansion coefficients for $\text{curl } \bar{E}$. If the curl of the expansion for $\bar{E}$ as given in Eq. 4.4.1 is computed, the surface term in Eq. 4.4.2 is not obtained. This result should not be surprising, since the expansion for $\bar{E}$ converges to the actual $\bar{E}$ at the boundary only for the case of perfectly walls. For this case the surface integral of Eq. 4.4.2 is zero, and the two methods of computing $\text{curl } \bar{E}$ yield the same result. Consequently,

in the general case it will not be possible to differentiate the expansions with respect to spatial coordinates, but rather it will be necessary to expand directly each expression desired.

By the same methods it is easily shown that

$$\text{curl } \bar{H} = \sum_a \bar{E}_a \left\{ k_a \int_V \bar{H} \cdot \bar{H}_a dV + \int_{S'} (\hat{n} \times \bar{H}) \cdot \bar{E}_a dA \right\} \quad (4.4.3)$$

The expansion for $\text{div } \bar{D}$ has the form

$$\text{div } \bar{D} = \sum_a \psi_a \int_V \psi_a \text{div } \bar{D} dV. \quad (4.4.4)$$

From vector analysis

$$\begin{aligned} \text{div}(\psi_a \bar{D}) &= \psi_a \text{div } \bar{D} + \text{grad } \psi_a \cdot \bar{D} \\ &= \psi_a \text{div } \bar{D} + k_a \bar{F}_a \cdot \bar{D}. \end{aligned}$$

Integration over V leads to

$$\int_V \text{div}(\psi_a \bar{D}) dV = \int_{S_0 S'} \psi_a (\bar{D} \cdot \hat{n}) dA = \int_V \psi_a \text{div } \bar{D} dV + k_a \int_V \bar{F}_a \cdot \bar{D} dV,$$

where the divergence theorem has been used to transform the volume integral into a surface integral. Now ψ_a is zero on S and S' ; therefore Eq. 4.4.4 becomes

$$\text{div } \bar{D} = \sum_a \psi_a \left\{ -k_a \int_V \bar{D} \cdot \bar{F}_a dV \right\}. \quad (4.4.5)$$

Equation 4.4.5 provides the last of the expansions that are required for consideration of Maxwell's equations. These expansions will now be substituted into Maxwell's equations and the coefficients of the several orthogonal functions will be equated to yield the differential equations satisfied by the coefficients.

From $\text{curl } \bar{E} = -\partial \bar{B} / \partial t$,

$$k_a \int_V \bar{E} \cdot \bar{E}_a \, dV + \mu \frac{d}{dt} \int_V \bar{H} \cdot \bar{H}_a \, dV = - \int_S (\hat{n} \times \bar{E}) \cdot \bar{H}_a \, dA. \quad (4.4.6)$$

The equation $\text{curl } \bar{H} = \partial \bar{D} / \partial t + \bar{J}$ leads to equations involving the coefficients of irrotational vectors and solenoidal vectors. The result is

$$k_a \int_V \bar{H} \cdot \bar{H}_a \, dV - \epsilon \frac{d}{dt} \int_V \bar{E} \cdot \bar{E}_a \, dV = \int_V \bar{J} \cdot \bar{E}_a \, dV - \int_S (\hat{n} \times \bar{H}) \cdot \bar{E}_a \, dA, \quad (4.4.7)$$

and

$$-\epsilon \frac{d}{dt} \int_V \bar{E} \cdot \bar{F}_a \, dV = \int_V \bar{J} \cdot \bar{F}_a \, dV. \quad (4.4.8)$$

The equation $\text{div } \bar{B} = 0$ is satisfied by Eq. 4.4.1. For $\text{div } \bar{D} = \rho$, it is found that

$$-k_a \epsilon \int_V \bar{E} \cdot \bar{F}_a \, dV = \int_V \rho \psi_a \, dV. \quad (4.4.9)$$

It is not difficult to show that Eqs. 4.4.8 and 4.4.9 are consistent with the equation of continuity. Combining the two equations,

$$k_a \int_V \bar{J} \cdot \bar{F}_a \, dV = \frac{d}{dt} \int_V \rho \psi_a \, dV. \quad (4.4.10)$$

With the identity

$$\text{div}(\psi_a \bar{J}) = \psi_a \text{div } \bar{J} + \bar{J} \cdot \text{grad } \psi_a,$$

it is found that

$$\int_V \text{div}(\psi_a \bar{J}) \, dV = \int_V \psi_a \text{div } \bar{J} \, dV + \frac{d}{dt} \int_V \rho \psi_a \, dV.$$

The left hand side may be transformed into a surface integral which

vanishes because $\psi_a = 0$ on S and S' . Consequently,

$$\int_V \psi_a \left(\operatorname{div} \bar{\mathbf{J}} + \frac{\partial \rho}{\partial t} \right) dV = 0.$$

Because the ψ_a 's form a complete set (this is an assumption), it is necessary that

$$\operatorname{div} \bar{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0,$$

which is the equation of continuity.

Equations 4.4.6 and 4.4.7 may be combined to give the equations satisfied separately by

$$\int_V \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}_a dV \quad \text{and} \quad \int_V \bar{\mathbf{H}} \cdot \bar{\mathbf{H}}_a dV.$$

The result is

$$\begin{aligned} \epsilon \mu \frac{d^2}{dt^2} \int_V \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}_a dV + k_a^2 \int_V \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}_a dV = & -\mu \frac{d}{dt} \left(\int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a dV - \int_{S'} (\hat{\mathbf{n}} \times \bar{\mathbf{H}}) \cdot \bar{\mathbf{E}}_a dA \right) \\ & - k_a \int_S (\hat{\mathbf{n}} \times \bar{\mathbf{E}}) \cdot \bar{\mathbf{H}}_a dA, \end{aligned} \quad (4.4.11)$$

and

$$\begin{aligned} \epsilon \mu \frac{d^2}{dt^2} \int_V \bar{\mathbf{H}} \cdot \bar{\mathbf{H}}_a dV + k_a^2 \int_V \bar{\mathbf{H}} \cdot \bar{\mathbf{H}}_a dV = & k_a \left(\int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a dV - \int_{S'} (\hat{\mathbf{n}} \times \bar{\mathbf{H}}) \cdot \bar{\mathbf{E}}_a dA \right) \\ & - \epsilon \frac{d}{dt} \int_S (\hat{\mathbf{n}} \times \bar{\mathbf{E}}) \cdot \bar{\mathbf{H}}_a dA. \end{aligned} \quad (4.4.12)$$

The solutions of Eqs. 4.4.11 and 4.4.12 will be considered in the next section.

4.5 Oscillations in a Hollow Cavity

The problem of oscillations in a hollow cavity will now be investigated using the orthogonal functions which were developed in the pre-

ceding sections. In practice, the walls of a cavity will not be perfectly conducting, and there will be a component of electric field, $\vec{E}$, tangential to the surface, S , of the cavity. If the walls are good conductors of conductivity σ , it can be shown that on the conducting surface

$$\hat{n} \times \vec{E} \approx (1 + j) \omega \mu \frac{\delta}{2} \vec{H} \quad (4.5.1)$$

where δ is the skin depth and is given by

$$\delta = \sqrt{2/(\omega \sigma \mu)}, \quad (4.5.2)$$

and $\vec{H}$ is the value computed for the case of infinite conductivity.⁴²

The surface integrals over the conducting surface S appearing in Eqs. 4.4.11 and 4.4.12 may be evaluated with the aid of Eq. 4.5.1:

$$\int_S (\hat{n} \times \vec{E}) \cdot \vec{H}_a \, dA \approx (1 + j) \omega \mu \frac{\delta}{2} \int_S \vec{H} \cdot \vec{H}_a \, dA. \quad (4.5.3)$$

The integral on the right may be further simplified if it is required that only one mode be excited. That is, the expansion of the field in normal modes is required to have the form

$$\begin{aligned} \vec{H} &\approx \vec{H}_a \int_V \vec{H} \cdot \vec{H}_a \, dV, \\ \vec{E} &\approx \vec{E}_a \int_V \vec{E} \cdot \vec{E}_a \, dV + \sum_{\alpha} \vec{F}_{\alpha} \int_V \vec{E} \cdot \vec{F}_{\alpha} \, dV. \end{aligned}$$

For this case it is readily shown that

$$\int_V (\hat{n} \times \vec{E}) \cdot \vec{H}_a \, dA \approx (1 + j) \omega \mu \frac{\delta}{2} \int_S H_a^2 \, dA \int_V \vec{H} \cdot \vec{H}_a \, dV.$$

It is customary to introduce the notation

$$\frac{1}{Q_a} = \frac{\delta}{2} \int_S H_a^2 dA. \quad (4.5.4)$$

With this notation Eq. 4.5.3 may be written

$$\int_S (\hat{n} \times \bar{E}) \cdot \bar{H}_a dA = (1+j) \frac{\omega \mu}{Q_a} \int_V \bar{H} \cdot \bar{H}_a dV. \quad (4.5.5)$$

If it is assumed that all of the field expressions depend on time as $e^{j\omega t}$ where the physical quantities are given by the real part of such expressions, it is readily found from Eq. 4.4.6 that

$$\int_V \bar{H} \cdot \bar{H}_a dV = - \frac{\int_V \bar{E} \cdot \bar{E}_a dV}{j\eta \left\{ \frac{\omega}{\omega_a} + \frac{1-j}{Q_a} \right\}}. \quad (4.5.6)$$

With the aid of Eqs. 4.5.5 and 4.5.6, Eqs. 4.4.11 and 4.4.12, which determine

$$\int_V \bar{E} \cdot \bar{E}_a dV \quad \text{and} \quad \int_V \bar{H} \cdot \bar{H}_a dV,$$

become for the assumed time dependence

$$\left\{ j \left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} \right) + \frac{1+j}{Q_a} \frac{1}{\left(\frac{\omega}{\omega_a} + \frac{1-j}{Q_a} \right)} \right\} \int_V \bar{E} \cdot \bar{E}_a dV = - \frac{\int_V \bar{J} \cdot \bar{E}_a dV - \int_{S'} (\hat{n} \times \bar{H}) \cdot \bar{E}_a dA}{\epsilon \omega_a} \quad (4.5.7)$$

and

$$\left\{ j \left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} \right) + \frac{\omega}{\omega_a} \frac{1+j}{Q_a} \right\} \int_V \bar{H} \cdot \bar{H}_a dV = \frac{\omega_a}{c} \left\{ \int_V \bar{J} \cdot \bar{E}_a dV - \int_{S'} (\hat{n} \times \bar{H}) \cdot \bar{E}_a dA \right\}, \quad (4.5.8)$$

where

$$\omega_a^2 = c^2 k_a^2 = k_a^2 / \mu \epsilon. \quad (4.5.9)$$

For the cases of practical interest it is usually found that Q_a is of the order of 10^3 to 10^4 . If Q_a is of this order of magnitude, Eqs. 4.5.8 and 4.5.9 may be further simplified to yield

$$\left\{ j \left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} + \frac{1}{Q_a} \right) + \frac{1}{Q_a} \right\} \int_V \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}_a \, dV = - \frac{1}{\epsilon \omega_a} \left\{ \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a \, dV - \int_{S'} (\hat{\mathbf{n}} \times \bar{\mathbf{H}}) \cdot \bar{\mathbf{E}}_a \, dA \right\} \quad (4.5.10)$$

and

$$\left\{ j \left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} + \frac{1}{Q_a} \right) + \frac{1}{Q_a} \right\} \int_V \bar{\mathbf{H}} \cdot \bar{\mathbf{H}}_a \, dV = \frac{\omega_a}{c} \left\{ \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a \, dV - \int_{S'} (\hat{\mathbf{n}} \times \bar{\mathbf{H}}) \cdot \bar{\mathbf{E}}_a \, dA \right\}, \quad (4.5.11)$$

for ω near ω_a . The solutions of the homogeneous equations are obtained by letting

$$\omega = \omega_a' + j \omega_2.$$

It is readily found that

$$\omega_a' \approx \omega_a \left(1 - \frac{1}{2Q_a} \right) \quad (4.5.12)$$

and

$$\omega_2 \approx \frac{\omega_a'}{2Q_a}. \quad (4.5.13)$$

This result indicates that the fields in the cavity are damped and that the resonant frequency is perturbed by the losses in the walls. The quantity Q_a , which is a measure of the losses in the walls, is usually called the unloaded Q for the a^{th} mode.

If there are currents in the cavity which have a frequency near one of the resonant frequencies of the cavity, it is possible for oscillations to be set up within the cavity. The power delivered by the currents to the field is given by

$$P = -\frac{1}{2} \operatorname{Re} \left\{ \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}^* \, dV \right\}, \quad (4.5.14)$$

where Re means the real part, and $\bar{E}^*$ is the complex conjugate of $\bar{E}$. It is also assumed that $\bar{J}$ and $\bar{E}$ both oscillate with the same frequency.

The negative sign is introduced because the work done on the field rather than the work done by the field is being considered. If $\bar{J}$ and $\bar{E}$ are expanded in normal modes, it is easily found that

$$P = -\frac{1}{2} \text{Re} \left\{ \sum_a \int_V \bar{J} \cdot \bar{E}_a dV \int_V \bar{E}^* \cdot \bar{E}_a dV + \sum_\alpha \int_V \bar{J} \cdot \bar{F}_\alpha dV \int_V \bar{E}^* \cdot \bar{F}_\alpha dV \right\}. \quad (4.5.15)$$

This expression may be simplified by investigating the second summation. Consider one term of this summation. With the aid of Eq. 4.5.7

$$-\frac{1}{2} \text{Re} \left\{ \int_V \bar{J} \cdot \bar{F}_\alpha dV \int_V \bar{E}^* \cdot \bar{F}_\alpha dV \right\} = \frac{1}{2} \text{Re} \epsilon \int_V \bar{E}^* \cdot \bar{F}_\alpha dV \frac{d}{dt} \int_V \bar{E} \cdot \bar{E}_\alpha dV \quad (4.5.16)$$

Now

$$\text{Re} \frac{d}{dt} \left| \int_V \bar{E} \cdot \bar{F}_\alpha dV \right|^2 = 2 \text{Re} \int_V \frac{d\bar{E}}{dt} \cdot \bar{F}_\alpha dV \int_V \bar{E}^* \cdot \bar{F}_\alpha dV,$$

if $\bar{F}_\alpha$ is real as it may be chosen. Equation 4.5.16 now becomes

$$-\frac{1}{2} \text{Re} \left\{ \int_V \bar{J} \cdot \bar{F}_\alpha dV \int_V \bar{E}^* \cdot \bar{F}_\alpha dV \right\} = \frac{1}{4} \epsilon \text{Re} \frac{d}{dt} \left\{ \left| \int_V \bar{E} \cdot \bar{F}_\alpha dV \right|^2 \right\}.$$

The assumption that $\bar{E}$ involves the time as $e^{j\omega t}$ makes the right hand side of this equation zero and consequently Eq. 4.5.15 becomes

$$P = -\frac{1}{2} \text{Re} \left\{ \sum_a \int_V \bar{J} \cdot \bar{E}_a dV \int_V \bar{E}^* \cdot \bar{E}_a dV \right\}.$$

If only one mode is excited, this expression simplifies to

$$P = -\frac{1}{2} \text{Re} \int_V \bar{J} \cdot \bar{E}_a dV \int_V \bar{E}^* \cdot \bar{E}_a dV. \quad (4.5.17)$$

The important conclusion to be drawn from this result is that only the solenoidal component of the current $\bar{J}$ is effective in the interchange of energy between the electric field and the current.

For the case of a cavity completely bounded with conducting walls, Eqs. 4.5.10 and 4.5.17 lead to

$$P = P(\omega) = \frac{1}{2\epsilon\omega_a} \frac{1}{Q_a} \frac{\left| \int_V \bar{J} \cdot \bar{E}_a dV \right|^2}{\left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} + \frac{1}{Q_a} \right)^2 + \left(\frac{1}{Q_a} \right)^2} \quad (4.5.18)$$

This expression is maximum at

$$\omega = \omega'_a = \omega_a \left(1 + \frac{1}{2Q_a} \right),$$

when it becomes

$$P(\omega'_a) = \frac{Q_a}{2\epsilon\omega_a} \left| \int_V \bar{J} \cdot \bar{E}_a dV \right|^2. \quad (4.5.19)$$

The power delivered by the electronic currents is proportional to the unloaded Q of the cavity. From Eq. 4.5.4 it is possible to determine the dependence of Q_a on frequency. If it is assumed that H_a on the surface is about equal to its average value throughout the volume, then

$$\frac{1}{Q_a} \approx \frac{\delta}{2} \frac{A}{V}. \quad (4.5.20)$$

where V is the cavity volume and A is the surface area. The resonant wavelengths of two cavities of the same shape but different in size are in the same proportion as the corresponding linear dimensions. Thus V is proportional to λ^3 and A is proportional to λ^2 . From Eq. 4.5.2 the

skin depth is readily seen to be proportional to $\lambda^{1/2}$. Consequently, it is found that Q_a is proportional to $\lambda^{1/2}$ and from Eq. 4.5.19 that the power generated is proportional to $\lambda^{1/2}$, a result which unhappily does not favor the effort to produce power at millimeter wavelengths by the excitation of resonant cavities with electronic currents.

4.6 Normal Modes for a Waveguide of Arbitrary Cross Section⁴³

If it is desired to extract power from a resonant cavity excited by an electron beam, for example, this may be done through a suitably attached waveguide output. For the purposes of this development a waveguide will be defined as a cylindrical tube of arbitrary cross section. The walls are to be highly conducting and the waveguide is to be either empty or filled with a dielectric of low loss. The results of the preceding section must be modified in this case to include losses in power out through the waveguide in addition to the power losses in the cavity walls. To do this, normal mode functions for the waveguide will be constructed. As in the cavity problem these normal mode functions will be solutions of the vector Helmholtz equation, but there will be a simplification in this case: There will be no currents in the waveguide output; therefore there will be no need for a set of irrotational functions as was required for the cavity problem.

It can be shown that three independent vector solutions of the vector Helmholtz equation

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0 \quad (4.6.1)$$

are given by

$$\begin{aligned}\bar{\mathbf{L}} &= \nabla u \\ \bar{\mathbf{M}} &= \text{curl}(\hat{\mathbf{i}}_z \varphi) \\ \bar{\mathbf{N}} &= 1/k \text{curl} \text{curl}(\hat{\mathbf{i}}_z \chi)\end{aligned}\tag{4.6.2}$$

where $\hat{\mathbf{i}}_z$ is a unit vector in the z direction, and u , φ , and χ are scalar functions which satisfy the scalar Helmholtz equation,

$$\begin{aligned}\nabla^2 u + k^2 u &= 0 \\ \nabla^2 \varphi + k^2 \varphi &= 0 \\ \nabla^2 \chi + k^2 \chi &= 0\end{aligned}\tag{4.6.3}$$

for the same value of k . In general these functions need not be the same, but in special cases they may be. The most general solution of Eq. 4.6.1 is then

$$\bar{\mathbf{A}} = \bar{\mathbf{L}} + \bar{\mathbf{M}} + \bar{\mathbf{N}}. \quad 44$$

From Eqs. 4.6.2 it is readily seen that

$$\begin{aligned}\text{curl } \bar{\mathbf{L}} &= 0, \\ \text{div } \bar{\mathbf{M}} &= \text{div } \bar{\mathbf{N}} = 0.\end{aligned}$$

In the waveguide the electric and magnetic fields will be solenoidal, since there are to be no currents within the guide. Consequently, the normal mode functions of interest for this problem will be generated from the $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ solutions.

A rectangular coordinate system will be assumed with the axis of the waveguide extending in the z direction. In rectangular coordinates the scalar Helmholtz equation is separable. If a general solution of

the form $\psi(x, y)e^{-\gamma z}$ is assumed, that is, if

$$\nabla^2 \{\psi(x, y)e^{-\gamma z}\} + k^2 \{\psi(x, y)e^{-\gamma z}\} = 0, \quad (4.6.4)$$

it is found that $\psi(x, y)$ must satisfy the equation

$$\nabla^2 \psi(x, y) + (k^2 + \gamma^2) \psi(x, y) = 0. \quad (4.6.5)$$

It is customary to define

$$k^2 = \omega^2/c^2 = -\gamma_0^2 \quad (4.6.6)$$

and

$$\beta_c^2 = \gamma^2 + k^2 = \gamma^2 - \gamma_0^2. \quad (4.6.6)$$

With this notation, Eq. 4.6.5 may be written

$$(\partial^2 \psi / \partial x^2) + (\partial^2 \psi / \partial y^2) + \beta_c^2 \psi = 0. \quad (4.6.7)$$

The vector solutions which will lead to the normal mode solutions can be generated by requiring that

$$\bar{\mathbf{M}} = \text{curl}(\hat{\mathbf{i}}_z \varphi e^{-\gamma z}) \quad (4.6.8a)$$

and

$$\bar{\mathbf{N}} = 1/k \text{ curl curl}(\hat{\mathbf{i}}_z \chi e^{-\gamma z}) \quad (4.6.8b)$$

where the functions φ and χ are solutions of Eq. 4.6.7.

The first solutions to be considered are those analogous to the $\bar{\mathbf{E}}_a$ solutions of Section 4.2. Therefore it is required that on the surface, which will be designated by S ,

$$\hat{\mathbf{n}} \times \bar{\mathbf{M}} = \hat{\mathbf{n}} \times \bar{\mathbf{N}} = 0.$$

From Eq. 4.6.8 the requirement for φ may be found:

$$\hat{\mathbf{n}} \times \bar{\mathbf{M}} = \hat{\mathbf{n}} \times \text{curl}(\hat{\mathbf{i}}_z \varphi e^{-\gamma z}) = \hat{\mathbf{n}} \times \nabla(\varphi e^{-\gamma z}) \times \hat{\mathbf{i}}_z = 0,$$

from which it may be shown readily that

$$\hat{\mathbf{n}} \cdot \nabla \varphi = 0, \quad \text{on } S. \quad (4.6.9)$$

The $\bar{\mathbf{M}}$ solutions are therefore generated from solutions φ of the two dimensional scalar Helmholtz equation for the cross section of the

waveguide which have no normal derivative. A particular function satisfying both Eqs. 4.6.7 and 4.6.9 will be denoted by φ_n , and the corresponding vector will be denoted by $\bar{M}_n$.

For the $\bar{N}$ solutions it is required that

$$\hat{n} \times \bar{N} = \hat{n} \times 1/k \text{ curl curl } (\hat{i}_z \chi \epsilon^{-Yz}) = 0 \text{ on } S.$$

From vector analysis this expression may be written

$$\hat{n} \times \frac{1}{k} \text{ curl } \{ \nabla (\chi \epsilon^{-Yz}) \times \hat{i}_z \} = \hat{n} \times \frac{1}{k} \{ -\hat{i}_z \nabla^2 (\chi \epsilon^{-Yz}) + \frac{\partial}{\partial z} \nabla (\chi \epsilon^{-Yz}) \} = 0, \quad (4.6.10)$$

since

$$\nabla \times (\bar{a} \times \bar{b}) = \bar{a} \nabla \cdot \bar{b} - \bar{b} \nabla \cdot \bar{a} + (\bar{b} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{b}.$$

With the aid of Eqs. 4.6.6 and 4.6.7, the right side of Eq. 4.6.10 becomes

$$\hat{n} \times \bar{N} = \frac{1}{k} \left\{ \hat{i}_x n_y \beta_c^2 \chi + \hat{i}_y n_x \beta_c^2 \chi + \hat{i}_z \gamma \left(n_y \frac{\partial \chi}{\partial x} - n_x \frac{\partial \chi}{\partial y} \right) \right\} \epsilon^{-Yz} = 0 \quad (4.6.11)$$

where n_x and n_y are the components of $\hat{n}$ in the x and y directions, respectively. Each component of Eq. 4.6.11 must be zero. From the z component,

$$\hat{i}_z \left(n_y \frac{\partial \chi}{\partial x} - n_x \frac{\partial \chi}{\partial y} \right) = -\hat{n} \times \nabla \chi = 0,$$

it is found that

$$\chi = \text{constant on } S.$$

From the x and y components it is seen that

$$\chi = 0 \text{ on } S. \quad (4.6.12)$$

It has now been shown that the $\bar{N}$ solutions are generated from solutions of the two dimensional scalar Helmholtz equation for the cross section of the waveguide which reduce to zero on the boundary. A particular function satisfying both Eqs. 4.6.7 and 4.6.12 will be denoted by χ_n , and the corresponding vector will be denoted by $\bar{N}_n$.

For each function φ_n and χ_n there will be a corresponding value of β_c^2 from Eq. 4.6.7 which will be denoted as β_{cn}^2 . From Eq. 4.6.6 the corresponding value of γ , γ_n , may be computed. Then using Eq. 4.6.8a for the φ_n 's and Eq. 4.6.8b for the χ_n 's, the vector functions may be determined. These functions, which are normal at the surface, represent all the possible electric field distributions. The corresponding magnetic field distributions may be found from Maxwell's equations.

If the fields are assumed to be the real part of expressions which involve the time as $e^{j\omega t}$, the equation

$$\text{curl } \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}$$

may be written

$$\bar{H} = -\frac{1}{j\mu\omega} \text{curl } \bar{E}. \quad (4.6.13)$$

Now, with the aid of Eq. 4.6.6

$$j\mu\omega = \gamma_0 \sqrt{\frac{\mu}{\epsilon}}, \quad (4.6.14)$$

since

$$c^2 = \frac{1}{\mu\epsilon}.$$

With the definition

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}}, \quad (4.6.15)$$

and with Eq. 4.6.14, it is found that

$$\bar{H} = -\frac{1}{\gamma_0 Z_0} \text{curl } \bar{E}. \quad (4.6.16)$$

Using Eqs. 4.6.8a and 4.6.16 the result for the $\bar{M}_n$ solutions is

$$\bar{E}_n = \bar{M}_n = -\hat{i}_z \times \nabla(\varphi_n e^{-\gamma_n z}) = (-\hat{i}_z \times \nabla \varphi_n) e^{-\gamma_n z}, \quad (4.6.17)$$

and

$$\bar{H}_n = -\frac{1}{\gamma_0 Z_0} \text{curl } \bar{E}_n = -\frac{\beta_{cn}^2}{\gamma_0 Z_0} \left\{ -\frac{\gamma_n}{\beta_{cn}^2} \nabla \varphi_n + \hat{i}_z \varphi_n \right\} e^{-\gamma_n z}. \quad (4.6.18)$$

From Eq. 4.6.17 these solutions have no electric field in the z direction. For this reason they are called the transverse electric (TE) modes. With the definition

$$-\frac{\beta_{cn}^2}{\gamma_0 Z_0} \varphi_n = H_{zn}, \quad (4.6.19)$$

these equations may be written

$$\bar{H}_n = (\bar{H}_{tn} + \hat{i}_z H_{zn}) e^{-\gamma_n z}, \quad (4.6.20)$$

$$\bar{E}_n = \bar{E}_{tn} e^{-\gamma_n z} \quad (4.6.21)$$

where

$$\bar{H}_{tn} = -\frac{\gamma_n}{\beta_{cn}^2} \nabla H_{zn}, \quad (4.6.22)$$

and

$$\bar{E}_{tn} = \frac{Z_0 \gamma_0}{\gamma_n} (\bar{H}_{tn} \times \hat{i}_z). \quad (4.6.23)$$

Equation 4.6.23 may also be written

$$\hat{i}_z \times \bar{E}_{tn} = \frac{Z_0 \gamma_0}{\gamma_n} \bar{H}_{tn}, \quad (4.6.24)$$

From Eqs. 4.6.22 and 4.6.23 it is clear that for TE modes knowledge of the z component of magnetic field is sufficient to permit determination of all other components of the electric and magnetic fields.

In a similar manner, for the $\bar{N}_n$ solutions it is found that

$$\bar{E}_n = \bar{N}_n = \frac{\beta_{cn}^2}{k} \left\{ -\frac{\gamma_n}{\beta_{cn}^2} \nabla \chi_n + \hat{i}_z \chi_n \right\} e^{-\gamma_n z} \quad (4.6.25)$$

and

$$\bar{H}_n = -\frac{1}{Y_0 Z_0} \text{curl } \bar{E}_n = -\frac{Y_0}{k Z_0} (\hat{i}_z \times \nabla \chi_n) e^{-Y_n z}. \quad (4.6.26)$$

These solutions have no z component of magnetic field and consequently are called the transverse magnetic (TM) modes. With the definition

$$\frac{\beta_{cn}^2}{k} \chi_n = E_{zn} \quad (4.6.27)$$

these equations may be written

$$\bar{E}_n = (\bar{E}_{tn} + \hat{i}_z E_{zn}) e^{-Y_n z}, \quad (4.6.28)$$

$$\bar{H}_n = \bar{H}_{tn} e^{-Y_n z} \quad (4.6.29)$$

where

$$\bar{E}_{tn} = -\frac{Y_n}{\beta_{cn}^2} \nabla E_{zn}, \quad (4.6.30)$$

and

$$\bar{H}_{tn} = \frac{Y_0}{Y_n Z_0} (\hat{i}_z \times \bar{E}_{tn}). \quad (4.6.31)$$

In Eqs. 4.6.30 and 4.6.31 it is shown that for TM modes knowledge of the z component of electric field is sufficient to permit determination of all other components of the electric and magnetic fields.

It has been shown that the electric and magnetic fields for both TE and TM modes are completely determined in terms of a solution of the scalar Helmholtz equation. For TE modes this solution plays the roll of the z component of the magnetic field, and the requirement is that its normal derivative be zero at the walls of the waveguide. For TM modes the solution of the scalar Helmholtz equation becomes the z component of the electric field, and the requirement is that it vanish at the walls. There will be an infinite number of such scalar functions leading to an

infinite number of vector solutions of the vector Helmholtz equation. In addition to those solutions which have been obtained thus far, there is a corresponding set for waves traveling along the z axis in the direction of decreasing z . These solutions may be obtained from those of this section by replacing γ_n by $-\gamma_n$ wherever it appears in the equations of this section. It will now be demonstrated that these vector solutions possess certain useful orthogonality properties:

$$(1) \quad \int_A \bar{E}_{tm} \cdot \bar{E}_{tn} \, dA = 0, \quad \int_A \bar{H}_{tm} \cdot \bar{H}_{tn} \, dA = 0,$$

$$(2) \quad \int_A E_{zm} E_{zn} \, dA = 0, \quad \int_A H_{zm} H_{zn} \, dA = 0,$$

$$(3) \quad \int_A \hat{i}_z (\bar{E}_{tm} \times \bar{H}_{tn}) \cdot \hat{i}_z \, dA = 0,$$

subject to the condition that $m \neq n$, where A is the cross section of the waveguide.

To prove these statements Green's theorem in two dimensions will be used:

$$\int_A (\varphi \nabla^2 \psi + \text{grad } \varphi \cdot \text{grad } \psi) \, dA = \int_S \varphi (\hat{n} \cdot \text{grad } \psi) \, dS.$$

For the first case let φ be H_{zn} and ψ be H_{zm} , so that two TE modes are being considered. Then, since

$$\nabla^2 H_{zn} + \beta_{cn}^2 H_{zn} = 0,$$

and

$$\nabla^2 H_{zm} + \beta_{cm}^2 H_{zm} = 0,$$

it is easily shown that

$$-\beta_{cm}^2 \int_A H_{zn} H_{zm} dA + \frac{\beta_{cm}^2 \beta_{cn}^2}{\gamma_m \gamma_n} \int_A \bar{H}_{tm} \cdot \bar{H}_{tn} dA = \int_S H_{zn} \frac{\partial H_{zm}}{\partial n} dS, \quad (4.6.32)$$

with the aid of Eq. 4.6.22. However, the boundary condition on H_{zm} is just that $\partial H_{zm} / \partial n = 0$. Consequently,

$$\int_A \bar{H}_{tm} \cdot \bar{H}_{tn} dA = \frac{\gamma_m \gamma_n}{\beta_{cn}^2} \int_A H_{zm} H_{zn} dA. \quad (4.6.33)$$

From Eq. 4.6.23 it is easily shown that

$$\bar{E}_{tm} \cdot \bar{E}_{tn} = \frac{Z_0 \gamma_0}{\gamma_m} \frac{Z_0 \gamma_0}{\gamma_n} \bar{H}_{tm} \cdot \bar{H}_{tn}, \quad (4.6.34)$$

and that

$$\hat{i}_z \cdot (\bar{E}_{tm} \times \bar{H}_{tn}) = \frac{Z_0 \gamma_0}{\gamma_m} \bar{H}_{tm} \cdot \bar{H}_{tn}. \quad (4.6.35)$$

All of the conditions of orthogonality may therefore be demonstrated for the case of two TE modes if it can be shown that

$$\int_A H_{zm} H_{zn} dA = 0,$$

because for two TE modes, $E_{zm} E_{zn} = 0$, and thus the remaining condition is satisfied also. To perform the proof, interchange the subscripts m and n in Eq. 4.6.32 and subtract the resultant equation from Eq. 4.6.32.

The result is

$$(\beta_{cn}^2 - \beta_{cm}^2) \int_A H_{zm} H_{zn} dA = 0. \quad (4.6.36)$$

If $\beta_{cn}^2 \neq \beta_{cm}^2$ the proof is completed. If this condition is not satisfied it is still possible by a suitable linear combination of the de-

generate functions to construct functions which satisfy the orthogonality requirements.

For the case of two TM modes the proof is very similar. In this case let ϕ be E_{zn} and ψ be E_{zm} . The line integral on the right hand side of Eq. 4.6.32 is zero because $E_{zn} = 0$ on the boundary, and the remainder of the proof proceeds as in the case of two TE modes.

The case of one TE mode and one TM mode will now be considered. Let the n^{th} mode be TE, and m^{th} mode be TM. From Eqs. 4.6.23 and 4.6.30 it is easily shown that

$$\vec{E}_{tn} \cdot \vec{E}_{tm} = - \frac{Z_0 \gamma_n}{\gamma_0} \frac{\gamma_n}{\beta_{cn}^2} \frac{\gamma_m}{\beta_{cm}^2} \{ -\hat{i}_z \times \nabla H_{zn} \cdot \nabla E_{zm} \}. \quad (4.6.37)$$

It is easily shown that

$$-\hat{i}_z \times \nabla H_{zn} = \text{curl}(\hat{i}_z H_{zn})$$

by simply expanding the quantity on the right. It is also easy to show that

$$\text{curl} \{ (\hat{i}_z H_{zn}) \} \cdot \nabla E_{zm} = \text{div} \{ E_{zm} \text{curl}(\hat{i}_z H_{zn}) \}.$$

Equation 4.6.37 may now be written

$$\vec{E}_{tn} \cdot \vec{E}_{tm} = - \frac{Z_0 \gamma_n}{\gamma_0} \frac{\gamma_n}{\beta_{cn}^2} \frac{\gamma_m}{\beta_{cm}^2} \text{div} \{ E_{zm} \text{curl}(\hat{i}_z H_{zn}) \}. \quad (4.6.38)$$

If Eq. 4.6.38 is integrated over the volume of a waveguide cylinder of unit length it can be shown that the integral vanishes yielding the desired result. This result is readily obtained by transforming the volume integral into a surface integral with the aid of the divergence theorem.

$$\int_V \text{div} \{ E_{zm} \text{curl}(\hat{i}_z H_{zn}) \} dV = \int_A \hat{n} \cdot E_{zm} \text{curl}(\hat{i}_z H_{zn}) dA.$$

Because E_{zm} is zero on the side walls and $\hat{n} \cdot \text{curl}(\hat{i}_z H_{zn}) = 0$ on the end walls, the surface integral vanishes. Thus it has been shown that

$$\int_A \bar{E}_{tn} \cdot \bar{E}_{tm} dA$$

is zero when one mode is TE and the other is TM. The remaining proofs follow simply from this result.

It can be shown that an arbitrary solenoidal field within a waveguide may be expressed as a superposition of the normal mode fields of the waveguide. In general it will be necessary to include solutions representing waves propagating in both directions through the waveguide. The orthogonality properties just demonstrated permit the determination of the coefficients of the various modes. For an arbitrary field oscillating sinusoidally with angular frequency, ω , the expansions are

$$\begin{aligned} \bar{E} &= \sum_n \{ \bar{E}_{tn} (A_n e^{-\gamma_n z} + B_n e^{\gamma_n z}) + \hat{i}_z E_{zn} (A_n e^{-\gamma_n z} - B_n e^{\gamma_n z}) \} e^{j\omega t} \\ \bar{H} &= \sum_n \{ \bar{H}_{tn} (A_n e^{-\gamma_n z} - B_n e^{\gamma_n z}) + \hat{i}_z H_{zn} (A_n e^{-\gamma_n z} + B_n e^{\gamma_n z}) \} e^{j\omega t} \end{aligned} \quad (4.6.39)$$

where the field expressions for the waves traveling in the negative z direction have been obtained from the results of this section by replacing γ_n by $-\gamma_n$ wherever it appears in those results. If the transverse components of $\bar{E}$ and $\bar{H}$, $\bar{E}_t$ and $\bar{H}_t$, are known at a particular plane, for example, $z = 0$, then from the transverse component of

Eq. 4.6.39

$$\bar{\mathbf{E}}_t = \sum_n \bar{\mathbf{E}}_{tn} e^{j\omega t} (A_n + B_n),$$

and

$$\bar{\mathbf{H}}_t = \sum_n \bar{\mathbf{H}}_{tn} e^{j\omega t} (A_n - B_n).$$

It will be convenient to normalize the normal mode functions so that $\bar{\mathbf{E}}_{tn}$ is real and

$$\int_A \bar{\mathbf{E}}_{tn} \cdot \bar{\mathbf{E}}_{tn} dA = 1.$$

It is also convenient to introduce the notation

$$Z_{on} = \begin{cases} \frac{Z_0 Y_0}{Y_n} & \text{for TE modes} \\ \frac{Z_0 Y_n}{Y_0} & \text{for TM modes} \end{cases} \quad (4.6.40)$$

Then it is easy to show that

$$\begin{aligned} A_n + B_n &= \int_A \bar{\mathbf{E}}_t \cdot \bar{\mathbf{E}}_{tn} dA, \\ A_n - B_n &= Z_{on}^{-2} \int_A \bar{\mathbf{H}}_t \cdot \bar{\mathbf{H}}_{tn} dA. \end{aligned} \quad (4.6.41)$$

This result shows that a knowledge of the transverse components of the electric and magnetic fields at a particular cross section of the waveguide is sufficient to determine the fields everywhere within the waveguide, because the coefficients A_n and B_n are determined by Eqs. 4.6.41 and, consequently, the expansion given in Eq. 4.6.39 is determined. It will be of interest later that Eqs. 4.6.41 permit the

expansion

$$\begin{aligned}\bar{E}_t \epsilon^{j\omega t} &= \sum_n V_n \bar{E}_{tn}, \\ \bar{H}_t \epsilon^{j\omega t} &= \sum_n i_n Z_{0n} \bar{H}_{tn},\end{aligned}\quad (4.6.42)$$

at an arbitrary cross section of the waveguide. The coefficients V_n and i_n depend on the time as $\epsilon^{j\omega t}$ and it is clear that

$$\begin{aligned}V_n &= (A_n + B_n) \epsilon^{j\omega t}, \\ i_n &= 1/Z_{0n} (A_n - B_n) \epsilon^{j\omega t}.\end{aligned}$$

If there is only one mode present in the waveguide, as is usually the case, sufficiently far from any discontinuities within the guide, it is readily shown that the coefficients V_n and i_n are analogous to voltage and current in ordinary circuit theory. For this case

$$\begin{aligned}\bar{E}_t \epsilon^{j\omega t} &= V_n \bar{E}_{tn} \\ \bar{H}_t \epsilon^{j\omega t} &= i_n Z_{0n} \bar{H}_{tn} \\ Z &= \frac{E_t}{H_t} = \frac{V_n}{i_n},\end{aligned}\quad (4.6.43)$$

because by Eqs. 4.6.23, 4.6.31, and 4.6.40

$$\frac{E_{tn}}{H_{tn}} = Z_{0n}.$$

Furthermore, the power flow in the positive z direction is given by

$$P = \frac{1}{2} \operatorname{Re} \left\{ \int_A \hat{i}_z (E \times H^*) \cdot dA \right\} = \frac{1}{2} \operatorname{Re}(V_n i_n^*), \quad (4.6.44)$$

where, as before, Re means the real part of what follows.

With the techniques presented in this section it will now be possible to modify the results of Section 4.5 to include the loss of power through the waveguide output. This modification will be carried out in the next section.

4.7 The Resonant Cavity with a Waveguide Output

In Section 4.5 it was assumed that the cavity was entirely enclosed by a conducting surface S , so that

$$\int_S (\hat{n} \times \bar{H}) \cdot \bar{E}_a \, dA$$

was zero. It is now possible to introduce a surface S' and to compute the resulting integral. The surface S' will be chosen as a plane across the waveguide output. At S' it is then possible to write

$$\bar{E}_a = \sum_n v_{an} \bar{E}_{tn} = \sum_n \left(\int_{S'} \bar{E}_a \cdot \bar{E}_{tn} \, dA \right) \bar{E}_{tn}. \quad (4.7.1)$$

If the surface S' is sufficiently far removed from any discontinuities in the waveguide, only propagated modes will appear in the summation of Eq. 4.7.1. It will now be assumed that over the surface S' an arbitrary magnetic field, $\bar{H}$, is impressed by some means, where

$$\bar{H} = \sum_n i_n Z_{on} \bar{H}_{tn}. \quad (4.7.2)$$

The positive direction in the waveguide is chosen to be into the cavity so that

$$\hat{n} = -\hat{i}_z.$$

From Eq. 4.7.2

$$\hat{n} \times \bar{H} = \sum_n i_n Z_{on} (\bar{H}_{tn} \times \hat{i}_z).$$

But for both TE and TM modes it was shown that

$$Z_{on} (\bar{H}_{tn} \times \hat{i}_z) = \bar{E}_{tn}.$$

Therefore

$$\int_{S'} (\hat{n} \times \bar{H}) \cdot \bar{E}_a \, dA = \int_{S'} \sum_n i_n \bar{E}_{tn} \cdot \sum_m v_{am} \bar{E}_{tm} \, dA.$$

From the orthonormality properties of the $\bar{E}_{tn}$, this integral becomes

$$\int_{S'} (\hat{n} \times \bar{H}) \cdot \bar{E}_a \, dA = \sum_n i_n v_{an}. \quad (4.7.3)$$

Substitution of this expression into Eq. 4.5.10 for the case of no currents within the cavity readily leads to

$$\int_V \bar{E} \cdot \bar{E}_a \, dV = \sum_n \frac{i_n v_{an} / \epsilon \omega_a}{j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right) + \frac{1}{Q_a}} \quad (4.7.4)$$

where, as before,

$$\omega_a' = \omega_a \left(1 - \frac{1}{2Q_a} \right).$$

The transverse electric field across S' may be expanded in terms of the normal modes in the waveguide:

$$\bar{E} = \sum_n v_n \bar{E}_{tn}$$

where

$$v_n = \int_{S'} \bar{E} \cdot \bar{E}_{tn} \, dA. \quad (4.7.5)$$

If $\bar{E}$ is expanded in terms of the cavity modes, this expression may be written

$$V_n = \sum_a \int_{S'} \left(\bar{E}_a \int_V \bar{E} \cdot \bar{E}_a dV \right) \cdot \bar{E}_{tn} dA. \quad (4.7.6)$$

By Eq. 4.7.1 this expression may be written

$$V_n = \sum_a v_{an} \int_V \bar{E} \cdot \bar{E}_a dV. \quad (4.7.7)$$

With the aid of Eq. 4.7.4,

$$V_n = \sum_m i_m \left(\sum_a \frac{v_{am} v_{an} / \epsilon \omega_a}{j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right) + \frac{1}{Q_a}} \right). \quad (4.7.8)$$

It is convenient to introduce the notation

$$Z_{mn} = \sum_a \frac{v_{am} v_{an} / \epsilon \omega_a}{j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right) + \frac{1}{Q_a}}. \quad (4.7.9)$$

If only one mode is present in the waveguide,

$$E = V_n E_{tn},$$

$$H = i_n Z_{on} H_{tn},$$

$$\frac{E}{H} = \sum_a \frac{v_{an}^2 / \epsilon \omega_a}{j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right) + \frac{1}{Q_a}} = Z_{nn}.$$

Thus Z_{nn} is the impedance looking into the cavity at S' .

It should be pointed out that this derivation has assumed that the cavity fields oscillate almost entirely in the a^{th} mode at angular frequencies near ω_a , because this assumption underlies the derivation of

Eq. 4.5.10. From Eq. 4.7.4 it can be shown that this condition will usually be satisfied.

If it is required that the waveguide propagate only in the dominant mode at the frequencies of interest, then, by Eq. 4.7.9, only Z_{11} is different from zero:

$$Z_{11} = \sum_a \frac{v_{a1}^2 / \epsilon \omega_a}{j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right) + \frac{1}{Q_a}}. \quad (4.7.10)$$

It is convenient to group together all of the terms except the a^{th} in Eq. 4.7.10 into a term Z_{a1} , so that

$$Z_{11} = \frac{Z_{01}/Q_{\text{ext}, a1}}{j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right) + \frac{1}{Q_a}} + Z_{a1}. \quad (4.7.11)$$

where Z_{01} is the guide impedance for the dominant mode, and

$$\frac{1}{Q_{\text{ext}, a1}} = v_{a1}^2 / \epsilon \omega_a Z_{01}. \quad (4.7.12)$$

The term Z_{a1} is a slowly varying, almost entirely reactive term. By a suitable choice of the surface S' , it is possible to make Z_{a1} negligible for most cases of practical interest. This amounts to choosing for S' the plane of a voltage standing-wave minimum at angular frequency ω_a' when the a^{th} resonance has been detuned. This plane is the so-called "plane of the detuned short."

The effect of terminating the waveguide output by some impedance, Z_2 , will now be considered. The natural frequency of the cavity and waveguide output may be found from the requirement that the impedance looking into the cavity at S' should be the negative of the impedance

looking out. That is,

$$Z_{11} = -Z_1.$$

From Eq. 4.7.11,

$$j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right) + \frac{1}{Q_a} + \frac{v_{a1}^2 / \epsilon \omega_a}{Z_1 + Z_{a1}} = 0. \quad (4.7.13)$$

With the introduction of the normalized admittance

$$g + jb = \frac{Z_{01}}{Z_1 + Z_{a1}}, \quad (4.7.14)$$

Eq. 4.7.13 becomes

$$j \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} + \frac{b}{Q_{\text{ext}, a1}} \right) + \left(\frac{1}{Q_a} + \frac{g}{Q_{\text{ext}, a1}} \right) = 0. \quad (4.7.15)$$

If

$$\omega = (\omega_a' + \delta\omega_a') + j\omega_2$$

is substituted in Eq. 4.7.15 and the real and imaginary parts are set equal to zero, it is found that

$$\frac{\delta\omega_a'}{\omega_a'} = \frac{1}{2} \frac{b}{Q_{\text{ext}, a1}}, \quad (4.7.16)$$

and

$$\frac{2\omega_2}{\omega_a'} = \frac{1}{Q} = \frac{1}{Q_a} + \frac{g}{Q_{\text{ext}, a1}}. \quad (4.7.17)$$

The conductive component of the normalized admittance introduced in Eq. 4.7.14 makes a contribution to Q and therefore to the damping, as might be expected. In addition, the reactive part of the normalized admittance causes a shift in resonant frequency.

If there are no losses in the cavity walls then the Q arises only from losses through the waveguide output. For $Z_1 + Z_{a1} = Z_{o1}$,

$$\frac{1}{Q} = \frac{1}{Q_{\text{ext}, a1}}$$

if the wall losses are neglected. If the term Z_{a1} is neglected, this means that the waveguide output is terminated in a matched load. The quantity $Q_{\text{ext}, a1}$ is called the external Q for the a^{th} mode and the first mode in the waveguide.

The problem of the resonant cavity with a waveguide output excited by electronic currents will now be considered. Once again the impedance at S' looking out of the cavity is Z_1 . The impedance looking into the cavity is therefore $-Z_2$ where

$$-Z_2 = \frac{V_1}{i_1} = \frac{\sum v_{a1} \int_V \bar{E} \cdot \bar{E}_a dV}{i_1}, \quad (4.7.18)$$

with the aid of Eq. 4.7.7. Near the a^{th} mode this expression may be written

$$-Z_2 = \frac{v_{a1}}{i_1} \int_V \bar{E} \cdot \bar{E}_a dV + Z_{a1}, \quad (4.7.19)$$

where Z_{a1} is a slowly varying term arising from all the terms except the a^{th} in Eq. 4.7.18. It is now easily shown that

$$i_1 v_{a1} = \frac{-v_{a1}^2 \int_V \bar{E} \cdot \bar{E}_a dV}{Z_1 + Z_{a1}} = -\frac{v_{a1}^2}{Z_{o1}} (g + jb) \int_V \bar{E} \cdot \bar{E}_a dV. \quad (4.7.20)$$

With the aid of Eqs. 4.7.3 and 4.7.20, Eq. 4.5.10 easily leads to

$$\int_V \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}_a dV = \frac{1}{\epsilon \omega_a} \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a dV \cdot \left[j \left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} + \frac{1}{Q_a} + \frac{b}{Q_{\text{ext}, a1}} \right) + \frac{1}{Q_a} + \frac{g}{Q_{\text{ext}, a1}} \right]. \quad (4.7.21)$$

The power flow out of the cavity is given by

$$\mathcal{P} = -\frac{1}{2} \text{Re}(V_{a1} i_1^*), \quad (4.7.22)$$

where the negative sign enters because the positive direction in the waveguide was chosen to be into the cavity. From Eqs. 4.7.7 and 4.7.19 this expression becomes

$$\mathcal{P} = -\frac{1}{2} \text{Re} \left\{ - \frac{v_{a1} v_{a1}^*}{Z_1^* + Z_{a1}^*} \right\} \left| \int_V \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}_a dV \right|^2. \quad (4.7.23)$$

From Eq. 4.7.21 it is easily seen that the power flowing out of the waveguide is a maximum when

$$\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} + \frac{1}{Q_a} + \frac{b}{Q_{\text{ext}, a1}} = 0$$

or when

$$\omega = \omega_a'' = \omega_a \left\{ 1 - \frac{1}{2Q_a} - \frac{b}{2Q_{\text{ext}, a1}} \right\}. \quad (4.7.24)$$

At this frequency

$$\int_V \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}_a dV = \frac{Q}{\epsilon \omega_a} \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a dV.$$

Substitution in Eq. 4.7.23 leads to

$$\mathcal{P}_{\text{max}}(Q) = -\frac{1}{2} \text{Re} \left\{ - \frac{v_{a1} v_{a1}^*}{Z_{01}^*} \frac{Z_{01}^*}{Z_1^* + Z_{a1}^*} \frac{Q^2}{\epsilon \omega_a} \right\} \left| \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a dV \right|^2. \quad (4.7.25)$$

Now

$$\frac{Z_{01}^*}{Z_{i1}^* + Z_{a1}^*} = g - jb.$$

Furthermore, if the E_a 's and the E_{tn} 's are chosen to be real, which they may be,

$$v_{a1} = v_{a1}^*.$$

Finally, for the waveguides of interest in this case,

$$Z_{01} = Z_{01}^*.$$

With the aid of these last relationships, it is found that

$$P(\omega_a'', Q) = \frac{gQ^2}{2\epsilon\omega_a Q_{\text{ext}, a1}} \left| \int_V \bar{J} \cdot \bar{E}_a dV \right|^2.$$

If the term Z_{a1} in the input impedance of the cavity is neglected and the waveguide output is terminated in a matched load, then

$$g = 1, b = 0$$

$$\frac{1}{Q} = \frac{1}{Q_a} + \frac{1}{Q_{\text{ext}, a1}},$$

$$\omega'' = \omega_a'$$

from which

$$P(\omega_a', Q) = \frac{Q_a Q_{\text{ext}, a1}}{(Q_a + Q_{\text{ext}, a1})^2} \frac{Q_a}{2\epsilon\omega_a} \left| \int_V \bar{J} \cdot \bar{E}_a dV \right|^2. \quad (4.7.26)$$

If this expression is differentiated with respect to $Q_{\text{ext}, a1}$, it is found that its maximum occurs when $Q_a = Q_{\text{ext}, a1}$, or when the losses in the walls just equal the losses through the waveguide output.

The final result is then

$$P_{\text{max}}(\omega_a') = \frac{1}{4} \left\{ \frac{Q_a}{2\epsilon\omega_a} \left| \int_V \bar{J} \cdot \bar{E}_a dV \right|^2 \right\} \quad (4.7.27)$$

which is just one quarter of the value obtained in Section 4.5 for the power generated in a cavity with no waveguide output but with wall losses.

A simple equivalent circuit picture may be used to explain the factor of $\frac{1}{4}$ which relates the results of this section with those of Section 4.5. The electronic currents are represented as a current generator, and the cavity at resonance by a resistance, R . The coupling iris between the waveguide output and the cavity is represented as an ideal transformer and the waveguide output is represented as a resistor, R_0 . This circuit is shown in Fig. 7. Now the requirement that $Q_a = Q_{\text{external}}$

Figure 7. Equivalent Circuit for a Resonant Cavity Excited by an Electron Beam

means that the cavity should be matched to the waveguide output on resonance, because

$$\frac{Z_{11}}{Z_{01}} = \frac{Q_a/Q_{\text{external}}}{1 + jQ_a \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right)}$$

if the term Z_{a1}/Z_{01} , which arises from other modes, is neglected. In order for the cavity to be matched to the waveguide output, it is necessary that

$$n^2 R_0 = R.$$

If the load is transformed, the circuit may be represented as shown in

Figure 8. Transformed Equivalent Circuit

Fig. 8. The power delivered to the resistor representing the load is

$$p = \frac{I^2 R}{4}$$

If there is no load on the output, the power delivered to the resistor representing the walls is

$$P = I^2 R,$$

which is four times the load power for a matched load.

This discussion concludes the general analysis of resonant cavities. The results of this analysis will now be used in the treatment of higher order modes in cylindrical cavities which follows.

5. THE RIGHT, CIRCULARLY CYLINDRICAL CAVITY

5.1 Introduction

In this chapter the general results of Chapter 4 will be applied to the special case of a right, circularly cylindrical cavity excited by an axial electron beam. The aim of this chapter is to evaluate Eq. 4.7.27 for the power delivered to a matched load on the waveguide output in terms of such parameters as, for example, the cavity shape and the beam velocity. As the first step in this program, the normal mode fields for the cavity will be computed.

5.2 Normal Modes for a Right, Circularly Cylindrical Cavity

The orientation of the cavity and the coordinate axes for the discussion which follows are shown in Fig. 9. It was shown in Section 4.2

Figure 9 Cylindrical Cavity of Radius R and Length L

that the normal mode functions $\bar{E}_a$, $\bar{H}_a$, and $\bar{F}_a$ are all solutions of the vector Helmholtz equation:

$$\nabla^2 \bar{A} + k^2 \bar{A} = 0. \quad (5.2.1)$$

In this section Eq. 5.2.1 will be solved for the case of a cavity entirely enclosed by a surface of the type S . There will be no surface S' in these calculations. The boundary conditions on the functions of interest are

$$\hat{n} \times \bar{E}_a = \hat{n} \cdot \bar{H}_a = \hat{n} \times \bar{F}_a = 0 \quad (5.2.2)$$

on S . In addition $\bar{E}_a$ and $\bar{H}_a$ must satisfy the relations

$$\text{curl } \bar{H}_a = k_a \bar{E}_a, \text{ curl } \bar{E}_a = k_a \bar{H}_a \quad (5.2.3)$$

There is also the requirement that

$$\text{div } \bar{E}_a = \text{div } \bar{H}_a = \text{curl } \bar{F}_a = 0 \quad (5.2.4)$$

As in Section 4:6, three independent vector solutions of Eq. 5.2.1 are

$$\begin{aligned} \bar{L} &= \text{grad } \psi, \\ \bar{M} &= \text{curl } (\hat{i}_z \varphi) \\ \bar{N} &= \frac{1}{k} \text{curl curl } (\hat{i}_z \chi) \end{aligned} \quad (5.2.5)$$

where, as before, $\hat{i}_z$ is a unit vector in the z direction, and ψ , φ , and χ are solutions of the scalar Helmholtz equation

$$\nabla^2 \psi + k^2 \psi = 0,$$

$$\nabla^2 \varphi + k^2 \varphi = 0,$$

$$\nabla^2 \chi + k^2 \chi = 0,$$

for the same value of k . The general solution is then

$$\bar{A} = \bar{L} + \bar{M} + \bar{N}. \quad (5.2.6)$$

From Eqs. 5.2.5 it is clear that

$$\begin{aligned} \text{curl } \bar{L} &= 0 \\ \text{div } \bar{M} &= \text{div } \bar{N} = 0 \end{aligned} \quad (5.2.7)$$

Consequently the functions $\bar{F}_a$ will be constructed from the $\bar{L}$ solutions and the functions $\bar{E}_a$ and $\bar{H}_a$ will be constructed from the $\bar{M}$ and $\bar{N}$ solutions. The functions $\bar{E}_a$ and $\bar{F}_a$ which are normal at the boundary will be considered first. The scalar Helmholtz equation will be solved, and from its solutions vector functions $\bar{L}_a$, $\bar{M}_a$, and $\bar{N}_a$ which are normal on S will be constructed in accordance with Eq. 5.2.5.

Solutions of the scalar Helmholtz equation which are finite on the z axis have the form

$$\psi = \frac{J_l(k_1 r)}{k_1} \begin{Bmatrix} \cos l\varphi \\ \sin l\varphi \end{Bmatrix} \begin{Bmatrix} \sin k_3 z \\ \cos k_3 z \end{Bmatrix} \quad (5.2.8)$$

where

$$k^2 = k_1^2 + k_3^2, \quad (5.2.9)$$

and $J_l(r)$ is a Bessel function of the first kind of integral order l .

In Eq. 5.2.8 any of the four combinations which can be formed by selecting either term in each bracketed pair is permissible, or a sum of such terms is permissible. To simplify the notation only scalar functions with even φ dependence will be considered in the development. To construct the vector functions arising from scalars with odd φ dependence it will be necessary only to replace $\cos l\varphi$ by $\sin l\varphi$ and $\sin l\varphi$ by

$\cos l\varphi$ in the final expressions. With this in mind, Eq. 5.2.8 may be specialized so that

$$\psi_e = \frac{J_l(k_1 r)}{k_1} \cos l\varphi \begin{Bmatrix} \sin k_3 z \\ \cos k_3 z \end{Bmatrix} \quad (5.2.10)$$

To determine $\bar{L}$, the gradient of ψ must be computed as indicated in Eq. 5.2.5. The requirement that $\bar{L}$ be normal at the surface leads to

$$v_{ae} = \frac{J_l'(k_1 r)}{k_1} \cos l\varphi \sin k_3 z \quad (5.2.11)$$

and

$$\begin{aligned} \bar{L}_{ae} &= \hat{e}_r J_l'(k_1 r) \cos l\varphi \sin k_3 z \\ &- \hat{e}_\varphi \frac{l J_l(k_1 r)}{k_1 r} \sin l\varphi \sin k_3 z \\ &+ \hat{e}_z \frac{k_3}{k_1} J_l(k_1 r) \cos l\varphi \cos k_3 z, \end{aligned} \quad (5.2.12)$$

where

$$J_l'(k_1 r) = d[J_l(k_1 r)]/d(k_1 r),$$

$$k_3 = \frac{n\pi}{L} \quad (n \text{ is integer} > 0),$$

$$k_1 = \frac{r_{lm}}{R}, \quad J_l(r_{lm}) = 0,$$

$$k^2 = k_a^2 = \left(\frac{r_{lm}}{R}\right)^2 + \left(\frac{n\pi}{L}\right)^2,$$

and $\hat{e}_r$, $\hat{e}_\varphi$, and $\hat{e}_z$ are the unit base vectors for the cylindrical coordinate system. In a similar way it is found that

$$v_{ae} = \frac{J_l(k_1 r)}{k_1} \cos l\varphi \sin k_3 z$$

and

$$\begin{aligned} \bar{M}_{ae} = & \hat{i}_r \frac{l J_l(k_1 r)}{k_1 r} \sin l\varphi \sin k_3 z \\ & - i\varphi J'_l(k_1 r) \cos l\varphi \sin k_3 z. \end{aligned} \quad (5.2.13)$$

where

$$k_3 = \frac{n\pi}{L}, \quad (n \text{ is integer } > 0)$$

and

$$k_1 = \frac{r'_{lm}}{R}, \quad J'_l(r'_{lm}) = 0.$$

It should be noted that the two scalar functions which have been constructed are different as a result of the difference in the radial dependence. It should also be noted that the associated eigen values for these two sets of functions are generally different.

In Section 4.3 it was shown that a possible solution for the electric field in a cavity completely enclosed with a perfectly conducting surface is

$$\bar{E} = \bar{E}_a e^{j\omega t}$$

The vector functions $\bar{M}_a$ of this discussion satisfy the same equation and the same boundary conditions as the functions $\bar{E}_a$ of Section 4.3. From Eq. 5.2.13 it is seen that the vectors $\bar{M}_a$ have no z component. It is therefore concluded that the functions $\bar{M}_a$ represent the spatial part of the electric field for the transverse electric (TE) modes.

By similar methods it is found that

$$\chi_{ae} = \frac{J_l(k_1 r)}{k_1} \cos l\varphi \cos k_3 z \quad (5.2.14)$$

and

$$\begin{aligned}\bar{N}_{ae} = & \hat{i}_r \frac{k_3}{k_a} J_l'(k_1 r) \cos l\varphi \sin k_3 z \\ & + \hat{i}_\varphi \frac{k_3 l}{k_a} \frac{J_l(k_1 r)}{k_1 r} \sin l\varphi \sin k_3 z \\ & + \hat{i}_z \frac{k_1}{k_a} J_l(k_1 r) \cos l\varphi \cos k_3 z,\end{aligned}\quad (5.2.15)$$

where

$$k_3 = \frac{n\pi}{L}, \quad (n \text{ is integer})$$

$$k_1 = \frac{r_{lm}}{R}, \quad J_l(r_{lm}) = 0.$$

Although the eigen values of the functions χ_a are the same as those of the functions ψ_a , the functions themselves are different. Indeed, the three scalars which have been employed to generate the independent vector solutions of the vector Helmholtz equation are all different.

From Eq. (5.2.15) it is seen that the vectors $\bar{N}_a$ all have a z component. By Section 4.3

$$\bar{E} = \bar{N}_a e^{j\omega t}$$

is an admissible solution for the electric field in a cylindrical cavity with perfectly conducting walls. It is clear that these functions must represent the spatial part of the electric field for the transverse magnetic(TM) modes.

Three independent sets of vector functions, all of which are solutions of the vector Helmholtz equation and are normal to the surface at the walls of the cylindrical cavity considered in this section, have been constructed. The irrotational functions $\bar{L}_a$

are, except for a normalization factor, the functions $\bar{F}_a$ of Section 4.2. Similarly, the functions $\bar{M}_a$ and $\bar{N}_a$ are, except for a normalization factor, the functions $\bar{E}_a$ of that section. The $\bar{H}_a$ functions may now be obtained from Eq. 5.2.3.

It has been shown that

$$\bar{E}_{a,TE} = c_a \bar{M}_a,$$

$$\bar{E}_{a,TM} = d_a \bar{N}_a$$

where c_a and d_a are constants such that

$$\int_V c_a^2 M_a^2 dV = \int_V d_a^2 N_a^2 dV = 1$$

From Eq. 5.2.3

$$\bar{H}_{a,TE} = \frac{c_a}{k_a} \text{curl } \bar{M}_a$$

$$\bar{H}_{a,TM} = \frac{d_a}{k_a} \text{curl } \bar{N}_a.$$

The computations are straightforward and lead to

$$\begin{aligned} \bar{H}_{a,TE} = c_a \left\{ \hat{i}_r \frac{k_3}{k_a} J'_l(k_1 r) \cos l\phi \cos k_3 z \right. \\ \left. - \hat{i}_\phi \frac{lk_3}{k_a} \frac{J_l(k_1 r)}{k_1 r} \sin l\phi \cos k_3 z \right. \\ \left. + \hat{i}_z \frac{k_1}{k_a} J_l(k_1 r) \cos l\phi \sin k_3 z \right\} \end{aligned} \quad (5.2.16)$$

$$\bar{H}_{a, TM} = d_a \left\{ \hat{i}_r \frac{l J_l(k_1 r)}{k_1 r} \sin l\varphi \cos k_3 z \right. \\ \left. - \hat{i}_\varphi J'_l(k_1 r) \cos l\varphi \cos k_3 z \right\}. \quad (5.2.17)$$

The functions with odd φ dependence may easily be obtained from these results, as indicated earlier in this section.

Examination of the electric field expressions shows that only those TM modes characterized by $l = 0$ have a non-zero axial electric field, because if l is not zero, $J_l(0) = 0$, and the z component of electric field on the axis will involve this term as a factor. For TM_{0mn} modes

$$\bar{E}_{0mn} = d_{0mn} \left\{ \hat{i}_r \frac{n\pi/L}{k_{0mn}} J_1(r_{0m} \frac{r}{R}) \sin(\frac{n\pi z}{L}) \right. \\ \left. + \hat{i}_z \frac{r_{0m}/R}{k_{0mn}} J_0(r_{0m} \frac{r}{R}) \cos(\frac{n\pi z}{L}) \right\} \quad (5.2.18)$$

and

$$\bar{H}_{0mn} = \hat{i}_\varphi d_{0mn} J_1(r_{0m} \frac{r}{R}) \cos(\frac{n\pi z}{L}) \quad (5.2.19)$$

where, from Eq. 5.2.9,

$$k_{0mn}^2 = (\frac{\omega_{0mn}}{c})^2 = (\frac{r_{0m}}{R})^2 + (\frac{n\pi}{L})^2, \quad (5.2.20)$$

and the relationship

$$J'_0(x) = -J_1(x)$$

has been used. The constant d_{0mn} is determined from the requirement

that

$$\int_V E_{0mn}^2 dV = \int_V H_{0mn}^2 dV = 1.$$

This normalization integration is straightforward and leads to

$$d_{0m0} = \left(\frac{1}{V}\right)^{1/2} \frac{1}{J_1(r_{0m})} \quad n=0$$

$$d_{0mn} = \left(\frac{2}{V}\right)^{1/2} \frac{1}{J_1(r_{0m})} \quad n \neq 0$$

where V is the volume of the cavity. With these values for the normalization constant, the field expressions are:

$$\left. \begin{aligned} \bar{E}_{0m0} &= \hat{i}_z \left\{ \left(\frac{1}{V}\right)^{1/2} \frac{r_{0m}/R}{k_{0m0}} \frac{J_0(r_{0m} \frac{r}{R})}{J_1(r_{0m})} \right\}, \\ \bar{H}_{0m0} &= \hat{i}_\phi \left\{ \left(\frac{1}{V}\right)^{1/2} \frac{J_1(r_{0m} \frac{r}{R})}{J_1(r_{0m})} \right\} \end{aligned} \right\} n=0 \quad (5.2.21)$$

$$\left. \begin{aligned} \bar{E}_{0mn} &= \hat{i}_r \left\{ \left(\frac{2}{V}\right)^{1/2} \frac{n\pi/L}{k_{0mn}} \frac{J_1(r_{0m} \frac{r}{R})}{J_1(r_{0m})} \sin\left(\frac{n\pi z}{L}\right) \right\} \\ &+ \hat{i}_z \left\{ \left(\frac{2}{V}\right)^{1/2} \frac{r_{0m}/R}{k_{0mn}} \frac{J_0(r_{0m} \frac{r}{R})}{J_1(r_{0m})} \cos\left(\frac{n\pi z}{L}\right) \right\} \\ \bar{H}_{0mn} &= \hat{i}_\phi \left\{ \left(\frac{2}{V}\right)^{1/2} \frac{J_1(r_{0m} \frac{r}{R})}{J_1(r_{0m})} \cos\left(\frac{n\pi z}{L}\right) \right\} \end{aligned} \right\} n \neq 0 \quad (5.2.22)$$

These expressions represent those modes which have a non-zero axial electric field. For an electron beam traveling down the axis of the cavity, they are the only modes for which the power generated within the cavity as given by Eq. 4.5.19 will be different from zero:

$$P(\omega'_a) = \frac{Q_a}{2\epsilon\omega_a} \left| \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a dV \right|^2. \quad (4.5.19)$$

If a particular current density, $\bar{\mathbf{J}}$, is assumed it will now be possible to evaluate this expression using the normalized normal mode fields which have just been determined.

5.3 The Unloaded Q of A Right, Circular Cylindrical Cavity

To evaluate the power generated in a cavity it is necessary to evaluate first the quantity Q_a which appears in Eq. 4.5.19. From Eq. 4.5.4

$$\frac{1}{Q_{0mn}} = \frac{\delta}{2} \int_S H_{0mn}^2 dA, \quad (5.3.1)$$

where

$$\delta = \sqrt{\frac{2}{\omega_{0mn}\sigma_w}} \quad (5.3.2)$$

is the skin depth. Evaluation of the integral in Eq. 5.3.1 is straightforward and with the aid of Eqs. 5.2.21 and 5.2.22 leads readily to

$$Q_{0m0} = \frac{\pi r_{0m}}{2R_s} \frac{1}{1 + R/L}, \quad n = 0 \quad (5.3.3)$$

$$Q_{0mn} = \frac{\eta r_{0m}}{2R_s} \frac{[1 + (\frac{n\pi R}{r_{01}L})^2]^{\frac{1}{2}}}{1 + 2R/L}, \quad n \neq 0 \quad (5.3.4)$$

where

$$R_s = \sqrt{\frac{\omega_{0mn} \mu}{2\sigma}}, \quad \eta = \sqrt{\frac{\mu}{\epsilon}}$$

These expressions are often given in the alternate form^{4,5}

$$Q_{0mn} \frac{\delta}{\lambda} = \frac{r_{0m}}{2\pi(1 + \frac{R}{L})}, \quad (n = 0) \quad (5.3.5)$$

$$Q_{0mn} \frac{\delta}{\lambda} = \frac{r_{0m} [1 + (\frac{n\pi R}{r_{01}L})^2]^{\frac{1}{2}}}{2\pi(1 + \frac{2R}{L})}, \quad (n \neq 0) \quad (5.3.6)$$

Curves plotted from these expressions are given in Figs. 11, 19 and 26.

5.4 Independence of Harmonically Related Oscillations

Before proceeding with the evaluation of $\int_V \vec{J} \cdot \vec{E}_a dV$ it will

be useful to make some preliminary observations. In all the work which has preceded this section it has been assumed that all of the field quantities oscillate sinusoidally with angular frequency, ω . It has also been assumed that the physical quantities are given by the real part of expressions which involve the time as $e^{j\omega t}$. That is, the actual current density $\vec{J}_r$ is given by

$$\vec{J}_r = \text{Re} \vec{J} = \text{Re} \vec{J}_1 e^{j(\omega t + \phi)} = \vec{J}_1 \cos(\omega t + \phi), \quad (5.4.1)$$

and the actual electric field $\bar{E}_r$ is given by

$$\bar{E}_r = \text{Re } \bar{E} = \text{Re } \bar{E}_1 e^{j(\omega t + \psi)} = \bar{E}_1 \cos(\omega t + \psi). \quad (5.4.2)$$

where $\bar{J}_1$ and $\bar{E}_1$ are real. To find the average work done on the current by the field, it is necessary to average the dot product $\bar{J}_1 \cdot \bar{E}_1$ over a cycle of oscillation and then to integrate the resulting expression over the volume, V . Thus,

$$P_{\text{avg}} = \frac{\omega}{2\pi} \int_V \int_0^{2\pi/\omega} \bar{J}_1 \cdot \bar{E}_1 \cos(\omega t + \phi) \cos(\omega t + \psi) dt dV.$$

This integration is straightforward and leads to

$$P_{\text{avg}} = \frac{1}{2} \int_V \bar{J}_1 \cdot \bar{E}_1 \cos(\phi - \psi) dV. \quad (5.4.3)$$

The time average of $\bar{J} \cdot \bar{E}^*$ is

$$(\bar{J} \cdot \bar{E})_{\text{avg}} = \bar{J} \cdot \bar{E}^* = \bar{J}_1 \cdot \bar{E}_1 e^{j(\phi - \psi)}.$$

The real part of this expression is just the integrand of Eq. 5.4.3, so it is clear that

$$P_{\text{avg}} = \frac{1}{2} \text{Re} \int_V \bar{J} \cdot \bar{E}^* dV, \quad (5.4.4)$$

which is just the negative of the expression given in Eq. 4.5.14.

The difference in sign arises because in Eq. 4.5.14 the work done on the field is considered rather than the work done by the field as in Eq. 5.4.4.

A more complicated case occurs when the current and the electric

field contain many frequencies at integral multiples of some fundamental frequency. For example, let

$$\bar{J}_r = \text{Re } \bar{J} = \text{Re } \sum_n \bar{J}_n e^{j(n\omega t + \phi_n)} = \sum_n \bar{J}_n \cos(n\omega t + \phi_n), \quad (5.4.5)$$

$$\bar{E}_r = \text{Re } \bar{E} = \text{Re } \sum_n \bar{E}_n e^{j(n\omega t + \psi_n)} = \sum_n \bar{E}_n \cos(n\omega t + \psi_n), \quad (5.4.6)$$

where $\bar{J}_n$ and $\bar{E}_n$ are real. In terms of the real functions the expression for the average power is

$$P_{\text{avg}} = \frac{\omega}{2\pi} \int_V \int_0^{2\pi/\omega} \left\{ \sum_n \bar{J}_n \cos(n\omega t + \phi_n) \right\} \cdot \left\{ \sum_m \bar{E}_m \cos(m\omega t + \psi_m) \right\} dt dV.$$

Because of the orthogonality properties of the sines and cosines, this expression readily reduces to

$$P_{\text{avg}} = \sum_n \frac{1}{2} \int_V \bar{J}_n \cdot \bar{E}_n \cos(\phi_n - \psi_n) dV. \quad (5.4.7)$$

This expression is just a sum of terms like Eq. 5.4.3, and it is concluded that the various harmonic components interact independently with the electron beam.

The time average of $\bar{J} \cdot \bar{E}^*$ for this case is

$$(\bar{J} \cdot \bar{E}^*)_{\text{avg}} = \sum_{m,n} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \bar{J}_m \cdot \bar{E}_n e^{j(m-n)\omega t} e^{j(\phi_m - \phi_n)} dt.$$

Only those terms for which $m = n$ contribute to the time average, and the result is

$$(\bar{J} \cdot \bar{E}^*)_{\text{avg}} = \sum_n \bar{J}_n \cdot \bar{E}_n e^{j(\phi_n - \psi_n)}.$$

The real part of this expression is just the integrand of Eq. 5.4.7, so it is clear that in complex notation

$$P_{avg} = \frac{1}{2} \operatorname{Re} \int_V (\bar{\mathbf{J}} \cdot \bar{\mathbf{E}}^*)_{avg} dV. \quad (5.4.8)$$

It is also interesting to consider the flow of energy for this special case of several oscillations at harmonically related frequencies. By the same methods used in this section it can be shown that the average power flow, $\bar{\mathbf{S}}_{avg}$, is given by

$$\bar{\mathbf{S}}_{avg} = \frac{1}{2} \operatorname{Re} (\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*)_{avg}.$$

It is clear that there are no cross terms in this expression, due to the averaging process, and it is concluded that in so far as energy flow is concerned the harmonically related fields are independent.

Returning now to the problem of resonant cavities, it is clear that the analysis of Chapter 4 applies to a cavity containing fields oscillating at harmonically related frequencies, subject to the restriction that at any one frequency the oscillation is to be predominately in only one normal mode. For this case each frequency is to be treated independently.

5.5 Evaluation of $\int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_a dV$ and $P(\omega'_{omn})$

It will be assumed that an electron beam of radius ρ_0 travels along the z axis in the direction of increasing z . Within the beam the electron density is to be a function only of z and t , and all electrons are

to travel with speed v parallel to the z axis. The current density is to be periodic along the z axis and in time and of such a form that it may be represented by the Fourier series

$$\bar{J} = \hat{i}_z \operatorname{Re} \sum_{n=0}^{\infty} J_n e^{-jn\omega_0 z/v} e^{-j\phi_n} e^{jn\omega_0 t}, \quad (5.5.1)$$

for $0 \leq r \leq \rho_0$ and $0 \leq z \leq L$. If the cavity dimensions are chosen so that the frequency of a particular current component, the s^{th} , corresponds to the resonant frequency, ω_{0mn} (wall losses are neglected), of the TM_{0mn} mode, this mode may be excited within the cavity. Therefore, if

$$s\omega_0 = \omega_{0mn} = c \frac{r_{0m}}{R} \sqrt{1 + \left(\frac{n\pi R}{r_{0m} L}\right)^2},$$

then, by Section 5.4, to find the power generated in the E_{0mn} mode only the s^{th} component of Eq. 5.5.1 need be considered and the analysis of Chapter 4 will apply so long as no other mode is excited at the same frequency. For this case, with the aid of Eqs. 5.2.21 and 5.2.22, it can be shown that

$$\left\{ \int_V \bar{J} \cdot \bar{E}_{0mn} dV \right\}_s = \left[2\pi\rho_0 J_1(r_{0m} \frac{\rho_0}{R}) \frac{R}{r_{0m}} \right] J_s \left\{ \frac{\frac{r_{0m}}{R} \sqrt{2/V}}{J_1(r_{0m}) \sqrt{(\frac{r_{0m}}{R})^2 + (\frac{n\pi}{L})^2}} \right\} L \\ \times e^{-j(\phi_s + \frac{s\omega_0 L}{2v})} \left\{ e^{j(n\pi/2)} \frac{1}{2} \frac{\sin\left(\frac{s\omega_0 L}{2v} + \frac{n\pi}{2}\right)}{\frac{s\omega_0 L}{2v} + \frac{n\pi}{2}} \right\}$$

$$+e^{-j(n\pi/2)} \frac{1}{2} \frac{\sin(\frac{s\omega_0 L}{2v} - \frac{n\pi}{2})}{\frac{s\omega_0 L}{2v} - \frac{n\pi}{2}} \left. \vphantom{\frac{1}{2}} \right\} e^{js\omega_0 t} \quad (n \neq 0) \quad (5.5.2)$$

$$\left\{ \int_V \mathbf{J} \cdot \mathbf{E}_{0m0} dV \right\}_s = [2\pi\rho_0 J_1(\frac{r_{0m}\rho_0}{R}) \frac{R}{r_{0m}}] J_s \frac{\sqrt{1/V}}{J_1(r_{0m})} L \quad (5.5.3)$$

$$\times e^{-j(\varphi_s + \frac{s\omega_0 L}{2v})} \frac{\sin \frac{s\omega_0 L}{2v}}{\frac{s\omega_0 L}{2v}} \quad (n=0)$$

If the beam diameter is small compared to the cavity diameter, that is if $\rho_0 \ll R$, the approximation

$$J_1(x) \approx \frac{x}{2}$$

for small x may be used. With this expression it is seen that

$$[2\pi\rho_0 J_1(\frac{r_{0m}\rho_0}{R}) \frac{R}{r_{0m}}] J_s \approx \pi\rho_0^2 J_s = I_s$$

where I_s is the total current associated with the s^{th} harmonic. It is convenient to introduce a quantity k_n called the radial coupling coefficient defined by

$$k_n = \frac{[2\pi\rho_0 J_1(\frac{r_{0m}\rho_0}{R}) \frac{R}{r_{0m}}]}{\pi\rho_0^2} \quad (5.5.4)$$

For an axial line beam $k_n = 1$. It is also convenient to introduce the longitudinal coupling coefficients k_+ and k_- defined by

$$k_{\pm} = \frac{1}{2} \frac{\sin(\frac{s\omega_0 L}{2v} - \frac{n\pi}{2})}{\frac{s\omega_0 L}{2v} - \frac{n\pi}{2}} \quad (5.5.5)$$

and

$$k_{\pm} = \frac{1}{2} \frac{\sin \left(\frac{s\omega_0 L}{2v} + \frac{n\pi}{2} \right)}{\frac{s\omega_0 L}{2v} + \frac{n\pi}{2}} \quad (5.5.6)$$

These coefficients measure the coupling to the wave traveling in the positive z -direction and the negative z -direction, respectively, if the standing wave within the cavity is thought of as being composed of two traveling waves. For the case $n = 0$,

$$k_{+} = k_{-} = \frac{1}{2} \frac{\sin \left(\frac{s\omega_0 L}{2v} \right)}{\left(\frac{s\omega_0 L}{2v} \right)} \quad (5.5.7)$$

The phase velocity, v_{0mn} , of the waves in the cavity is given by

$$v_{0mn} = \frac{\omega_{0mn}}{2\pi} \lambda_{0mn},$$

where λ_{0mn} is the wavelength in the cavity defined by

$$\frac{n}{2} \lambda_{0mn} = L.$$

It now follows that

$$\frac{s\omega_0 L}{2v} - \frac{n\pi}{2} = \frac{n\pi}{2} \left(\frac{v_{0mn}}{v} - 1 \right), \quad (n \neq 0) \quad (5.5.8)$$

and

$$v_{0mn} = c \sqrt{1 + (r_{0m} L / n\pi R)^2} \quad (n \neq 0)$$

With the aid of Eq. 5.5.8, it is found that

$$k_{\pm} = \frac{1}{2} \frac{\sin \left(\frac{n\pi}{2} \left[\frac{v_{0mn}}{v} \pm 1 \right] \right)}{\frac{n\pi}{2} \left[\frac{v_{0mn}}{v} \pm 1 \right]} = \frac{1}{2} \frac{\sin \left[\frac{n\pi}{2} \left(\frac{1}{\beta} \sqrt{1 + (r_{0m} L / n\pi R)^2} \pm 1 \right) \right]}{\frac{n\pi}{2} \left(\frac{1}{\beta} \sqrt{1 + (r_{0m} L / n\pi R)^2} \pm 1 \right)} \quad (n \neq 0) \quad (5.5.9)$$

To simplify the notation, let

$$\theta_{\pm} = \frac{n\pi}{2} \left(\frac{1}{\beta} \sqrt{1 + (r_{0m}L/n\pi R)^2} \mp 1 \right), \quad (n \neq 0) \quad (5.5.10a)$$

$$\theta_{\pm} = \frac{r_{0m}L}{2\beta R}, \quad (n = 0) \quad (5.5.10b)$$

Then

$$h_{\pm} = \frac{1}{2} \frac{\sin \theta_{\pm}}{\theta_{\pm}}, \quad (\text{all } n) \quad (5.5.11)$$

From Eqs. 5.5.10 it is readily shown that

$$\theta_{+} = \theta_{-} - n\pi, \quad (\text{all } n)$$

and consequently,

$$\sin \theta_{+} = \sin \theta_{-} \cos n\pi = (-1)^n \sin \theta_{-}. \quad (5.5.12)$$

From Eqs. 5.5.10 it is clear that β must be close to 1 if θ_{+} is to be less than π . From Eqs. 5.2.21 and 5.2.22 it is seen that the maximum amplitude of the normal mode electric field in the z direction $(E_z)_{0mn}^0$ is given by

$$(E_z)_{0m0}^0 = \left(\frac{1}{V}\right)^{1/2} \frac{1}{J_1(r_{0m})} \quad (n=0) \quad (5.5.13)$$

and

$$(E_z)_{0mn}^0 = \left(\frac{2}{V}\right)^{1/2} \frac{r_{0m}}{Rk_{0mn}} \frac{1}{J_1(r_{0m})} \quad (n \neq 0) \quad (5.5.14)$$

With the notation thus introduced Eqs. 5.5.2 and 5.5.3 may be written

in the same form:

$$\left\{ \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_{0mn} dV \right\}_s = k_n I_s (E_z)_{0mn}^0 L e^{-j(\varphi_s + \frac{s\omega_0 L}{2v})} \times \left\{ e^{j(n\pi/2)} k_- + e^{-j(n\pi/2)} k_+ \right\} e^{js\omega_0 t}. \quad (5.5.15)$$

It is now easily shown that

$$\left| \left\{ \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_{0mn} dV \right\}_s \right|^2 = \left\{ k_n I_s (E_z)_{0mn}^0 L \right\}^2 \left\{ k_-^2 + 2(-1)^n k_+ k_- + k_+^2 \right\}. \quad (5.5.16)$$

If the longitudinal coupling coefficient, k_l , defined by

$$k_l^2 = k_+^2 + 2(-1)^n k_+ k_- + k_-^2, \quad (5.5.17)$$

and the total coupling coefficient, K , defined by

$$K^2 = k_n^2 k_l^2, \quad (5.5.18)$$

are introduced, Eq. 5.5.16 becomes

$$\left| \left\{ \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_{0mn} dV \right\}_s \right|^2 = \left\{ K I_s L (E_z)_{0mn}^0 \right\}^2. \quad (5.5.19)$$

From Eq. 4.5.19, the power delivered to the cavity by the electron beam is, if there is no waveguide output,

$$P = (\omega'_{0mn}) = \frac{Q_{0mn}}{2\epsilon\omega_{0mn}} \left| \int_V \bar{\mathbf{J}} \cdot \bar{\mathbf{E}}_{0mn} dV \right|^2.$$

With the aid of Eq. 5.5.19 this expression may be written

$$P(\omega'_{0mn}) = \frac{1}{2} I_s^2 K^2 \left[\frac{Q_{0mn}}{\epsilon\omega_{0mn}} \left\{ L (E_z)_{0mn}^0 \right\}^2 \right] \quad (5.5.20)$$

The quantity in the square brackets is defined as R_{shunt} , the shunt impedance of the cavity for the TM_{0mn} mode, a property of the physical structure of the cavity. With the aid of Eqs. 5.3.3 and 5.5.13,

$$R_{\text{shunt}} = \frac{n^2}{\pi R_s J_1^2(r_{0m})} \frac{1}{\frac{2R}{L} (1+R/L)} \quad (n=0) \quad (5.5.21)$$

Equations 5.3.4 and 5.5.14 lead to

$$R_{\text{shunt}} = \frac{n^2}{\pi R_s J_1^2(r_{0m})} \frac{1}{\frac{R}{L} (1+2R/L)[1+(\frac{n^2 R}{r_{01}^2 L})^2]} \quad n \neq 0 \quad (5.5.21)$$

An equivalent circuit to represent the cavity and electron beam may be formulated on the basis of Eq. 5.5.20. It is clear that a current generator of peak amplitude $(I_s K)$ working into a resistance of value R_{shunt} would deliver the same power to the resistance as found in Eq. 5.5.20 for the power delivered to the cavity. It is just this fact that permits the introduction of the equivalent circuit used in Section 4.7.

Curves of the quantities introduced in this section are given in Figs. 10-29 for several TM_{01n} , TM_{02n} , and TM_{03n} modes.

5.6 The Choice of a Cavity Mode

In general, the choice of a cavity mode and of cavity dimensions will be influenced by such factors as the diameter of the electron beam, the velocity of the electrons in the beam, and the problems of fabrication. If the wavelengths involved are sufficiently

Figure 10 Mode Chart for TM_{omn} Modes in a Cylindrical Cavity

Figure 11 Universal Plot of Q_{01n} for Four TM_{01n} Modes

Figure 12 Universal Curves of Shunt Impedance, R_{shunt} , for Five TM_{0ln} Modes

Figure 13 Radial Coupling Coefficient for TM_{01n} Modes

Figure 14 Longitudinal Coupling Coefficients k_z^2 for Three TM_{0in} Modes ($\theta_+ < \pi$)

Figure 15 Universal Power Curves for Silver Cylindrical Cavities Operating in TM_{0in} Modes ($\theta_+ < \pi$)

Figure 16. Universal Plot of Q_{02n} for Three TM_{02n} Modes

Figure 17 Universal Plot of the Shunt Impedance, R_{shunt} , for Three TM_{02n} Modes

Figure 18 Radial Coupling Coefficient k_n^2 for TM_{02n} Modes

Figure 19. Longitudinal Coupling Coefficients k_L^2 for Two TM_{02n} Modes ($\Theta_+ < 2\pi$).

Figure 20 Universal Power Curves for Silver Cylindrical Cavities Operating in Two TM_{02n} Modes ($\theta_+ < 2\pi$)

Figure 21. Longitudinal Coupling Coefficients k_L^2 for the TM_{028} Mode ($\theta_+ < 2\pi$).

Figure 22 Universal Power Curves for Silver Cylindrical Cavities Operating in the TM_{026} Mode ($\theta_+ < 2\pi$).

Figure 23. Universal Power Curves for Silver Cylindrical Cavities Operating in the TM_{028} Mode ($\theta_+ < 2\pi$)

Figure 24. Universal Power Curves for Silver Cylindrical Cavities Operating in the TM_{028} Mode ($\pi < \theta_+ < 2\pi$)

Figure 25 Universal Power Curves for Silver Cylindrical Cavities Operating in the $TM_{02(10)}$ Mode ($\theta_+ < \pi$)

Figure 26. Universal Plot of Q_{03n} for Four TM_{03n} Modes

Figure 27 Universal Plot of the Shunt Impedance, R_{shunt} , for Four TM_{03n} Modes

Figure 28 Radial Coupling Coefficient, k_A^2 for TM_{03n} Modes

Figure 29. Universal Power Curves for Silver Cylindrical Cavities Operating in Three TM_{03n} Modes ($\theta_+ < 2\pi$)

long that the problems of fabrication are not too difficult, and the beam diameter is small enough to give good coupling to the fields within the cavity, it is not advisable to use a higher order mode resonance. At wavelengths of the order of a few millimeters, however, it becomes difficult to build the structures required, and electron beams of very small diameter are required to excite these structures effectively.

As an example, consider the case of a cylindrical cavity operating in the TM_{010} mode at a wavelength of one millimeter. The diameter of such a cavity is

$$D = \lambda(r_{01}/\pi) = 0.765 \text{ mm.}$$

The problem of injecting an electron beam of the order of 100 milliamperes through such a cavity is rather obvious. A reasonable diameter for such a beam at high energy is about one millimeter. If this diameter is assumed, only 59% of the current could enter the cavity even if the entire end walls were removed to admit the beam. For the current entering the cavity under these conditions, it is seen from Fig. 13 that $k_n^2 = 0.2$. Therefore, assuming a 100 ma beam with $\beta = 0.95$,

$$[k_n I_s (\text{amps})]^2 \sqrt{\lambda (\text{mm})} = 0.2 (0.59 \times 0.1)^2 \sqrt{1} = 6.96 \times 10^{-4} \text{ amp}^2 \text{ mm}^{1/2}.$$

If $R/L = 0.9$ so that operation is on the peak of the power curve in Fig. 15, it is found that

$$L = 0.425 \text{ mm.}$$

$$\frac{P(\text{watts})}{[k_n I_s (\text{amps})]^2 \sqrt{\lambda (\text{mm})}} = 8.7 \times 10^4$$

from which

$$P(\text{watts}) \approx 61 \text{ watts},$$

and the maximum output power is

$$P = 1/4 P \approx 15 \text{ watts}.$$

It is instructive to compare this result to that obtained for a cavity operating in the TM_{028} mode. From Fig. 20, choose $R/L = 0.6$, to permit operation near the peak of the power curve. For this situation

$$\frac{P(\text{watts})}{[k_n I_S(\text{amps})]^2 \sqrt{\lambda(\text{mm})}} = 1.6 \times 10^4.$$

The dimensions of the cavity are found to be

$$D \approx 5.11 \text{ mm}$$

and

$$L \approx 4.26 \text{ mm}$$

From Fig. 18,

$$k_n^2 \approx 0.74$$

and

$$P(\text{watts}) = (1.6 \times 10^4)(0.74)(10^{-2})(1) = 118 \text{ watts}$$

from which

$$P = 1/4 P \approx 29 \text{ watts}.$$

These results are tabulated below in Table 4.

Mode	λ mm	D mm	L mm	P watts	ρ watts
TM ₀₁₀	1	0.765	0.425	61	15
TM ₀₂₈	1	5.11	4.26	118	29

Table 4 A Comparison of Two Cavities for the Generation of Millimeter Wavelength Radiation ($\rho_0 = 0.5$ mm, $\beta = 0.95$, $I_s = 10$ ma.,

These results clearly indicate that in certain cases the use of a higher order mode cavity offers definite advantages over the use of a TM₀₁₀ mode cavity:

From the curves of Figs. 10 to 29, the following observations may be made:

1. To obtain as large a cavity as possible, it is desirable to choose m and n large and to operate at a large value of R/L or, equivalently, $(D/L)^2$. In practice some compromise must be made between cavity size and power output.
2. The shunt impedance, R_{shunt} , increases with increasing m but decreases with increasing n .
3. In any practical case the diameter of the electron beam must be considered. An apparent advantage resulting from an increase in m may be more than offset by an accompanying reduction in the radial coupling coefficient.

4. As m increases, the curves of k_{ℓ}^2 , the longitudinal coupling coefficient, move to the right; as n increases, they move down and to the left. As β increases, these curves move up and to the left.
5. As m increases, the power curves move down and to the right. As n increases, they move down and to the left. As β increases, these curves move up and to the left.
6. Operation in the range $\pi < \theta_+ < 2\pi$ is possible and, indeed, for $\beta \approx 0.85$, would appear to be quite desirable if the TM_{028} mode were to be used, for example. Larger values of θ_+ have not been investigated.

The problem of mode interference remains to be discussed. A complete mode chart, similar to that of Fig. 10, would contain lines representing all TM_{lmn} and TE_{lmn} modes. In order for the analysis of Chapter 4 to apply it is necessary that only one mode be excited at the frequency of interest in the cavity. To satisfy this requirement, an operating point away from any intersection of the desired mode with any other mode must be chosen. If the chosen mode is the TM_{0mn} mode, then the requirement that no other mode be excited leads readily from Eq.

4.7.21 to the condition

$$\frac{\omega_{0mn}''}{\omega_a''} - \frac{\omega_a''}{\omega_{0mn}''} \gg \frac{1}{Q_a} + \frac{g}{Q_{ext,a}}$$

for the particular cavity dimensions chosen and for the subscript a corresponding to all modes other than the TM_{0mn} mode. The term ω_a'' is defined by Eq. 4.7.24. A brief examination of the mode chart

will show that the problem of mode intersections and consequent complications is kept to a minimum by choosing relatively low m and n and a relatively large value for $(D/L)^2$.

This section concludes Chapter 5 and the application of the general methods of Chapter 4 to the special case of a right, circularly cylindrical cavity. In the next chapter, a device capable of producing an electron beam suitable for the excitation of higher order mode resonant cavities at millimeter wavelengths will be discussed.

6. THE SINGLE CAVITY ELECTRON ACCELERATOR AND BUNCHER

6.1 Introduction

The single cavity electron accelerator and buncher is a device which is capable of producing a bunched electron beam suitable for the excitation of higher order mode cavities at millimeter wavelengths. This device has been reported on elsewhere in great detail^{46,47,48} and consequently will be described only briefly here. It consists principally of a convergent beam injection tube and cylindrical accelerating cavity. In a more refined form it may also include a modulating cavity between the injection tube and the accelerating cavity. A schematic diagram of the simpler device is shown in Fig. 30. Figure 31 is a block diagram of the accelerator and Fig. 32 is a photograph of the machine.

Figure 30. Schematic Diagram of Single Cavity Accelerator and Buncher

6.2 The Injection Tube

The accelerator beam current originates in a convergent beam, Pierce gun diode⁴⁹. The tube is designed to inject 300 milliamperes through an entrance aperture one millimeter in diameter. In practice it has

Figure 31 Block Diagram of the Single Cavity Electron Accelerator

Figure 32 Photograph of the Single Cavity Electron Accelerator

been possible to inject 270 milliamperes through a one millimeter aperture. Figure 33 is a diagram of the injection tube currently in use. It was designed by R.J. Kenyon, and embodies contributions from several earlier models designed by P.D. Coleman, I. Kaufman, and M.D. Sirkis, as well as numerous improvements. This latest tube can be adjusted on the accelerator to yield maximum accelerated current. It is demountable so that the oxide-coated cathode can be readily cleaned and resprayed following exposure to air. This assembly also permits easy replacement of defective heaters.

6.3 The Microwave Circuit

A cross sectional scale drawing of the accelerating cavity is shown in Fig. 34. Figure 35 is a photograph of the cavity. The cavity was machined from O.F.H.C. copper and soldered together in a hydrogen furnace.

The waveguide outputs 1 and 2 admit power to the cavity through the circular coupling irises from the two RK5586 magnetrons which supply RF power to the accelerator. The third waveguide output connects to the cavity through a smaller iris and permits monitoring of the RF power in the cavity. Two vacuum connections are provided to facilitate effective evacuation of the cavity. The waveguide outputs are sealed with Litton waveguide windows.

Figure 36 is a drawing of the microwave circuit for the accelerator. The reference planes of the cavity and T junctions have been placed one-half guide wavelength apart, whereas those of the T junctions and door-knob feeds have been placed an integral multiple of a half guide wavelength apart.⁵⁰

Figure 33 Assembly Drawing of the 20 kv Injection Tube

Figure 34 Scale Drawing Of Accelerating Cavity

Figure 35 Photograph of the Accelerating Cavity

Figure 36 Microwave Circuit for Two Magnetrons Feeding a Single Cavity

6.4 Performance of the Single Cavity Electron Accelerator

The single cavity electron accelerator used for this work was first investigated by P.D. Coleman⁵¹. He has found that a cavity operating in the TM_{010} mode at a wavelength of 10.75 centimeters is suitable for the device. He has chosen the length of the cavity to be four centimeters.⁵² For a power level of 0.8 megawatt in the cavity he finds that 41% of the accelerated electrons (15% of the injected electrons) have energies of 1.53 ± 0.05 mev, and that these electrons are distributed over a distance of 4.5 millimeters, if they are injected into the cavity with the constant energy of 20 kev.⁵³ In this simple form such a machine would not be suitable for the generation of millimeter and submillimeter wavelength radiation through the excitation of resonant structures, because the bunch length would be longer than the wavelength of the radiation to be generated. This, in turn, means that the Fourier current component associated with the frequency to be generated would be small. However, the addition of a modulating cavity between the injection tube and the accelerating cavity as proposed by Coleman improves the performance of the machine considerably. For a machine with a modulating cavity, Coleman has shown that 26% of the electrons emitted from the cathode of the injection tube are accelerated to a velocity of $0.97c$ and emerge from the accelerator distributed over a distance of only 80 microns. This theoretical result was computed for a power level of 0.8 megawatt in the accelerating cavity and for a modulation voltage of 6.34 kilovolts peak.⁵⁴ Such a charge distribution has high harmonic content into the submillimeter range of wavelengths.

More extensive analysis of the accelerator is currently being carried out by I. Kaufman with the aid of the Illiac, the University of Illinois automatic computer. Figure 37 is a curve computed by Kaufman for the case of constant velocity injection with the injection velocity equal to $0.26c$, corresponding to an injection voltage of 18.2 kilovolts. It can be shown that this curve is not very sensitive with respect to changes in injection voltage in so far as its shape is concerned. Kaufman's data indicates that over 80% of the electrons composing the waveform of Fig. 37 have velocities greater than $0.92c$, for a cavity power of 0.6 megawatt. These electrons occur for $\theta_h < 4$ in Fig. 37.

At the time this investigation was being carried out, it was not possible to obtain electron velocities greater than $0.93c$ with the accelerator. This difficulty was due to the low output power of the RK 5586 magnetrons which were available for this work. The best of the tubes on hand generated only one-half of its rated output power, and financial limitations prevented the purchase of replacements. For this reason power measurements with the higher order mode cavities were limited to $\beta < 0.93$.

A curve of accelerator output current as a function of phase corresponding to a maximum electron velocity of $0.901c$ is given in Fig. 38. This curve corresponds to a constant injection current at a potential of 20 kv. The current waveform has been computed at a position 10 centimeters from the accelerating cavity beam exit aperture because the physical design of the accelerating cavity prevents mounting apparatus any closer. Approximately 70% of the emergent electrons have velocities

Figure 37 Accelerator Output Current Waveform for Constant Velocity Injection.

Figure 38 Computed Waveform of Current 10 cm from Accelerating Cavity Exit Aperture With Percentages of Injected Electrons Included in Four Intervals

greater than 0.83c. The theoretical power dissipation in the accelerating cavity for this case is 0.2 megawatt. This curve was computed by R. W. Liu, under Kaufman's direction. The computed curve of Fig. 38 was approximated by the curve of Fig. 39 to facilitate harmonic analysis. The curve of Fig. 39 is given by

$$I_x/I_e = 5.46 \quad 9.79 < \theta_x < 9.90 \quad (6.4.1a)$$

$$I_x/I_e = 2.66e^{-1.4(\theta_x - 9.90)} \quad 9.90 < \theta_x < 16.07 \quad (6.4.1b)$$

The amplitudes of the two approximating functions are chosen so that the number of electrons in each section of the approximating function is equal to the number in the corresponding section of the computed function. A Fourier analysis of the approximating function is easily carried out. The Fourier series is

$$I_x/I_e = \sum_{s=0}^{\infty} A_s \cos(s\theta) + B_s \sin(s\theta). \quad (6.4.2)$$

The coefficients in Eq. 6.4.2 are given by

$$A_0 = 1/2\pi \int_{\theta_0}^{\theta_0+2\pi} I_x/I_e d\theta, \quad (6.4.3a)$$

$$B_0 = 0, \quad (6.4.3b)$$

if $s = 0$, and

$$A_s = 1/\pi \int_{\theta_0}^{\theta_0+2\pi} I_x/I_e \cos(s\theta) d\theta, \quad (6.4.4a)$$

Figure 39 Approximate Waveform of Current 10 cm from Accelerating Cavity Exit Aperture for Harmonic Analysis.

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$$B_s = 1/\pi \int_{\theta_0}^{\theta_0+2\pi} I_x/I_e \sin(s\theta) d\theta, \quad (6.4.4b)$$

if $s \neq 0$.

The ratio of the total current associated with the s th harmonic, I_s , is given by

$$I_s/I_{avg} = \sqrt{A_s^2 + B_s^2}/A_0. \quad (6.4.5)$$

The relationship between θ and θ_x is arbitrary. Although A_n and B_n will depend on the particular choice made, the sum of their squares and consequently Eq. 6.4.5 will not. It is convenient to choose

$$\theta = \theta_x - 9.90,$$

and to integrate over the range

$$-0.11 < \theta < 6.17.$$

The corresponding coefficients are given by

$$A_0 = 0.398. \quad (6.4.6a)$$

$$A_s = 1.74 \frac{\sin(0.11s)}{s} + \frac{1.19}{(1.4)^2 + s^2} \quad (6.4.6b)$$

$$B_s = -1.74 \frac{1 - \cos(0.11s)}{s} + 0.847 \frac{s}{(1.4)^2 + s^2} \quad (6.4.6c)$$

Values of Eq. 6.4.5 computed for five harmonics with the aid of Eqs. 6.4.6 are presented in Table 5.

s	10	13	17	22	26
I_s/I_{avg}	.42	.37	.34	.29	.25

Table 5. Approximate Harmonic Current Amplitudes

An experimental measurement of the operating wavelength of the accelerator resulted in a value of 10.803 centimeters for this parameter. The corresponding harmonic wavelengths are given by

$$\lambda_s = \frac{10.803 \text{ cm.}}{s}$$

7. EXCITATION OF HIGHER ORDER MODES WITH THE SINGLE CAVITY ACCELERATOR

7.1 Introduction

The analysis developed in Chapters 4 and 5 treats the case of a resonant cavity excited by a steady electronic current which oscillates with a frequency at or near the resonant frequency of a particular mode. If a bunched electron beam produced by a pulsed electron accelerator is used to excite the oscillations in the cavity at the s^{th} harmonic of the fundamental bunching frequency, the Fourier current component for that harmonic is to be used in the theoretical expressions of Chapters 4 and 5. There are, however, several effects which must be considered. These will be discussed in this chapter.

7.2 Damping Due to Harmonic Excitation

If the cavity is tuned to the s^{th} harmonic of the fundamental bunching frequency, the cavity fields go through s cycles for each bunch of electrons which passes through it. For this reason there will be some damping of the oscillations between successive bunches. From Eq. 4.7.17 it is seen that this damping is dependent on Q_0 , the loaded Q , which includes both the wall losses in the cavity and the losses out through the waveguide output. The ratio of P_{min} , the power in the cavity just before a bunch enters, to P_{max} , the power in the cavity just after a bunch leaves, is approximately

$$\frac{P_{\text{min}}}{P_{\text{max}}} = e^{-s\omega_0 T/Q} = e^{-2\pi s/Q}, \quad (7.2.1)$$

where T is the period associated with the accelerator frequency:

$$\omega_0 T = 2\pi.$$

If the fluctuation in power between bunches is to be small, it is necessary that

$$2\pi s \ll Q$$

For power generation at one millimeter with the machine of Chapter 6,

$$s \approx 110.$$

The calculations of Chapter 4 show that theoretical values of Q_a at one millimeter are of the order of 6,000. For a matched cavity this value leads to

$$Q = \frac{Q_a}{2} \approx 3,000.$$

Substituting these values in Eq. 7.2.1 it is found that

$$\frac{P_{\max} - P_{\min}}{P_{\max}} \times 100\% = 20\%.$$

For an electron accelerator operating at a wavelength of three centimeters, only the 30th harmonic would be required to generate one millimeter. For this case

$$\frac{P_{\max} - P_{\min}}{P_{\max}} \times 100\% = 6\%.$$

From this calculation it is clear that the electron accelerator should be built to operate at as high a frequency as is possible if the ripple in the output power is to be small.

7.3 Build-Up Time of Oscillations

The time required for the oscillations in the higher order mode cavity to build up will now be considered. The power in the cavity is given by

$$P(t) = P_{\max} (1 - e^{-s\omega_0 t/Q})$$

where t is the time elapsed after the first electron bunch of the video pulse enters the cavity. The power in the cavity will reach its steady state when

$$t \gg Q / s\omega_0 = Q / \omega_s$$

This condition is most restrictive at low frequencies, where Q is large and s is small. At a wavelength of 8.31 millimeters, where the measurements to be discussed in Chapters 9 and 10 were made,

$$Q_a \approx 10^4,$$

$$\omega_s \approx 2\pi(36.1 \times 10^9).$$

Therefore, the requirement is

$$t \gg 0.04 \text{ microseconds.}$$

Thus it is seen that a pulse length of the order of a microsecond is sufficient for the oscillations in the higher order mode cavity to build up.

7.4 Frequency Stability Problems

Another question which should be considered is the effect of small shifts in the accelerator frequency. Once again it will be assumed that

the cavity is operating at the s^{th} harmonic of the accelerator frequency.

For this case

$$s\omega_0' = s(\omega_0 + \delta\omega_0).$$

For the higher order mode cavity to be excited effectively at the changed frequency it is necessary that

$$\left| \frac{s\omega_0' - s\omega_0}{s\omega_0} \right| = \left| \frac{\delta\omega_0}{\omega_0} \right| << \frac{1}{Q}$$

Now the accelerating cavity in the electron accelerator has a measured loaded Q of the order of 9,000. This means that the accelerator itself will function properly only if

$$\frac{\delta\omega_0}{\omega_0} << \left(\frac{1}{Q} \right)_{\text{acc}} \approx \frac{1}{9,000}.$$

In practice, the higher order mode cavities will have loaded Q 's less than 9,000; consequently, it is concluded that the frequency stability of the accelerator is adequate.

7.5 Upper Frequency Limit

No attempt has been made during this investigation to determine the theoretical upper frequency limit associated with this approach to the millimeter wave generation problem. A practical complication, resulting from the simultaneous excitation of the same mode at two or even more adjacent harmonic frequencies, may be expected to arise at sufficiently

high frequencies. This condition will arise for the s^{th} harmonic when

$$\left| \frac{(s \pm 1) \lambda_o - s \lambda_o}{s \lambda_o} \right| \leq \frac{1}{Q}$$

where Q is the loaded Q for the mode in question. From the results of Chapter 5, it is seen that this complication is not to be expected until s is at least of the order of 1000. Thus it is not of any practical consequence for generation at wavelengths longer than 0.1 millimeter.

8. DESIGN OF HIGHER ORDER MODE RESONATORS FOR USE WITH THE SINGLE CAVITY ELECTRON ACCELERATOR

8.1 Introduction

In designing a resonator to operate on the Single Cavity Accelerator, attention must be given to several factors. The choice of an operating mode will depend on such things as the beam diameter, the beam velocity, and the wavelength, as discussed in Chapter 5. Once the mode has been selected the cavity dimensions must be chosen with consideration of such things as mode interference. The mode interference problem has two aspects: There may be mode interference at the frequency of operation or there may be interference from some other oscillation at a different frequency. The first situation must be avoided, and it is desirable to avoid the second also. Some means must also be provided for tuning the cavity, and the cavity must be vacuum tight.

The experimental work on higher order mode resonators was carried out at 8.310 millimeters, corresponding to the thirteenth harmonic of the accelerator frequency. This wavelength was chosen because it was the shortest for which the laboratory could be well equipped for bench measurements at the time the work was initiated. These bench measurements, for the purpose of mode identification, iris matching, and Q estimation, will be discussed in this chapter.

8.2 Choice of Cavity Dimensions

The modes of oscillation chosen for this work were the TM_{018} and

TM₀₂₈ modes. The choice made was rather arbitrary in this case because at 8.31 millimeters almost any mode would have been satisfactory. Rather than choose a lower order mode which would have yielded more power, it was decided to use modes and cavity dimensions which could be scaled down to wavelengths less than one millimeter without presenting hopeless difficulties with regard to fabrication.

Once a mode has been chosen the next problem is to choose the cavity dimensions. In the analysis of Chapter 5, it was shown that

$$(D/\lambda)^2 = (r_{om}/\pi)^2 + (nD/2L)^2. \quad (8.2.1)$$

This relation determines the dimensions of the cavity if the wavelength, λ , and the ratio of the cavity diameter to the cavity length is fixed. The choice of D/L is best made from a consideration of the appropriate power curve. It will generally be desirable to operate near the peak of the power curve corresponding to the appropriate beam velocity. In some cases another consideration may also be important, and that is the effect of small variations in beam velocity on power output. Consider the case of Fig. 24 for β near $\beta_0 = 0.935$. It is clear that if $R/L = 0.23$, the power increases markedly when β becomes greater than β_0 , whereas if $R/L = 0.31$, the power decreases sharply as β increases past β_0 . For $R/L = 0.27$, small changes in β around β_0 cause only a small change in the power output. The cavity size must also be considered before a value of D/L is finally chosen. The larger D/L is, the larger the cavity is, as is seen from Eq. 8.2.1.

Before the final value of D/L is chosen, the problem of mode interference at the operating frequency must be given consideration. This can be done by determining from Eq. 8.2.1 the values of $(D/L)^2$ at which the resonant frequency of the desired mode corresponds to the resonant frequency of some other mode. The results of a calculation of this type made with the aid of Eq. 8.2.1 for intersections of the TM_{028} mode line with other mode lines in the range $0.30 \leq (D/L)^2 \leq 0.35$ are given in Table 5.

Intersection with	$(D/L)^2$
TM_{032}	0.30002
$TE_{11}(10)$	0.30488
TE_{319}	0.30566
TE_{133}	0.31242
TM_{512}	0.31387
TE_{713}	0.31765
TM_{033}	0.32729
TM_{224}	0.34093
TM_{513}	0.34240

Table 5. Intersections of TM_{028} Mode Line with Other Mode Lines in the Range $0.30 \leq (D/L)^2 \leq 0.35$.

From a tabulation of this sort a value of $(D/L)^2$ may be chosen to lie between two intersections. For example, from the data in Table 5, $(D/L)^2 = 0.334$ appears to be a satisfactory value. With this value for $(D/L)^2$ it is then necessary to compute the resonant frequencies for

the nearby modes to be certain that they will not be excited.

With the aid of Eq. 8.2.1, this computation is readily made.

Mode	$(D/L)^2$	$(D/\lambda)^2$	$\frac{\lambda_{l_{mn}}}{ \lambda_{l_{mn}} - \lambda_{028} }$
TM ₀₃₃	0.334	8.3391	182
TM ₀₂₈	0.334	8.4314	∞
TM ₂₂₄	0.334	8.5141	204

Table 6. Mode Separation at Operating Frequency

From Eq. 4.5.18 it is seen that if the a^{th} mode is to be excited appreciably, it is necessary that

$$\left| \frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right| < \frac{1}{Q_a} \quad (8.2.2)$$

where as before

$$\omega_a' = \omega_a \left(1 \pm \frac{1}{2Q_a} \right)$$

Equation 8.2.2 may be written

$$\left| \frac{(\omega + \omega_a')(\omega - \omega_a')}{\omega_a' \omega} \right| \approx 2 \left| \frac{\omega - \omega_a'}{\omega} \right| < \frac{1}{Q_a}$$

from which

$$Q_a < \left| \frac{\lambda}{2(\lambda - \lambda_a')} \right| \quad (8.2.3)$$

From Figs. 11, 16, and 26, it is seen that Q_a is of the order of 25 times the values listed in the third column of Table 6 where the resonant frequencies have been computed neglecting losses, even at a wavelength of one millimeter. Consequently, it is concluded that these other modes will not be excited appreciably.

The remaining problem to be considered is the excitation of two frequencies simultaneously within the cavity by different current frequency components of the electron beam. It was shown in Section 5.4 that such oscillations would act independently, but nevertheless it may be undesirable to have more than one frequency present in the output waveguide. Again use is made of Eq. 8.2.1, and a calculation similar to that given in this section for the case of mode interference at the operating frequency is carried out. In this calculation, however, the comparison is made between the resonant frequencies of the various TM_{0mn} modes at the particular value of $(D/L)^2$ chosen and the harmonic frequencies present in the electron beam.

By these methods the cavity dimensions tabulated in Table 7 were chosen for operation at the thirteenth harmonic with a wavelength of 8.310 millimeters.

Cavity No.	Mode	$(D/L)^2$	D Centimeters	L Centimeters
1	TM_{018}	0.3000	1.929	3.521
2	TM_{018}	0.3576	2.087	3.490
3	TM_{018}	0.4904	2.413	3.446
4	TM_{018}	0.8100	3.058	3.398
5	TM_{018}	0.3006	1.930	3.521
6	TM_{028}	0.3465	2.441	4.148
7	TM_{028}	0.3610	2.474	4.118

Table 7 Cavity Dimensions for Seven Higher Order Mode Cavities
Resonant at a Wavelength of 8.310 Millimeters

8.3 The "Magic Tee" Impedance Bridge

The bench investigation of mode configuration and the approximate Q measurements were made with the aid of a "magic tee" impedance bridge which will be described at this time. Figure 40 is a schematic diagram of a "magic tee" impedance bridge. If unit power from a matched generator is incident on the impedance bridge, it can be shown that the crystal detector power is

$$P = 1/4 |\Gamma_t|^2$$

where Γ_t is the reflection coefficient of the test piece.⁵⁵ Therefore when the test piece is matched to the impedance of the transmission line, no power reaches the detector. On the oscilloscope shown in Fig. 40, this condition is shown by the return of the oscilloscope trace to the base line at A. The output voltage from the sawtooth oscillator is applied to the reflector of the klystron and causes the frequency of the klystron to vary periodically at the frequency of the sawtooth oscillator. Thus the oscilloscope pattern is essentially a plot of $|\Gamma_t|^2$ as a function of frequency over any range for which the power output of the klystron is constant.

8.4 Mode Identification

A non-tunable cavity with dimensions

$$D = 2.088 \text{ centimeters}$$

$$L = 3.482 \text{ centimeters}$$

designed to resonate in the TM_{018} mode was made. The waveguide output for this cavity was placed in the midplane of the cavity as shown in Fig. 41. There was also a No. 39 drill hole in the center of each end plate, and a slot 0.028 inch wide and parallel to the axis of the cavity had been

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Figure 40. "Magic Tee" Impedance Bridge

Figure 41 Test Cavity

machined through the cylindrical wall as shown in Fig. 41. This cavity was made so that a small probe could be inserted through the slot and holes, and in this way the field configuration within the cavity could be investigated.

The experimental apparatus for this measurement is shown in Fig. 42. For simplicity the field configuration shown in the cavity corresponds to the TM_{012} mode. A Tektronix Type 531 dual beam oscilloscope was used so that the probe power and the impedance bridge detector power could be observed simultaneously. The upper trace shown on the oscilloscope corresponds to the power to the bridge detector; the lower trace corresponds to the power to the probe detector. The situation illustrated is for a radial field maximum. Photographs of oscilloscope traces illustrating this situation and also the case of a radial field null are shown in Figs. 43 and 44. Figure 45 is a drawing of the probe used for this measurement.

The result of this measurement for the TM_{018} mode at $\lambda = 8.29$ millimeters is given in Fig. 46. The theoretical wavelength for this mode was computed to be $\lambda = 8.296$ millimeters. A mode identified as the TM_{026} mode was located at $\lambda = 8.31$ millimeters. The theoretical value for this mode is $\lambda = 8.303$ millimeters.

The presence of an axial electric field was indicated in both of these cases when the probe was inserted through the axial hole in the end wall. In Section 5.2 it was shown that only the TM_{0mn} modes possess a non-zero axial electric field. It would appear that a measurement of the axial electric field would be sufficient to determine

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Figure 42 Determination of Mode Configuration. Radial Field for TM_{012} Mode Illustrated

Figure 43. Radial Field Measurement at Maximum

1. Upper trace - bridge detector power
2. Lower trace - radial field probe power

Figure 44. Radial Field Measurement at Minimum

1. Upper trace - bridge detector power
2. Lower trace - radial field probe power

Figure 45 Probe Used for the Field Configuration Measurements

Figure 46 Experimental Measurement of the Radial Electric Field

whether or not the mode in question was a TM_{0mn} mode. It was found that the apparatus used for these measurements was not sufficiently refined to determine definitely whether an unknown mode was a TM_{0mn} mode.

Although any mode which was identified as a TM_{0mn} mode seemed to possess an axial electric field, there were other modes which did not appear to be TM_{0mn} modes, but nevertheless seemed to have an axial field. It was felt that the presence of the probe and the hole in the end wall of the cavity caused enough distortion in the field patterns near the hole to give these inconclusive results. Subsequent investigations of the axial field for modes identified as other than TM_{0mn} by methods to be discussed confirmed the fallibility of this method.

Although it was felt that the mode identity, at least for the TM_{0mn} modes, could be established with the measurements along the cavity and at the cavity end wall, this method was not desirable because a special cavity would have to be made for the purpose of mode identification studies for each new set of cavity dimensions. The identification would not be made in the cavity to be used, so there might still be a chance for error in going from the test cavity to the unslotted cavity. That is, it still might be possible to confuse the modes in the unslotted cavity, even though they were properly identified in the slotted cavity. Consequently, some other method of mode identification was desirable.

It has been shown that a dielectric bead can be used to probe the field inside a resonant cavity.⁵⁶ As the bead is moved around within the cavity, it causes changes in the resonant frequency of the cavity. These changes depend on the material of which the bead is

made, on its size and shape, and on its location within the cavity. The measured changes in the resonant frequency and the physical properties of the dielectric bead can then be used to compute the electric field distribution within the cavity. For the present problem, this method could be used in a simplified way to provide a means for identifying the subscript n in the mode designation. The method is illustrated in Fig. 47. As the bead is drawn through the cavity, the pip on the oscilloscope trace due to absorption of power by the cavity at its resonant frequency is displaced alternately to the left and right. The distance, Δ , which the bead moves between successive maximum displacements to the left or to the right can be used to determine n by the relationship:

$$n = L/\Delta.$$

With the dimensions of the cavity known and with n known, the mode identity can then be established with the aid of a mode chart.

Data obtained in this way for a cavity which could be tuned by varying its length are displayed in Table 8. (The cavity designs will be discussed in Section 8.5) The cavity used for this measurement was of the type shown in Fig. 52.

Mode	L Theoretical cm.	L Measured cm.	Δ Measured cm.	L/n cm.
TM ₁₁₇	3.419	3.42	0.50	0.49
TE ₁₁₈	3.436	3.43	0.42	0.43
TM ₀₁₈	3.521	3.52	0.43	0.44
TM ₀₂₆	3.815	3.83	0.65	0.64

Table 8. Mode Identification with a Dielectric Bead for Cavity No. 1. Diameter = 1.929 centimeters, $\lambda = 8.31$ millimeters.

Figure 47. Determination of Mode Configuration with Dielectric Bead Probe. TM_{012} Mode Illustrated.

A third method of mode identification was also investigated. The cavity shown in Fig. 52 is tuned by changing its length, the cavity retaining its cylindrical shape. It is therefore to be expected that a plot of measured values of $(D/\lambda)^2$ as a function of $(D/L)^2$ for an arbitrary mode should lie along a straight line. It should then be possible to relate the straight line thus obtained to one of the lines on a mode chart of the form of Fig. 10, but including all TE and TM modes. In practice it was found that measurements could be made only over a very limited range of $(D/L)^2$, because the coupling to the modes of interest was appreciable only over a small region. In general, complications arise whenever two modes have resonant frequencies close enough together so that interaction between them may occur.

Nevertheless, it is still possible to obtain satisfactory results from the measurement of these tuning curves, if reasonable care is used in measuring the resonant frequency and the resonant length of the cavity. Tuning curves for the same cavity for which the data of Table 8 were obtained for the TM_{018} and TE_{118} modes are given in Figs. 48 and 49. Tuning curves for two other cavities are shown in Figs. 50 and 51. On the basis of these results it was decided to rely primarily on tuning curves for mode identification.

The final test of the success or failure of the attempts at mode identification was provided by the accelerator. Of the seven cavities which were actually placed on the machine, only Cavity No. 1, for which the data presented in Table 8 and Figs. 48 and 49 were obtained, did

Figure 48 Tuning Curve for TM_{018} Mode (Note Limited Range of Scale)

Figure 49 Tuning Curve for TE_{118} Mode (Note the Limited Range of Scale)

Figure 50 Tuning Curve for Cavity No. 3, Designed to Operate in TM_{018} Mode

Figure 51, Tuning Curve for Cavity No. 5, Designed to Operate in TM_{028} Mode

not function properly because of effects at the operating wavelength of 8.31 millimeters.

This difficulty occurred in the following way: A measurement of the accelerator wavelength with a Mico Wavemeter resulted in

$$\lambda_0 = 10.803 \text{ centimeters.}$$

From this number,

$$\lambda_0/13 = \lambda_{13} = 8.3100 \text{ millimeters,}$$

and

$$f_{13} = c/\lambda_{13} = 36.076 \times 10^9 \text{ cycles/second.}$$

According to the calibration curve supplied with the Sperry Model 350 frequency meter, this frequency should have corresponded to a reading of 0.0872 inch on the frequency meter micrometer. At this setting of the frequency meter the mode behavior was quite satisfactory during the bench tests. When the cavity was placed on the accelerator it was found that the power generated corresponded to a frequency meter setting of 0.0875 inch, and was well below the value expected. The cavity was then rechecked on the impedance bridge and it was found that at a frequency meter setting of 0.0875 inch there was serious mode interference, resulting in a greatly decreased coupling between the waveguide and the TM_{018} mode. At 0.0872 inch the behavior was checked again and found to be satisfactory. In all subsequent work the cavities were checked for mode interference at a frequency corresponding to 0.0875 inch on the frequency meter, and no further difficulties of this nature were experienced.

8.5 Physical Design of Higher Order Mode Resonators for Use on the Single Cavity Electron Accelerator

The physical designs of the cavities which were used on the accelerator are shown in Figs. 52 and 53. In both of these designs tuning is accomplished by changing the length of the cavity through turning the movable end plate assembly which is threaded onto the barrel. In both designs the vacuum seal for the tuning mechanism was made with a sliding O-ring joint, as shown in the drawings.

In the design of Fig. 52 a half wave choke was incorporated in the design because the junction of the cylindrical wall and the end plate is a high current point for TM_{0mn} modes, with the currents flowing across the joint.⁵⁷ The waveguide output is shown in the midplane of the cavity. Cavities 1, 2, and 3 were made in accordance with the design of Fig. 52. Cavity No. 4 differed only in the position of the waveguide output, which was moved to a position one eighth of the theoretical cavity length from the fixed end. This was done to make the coupling between the waveguide and the cavity less frequency sensitive.

The cavity shown in Fig. 53 was designed to operate with a 0.010 inch gap separating the movable end cap from the body of the cavity. The center of this gap was placed one sixteenth of the theoretical cavity length from the end wall of the movable end cap. This position corresponds to a current node for the design mode. The 0.010 inch gap was included to allow the cavity to be tuned above as well as below the resonant frequency, and also to allow for the possibility that the resonant cavity length might be up to 0.010 inch less than the theoretical

Figure 52. Choke Plunger Higher Order Mode Cavity

Figure 53. Split Wall Higher Order Mode Cavity

length. Cavities 5, 6, and 7 were built according to this design.

All of these cavities were machined from O.F.H.C. copper and were assembled with high temperature solder in a hydrogen furnace.

8.6 Q Measurements on Higher Order Mode Resonators

Approximate Q measurements may be made with the aid of the "magic tee" impedance bridge. If it is assumed that unit power from the source is incident on the bridge, the power to the bridge detector is

$$P = \frac{|\Gamma_t|^2}{4}$$

where Γ_t is the reflection coefficient of the test piece. From Eq. 4.7.11 the normalized input impedance of a resonant cavity may be written

$$Z_{11}/Z_{01} = \frac{1}{j\left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega}\right) + \frac{1}{Q_a}} \quad (8.5.1)$$

near the a^{th} resonance, where the term Z_a/Z_{01} due to all other resonances has been neglected. If a cavity is used as the test piece, then the reflection coefficient is

$$\Gamma_t(\omega) = \frac{Z/Z_{01} - 1}{Z/Z_{01} + 1} = \frac{(Q_a/Q_{\text{ext}, a1} - 1) - j Q_a \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega}\right)}{(Q_a/Q_{\text{ext}, a1} + 1) + j Q_a \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega}\right)}$$

and

$$|\Gamma_t(\omega)|^2 = \frac{(Q_a/Q_{\text{ext}, a1} - 1)^2 + Q_a^2 \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega}\right)^2}{(Q_a/Q_{\text{ext}, a1} + 1)^2 + Q_a^2 \left(\frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega}\right)^2} \quad (8.5.2)$$

At resonance

$$\omega = \omega_a'$$

and

$$|\Gamma_t(\omega_a')|^2 = \frac{(Q_a/Q_{ext,a} - 1)^2}{(Q_a/Q_{ext,a} + 1)^2} \quad (8.5.3)$$

Off resonance it is seen that

$$|\Gamma_t|^2 \approx 1.$$

Consequently it is possible to relate the oscilloscope trace drawn in Fig. 54 to the reflection coefficient as shown there, if it is assumed that the power incident on the bridge over the frequency range of the

Figure 54 Relationship Between Oscilloscope Deflection and Reflection Coefficient

resonance curve is constant, and also that the detector is square law.

The power absorbed by the cavity is proportional to $(1 - |\Gamma_t|^2)$, so the distance of the trace below the level corresponding to $|\Gamma_t| = 1$ is a measure of the power absorbed by the cavity. The frequencies at which the cavity absorbs one half of the power absorbed at resonance

may be found from Eqs. 8.5.2 and 8.5.3. The requirement is

$$1 - |\Gamma_t(\omega)|^2 = \frac{1}{2}[1 - |\Gamma_t(\omega_a')|^2]. \quad (8.5.4)$$

It is convenient to introduce the notation

$$\delta = \left| \frac{\omega - \omega_a'}{\omega_a'} \right| \approx \left| \frac{\omega}{\omega_a'} - \frac{\omega_a'}{\omega} \right| \quad (8.5.5)$$

The solution of Eq. 8.5.4. may then be written

$$2\delta = \frac{1}{Q_a} + \frac{1}{Q_{\text{ext}, a1}} \quad (8.5.6)$$

If the impedance seen looking toward the generator is the characteristic impedance as it will be on the "magic tee" impedance bridge, Eq. 8.5.6 becomes

$$2\delta = \frac{1}{Q} \quad (8.5.7)$$

where Q is the total or loaded Q introduced in Section 4.7. If the cavity is to be matched to the waveguide on resonance, it is clear from Eq. 8.5.3 that

$$Q_a = Q_{\text{ext}, a1} \quad (8.5.8)$$

and Eq. 8.5.7 may be written

$$2\delta = 2/Q_a \quad (8.5.9)$$

The method of determining the points at which the frequency is to be measured is shown in Fig. 55. Photographs which illustrate this

Figure 55 Location of Points at which the Frequency is to Be Measured to Determine δ

procedure for Cavity No. 5 are shown in Fig. 56. The difference in frequency between the two wavemeter positions two centimeters apart is

$$f_2 - f_1 \approx c \left\{ \frac{1}{8.310 \text{ mm}} - \frac{1}{8.316 \text{ mm}} \right\} \approx 2.6 \times 10^7 \text{ cps.}$$

The width of the resonance curve between the half power points is 0.45 centimeters. The corresponding frequency increment is

$$2 \left(\frac{\omega_0'}{2\pi} \right) \delta \approx 0.45 \text{ cm} (1.3 \times 10^7 \text{ cps/cm}) \approx 0.58 \times 10^7 \text{ cps.}$$

From Eq. 8.5.9, since Fig. 56 shows that $Q_a \approx Q_{\text{ext}, a1}$,

A. Power Absorption for Cavity No. 5

B. Frequency Meter Setting 0.0875 Inch
 $\lambda = 8.310$ Millimeters

C. Frequency Meter Setting 0.0877 Inch
 $\lambda = 8.316$ Millimeters

Figure 56. Q Measurement for Cavity No. 5

$$Q_a \approx \frac{1}{\delta} = \frac{2(\omega_a/2\pi)}{0.58 \times 10^7 \text{ cps}} \approx 12,000$$

A tabulation of the theoretical and measured Q 's for the five cavities for which the generated power was measured is given in Table 9. A brief discussion of the deviation between the theoretical Q_a and the measured Q_a will now be given.

From Maxwell's equations it is easily shown that the currents in the conducting walls of a cavity flow in a direction perpendicular

Cavity No.	Mode	$(D/L)^2$	Q_a Theoretical	Q_a Measured
2	TM ₀₁₈	0.3576	18,770	12,000
4	TM ₀₁₈	0.8100	23,130	20,000
5	TM ₀₁₈	0.3006	17,920	12,000
6	TM ₀₂₈	0.3465	22,080	13,000
7	TM ₀₂₈	0.3610	22,210	15,000

Table 9 Comparison of Measured Q_a and Theoretical Q_a for the Five Cavities Used for Power Measurements

to the direction of the tangential magnetic field at the surface. For TM_{0mn} modes, Eq. 5.2.22 shows that the magnetic field has only an azimuthal component. Consequently, on the cylindrical surface of the cavity the currents flow in the z direction whereas on the end walls they flow in a radial direction.

The cavities used in these experiments were machined from solid copper stock on a lathe. The tool marks from the machining operations are perpendicular to the current lines. The depth of these grooves

is at least of the order of the skin depth of copper, which at a wavelength of 8.31 millimeters is only about 4×10^{-5} centimeters. Under these conditions Morgan ⁵⁸ has shown that the losses are proportional to the surface area. If it is assumed that the profile of the tool marks is similar to that shown in Fig. 57, it is seen that the area of the cavity

Figure 57 Profile of Tool Marks on the Cavity Walls

walls is increased by approximately 40%, which means that the losses in the walls are increased by about 40%. Because $\frac{1}{Q_a}$ is proportional to the losses in the walls, it follows that Q_a is reduced by the factor $1/1.4$ or to approximately 70% of the theoretical value. This same conclusion could have been reached with the aid of Eq. 4.5.20.

8.7 Iris Matching

It was shown in Section 4.7 that maximum power is delivered to a waveguide matched load when $Q_a = Q_{ext,al}$ or equivalently, when the cavity appears matched to the waveguide output on resonance. The cavities used in this work were coupled to their waveguide outputs by means of a circular iris in the cylindrical cavity wall. $Q_{ext,al}$

is a function of the size of the iris and decreases as the iris size increases. With the aid of the "magic tee" impedance bridge each iris was carefully enlarged until the reflected power from the cavity was not more than about three percent of the incident power. A satisfactory match was obtained with a No. 34 drill (0.1135 inch diameter) in all five of the cavities listed in Table 9. The result of this procedure for Cavity No. 2 is shown in Fig. 56.

The effect of a slight mismatch on the power delivered to a matched waveguide load will now be considered. If Eq. 4.7.26 is expanded in powers of $Q_a/Q_{\text{ext},a1}$ near

$$Q_a/Q_{\text{ext},a1} = 1,$$

it is found that

$$P(\omega'_a, Q) \approx 1/4 \{1 - 1/4 (Q_a/Q_{\text{ext},a1} - 1)^2\} \frac{Q_a}{2\epsilon\omega_a} \left| \int_V \mathbf{J} \cdot \bar{\mathbf{E}}_a dV \right|^2. \quad (8.6.1)$$

Expansion of Eq. 8.5.3 in the same way yields

$$|\Gamma_t(\omega'_a)|^2 \approx 1/4 (Q_a/Q_{\text{ext},a1} - 1)^2$$

so that Eq. 8.6.1 may be written

$$P(\omega'_a, Q) = 1/4 \{1 - |\Gamma_t(\omega'_a)|^2\} \frac{Q_a}{2\epsilon\omega_a} \left| \int_V \mathbf{J} \cdot \bar{\mathbf{E}}_a dV \right|^2, \quad (8.6.2)$$

if

$$|\Gamma_t(\omega'_a)|^2 \ll 1.$$

The power delivered to a matched load is reduced by a slight mismatch of the cavity to the waveguide output. The amount of this reduction is easily determined from an oscilloscope pattern of the type shown in Fig. 58. If the deflection on the oscilloscope is assumed to be proportional to the power incident on the detector, then

$$|\Gamma_t(\omega'_a)|^2 = \frac{B}{A+B} \quad .$$

Figure 58. Oscilloscope Pattern Determination of $|\Gamma_t(\omega'_a)|^2$

9. THE MEASUREMENT OF BEAM VELOCITY

9.1 Introduction

The results of Chapter 5 indicate that a measurement of the power generated in a higher order mode cavity should be accompanied by a measurement of the beam velocity, because in most cases the power generated is sensitively dependent on the beam velocity. It was therefore necessary to include in the experimental apparatus a device which was capable of measuring the beam velocity. A magnetic deflection electron spectroscopy was built using an RCA Type 219D1 kinescope deflecting coil as the dispersive element. The design of this instrument will be discussed in this chapter.

9.2 Empirically Determined Properties of the RCA Type 219D1 Deflecting Coil

A drawing of the RCA Type 219D1 deflecting coil is shown in Fig. 59. Because of its complex geometry, the magnetic field produced by it was determined empirically. A small coil mounted on the extended shaft of a synchronous motor was used to probe the field. This coil had been previously calibrated, one volt output corresponding to 997 gauss.

For the following discussion of the dependence of field strength on position, refer to Fig. 59. The deflecting coil consists of two pairs of coils. These pairs are not identical, and the pair producing the larger field for a given current was selected. This pair is labeled A-B in Fig. 59. The plane $z = 0$ is a symmetry plane for the pair A-B. For small

Figure 59 Deflecting Coil Showing the Orientation of the Axes

the field in this plane is approximately

$$\vec{B} = \hat{i}_z B e^{-(x/a)^2}, \quad (9.2.1)$$

where $\hat{i}_z$ is, as usual, a unit vector in the z direction, B , is a constant, and $a \approx 3.5$ centimeters. A graph of the magnetic field as measured and as given by Eq. 9.2.1 is shown in Fig. 60. In the spectroscope the deflection of the electron beam from the x -axis is sufficiently small that the approximation of Eq. 9.2.1 may be used.

The dependence of the magnetic field amplitude, B on the current through the coil was also determined with the aid of the probe coil. The results of this measurement are given in Fig. 61. It was found that

$$B \text{ (gauss)} = 0.2 I \text{ (milliamperes)}, \quad (9.2.2)$$

9.3 Solution of Equations of Motion for the Trajectory and the Deflection Angle

The equation to be solved is

$$e\vec{r} \times \hat{i}_z B e^{-(x/a)^2} = m\ddot{\vec{r}}, \quad (9.3.1)$$

where e is the electronic charge, m is the electronic mass, and $\vec{r}$ is the position vector. In component form this equation is

$$ev_x B e^{-(x/a)^2} = -m\ddot{y},$$

and

$$ev_y B e^{-(x/a)^2} = m\ddot{x}, \quad (9.3.2)$$

Figure 60. Magnetic Field of the RCA Type 219D1 Deflecting Coil

Figure 61 Magnetic Field, B , for RCA Type 219D1 Deflecting Coil

since

$$B_x = B_y = 0. \quad (9.3.3)$$

From Eq. 9.3.2 it is readily found that

$$\dot{y} = v_y = \frac{eB}{m} \int_{-\infty}^x e^{-(t/a)^2} dt, \quad (9.3.4)$$

where the initial condition $v_y(-\infty) = 0$ has been used. The velocity in the x direction may be found from the conservation of energy relation

$$v_x = \sqrt{v^2 - v_y^2}, \quad (9.3.5)$$

where v is the beam speed, which remains constant. Because

$$\frac{dy}{dx} = \frac{v_y(x)}{v_x(x)},$$

$$\frac{dy}{dx} = \frac{\int_{-\infty}^x e^{-(t/a)^2} dt}{\sqrt{\left(\frac{mv}{eB}\right)^2 - \left[\int_{-\infty}^x e^{-(t/a)^2} dt\right]^2}}. \quad (9.3.6)$$

The final deflection angle θ is defined by

$$\tan \theta = \lim_{x \rightarrow \infty} \frac{dy}{dx} = \frac{\int_{-\infty}^{\infty} e^{-(t/a)^2} dt}{\left(\frac{p}{eB}\right)^2 + \left[\int_{-\infty}^{\infty} e^{-(t/a)^2} dt\right]^2} \quad (9.3.7)$$

where

$$p = mv. \quad (9.3.8)$$

It can be shown that

$$\int_{-\infty}^{\infty} e^{-(t/a)^2} dt = a\sqrt{\pi}.$$

With this substitution, Eq. 9.3.7 becomes

$$\sin \theta = \frac{eB}{p} a \sqrt{\pi}. \quad (9.3.9)$$

In order to keep the current through the deflecting coil low enough to prevent damage to the coil, a 10° deflection angle was chosen. A graph of Eq. 9.3.6 for this case is shown in Fig. 62. From Eqs. 9.2.2 and 9.3.9 it follows readily that

$$\frac{p/e}{I} \left(\frac{\text{gauss-cm}}{\text{ma}} \right) = 7.14 \quad (9.3.10)$$

For a 1.5 mev electron $p/e = 6500$ gauss-cm, and a coil current of 910 milliamperes is required.

The equation of the orbit must be found from integration of Eq. 9.3.6. Because this equation appeared rather difficult to integrate, an approximate method was used. The expression for dy/dx was approximated by

$$\begin{aligned} \frac{dy}{dx} &= 0, & u &\leq -\pi/2 \\ \frac{dy}{dx} &= 0.087 (1 + \sin u), & -\pi/2 &\leq u \leq \pi/2 \quad (9.3.11) \\ \frac{dy}{dx} &= 0.174, & \pi/2 &\leq u, \end{aligned}$$

where $u = \pi x/3a$. This approximate function and the exact function of Eq. 9.3.6 are shown in Fig. 62. Equation 9.3.11 may be readily integrated to yield the equation for the trajectory:

$-x/a$	$\tan \theta$	$.087(1 + \sin \pi x/3a)$
$-3/2$	.003	.000
-1	.014	.012
$-1/2$	.042	.044
0	.087	.087
$1/2$	.133	.131
1	.162	.162
$3/2$	.173	.174

Figure 62. Analytical and Approximate Expressions For The Deflection Angle θ

$$y = 0, \quad u \leq -\pi/2$$

$$y = 0.291 \left\{ \frac{\pi}{2} + u - \cos u \right\}, \quad -\pi/2 \leq u \leq \pi/2 \quad (9.3.12)$$

$$y = 0.914 + 0.584 \left\{ u - \frac{\pi}{2} \right\}, \quad \frac{\pi}{2} \leq u.$$

A graph of the orbit showing its orientation with respect to the deflecting coil and spectroscope is given in Fig. 63.

The spectroscope was designed to permit deflection in two directions. This was done to simplify the alignment problem, arising from geometrical factors and stray magnetic fields. In practice, the average of the currents required for right and left deflection was used to determine the corresponding momentum. It is not difficult to show that the effect of stray magnetic fields is cancelled out in this way, if these stray fields are essentially the same along both trajectories.

The slits used in the spectroscope are 1/16 inch wide. As the beam enters the spectroscope, it must pass through a circular aperture 1/16 inch in diameter. Therefore, if beam spreading is neglected (in the experimental work it was found that the beam spread appreciably), the beam diameter is equal to the slit width.

Experimentally the relationship between deflecting coil current and electron momentum was found to be

$$\frac{p/e}{I} \left(\frac{\text{gauss-cm}}{\text{ma}} \right) = 6.43 \quad (9.3.13)$$

which is about 10 percent less than the theoretical value. The experimentally determined calibration curve is shown in Fig. 64.

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Figure 63. Computed Trajectory for 10° Spectroscope Using RCA Type 2J9D1 Deflecting Coil.

Figure 64. Experimental Calibration of Spectroscope.

9.4 Resolution of the Spectroscope

A calculation of the spectroscope resolution must be carried through if measurements made with the spectroscope are to be interpreted correctly. The calculation will be given here for the case of a line beam and for the case of a circular beam with a diameter equal to the slit width.

From Fig. 63 it is seen that the electrons initially moving along the $-x/a$ axis appear to come from a region near $x/a = 0$, $y = 0$, after being deflected. For this reason it will be possible to consider the trajectories as straight lines broken at the origin, for the purposes of the present investigation. The situation for the line beam case is shown in Fig. 65.

Figure 65 Condition for Resolution in the Line Beam Case.

It is apparent from this drawing that the first slit plays no role for the line beam case. All of the current passes through the second slit for a range of angular variation, $\delta\theta$, about θ_0 . For larger variations, no current

passes through the second slit. From Fig. 65,

$$\theta_b - \theta_o = \theta_o - \theta_a = \frac{\delta\theta}{2} = \frac{s}{4d} \quad (9.4.1)$$

It is possible to relate the quantity $\delta\theta$ to the magnetic field strength by means of Eq. 9.3.9. From that equation

$$\sin \theta_o = \frac{e\beta_o}{p_o} a\sqrt{\pi} \quad (9.4.2)$$

where $\theta_o = 10^\circ$. A Taylor series expansion of Eq. 9.4.2 near $p = p_o$ yields

$$(\theta - \theta_o) \approx \tan \theta_o \left(1 - \frac{p}{p_o}\right) \quad (9.4.3)$$

The current collected falls to zero at $\theta = \theta_a$ and $\theta = \theta_b$. This occurs when

$$\left|1 - \frac{p}{p_o}\right| \tan \theta_o = \frac{s}{4d} \quad (9.4.4)$$

The slits used in the spectroscope are 1/16 inch wide. The distance, d , between the slits is five inches. With these values, it is found that the current collected goes to zero when

$$\left|1 - \frac{p}{p_o}\right| = 0.018 \quad (9.4.5)$$

A graph of this result is shown in Fig. 67.

The case of a circular beam will now be considered. The assumptions for this calculation are:

1. The beam as it enters the spectroscope is axially symmetric with a radius, p_o , which satisfies the condition

$$s/2 \geq p_o > 0,$$

where s is the slit width.

2. The charge density and momentum density within the beam are not functions of the radius.
3. Space charge effects are neglected.
4. The trajectories of all the electrons are parallel. This follows from the approximation, which led to Eq. 9.2.1, that the magnetic field does not depend on y . This assumption immediately leads to the conclusion that the intersection of the electron beam with planes parallel to the y - z plane is a circle of radius ρ_0 .

The situation is illustrated in Fig. 66. It is assumed that the center

Figure 66. Condition for Resolution for the Narrow Beam Case.

of the beam strikes the plane of the slit at a distance, x_0 , from the center of the slit. The fraction of the electrons which pass through the slit is just equal to the ratio of the beam area within the slit to the total beam area by assumption 2 above. It is convenient to

make this computation in a plane perpendicular to the original direction of the electron beam and passing through the edge of the slit, as, for example, the plane pp' of Fig. 66. If this is done, it is found, for intersection with the left side of the slit ($\theta > \theta_0$), that

$$\frac{\delta A}{A} = \frac{2}{\pi \rho_0^2} \int_{\frac{x_0 - s/2}{\cos \theta}}^a \sqrt{\rho_0^2 - x^2} dx, \quad (9.4.6)$$

where δA is the area within the slit, and the total area, A , is given by

$$A = \pi \rho_0^2.$$

Because $\theta \approx 10^\circ$, $\cos \theta = 0.98$. Consequently Eq. 9.4.6 is approximately

$$\frac{\delta A}{A} \approx \int_{x_0 - s/2}^a \sqrt{\rho_0^2 - x^2} dx, \quad (9.4.7)$$

because $\cos(\theta - \theta_0) \approx 1$. Integration yields

$$\frac{\delta A}{A} \approx \frac{1}{\pi \rho_0^2} \left[\frac{\pi \rho_0^2}{2} - (x_0 - s/2) \sqrt{\rho_0^2 - (x_0 - s/2)^2} - \rho_0^2 \sin^{-1} \frac{x_0 - s/2}{\rho_0} \right]. \quad (9.4.8)$$

From Fig. 66

$$x_0 = 2d \tan(\theta - \theta_0).$$

With the aid of Eq. 9.4.3, this expression becomes

$$x_0 \approx 2d \tan \theta_0 \left(1 - \frac{p}{p_0}\right). \quad (9.4.9)$$

Substitution of Eq. 9.4.9 into 9.4.8 leads to

$$\begin{aligned} \frac{I(p)}{I_o} = & \frac{1}{\pi} \left(\frac{d}{\rho_o} \right)^2 \left\{ \left(\frac{\rho_o}{d} \right)^2 \frac{\pi}{2} - \left[2 \tan \theta_o \left(1 - \frac{p}{\rho_o} \right) - \frac{s}{2d} \right] \right. \\ & \times \frac{\rho_o}{d} \sqrt{1 - \left\{ \left[2 \tan \theta_o \left(1 - \frac{p}{\rho_o} \right) - \frac{s}{2d} \right] \frac{d}{\rho_o} \right\}^2} \\ & \left. - \left(\frac{\rho_o}{d} \right)^2 \sin^{-1} \left[\frac{2 \tan \theta_o \left(1 - \frac{p}{\rho_o} \right) - \frac{s}{2d}}{\rho_o/d} \right] \right\} \end{aligned} \quad (9.4.10)$$

where

$$\rho_o/d + s/d \geq x_o/d \geq -\rho_o/d + s/d \quad (9.4.11)$$

and I_o is the total beam current. For magnetic fields outside these limits, the beam does not strike the left edge of the slit. The case of intersection with the right side of the slit ($\theta < \theta_o$) proceeds in a similar manner and leads to the results

$$\begin{aligned} \frac{I(p)}{I_o} = & \frac{1}{\pi} \left(\frac{d}{\rho_o} \right)^2 \left\{ \left(\frac{\rho_o}{d} \right)^2 \frac{\pi}{2} + \left[2 \tan \theta_o \left(1 - \frac{p}{\rho_o} \right) + \frac{s}{2d} \right] \right. \\ & \times \frac{\rho_o}{d} \sqrt{1 - \left\{ \left[2 \tan \theta_o \left(1 - \frac{p}{\rho_o} \right) + \frac{s}{2d} \right] \frac{d}{\rho_o} \right\}^2} \\ & \left. + \left(\frac{\rho_o}{d} \right)^2 \sin^{-1} \left[\frac{2 \tan \theta_o \left(1 - \frac{p}{\rho_o} \right) + \frac{s}{2d}}{\rho_o/d} \right] \right\}, \end{aligned} \quad (9.4.12)$$

where

$$\frac{\rho_o}{d} - \frac{s}{2d} - \frac{x_o}{d} \geq - \frac{\rho_o}{d} - \frac{s}{2d} \quad (9.4.13)$$

Figure 67 includes a graph of Eqs. 9.4.10 and 9.4.12 for the case

$$\rho_o = s/2 = 1/32 \text{ inch}$$

$$d = 5 \text{ inches}$$

$$\theta_o = 10^\circ.$$

The range of momentum for which at least some current is collected is found from Eqs. 9.4.9, 9.4.10 and 9.4.12.

For this case,

$$\left| 1 - \frac{p}{p_o} \right| \leq \frac{s}{2d} \tan \theta_o = 0.036. \quad (9.4.14)$$

The current falls to one-half its maximum value when

$$|x_o| = s/2, \quad (9.4.15)$$

or when

$$\left| 1 - \frac{p}{p_o} \right| \leq \frac{s}{4d} \tan \theta_o = 0.018. \quad (9.4.16)$$

The results of this calculation appear in Fig. 67.

Figure 67. Relative Current Collected as a Function of Momentum for a Fixed Magnetic Field

The ability of the spectroscope to resolve two momenta $(p_0)_1$ and $(p_0)_2$ will now be calculated. The situation for a beam with diameter equal to the slit width is shown in Fig. 68. The two current peaks corresponding to momenta $(p_0)_1$ and $(p_0)_2$, where $(p_0)_1 < (p_0)_2$ will be resolved as long as

$$0.982 (p_0)_2 > 1.018 (p_0)_1,$$

Figure 68. Resolution of Spectroscopy When Beam Diameter Equals Slit Width

which leads to

$$\frac{(p_o)_2 - (p_o)_1}{(p_o)_1} > \left(\frac{1.018}{0.982} - 1 \right) \approx 0.036$$

This last expression may be written

$$\frac{\delta p_o}{p_o} > 0.036.$$

The same result is obtained for the case of a line beam. The momentum and the normalized beam velocity $\beta \approx v/c$ are related by

$$p = \frac{mc\beta}{\sqrt{1-\beta^2}}.$$

Differentiation leads to

$$\frac{d\beta}{\beta} = (1-\beta^2) \frac{dp}{p}$$

At 1 mev $\beta = 0.94$, and

$$\frac{d\beta}{\beta} = \{ 1-(0.94)^2 \} 0.036 = 0.004.$$

This resolution is adequate for the purposes of the experiments which follow. In these experiments, it is the beam velocity which enters through its effect on the longitudinal coupling coefficient.

9.5 Effect of Spectroscope Resolution on Measurements

For any particular value of the maximum magnetic field amplitude, $B = B_0$, electrons in a small range of momentum about p_0 where

$$\sin \theta_0 = ea\sqrt{\pi} \frac{B_0}{p_0}$$

will pass through the slit. The fraction of the electrons with momentum p which pass through the slit was computed in the preceding section.

Computations on the single cavity accelerator yield a value for dN/dp , the number of electrons per unit momentum per RF cycle. The average current, $I(B_0)$, collected by the collector at a magnetic field setting, B_0 , is related to dN/dp by

$$I(B_0) \propto \int_0^{\infty} \frac{dN}{dp} R(p) dp \quad (9.5.1)$$

where $R(p)$ is the response function of the spectroscope, as shown for two cases in Fig. 67.

The effect of the spectroscope resolution on measurements for the line beam case is demonstrated in Fig. 69. Figure 70 is the theoretical momentum distribution corresponding to the theoretical accelerator output current waveform of Fig. 38. Figure 71 shows how this curve would be altered by a spectroscope with a response function given by curve A of Fig. 67.

Figure 69 The Effect of Spectroscopy Resolution on Measurements for the Line Beam

Figure 70 Theoretical Momentum Distribution of Electrons Emerging from Single Cavity Electron Accelerator

Figure 71 The Curve of Figure 70 as Modified by a Spectroscope with the Response Function as Shown

10. POWER MEASUREMENTS

10.1 Introduction

Apparatus for the generation of power at harmonics of the accelerator frequency has been discussed in Chapters 5 to 8. In Chapter 9, the design of a device for measuring the velocity of the electrons in the electron beam was considered. With this equipment it is possible to relate the generated power to the beam velocity, and to compare these experimental results with the theoretical values. A schematic diagram of the apparatus used for these measurements is shown in Figure 72. In that figure, the switch box positions permit measurement of the accelerated current at position 1, the current into the test cavity at position 2, the current out of the test cavity at position 3, the current down the left spectroscopy pipe at position 4, the current down the center spectroscopy pipe at position 5, and the current down the right spectroscopy pipe at position 6. Figure 73 is a photograph of the spectroscopy and cavity no. 5 mounted on the accelerator. Figure 74 is a drawing of this assembly. The power generated was measured with a calibrated crystal and a calibrated attenuator. This chapter will begin with a discussion of the calibration of the crystal and the attenuator.

10.2 Calibration of the Crystal and the Attenuator

The attenuator used in these experiments was designed by A.W. Swago. The calibration was made with the aid of a PRD Type 650-A self-balancing power bridge, and a PRD Type 617 bolometer element in a PRD Type 618

Figure 72. Schematic Diagram of the Apparatus Used for the Measurement of Generated Power

Trace 1. Current through Test Cavity

Trace 2. Power in Accelerating Cavity

Trace 3. Generated Power

Figure 73 Cavity No. 5 and Electron Spectroscope Mounted on the Single Cavity Electron Accelerator. The lead shielding has been removed.

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Figure 74 Drawing of the Cavity and Spectroscope Assembly

bolometer mount. A schematic diagram of the apparatus is shown in Fig.

75.

Figure 75 Schematic Diagram of Apparatus for Attenuator Calibration

Continuous operation of the oscillator was used to avoid errors in power readings which may arise when a bolometer is used for power measurement under other than CW conditions.⁵⁹ The power input to Attenuator 2 was kept constant with the aid of the monitoring crystal and the Tektronix Type 512 oscilloscope, which has a direct coupled amplifier. Attenuator 2 may be calibrated by determining the power incident on the bolometer as a function of the setting of Attenuator 2 for constant power incident on the attenuator. A calibration curve obtained in this way is shown in Fig. 76.

The apparatus for calibrating the IN53 crystal used for measurement of the power generated in the higher order mode cavities is shown in Fig.

77.

Figure 76. Attenuator Calibration Curve

Figure 77 Schematic Diagram of the Apparatus for Crystal Calibration

For a particular setting of the calibrated attenuator, it is possible to compute the ratio of crystal power to bolometer power, since the attenuation through the directional coupler is known. A crystal calibration curve obtained with this apparatus is given in Fig. 78.

To obtain the curves shown in Figs. 76 and 78 it was necessary to measure power on the bolometer bridge in the range from 10 to 100 microwatts. In this range the temperature changes of the bolometer caused by the RF power are extremely small. Consequently, any small temperature changes of the bolometer element, caused, for example, by an air current arising from someone walking briskly by the bolometer mount, affect the power reading. Care must be taken to avoid errors arising in this way,

Figure 78 Calibration Curve for the IN53 Crystal Used for the Measurement of Generated Power

and it is also prudent to check the bridge balance before and after each power measurement.

The crystal calibration depends on the accuracy of the bolometer assembly for absolute power measurements. According to the manufacturer,

these units are for approximate power measurements only. All of the power absorbed by the bolometer mount is not, in general, absorbed by the bolometer element. The bolometer mount efficiency is defined as

$$\text{Efficiency} = \frac{\text{power absorbed by bolometer element}}{\text{total power absorbed}} \times 100.$$

The efficiency of the bolometer mount used in these experiments is estimated at about 80 to 90 percent. No correction has been made for this effect, however, so all power readings are probably about 10 to 20 percent low.

10.3 Preliminary Investigations of the Power Generated on the Single Cavity Electron Accelerator

A schematic diagram of the apparatus used for preliminary power measurements is shown in Fig. 79. With this apparatus it was possible to investigate:

1. The wavelengths generated and the possibility of the simultaneous excitation of more than one wavelength.
2. The relationship between the wavelengths generated and the cavity dimensions.
3. The spreading of the electron beam.
4. The effect of variations in the relative time phase between the pulse to the accelerator magnetrons and the pulse to the injection tube.

These points will be discussed in order.

The wavelengths generated were measured in two ways. The first

Figure 79. Schematic Diagram of Apparatus For Preliminary Investigation of Generated Power
 Trace 1. Generated Power
 Trace 2. Power In Accelerating Cavity
 Trace 3. Injection Tube Voltage

method involved an investigation of the standing waves between the two horns shown in Fig. 79. If a small sheet of metal is placed perpendicular to the line joining the two horns so that it intercepts some of the power, it will affect the power reaching the crystal detector. It is found that the wavelength may be measured by noting how far the metal must be moved along the axis of the horns between successive minima of the crystal output voltage. If only one wavelength is present this measurement readily leads to a value for the wavelength. If more than one wavelength is present, the distance between successive minima will not be equal. In this way it is possible to determine whether more than one wavelength is present in the generated power in appreciable amount. This method of measurement provides only an approximate value for the generated wavelength.

A more accurate value is obtained with a frequency meter, which provided the second method for measurement of the generated wavelength. The frequency meter used for these measurements was a Sperry Model 350 equipped with a side outlet through which power flows only when the frequency meter is tuned to the incoming frequency. By investigating the power flow out of this side arm, the presence of more than one frequency can also be detected.

In only one case were two wavelengths generated simultaneously. Cavity No. 3 produced power simultaneously at 10.803 millimeters and at 8.310 millimeters. This cavity was not used for the absolute power measurements because of the problems involved in sorting out the two frequency components. The interference in this case arose from either

the TM_{016} mode or the TM_{024} mode. The calculated wavelengths for these modes for the theoretical design dimensions of the cavity are given in Table 10.

Mode	$(D/L)^2$	λ	$\lambda/ \lambda - 10\lambda_0 $
TM_{024}	.49035	10.739	168
TM_{016}	.49035	10.792	981

Table 10. Computed Wavelengths for Interfering Modes

It would appear that these modes should not have been excited. Consequently, it is concluded that the systematic approach to the choice of cavity dimensions outlined in Chapter 8 is not infallible, in practice, due, undoubtedly, to the deviations of the actual cavity from the idealized cavity because of the beam irises, the coupling iris, and small differences in physical dimensions.

A counter was incorporated in the gear drive mechanism used for tuning the test cavities. With this counter it was possible to determine the length of the cavities at which power was generated. From these lengths the corresponding values of $(D/L)^2$ may be computed. If the wavelengths are determined, it becomes possible to compute the corresponding values of $(D/\lambda)^2$. These pairs of values determine points on a mode chart. It is then usually possible to determine the mode of oscillation from the nearness of this point to an appropriate mode line. Results for two cavities are shown in Figs. 80 and 81.

For Cavity No. 2, it was possible to relate all of the observed cavity lengths and generated wavelengths to TM_{0mn} modes, as one might expect to be the case. For Cavity No. 4, however, power was generated

Figure 80. Mode Chart for Cavity No. 2 Illustrating Wavelengths Obtained and Modes Excited

Figure 81. Mode Chart for Cavity No. 4, illustrating Wavelengths Obtained and Modes Excited

for five cavity lengths which could not be related to TM_{0mn} modes. The power level for these five resonances was low, to be sure, but a measurement of wavelength was made in all five cases. In four cases it was possible to match the observed point with a TM mode. In the fifth case the only mode corresponding to the observed point was the TE_{118} . If the cavity fields were the pure normal mode fields for a perfectly conducting, completely enclosed cylinder, and if the electron beam were axial and cylindrically symmetrical, only TM_{0mn} modes could be generated. However, it is to be expected that deviations of the electron beam from this ideal case and deviations of the fields from the ideal case, because of asymmetries of construction and the presence of the beam irises and the waveguide coupling iris, would make the excitation of modes other than the TM_{0mn} possible.

It was found that the beam from the electron accelerator spread in a radial direction much more rapidly than had been anticipated. On the basis of earlier calculations,⁶⁰ it was expected that the electron beam would travel approximately 50 centimeters before its diameter would double. In practice it was found that the beam diameter more than doubled in traveling 10 centimeters from the exit aperture of the accelerator. The situation is illustrated in Fig. 82. In this drawing I_1 is the average pulse current out of the accelerating cavity, I_2 is the average pulse current into the test cavity, and I_3 is the average pulse current out of the test cavity. Typical data for these currents with the accelerator in its present form are:

Figure 82 Sizes and Positions of Electron Beam Apertures

$$I_1 \approx 25 \text{ milliamperes,}$$

$$I_2 \approx 14 \text{ milliamperes,}$$

$$I_3 \approx 8 \text{ milliamperes,}$$

$$\beta_{\max} \approx 0.91,$$

It was not possible to place the test cavities closer to the accelerator exit aperture than eight centimeters because of a vacuum connection on the accelerator output to facilitate more rapid evacuation of the test apparatus than would be possible if pumping were done only through the accelerator exit aperture.

Three methods were tried in an attempt to increase I_2 and I_3 relative to I_1 . A mechanical aligner incorporating a metal bellows was designed and built to permit lateral and angular adjustment of the test apparatus. A magnetic deflector, using an RCA Type 219D1 kinescope deflecting coil, and a magnetic focusing coil were also designed and built. These devices were not used in the final experiments because it was found that the disadvantages resulting from the necessarily greater distance between the

accelerator and the test cavity effectively negated the advantages resulting from the adjustability.

The effect of variations in the relative time phase between the pulse to the accelerator magnetrons and the pulse to the injection tube is shown in Fig. 83. In these photographs, the upper trace represents the generated 8.31 millimeter power, the middle trace represents the voltage pulse on the injection tube, and the lower pulse represents the power in the accelerating cavity. It is clear that power is generated only when the injection tube voltage pulse occurs simultaneously with the magnetron pulse.

10.4 The Measurement of Generated Power as a Function of Beam Velocity

The apparatus shown in Fig. 74 was used to measure the generated power as a function of beam velocity and exciting current. The quantities measured were:

1. The injection tube voltage
2. The average pulse current out of the accelerator
3. The average pulse current into the test cavity
4. The average pulse current out of the test cavity
5. The current through the spectroscopy coil for which maximum beam current passed through the left slit system
6. The current through the spectroscopy coil for which maximum beam current passed through the right slit system.
7. The setting of the calibrated attenuator
8. The output of the calibrated crystal

Figure 83 Oscillograms Illustrating the Effect of Injection Tube-Magnetron Time Phasing on the Generated Power

Upper Trace: Generated Power at 8.31 Millimeters
Center Trace: Injection Tube Voltage Pulse
Lower Trace: Accelerating Cavity Power

9. The wavelength

10. The cavity length.

The harmonic current component was computed from the average pulse current into the test cavity by the methods of Chapter 6, using as a basis the waveform of current computed for a maximum electron velocity of $0.9c$ and an injection tube potential of 20 kilovolts.

The velocity associated with the harmonic current component corresponded to the momentum for which maximum current through the left and right spectroscopy slit systems was obtained. The electrons which emerge from the accelerator certainly do not all emerge with the same velocity. However, in order to compute a theoretical generated power, it is necessary to assume a value for the beam velocity so that the longitudinal coupling coefficient may be computed. For the range of beam velocities in which power measurements were made, and for the particular cavity dimensions chosen, the longitudinal coupling coefficient increases with increasing beam velocity. Hence it is the faster electrons in the beam which are most effective in exciting these test cavities. From Figs. 70 and 71, it is seen that the maximum current through the spectroscopy slit systems will occur for a value of momentum approximately 2% less than the maximum momentum in the beam. It is this value of momentum, slightly less than the maximum momentum present in the beam, which has been used to determine the velocity for use in the theoretical power expressions. That is, the entire current waveform of Fig. 38 is assumed to move with this velocity. Certainly this approximation is quite arbitrary, but if it is made, reasonable agreement between

theory and experiment is obtained.

From measurements of I_2 and I_3 it was possible to estimate values of the radial coupling coefficient. For example, using the data of Section 10.3 and Fig. 82, the radius of the beam after traversing the test cavity, ρ_{02} , is approximately

$$\rho_{02} \approx \left(\frac{I_2}{I_3}\right)^{1/2} \frac{0.0995}{2} \text{ inch} = 0.065 \text{ inch},$$

if it is assumed that the current density in the beam is uniform. The average radius is approximately

$$\bar{\rho}_0 = \frac{\rho_{01} + \rho_{02}}{2} = \frac{0.037 + 0.065}{2} = .051 \text{ inch},$$

where ρ_{01} is the radius of the beam as it enters the test cavity. Cavity No. 2, which operates in the TM_{018} mode, has a radius of 0.411 inch. For this case

$$\frac{\bar{\rho}_0}{R} = 0.124.$$

From Fig. 13 it is seen that $k_v^2 = 0.97$.

Figures 84 to 88 present graphically the results of these power measurements and include also the corresponding theoretical curves. From this data it is seen that the experimentally determined dependence of the generated power on the beam velocity agrees reasonably well with the theoretical dependence, considering the assumptions that have been made. For $\beta \approx 0.90$, it is seen that the generated power is approximately 20% of the theoretical value, with the exception of Cavity No. 2, for which the generated power is about 50% of the theoretical value. Cavity No. 2 is of the same design as Cavity No. 4 except for the

Figure 84. Power Generation with Cavity No. 1.

Figure 85. Power Generation with Cavity No. 2.

Figure 86. Power Generation with Cavity No. 4.

Figure 87. Power Generation with Cavity No. 6.

Figure 88. Power Generation with Cavity No. 7.

position of the waveguide output. This design is shown in Fig. 52. In Cavity No. 2 the waveguide is located at the midplane of the cavity, whereas all of the other cavities have the waveguide output located at a position $L/8$ from the fixed end. The effect of the waveguide output position on generated power will be investigated at some later date.

Approximately one meter of RG 96/U silver waveguide was used between the test cavity and the crystal used for measuring the generated power. The attenuation constant for the TE_{10} mode in a waveguide of broad dimension a and narrow dimension b is

$$\alpha_{10} = \frac{1}{\mu_0 c b} \frac{a + 2b(\lambda/\lambda_{10})^2}{ab[1 - (\lambda/\lambda_{10})^2]^{\frac{1}{2}}},$$

where λ is the free space wavelength and λ_{10} is the cutoff wavelength. For RG 96/U waveguide at a wavelength of 8.31 millimeters it is found that

$$\alpha_{10} = 0.058 \text{ meter}^{-1}.$$

For one meter of guide the generated power, P_1 , is related to the detected power, P_2 , by

$$P_1 = P_2 e^{2\alpha_{10}} = P_2 e^{0.116} = 1.12 P_2.$$

The experimentally measured powers have been multiplied by 1.12 to obtain the corresponding values for the generated powers. It should be noted that the actual attenuation is probably of the order of 30% greater than the theoretical value computed here because of the imperfect interior surfaces of the waveguide.⁶¹

In plotting the theoretical curves the theoretical value for the

unloaded Q , Q_a , has been used. The preliminary measurements on the test cavities indicated that the actual Q_a was only about 60% of the theoretical value, as is usually the case unless the cavity walls are carefully polished. Because the generated power is proportional to Q_a , the expected generated power was only about 60% of the values plotted in Figs. 84 to 88. The measured values are of the order of 35% of this amount, with the exception of Cavity No. 2, for which the number is about 80%.

It is felt that the agreement between theory and experiment is as good as might be expected, considering the unknown quantities involved, such as the actual thirteenth harmonic current component, and the accuracy of the bolometer and mount that were used for the crystal calibration. It has been assumed that the 13th harmonic current component is given by 37% of the average pulse current for the range of beam velocities used in these experiments. Undoubtedly there is some dependence of harmonic amplitudes on beam velocity. The errors in the measurement of the generated power may well be of the order of 20%, due to errors resulting from the bolometer and its mount, from the attenuator and crystal calibrations, and from the measurements with these units on the accelerator. The measurement of β was probably accurate to $\pm 1\%$. The measurement of the average pulse accelerator current was accurate to about $\pm 5\%$.

11. CONCLUSIONS

A promising new method for the generation of power at the watt level at submillimeter wavelengths through the excitation of a single selected TM_{0mn} mode in a cylindrical cavity has been theoretically investigated with a field theory approach and successfully demonstrated at a wavelength of 8.310 millimeters. The electromagnetic field in the cavity is excited by a high energy (1 mev range) bunched electron beam, produced by a pulsed single cavity electron accelerator operating at 2775 megacycles per second. This bunched electron beam contains current frequency components at high harmonics of the 2775 megacycles per second fundamental frequency. The power generated with five cavities operating at the thirteenth harmonic, corresponding to a wavelength of 8.310 millimeters, has been investigated, and the behavior is essentially as predicted by the theory. The physical size of the resonant cavities used is of the order of three to four wavelengths for both the length and the diameter, so that structures for use at submillimeter wavelengths can be built.

The theoretical relationship between the generated power, P , the wavelength, λ , and the harmonic current component, I_s , is typically of the order of

$$P(\text{watts}) \approx 3,000 [I_s (\text{amps})]^2 \sqrt{\lambda(\text{mm})}$$

for a 1 mev electron beam. If a harmonic current amplitude of 100 milliamperes (a reasonable value) is assumed, the theory predicts a pulsed generated power of approximately 30 watts at a wavelength of one millimeter. This power level is of the order of 10^7 (ten raised to the seventh power)

times that available in a one percent bandwidth from a black body with a surface area of one square centimeter at a temperature of 10^4 degrees Kelvin. It is of the order of 10^{13} (ten raised to the thirteenth power) times the power output of crystal harmonic generators. At the present time no sources better than these exist.

It has been shown that the harmonic current components of which the output current pulse from the single cavity electron accelerator is composed may be calculated from the theoretical accelerator output current pulse, and that for the purposes of the experiments considered here, these components may be thought to comprise electrons traveling with essentially the same velocity as the fastest electrons in the beam.

An improved electron accelerator, incorporating a modulating cavity between the electron injection tube and the accelerating cavity shows great promise of producing a high current, 1-1.5 mev electron beam with high harmonic content into the range of submillimeter wavelengths.⁶² If such an electron beam can be produced, it appears at this time that the principle of harmonic excitation of higher order oscillations in resonant cavities can be used successfully to generate relatively high power at wavelengths shorter than one millimeter. Work currently in progress in this laboratory has as its goal the production of an electron beam suitable for this purpose.

An effort to extend these techniques to shorter wavelengths is now underway. The current work is at wavelengths in the four millimeter region. Primary bench investigations are being made with the aid of a crystal harmonic generator excited by an eight millimeter klystron. It

is planned to continue work along this approach until practical difficulties either in the production of the electron beam or in the construction of the resonant cavities halt progress. It is felt that these difficulties will not become insurmountable until entry into the submillimeter wavelength region of the electromagnetic spectrum has been achieved.

The success of this approach, or any other approach, to the submillimeter wave generation problem will mean that research which is impossible today because of frequency limitations in the fields of microwave spectroscopy, gaseous discharge, and superconductivity can be undertaken. Adequate tools are necessary for the progress of scientific research, and the goal of this research has been to provide such a tool for work in the range of submillimeter wavelengths.

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