APPLICATION OF THE CLASSICAL THEORY OF ELECTRONS
IN GASES TO MULTIWIRE PROPORTIONAL AND DRIFT CHAMBERS

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ABSTRACT

In this paper we review the classical theory of electrons in gases and some of its results which may be useful in the design of multiwire proportional and drift chambers. We discuss especially how low diffusions and constant drift velocity may be obtained in certain gas mixtures, and the multiplication process.

INTRODUCTION

Among the various particle detection devices multiwire proportional counters¹ and drift chambers²⁻⁵ appear to be very promising. They have been the object of numerous recent experimental studies that have resulted in many cooking recipes, most of which are, however, of doubtful fundamental significance. In fact, very few high energy physicists working in this field seem to be aware -- at least we were not -- of the existence of a rather classical and successful theory of the behavior of electrons in gases. After the pioneering work of Townsend⁶ in the twenties, Trayvstein⁷, Morse, Allis, and Lamar⁸ and others were able to construct a successful theory just before World War II. The development of radar and gaseous switching tubes triggered some new theoretical developments from Holstein⁹, Margenau¹⁰, Golant¹¹, etc. after the war. The theory was then good enough to be used by Phelps¹² and co-workers in 1962 not only for the qualitative explanation of the observed behavior of the drift velocity and the diffusion, but to extract from measurement of these quantities detailed information on the various elastic and excitation cross sections of interest for atomic physicists.

In this paper we give a summary of this theory for high energy and nuclear physicists, since we think that in spite of the complexity of the gas mixtures we are using, some knowledge of the basic phenomena may be useful in understanding their properties. The first section of our work deals with the definition of the so called transport coefficients (drift velocity, diffusion coefficient, characteristic energy) describing the behavior of electrons in gases under the action of external fields. We will give some experimental results concerning these coefficients for the gases usually used in multiwire proportional chambers.

In the second section we develop a simplified theory for the calculation of these coefficients; although it uses only intuitive arguments and rough approximations, it shows how the different contributions appear. The action of a magnetic field is (which is important because detectors are often located inside strong magnetic fields, as in the new generation of large magnetic spectrometers for particle physics) is also considered.

The third section introduces a more rigorous theory of gaseous kinetics, starting from general transport equations, which enables computation of the energy distribution of the free electrons in the gas and therefore the various transport coefficients. As an example we show explicitly that the numerical solution of those equations describes reasonably well the behavior of electrons in pure argon. We then apply this theory to gas mixtures commonly used, namely argon-isobutane and argon-methane mixtures. We try to give an intuitive interpretation of the behavior of drift velocities.

These results allow us to comment in the fourth section on the desirable properties of gases to be used in drift chambers and the way to achieve them. We discuss especially the diffusion, the behavior in magnetic fields and the constancy of drift velocities.

The fifth section deals with gaseous amplification phenomena. We first show that the classical theory developed in Section 3 can describe quantitatively the multiplication processes in homogeneous fields. Considering then the case of multiwire proportional counters, where the electric field is strongly varying, we show through computer simulation that we can use the results obtained for homogeneous fields. This allows us to understand the various empirical formulae

proposed for the amplification in proportional counters. We finally comment on various parasitic phenomena and the way they are eliminated in modern proportional counters.

None of our results are entirely original. Section 2 is a generalization of Townsend's approach in "Electrons in Gases".⁶ The proof of the capability of the theory developed in Section 3 to accurately describe experimental data was already given by Golant, Phelps and co-workers. However, we have developed a program for computing the transport coefficients and will send it to any interested physicist. Sections 4 and 5 are more original, at least in the attempt to explain phenomena of great importance for design and operation of multiwire proportional and drift chambers.

For readers uninterested in mathematical details, we would recommend the reading of Sections 1, 2, 3.5 and 3.6, 4, 5.3.2, and 5.4.

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SECTION 1

TRANSPORT COEFFICIENTS CHARACTERIZING THE BEHAVIOR OF AN ELECTRON SWARM IN ELECTRIC AND MAGNETIC FIELDS

1.1 Drift of Electrons in Electric Field

The ionization of a gas by an ionizing radiation, as is well known, produces free electrons (and ions). If the ionized gas is not subject to any electric field, these electrons will move randomly with an average agitation energy that is given by the Maxwell formula $3/2 kT$ (about $3.7 \cdot 10^{-2} \text{ eV}$ at 18°C). If an electric field is applied, the electrons will continue to have a thermal non-directional velocity v (and energy ϵ), but they will exhibit a general overall drift along the field direction, with a mean drift velocity $w = \langle v_x \rangle$. Of course one gets

$$w = \langle v_x \rangle = \iiint v_x F(\vec{r}, \vec{v}, t) d\vec{v} \quad 1.1(1)$$

where $F(\vec{r}, \vec{v}, t)$ is the velocity distribution of the electrons at the point $\vec{r}$ and at time t ; in presence of external fields F is not the Maxwellian distribution. The mobility is defined as $\mu = w/E$ where E is the applied electric field.

Let us consider, for the sake of simplicity, a region where no ion or electron is formed and none disappears by attachment or recombination. The motion of the electrons has to be determined by a diffusion equation. If ρ is the spatial density of particles, the conservation of the number of particles yields

$$\int \rho dV = N \text{ or } \partial \rho / \partial t + \text{div}(\rho \vec{v}) = 0$$

The diffusion coefficient D is defined by

$$\vec{\nabla} \rho = -\frac{1}{D}(\rho \vec{v})$$

and the diffusor equation is

$$\partial \rho / \partial t = D \nabla^2 \rho \quad 1.1(2)$$

D describes the mean spread of a swarm of electrons

$$\frac{\partial \langle x^2 \rangle}{\partial t} = \frac{\partial}{\partial t} \int \frac{x^2 \rho(x, y, z)}{N} dV = 2D,$$

where we have used Eq. 1.1(2) and integrated by parts. So we get

$$\frac{\partial \langle R^2 \rangle}{\partial t} = 6D.$$

Similarly the quantity $\sqrt{2D/w}$ gives a measure for the lateral and longitudinal spread that the electrons undergo while traveling a unit distance in the gas, since

$$\frac{\partial \langle x^2 \rangle}{\partial x} = \frac{\partial \langle x^2 \rangle}{\partial t} \frac{\partial t}{\partial x} = \frac{2D}{w} = \frac{\partial \langle y^2 \rangle}{\partial t} = \frac{\partial \langle z^2 \rangle}{\partial t} \quad 1.1(3)$$

Note that in the presence of electric or magnetic fields the diffusion coefficients are not necessarily the same for all directions of space (see Section 2.4.3). The mean energy of electrons is of course

$$\langle \epsilon \rangle = \int \epsilon F(\vec{r}, \epsilon, t) d\vec{v}$$

where $\epsilon = \frac{1}{2} m v^2$. Another similar parameter, which is directly measurable, is the so called characteristic energy, defined as

$$\epsilon_k = \frac{eD}{\mu}$$

1.1(4)

Fig. 1. Experimental behavior of drift velocity and some characteristic energies in noble gases (Ar, He, Ne, Kr, Xe). In Ar the "magnetic drift velocity, w_M " is also shown.

where e is the charge of the electron, μ is the mobility, and D is again the diffusion coefficient. Note that

$$\frac{\partial \langle x^2 \rangle}{\partial x} = 2\epsilon_k / eE \quad 1.1(5)$$

Fig. 1 gives, as an example, the experimental behavior* of w and ϵ_k for the usual noble gases. It is seen that the behavior of these quantities is by no means simple. Note that they are universal functions of E/N , where N is the number of molecules per unit volume, or E/P_{300} , where P_{300} is the atmospheric pressure at 300° K. Drift velocities are typically 10^6 cm/s. It must be noticed that they are very much lower than the "thermal" random velocity. Mean energy and characteristic energy are in fact usually in the region of few electron volts, that is, we get a thermal velocity of the order of 10^8 cm/sec ($v = \sqrt{1.6 \times 10^{-19} \times 2\epsilon / m}$, where m is the mass of the electron in kilograms and ϵ is in eV). The diffusion coefficient has the dimension of cm^2/sec . For diatomic and polyatomic molecules the characteristic energies for $E/P_{300} \lesssim 1$ are much lower (Fig. 2 gives w and ϵ_k for N_2 , H_2 , and CO_2). Another striking experimental fact is that the

Fig. 2. Experimental behavior of drift velocities and characteristic energies in more complex molecules than monoatomic gases (H_2 , CO_2 , N_2) showing in particular smaller values of ϵ_k at low and intermediate electric fields.

*For the experimental method of measurement we refer the reader to Loeb¹³ and Massey and Burhop¹⁴.

mixture of noble gas with polyatomic molecules (N_2 , CO_2 or organic vapor, even in very small quantities) strongly modifies the behavior of the transport coefficients: w is increased by as much as a factor of 10 and is nearly independent of the electric field (Fig. 3). And, as we shall see, ϵ_k is very much reduced (see Section 3.5).

Fig. 3. Experimental behavior of drift velocity in different argon- CO_2 mixtures (from English and Hanna, Canadian Journal of Physics 31 (1953), 768).

1.2 Effect of the Magnetic Field

In the presence of a magnetic field, perpendicular to an electric field, the electrons drift along a path which is different from the electric field lines. We define $w_{\parallel}$ as the drift velocity component along the electric field lines and $w_{\perp}$ as the drift velocity component along the normal to the electric field and magnetic field. If the electric field is along $O\vec{x}$, and the magnetic field along $O\vec{z}$,

$$w_{\parallel} = \langle v_x \rangle = \int v_x F(\vec{r}, \vec{v}, t) d\vec{v}$$

$$w_{\perp} = \langle v_y \rangle = \int v_y F(\vec{r}, \vec{v}, t) d\vec{v}$$

The magnetic drift velocity is defined as

$$w_M = \frac{w_{\perp}}{w_{\parallel}} \frac{E}{B}.$$

Notice that the lateral displacement after a drift length is

$$\Delta y = w_M \frac{B}{E} x.$$

Experimentally w_M is usually close to w (see Fig. 1 for pure argon).

1.3 Life History of an Electron in a Multiwire Proportional Counter

In order to clarify the following discussions, let us follow the life history of an electron and give some relevant formulas for completeness.

1.3.1 Production of Electrons¹⁵

When a high energy particle of velocity βc and charge unity goes through a proportional chamber, it deposits in the chamber a mean amount of energy given by the Bethe-Bloch formula.¹⁶

$$\frac{dE}{dz} = -2\pi\rho m_e r_e^2 \frac{NZ}{A} \frac{c^2}{\beta^2} \left[\ln \frac{2m_e c^2}{I^2} \frac{\beta^2}{1-\beta^2} \epsilon_{\max}^{-2\beta^2 - \delta(\beta)} \right] \quad 1.3(1)$$

where ρ is the density of the gas, A and Z its atomic weight and atomic number, m_e is the electron mass, r_e is the classical electron radius, N the Avogadro number, $\epsilon_{\max}$ is the maximum energy allowed for the electrons to stop in the chamber (typically 22 keV for a 1-cm thick chamber filled with argon), I is the mean ionization potential which is given roughly by:

$$I = 13 \text{ eV} \times Z$$

$\delta(\beta)$ is a function of the velocity which accounts for the density effect responsible for the Fermi plateau.

However the mean number of electrons is not given by Eq. 1.3(1). One should distinguish between (a) the primary electrons, namely those directly produced by the high energy particles (Table I, taken from Korff¹⁷ gives their number, S , per cm per atmosphere. This number is important because it gives the basic inefficiency of a chamber of thickness l , assuming a Poisson law $\eta = e^{-Sl}$.) and (b) the secondary electrons some of these primary electrons are very energetic

and will ionize the gas. Moreover they also rapidly lose their energy through elastic and excitation collisions. One introduces the mean energy $\langle \epsilon \rangle_p$ per electron pair such that $\frac{dE}{\langle \epsilon \rangle_p dx}$ is the total number of electrons arriving in the avalanche region.

Table 1 gives experimental values of $\langle \epsilon \rangle_p$ for various gases, and the total number of ion pairs per cm at its minimum. Figure 4 gives the latter number as a function of pion energy.

	(No. of primary ion pairs) per cm	$\langle \epsilon \rangle_p$	Total no. of ion pairs per cm
Hydrogen	6.	33	~ 10
Helium	5.9 or 6.5	27.8	~ 12
Argon	29.4	25.4	~ 90
Methane (CH_4)	16.	27.3	~ 50
Neon	12.	27.4	~ 50
Xenon	44	20.8	~ 300

Fig. 4. Number of ion-electron pairs per centimeter deposited by pions in different gases as a function of the energy of the pions.

1.3.2 Drift Region

These approximately 100 electrons begin to drift in the electric field of the gap towards the sense wires. The electric field in a multiwire proportional counter has been given by G. A. Erskine¹⁸. The complex potential w at an arbitrary point $z = x+iy$ arising from a single wire with linear charge density q_0 , at position z_0 with $\text{Im } z_0 = 0$

$$w(z, z_0) = -\frac{2q_0}{4\pi\epsilon_0} \ln \left\{ \frac{\sinh \left[\left(\frac{\pi}{4L} \right) (z - z_0) \right]}{\cosh \left[\left(\frac{\pi}{4L} \right) (z - z_0) \right]} \right\}.$$

For a set of n wires at positions z_i and potentials V_i , the resulting potential at z (if the wire radius is negligible)¹⁹ is

$$w(z) = \sum_{i=1}^n q_i w(z, z_i)$$

where the q_i are determined by

$$\operatorname{Re}[w(z_i)] = V_i$$

The case where there are no ground planes (cathode made out of wires) is treated by letting L go to infinity. When the arrangement of wires is periodic and the wires are in the middle plane between two grounded cathode planes, the potential at position x, y is given a closed form

$$V = \frac{q}{4\pi\epsilon_0} \left[2\pi \frac{L}{S} - \log \left(4 \sin^2 \frac{\pi x}{S} + 4 \sinh^2 \frac{\pi y}{S} \right) \right]$$

where the origin is at one wire, S is the spacing between wires, and $L/S \geq 5$ has been assumed. The charge density q is given by

$$q = \frac{4\pi\epsilon_0 V_0}{2 \left[\frac{\pi L}{S} - \log \frac{\pi r_0}{S} \right]}$$

where V_0 is the applied voltage and r_0 is the radius of the wires. Figure 5 shows the mechanical structure

Fig. 5. Mechanical structure of a multiwire proportional chamber and of two types of drift chambers (Charpak et al. and Valenta et al.)

construction of usual multiwire proportional chambers¹ and drift chambers²⁻⁵. Figure 6 and 7 show the corresponding fields for the dimensions and applied potentials shown. Section 2, 3, and 4 will be devoted to the study of the behavior of the electrons in this region.

Fig. 6. Shape of the electric field in a multi-wire proportional chamber along the incoming particle direction (E_y) and along the normal to the beam (E_x). The field increases as $1/r$ in the region close to the wire.

Fig. 7. (a) Shape of the electric field in a drift chamber of the type built by Charpak et al.
(b) Shape of the electric field in a drift chamber of the type built by Valenta et al.

1.3.3 Avalanche Region

Finally when the electrons arrive at some 100μ from the wire, their energy is high enough to ionize the gas (see Section 5). The movement of negative charges towards the wire and of positive charge away from the wire produces a pulse on the wire. Because the electrons are in the majority produced on the last mean free path, they do not contribute very much to the pulse, and the positive ions which have to move down along the entire potential will be responsible for most of the induced charge¹. The time build up of the charge has been given by Wilkinson²⁰ and the pulse shape by Mitra²⁰. Assuming that a resistor R is attached to the wire and that the total (parasitic) capacity in parallel to the wire is C , the difference of potential across the resistance is

$$v(t) = \frac{Q}{C} \frac{q}{4\pi\epsilon_0 V} e^{-\frac{t+t_0}{RC}} \left[E_i\left(\frac{t+t_0}{RC}\right) - E_i\left(\frac{t_0}{RC}\right) \right] \quad 1.3(2)$$

where q is the charge per unit length of wire, V is the difference of potential across the chamber, Q is the total charge of the electrons in the avalanche, and t_0 is given by

$$t_0 = \frac{4\pi\epsilon_0 d^2}{16q\mu}$$

where d is the diameter of the wire, and μ the mobility of the positive ions (typically $1.6 \text{ cm}^2/\text{s}/V$ for Argon). E_i is the exponential integral function.

SECTION 2

SIMPLIFIED THEORY OF ELECTRONS IN GASES

Before presenting a rigorous theory of the behavior of electrons in gases, we would like to give an elementary derivation of the rigorous expressions of the transport coefficients in terms of the thermal velocity v , the collision length l with the molecules of the gas, and the fraction of energy Λ lost at each collision.

2.1 Drift Velocity

Let us first start with the drift velocity in an electric field. Consider an electron which has undergone a collision with a molecule of the gas at the time 0 and which has emerged at an angle θ_0 with the field direction, and a velocity v_0 (Fig. 8a). If the field is along the x direction the path covered along x direction by this electron in a time δt is

$$\delta x = \frac{1}{2} \frac{eE}{m} (\delta t)^2 + v_0 \cos \theta_0 \delta t \quad 2.1(1)$$

We will evaluate the drift velocity w along the field direction first in the following three approximations (relaxed later):

- a) the differential cross section for electron-molecule scattering is isotropic.
- b) the time δt between two collisions is independent of the first collision emission angle θ_0 (which is wrong, as we will see).
- c) the mean free path l is constant over the whole energy spectrum.

After averaging over θ_0 we get $(\int_0^\pi \cos \theta \, d \cos \theta = 0)$

$$\langle \delta x \rangle_{\cos \theta_0} = 1/2 \frac{eE}{m} (\delta t)^2$$

We now have to make the average over the distribution $g(\delta t)$ of the collision time δt , to get the mean path along the x direction between two collisions. If l is the mean free path and v the random velocity, then $\tau = l/v$ is the mean free time

$$g(\delta t) = \frac{1}{\tau} e^{-\delta t/\tau}$$

The mean of $(\delta t)^2$ is

$$\langle \delta t^2 \rangle_{\text{coll.}} = 2\tau^2/v^2,$$

where v , in a first approximation, is not different from v_0 . Therefore we get

$$\langle \delta x \rangle_{\cos \theta_0} = \frac{eE}{m} \frac{\tau^2}{v^2}$$

Since we have v/l collisions in a second, the drift velocity of an electron which has an energy $\epsilon = 1/2 m v^2$ is

$$w = \langle v_x \rangle = \langle \delta x \rangle \frac{v}{l} = \frac{eE}{m} \frac{l}{v}$$

We have still to average over the different electrons that have a different velocity v to get

$$w = \frac{eE}{m} \left\langle \frac{f}{v} \right\rangle_v \quad 2.1(2)$$

Remark: It is very easy to construct wrong arguments; for instance, because

$$v_x = \frac{eE}{m} \delta t + v_0 \cos \theta_0, \quad \text{one could write down}$$

$$w = \langle v_x \rangle_{\text{coll.}} = (v_{\text{max}} - v_{\text{min}})/2 = \frac{1}{2} \frac{eE}{m} \langle \delta t \rangle = \frac{1}{2} \frac{eE}{m} \left\langle \frac{f}{v} \right\rangle$$

But that is wrong because the time between collisions is not a constant.

We now go to a more exact evaluation of w . We retain the hypothesis (a) but we drop hypotheses (b) and (c). Now $f = f(\epsilon)$ and, as we will see, $\delta t = \delta t(\theta_0)$. We have again

$$x = \frac{1}{2} \frac{eE}{m} t^2 + v_0 \cos \theta_0 t$$

The real path of the electron is drawn in Fig. 8b. The path covered is given by

$$\left(\frac{ds}{dt} \right)^2 = v_x^2 + v_y^2 + v_z^2,$$

where

$$v_x = \frac{eE}{m} t + v_0 \cos \theta_0$$

$$v_y^2 + v_z^2 = v_0^2 \sin^2 \theta_0$$

Now let us expand these expressions in Taylor series and disregard the terms at the orders greater than one in E because, for not too high fields, $\Delta v/v_0 \ll 1$. We get

$$\frac{ds}{dt} = v_0 \left(1 + \frac{eE}{mv_0^2} v_0 \cos \theta_0 t \right) + \dots$$

By integrating one obtains

$$s = v_0 t \left(1 + \frac{eE}{2mv_0^2} v_0 \cos \theta_0 t \right) + \dots$$

and, always to the first order in E

$$t(s) = \frac{s}{v_0} \left(1 - \frac{eE}{2mv_0^2} v_0 \cos \theta_0 \frac{s}{v_0} \right) + \dots \quad 2.1(3)$$

So we get

$$v(s) = \frac{ds}{dt} = v_0 \left(1 + \frac{eE}{mv_0^2} v_0 \cos \theta_0 \frac{s}{v_0} \right) + \dots$$

for the dependence of the random velocity on the path covered. Now the number of collisions in the element of path ds is given by

$$\frac{ds}{l(v)} = ds \left(\frac{1}{l(v_0)} - (v-v_0) \frac{dl}{dv} \frac{1}{l^2(v_0)} \right) = \frac{ds}{l(v_0)} - \frac{eE}{mv_0^2} v_0 \cos \theta_0 \frac{s}{l^2(v_0)} \frac{dl}{dv} ds$$

where we have expanded $[l(v)]^{-1}$. Taking into account that, for events with mean free path $l(s)$, the probability of collision between s and $s+ds$ is

$$g(s) ds = \frac{ds}{l(s)} e^{-\int_0^s dz/l(z)} \quad 2.1(4)$$

we get

$$g(s) ds = \left(\frac{1}{l_0} - \frac{eE}{mv_0^2} v_0 \cos \theta_0 \frac{s}{l^2} \frac{dl}{dv} \right) ds \exp \left(-\frac{s}{l_0} + \frac{eE}{2mv_0^2} v_0 \cos \theta_0 \frac{s^2}{l^2} \frac{dl}{dv} \right)$$

Expanding the exponential factor and retaining only the first term in E we arrive at

$$g(s) ds \approx e^{-s/l_0} ds \left[\frac{1}{l_0} - \frac{eE}{mv_0^2} \cos \theta_0 \frac{dl}{dv} \left(\frac{s}{l_0} - \frac{1}{2} \frac{s^2}{l_0^2} \right) \right]$$

The mean path is then

$$\begin{aligned} \langle x \rangle_{\text{coll.}} &= \int_0^\infty x g(s) ds = \int \left(\frac{1}{2} \frac{eE}{m} t^2(s) + v_0 \cos \theta_0 t(s) \right) g(s) ds = \\ &= \int \left[\frac{1}{2} \frac{eE}{m} \frac{s^2}{v_0^2} \left(1 - \frac{eE}{mv_0^2} v_0 \cos \theta_0 \frac{s}{v_0} \right) + v_0 \cos \theta_0 \frac{s}{v_0} \left(1 - \frac{1}{2} \frac{eE}{mv_0^2} v_0 \cos \theta_0 \frac{s}{v_0} \right) \right] \\ &\quad \times \frac{1}{l_0} \left[1 - \frac{eE}{mv_0^2} v_0 \cos \theta_0 \frac{dl}{dv} \left(\frac{s}{l_0} - \frac{1}{2} \frac{s^2}{l_0^2} \right) \right] e^{-s/l_0} ds \end{aligned}$$

Using $\int_0^\infty s^n e^{-s} ds = n!$ and taking the average over $\cos \theta_0$ we obtain

$$\langle x \rangle_{\text{coll.}} = \frac{2}{3} \frac{eE}{m} \frac{l^2}{v_0^2} + \frac{1}{3} \frac{eE}{m} \frac{l}{v_0} \frac{dl}{dv} \Big|_{v=v_0}$$

and therefore

$$w = \frac{2}{3} \frac{eE}{m} \frac{l}{v_0} + \frac{1}{3} \frac{eE}{m} \frac{dl}{dv} \Big|_{v=v_0}$$

We should now introduce v_0 as a function of the mean random velocity v of the considered electron. But the two values differ by terms in $\frac{eE}{m}$, introducing correction of second order only. So we get at last

$$w = \frac{2}{3} \frac{eE}{m} \left\langle \frac{l}{v} \right\rangle + \frac{1}{3} \frac{eE}{m} \left\langle \frac{dl}{dv} \right\rangle \quad 2.1(5)$$

where the averages are to be performed over the energy distributions of the electrons. Note that for constant collision frequency ($\nu/l = \text{constant}$) $w = \frac{eE}{m} \langle \frac{l}{v} \rangle$.

Remarks:

(1) A simpler (and wrong) argument that might have been used is the following: at the time t the velocity of an electron along the field direction is

$$v_x = \frac{eE}{m} t + v_0 \cos \theta_0$$

From our function $t(s)$ [eq. 2.1(3)] and the distribution $g(s)$ [eq. 2.1(4)], we derive

$$\langle v_x \rangle_{\text{coll}} = \int v_x(s) g(s) ds = \frac{eE}{m} \frac{l}{v} + v_0 \cos \theta_0$$

Averaging now over $\cos \theta_0$ in the isotropy hypothesis, we get again the old result

$$\langle v_x \rangle = \frac{eE}{m} l/v.$$

This is wrong and shows the importance of dropping the hypothesis (b), that there was no dependence of δt from θ_0 . In this last simplified derivation we have in fact forgotten that at the time t , the distribution in $\cos \theta_0$ is no more uniform because the electrons that started with $\cos \theta_0 > 0$ are now less numerous than the electrons that started with $\cos \theta_0 < 0$, because for them the time between collision is shorter and a greater part of them has already undergone another collision.

(2) Our expressions are rigorous (to the first order in eE/m) and are those obtained in the general theory (Section 3). If the hypotheses (1) that the differential cross section for the electron-molecule scattering is isotropic is dropped, one should make the difference between the mean free path $\lambda_e(v, \cos \theta)$ for deflection at an angle θ and the "momentum" mean free path $l(v)$ defined as

$$\frac{1}{l(v)} = \int_{-1}^{+1} \frac{1 - \cos \theta}{\lambda_e(v, \cos \theta)} \frac{d \cos \theta}{2}$$

Then the quantity to use in computing w (and so on) can be shown to be l and not λ_e (see Section 3).

2.2 Diffusion

Let us now compute D in a similar way. After a time t , one electron will move by $R = v\delta t$ and the dispersion between two collisions is

$$\langle R^2 \rangle_{\text{coll}} = v^2 \langle \delta t^2 \rangle = 2v^2 \frac{l^2}{v^2} = 2l^2$$

where we integrated over $\cos \theta_0$ too. Because of the ν/l collisions in the unit time, $\frac{\partial \langle R^2 \rangle}{\partial t} = 2\nu l$ which yields, after the usual average over the energy distribution,

$$D = \langle l v \rangle / 3.$$

2.3 Effect of Magnetic Field

A similar computation can be made when the electrons are drifting in an electric field E normal to a magnetic field B . Let us take (Fig. 8c) the field E along the x direction, B along the z direction. Now there is no more symmetry in ϕ , because of the presence of B and averages over ϕ are to be made as well. The dynamical equations for the motion of an electron in the fields are

$$\frac{d^2 x}{dt^2} = \frac{eE}{m} + \frac{eB}{m} \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = -\frac{eB}{m} \frac{dx}{dt}$$

$$\frac{d^2 z}{dt^2} = 0$$

which implies

$$\frac{d^3 x}{dt^3} = -\left(\frac{eB}{m}\right)^2 \frac{dx}{dt}$$

Solving this equation in dx/dt and taking in account that $v_x^0 = v_0 \sin \theta_0 \cos \phi_0$ and $v_y^0 = v_0 \sin \theta_0 \sin \phi_0$ we get the velocity components

a)

b)

c)

d)

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Fig. 8. Various axis systems helpful in the reading of Sections 2 and 3 of this report.

$$\begin{aligned} dx/dt &= v_0 \sin \theta_0 \cos \phi_0 \cos \left(\frac{eBt}{m} \right) + (v_0 \sin \theta_0 \sin \phi_0 + E/B) \sin \left(\frac{eB}{m} t \right) \\ dy/dt &= v_0 \sin \theta_0 \cos \phi_0 \sin \left(\frac{eBt}{m} \right) + (v_0 \sin \theta_0 \sin \phi_0 + E/B) \cos \left(\frac{eBt}{m} \right) - E/B \\ dz/dt &= v_0 \cos \theta_0 \end{aligned}$$

By integrating we now can get the three components of the displacement

$$\begin{aligned} x &= x_0 + v_0 \sin \theta_0 \cos \phi_0 \frac{m}{eB} \sin \left(\frac{eBt}{m} \right) - (v_0 \sin \theta_0 \sin \phi_0 + E/B) \frac{eB}{m} \left(\cos \left(\frac{eBt}{m} \right) - 1 \right) \approx \\ &\approx x_0 + v_0 \cos \phi_0 \sin \theta_0 t + v_0 \sin \theta_0 \sin \phi_0 + E/B \frac{m}{2eB} \left(\frac{eB}{m} \right)^2 t^2 \\ y &= y_0 + v_0 \sin \theta_0 \cos \phi_0 \frac{m}{eB} \left(\cos \left(\frac{eBt}{m} \right) - 1 \right) + (v_0 \sin \theta_0 \sin \phi_0 + E/B) \frac{m}{eB} \sin \left(\frac{eBt}{m} \right) \\ &- \frac{E}{B} \approx y_0 - v_0 \sin \theta_0 \cos \phi_0 \frac{eB}{2m} t^2 + v_0 \sin \theta_0 \sin \phi_0 t - (v_0 \sin \theta_0 \sin \phi_0 + E/B) \\ &\cdot \left(\frac{eB}{m} \right)^2 t^3/6 \\ z &= z_0 + v_0 \cos \theta_0 t \end{aligned}$$

We assume as above that the differential cross section of collision is isotropic and take explicitly into account the effects of the lengthening of the trajectory and the increase of velocity. To the first order in eB/m , eE/m , e^2EB/m^2 one gets

$$\begin{aligned} s &= v_0 t \left(1 + \frac{eE}{2mv_0^2} v_0 \sin \theta_0 \cos \phi_0 t + \frac{1}{6} \frac{eE}{mv_0^2} \left(\frac{eB}{m} \right) t^2 v_0 \sin \theta_0 \sin \phi_0 \right) + \dots \\ t &= \frac{s}{v_0} \left(1 - \frac{eE}{2mv_0^2} v_0 \sin \theta_0 \cos \phi_0 \frac{s}{v_0} - \frac{1}{6} \frac{eE}{mv_0^2} \frac{eB}{m} \frac{s^2}{v_0^2} v_0 \sin \theta_0 \sin \phi_0 \right) + \dots \\ v(s) &= v_0 \left(1 + \frac{eE}{mv_0^2} \sin \theta_0 \cos \phi_0 s + \frac{eE}{2mv_0^3} \left(\frac{eB}{m} \right) s^2 \sin \theta_0 \sin \phi_0 + \dots \right) \end{aligned}$$

and the probability of collision between s and $s+ds$ is

$$\begin{aligned} g(s)ds &= \frac{ds}{l(s)} \exp \left(- \int_0^s dz/l(z) \right) \approx e^{-s/l_0} ds \left[\frac{1}{l_0} - \frac{1}{l_0} \frac{dl}{dv} \frac{eE}{mv_0} \sin \theta_0 \cos \phi_0 \left(\frac{s}{v_0} - \frac{s^2}{2l_0^2} \right) \right. \\ &\left. - \frac{dl}{dv} \frac{eE}{2mv_0^2} \frac{eB}{m} \sin \theta_0 \sin \phi_0 \left(\frac{s^2}{l_0^2} - \frac{s^3}{3l_0^3} \right) \right] \end{aligned}$$

We find, at last, after long but straightforward computation

$$\begin{aligned} \langle \delta x \rangle_{\theta, \phi, \text{coll}} &= \frac{2}{3} \frac{eE}{m} \frac{l^2}{v^2} + \frac{1}{3} \frac{eE}{m} \frac{l}{v} \frac{dl}{dv} \\ \langle \delta y \rangle_{\theta, \phi, \text{coll}} &= -\frac{1}{3} \frac{eE}{m} \frac{eB}{m} \frac{l^3}{v^3} - \frac{2}{3} \frac{eE}{m} \frac{eB}{m} \frac{l^2}{v^2} \frac{dl}{dv} \end{aligned}$$

So we obtain, after the usual average over energy distribution

$$w = \langle v_x \rangle = \frac{2}{3} \frac{eE}{m} \langle \frac{l}{v} \rangle + \frac{1}{3} \frac{eE}{m} \langle \frac{dl}{dv} \rangle \quad 2.3(1)$$

$$w_{\perp} = \langle v_y \rangle = -\left(\frac{1}{3} \frac{eE}{m} \frac{eB}{m} \langle \frac{l^2}{v^2} \rangle + \frac{2}{3} \frac{eE}{m} \frac{eB}{m} \langle \frac{l}{v} \frac{dl}{dv} \rangle \right) \quad 2.3(2)$$

D keeps the same expressions, as might be shown,

$$\langle l v \rangle / 3$$

Remarks:

(1) We have everywhere expanded sinuses and cosines to the first order in $(eB/m)t$. That is not correct if the collision time is long; in this case the particle would turn over an appreciable angle and our approximation would be wrong. We would have, in this case, to integrate expressions like the following ones

$$\int \cos\left(\frac{\omega s}{v}\right) e^{-s/l} \frac{ds}{l} = \frac{1}{1 + \omega^2 l^2 / v^2} \neq 1$$

$$\int \sin\left(\frac{\omega s}{v}\right) e^{-s/l} \frac{ds}{l} = \frac{\omega l / v}{1 + \omega^2 l^2 / v^2} \neq \omega l / v$$

where $\omega = eB/m$. We then have to divide the expressions we got before for w , w_M , D_T . (The diffusion coefficient transverse to the magnetic field)

$$1 + \omega^2 \frac{l^2}{v^2} = 1 + \frac{e^2 B^2 l^2}{2m^2 v^2}$$

An order of magnitude of the connection is obtained by noticing that

$$\frac{l}{v} \approx \frac{3}{2} \frac{wm}{eE}$$

Therefore, the connection factor is roughly

$$\frac{1}{1 + \frac{3}{2} \frac{B^2}{E^2} w^2}$$

For electric fields greater than 10 kV/cm and magnetic fields smaller than 2 tesla, these corrections are usually small (for $w < 10^7$ cm/s). We will treat this question quantitatively in Sections 3.4.2 and 4.3.

(2) In the general case where the differential cross section is not isotropic, l should be the momentum transfer mean free path.

2.4 Energy Conservation: A Simplified Theory

In order to compute the various transport coefficients one has to know the energy distribution of electrons. That will be done in Section 3. We present here, however, a very

simple argument based on the energy conservation that allows a rough determination of these coefficients and a qualitative discussion. An electron, continuously accelerated by a constant field E , reaches very quickly (in $\sim 10^{-11}$ sec) a stable drift velocity, exactly as any body accelerated in a viscous medium does. If we call $\Lambda(\epsilon)$ the mean fractional energy loss in a collision at energy ϵ , the equilibrium between the energy gained from the field acceleration and the energy lost in atomic collisions can be written

$$eEw = \langle \Lambda \epsilon v / l \rangle$$

If there are excitation collisions for which the mean free path is $l_h(\epsilon)$ and the excitation energy is ϵ_h

$$eEw = \langle \Lambda \epsilon \frac{v}{l} \rangle + \sum_h \epsilon_h \langle \frac{v}{l_h(\epsilon)} \rangle \quad 2.4(1)$$

This is equivalent* to the hypothesis of a frictional force

$$\vec{F} = -\gamma \vec{v}$$

with

$$eEw = \langle -\vec{F} \cdot \vec{v} \rangle, \quad \gamma = \frac{1}{2} \frac{\Lambda m v}{l}$$

and

$$w_{||} = \left\langle \frac{(4/3) eE \Lambda / \gamma}{1 + (1/4) l^2 B^2 (\Lambda / \gamma)^2} \right\rangle$$

$$w_{\perp} = \left\langle \frac{(1/12) l^2 B E (\Lambda^2 / \gamma^2)}{1 + (1/4) l^2 B^2 (\Lambda / \gamma)^2} \right\rangle.$$

2.4.1 Constant Cross Section

In order to make a rough estimation of transport coefficients, let us assume that the momentum transfer mean free path is independent of v and that the distribution in energy is narrow (this is not too true, in general). Then

$$eEw = \frac{1}{2} \Lambda m \frac{\langle v^3 \rangle}{l}$$

*cf F. Bourgeois, J. P. Dufey¹⁹; they adopt a completely classical approach (without considering the random motion). From $\vec{F} = -k\vec{v}$ they get

$$w_{||} = \frac{eE}{k} \frac{1}{1 + \frac{l^2 B^2}{k^2}} \quad w_{\perp} = \frac{l^2 B E}{k^2 (1 + \frac{l^2 B^2}{k^2})}$$

which does not differ too much from our expression, if $k = 3 \frac{\gamma}{\Lambda}$.

and

$$w \approx \sqrt{\frac{2}{3}} \sqrt{\frac{\Lambda}{3}} \frac{eE\ell}{m}$$

$$v \approx \sqrt{2} \sqrt{\frac{1}{3\Lambda}} \frac{eE\ell}{m}$$

$$\langle \epsilon \rangle \approx \sqrt{\frac{1}{3\Lambda}} eE\ell$$

$$D \approx \sqrt{\frac{2}{9}} \sqrt{\frac{1}{3\Lambda}} \frac{eE\ell}{m}$$

2. 4(2)

$$\epsilon_k \approx \frac{eD}{\mu} = \langle \epsilon \rangle$$

$$w_{\perp} \approx \frac{1}{2\sqrt{3}} \frac{eB}{m} \sqrt{\Lambda} \ell$$

$$w_M \approx \frac{1}{2} \frac{eE}{m} \frac{\ell}{v} = \frac{3}{4} w$$

We see that the shorter the mean free path is the lower are the drift velocity, the mean thermal velocity, and the diffusion coefficient. On the other hand with a great mean fractional energy loss per collision we get a high drift velocity, a small thermal velocity and diffusion.

2. 4. 2 Application to Pure Argon

As an example of the problems involved in the application of the foregoing simplified theory to real life cases, let us take the example of argon. Figure 9 gives the experimental elastic cross section.

Fig. 9. Momentum transfer cross sections for electrons in argon. Notice the Ramsauer dip at 0.3 eV of electron energy. The dotted line shows the fit we have made for practical computer application.

Let us first note that in our approximation of narrow energy distribution (from 2.4(5)):

$$\begin{aligned} w &\approx \frac{2}{3} \frac{eE}{m} \frac{f}{v} + \frac{1}{3} \frac{eE}{m} \frac{df}{dv} = \\ &= \frac{1}{3} v eE \frac{f}{\epsilon} + \frac{df}{d\epsilon} \end{aligned}$$

which is zero for a cross section rising linearly with energy. This happens in argon from 1 to 10 eV and in that range the approximate equation 2.4(1) in ϵ

$$\frac{1}{3} v(eE)^2 \left[\frac{f(\epsilon)}{\epsilon} + \frac{df(\epsilon)}{d\epsilon} \right] = \Lambda \frac{\epsilon v}{f(\epsilon)} + \sum_h \frac{\epsilon_h v}{f_h(\epsilon)}$$

has no solution. For practical application we are therefore obliged to use an even cruder approximation wherein we neglect df/dv

$$w = \frac{1}{3} v eE f(\epsilon)/\epsilon$$

$$eEw = \frac{\Lambda \epsilon v}{f(\epsilon)} + \epsilon_h \frac{v}{f_h(\epsilon)}$$

The result of this crude model for argon is given in Fig. 10. The general behavior of ϵ_k and w is reproduced only qualitatively, and $w_M \approx 3/4 w$ in contradiction with experiment. This shows the necessity of taking into account the distribution of energy, as will be done in the following section.

Fig. 10. Results of the crude model of Section 2.4.2 applied to argon compared to experimental results.

2.4.3 Non-uniform Electron Density

When the electron density is non-uniform, because of equation 1.1(2) there are diffusion currents which contribute to the energy balance. For an electric field along Ox , equation 2.4(1) becomes (omitting for simplicity inelastic collisions)

$$eE w - eE D_L \frac{1}{\rho} \frac{\partial \rho}{\partial x} = \langle \Lambda e \frac{v}{l} \rangle$$

where D_L is the diffusion along the Ox axis. Parker and Lowke²¹ have shown that in that case D_L is different from the diffusion coefficient D along the two other axes.*

What is happening is in fact easy to understand: consider an electron pulse in an electric field in a gas with constant cross section. In the leading edge $\partial \rho / \partial x < 0$ and because

$$eE \left(\frac{2}{3} eE \frac{l}{v} \right) + \left| eE D_L \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right| = \Lambda e \frac{v}{l}$$

ϵ has to increase and w to decrease. In the trailing edge, the opposite happens and we have therefore a "bunching" effect, decreasing the value of the longitudinal diffusion coefficient. Obviously, depending on the behavior of the cross section, the effect may be opposite.

*We thank D. Nygren for having brought this fact to our attention.

SECTION 3

CLASSICAL THEORY OF ELECTRONS IN GASES

The above approximations, if they give an intuitive picture of the phenomena, are too rough to describe quantitatively the experimental behavior of transport coefficients. The history of the various attempts for a rigorous theory is very intricate and we refer the reader to Loeb¹³ for an historical account. The real breakthrough was made by Morse, Allis and Lamar⁸ in 1935 and their theory was improved by Mazgenau¹⁰, Holstein⁹ and Phelps.¹² This classical theory is based on the Boltzmann transport equation which expresses the conservation of the number of electrons in the absence of ionization.

3.1 Boltzmann Equation

One introduces a distribution function $f(\vec{v}, \vec{r}, t)$ such that $f(\vec{v}, \vec{r}, t) d\vec{v} d\vec{r}$ gives the number of electrons in the element of six-dimensional volume around the point $\vec{r}, \vec{v}$ in phase space. We take the x axis along the electric field and consider stationary states. $f(\vec{v}, \vec{r}, t)$ is then only a function of $x, v, v_x = v \cos \theta$ for obvious symmetry reasons and we will consider the density of probability $v^2 f(x, v, \cos \theta) dx dv d\cos \theta d\phi$. Since $dx/dt = v_x$, $dv/dt = eE \cos \theta/m$, $d\cos \theta/dt = eE \sin^2 \theta/mv$, the Boltzmann transport equation may be written

$$\frac{\partial f}{\partial t} = v_x \frac{\partial f}{\partial x} + \frac{eE}{m} \cos \theta \frac{\partial f}{\partial v} + \frac{eE}{mv} \sin^2 \theta \frac{\partial f}{\partial \cos \theta} - \left(\frac{\partial f}{\partial t} \right)_{\text{coll.}} = 0 \quad 3.1(1)$$

where $\partial f / \partial t)_{\text{coll.}}$ is the difference between the gain and the loss of electrons because of collisions. We will assume that only two kinds of collisions are important:

a) Elastic collisions. If $d\sigma/d\Omega = q_{ef}(\psi, v)$ is the cross section for elastic diffusion at an angle ψ of an electron of velocity v in our gas, the difference between the gain and loss per unit time of electrons in the six-volume element $v^2 dv d\Omega$ is

$$\begin{aligned} \partial f / \partial t)_{\text{elastic}} &= N \int_{\Omega'} \left[f(\theta', \phi', v', x) \left(\frac{v'}{v} \right)^3 v' q_{ef}(\psi', v') - f(\theta, \phi, v, x) v q_{ef}(\psi, v) \right] d\Omega' = \\ &= N v \int_{\Omega'} \left[f(\theta', \phi', v', x) \left(\frac{v'}{v} \right)^4 q_{ef}(\psi', v') - f(\theta, \phi, v, x) q_{ef}(\psi, v) \right] d\Omega' \end{aligned}$$

where $f(x, \vec{v})$ has become, by simple change of variables $f(v, \theta, \phi, x)$, ψ is the angle between the direction θ', ϕ' before the impact and the direction θ, ϕ after the impact, v' is the velocity before the impact, N is the number of diffusion centers per unit volume, and $d\Omega'$ is the solid angle of diffusion (Fig. 8d). Introducing the azimuthal angle ν in the axes $\vec{v} \times (\vec{Ox} \times \vec{v}')$, $(\vec{Ox} \times \vec{v}')$, $\vec{v}'$

$$\begin{aligned} d\Omega' &= d\cos \psi d\nu \\ \cos \theta &= \cos \theta' \cos \psi + \sin \theta' \sin \psi \cos \nu \end{aligned} \quad 3.1(2)$$

and

$$\begin{aligned} v &= v' \left[1 - \frac{m}{M} (1 - \cos \psi) \right] \\ \epsilon &= \epsilon' \left[1 - \frac{2m}{M} (1 - \cos \psi) \right] \end{aligned}$$

Note that the mean fractional energy loss is $\Lambda = \frac{2m}{M}$. The term $(v'/v)^3$ in the first equation is the Jacobian $v'^2 dv'/v^2 dv$. Because v' is very close to v , one can approximate the functions of v' by a Taylor expansion around v , obtaining

$$\begin{aligned} \left. \frac{\partial f}{\partial t} \right|_{\text{elastic}} &= Nv \int_{\Omega'} [f(\theta', \phi', v, x) - f(\theta, \phi, v, x)] q_{el}(\psi, v) d\Omega' \\ &+ \frac{Nv}{v^3} \frac{m}{M} \frac{\partial}{\partial v} \int_{\Omega'} (1 - \cos \theta) q_{el}(\psi, v) v^4 f(\theta', \phi', v, x) d\Omega' \end{aligned} \quad 3.1(3)$$

We will assume here that the electron temperature, that is, their energy, is high enough to allow to discard the energy transferred to the electrons by the molecules of the gas (see Section 3.4.1).

b) Excitativ collisions. Their contribution is

$$\begin{aligned} \left. \frac{\partial f}{\partial t} \right|_{\text{excitation}} &= \sum_h N \int_{\Omega'} [f(\theta, \phi', v', x) \left(\frac{v'}{v}\right) v' q_h(\psi', v') - f(\theta, \phi, v, x) v q_h(\psi, v)] d\Omega' = \\ &= \sum_h N \int_{\Omega'} [f(\theta', \phi', v', x) \left(\frac{v'}{v}\right)^2 q_h(\psi', v') - f(\theta, \phi, v, x) q_h(\psi, v)] d\Omega' \end{aligned} \quad 3.1(4)$$

where $1/2mv'^2 - 1/2mv^2 = 1/2mv_h^2 = \epsilon_h$ is the excitation energy of the h -th level and $q_h(\psi, v)$ the excitation cross section for that level. The Jacobian is now v'/v since $dv'/dv = v/v'$. Because of the large values of ϵ_h no Taylor expansion is now possible. Here again if the temperature of the electrons is high enough one can neglect the collisions of second kind where excited molecules give back their excitation energy to the electrons.

We neglect for the time being ionization collisions, which we will treat in Section 5. Let us assume also that the distribution of electrons is uniform. It will be independent of x .

In order to solve this Boltzmann equation we expand $f(v, \psi_x) = f(v, \cos \theta)$ in Legendre polynomials. It can be shown a posteriori that the first two terms are usually sufficient and we write

$$f(v, \cos \theta) = f_0(v) + \cos \theta f_1(v)$$

Taking the averages $\int \frac{d \cos \theta}{2}$ and $\int \frac{\cos \theta}{2} d \cos \theta$ we get the two equations

$$-\left(\frac{a}{3v}\right) \frac{\partial}{\partial v} (v^2 f_1) + v^{-2} \left(\frac{m}{M}\right) \frac{\partial}{\partial v} (v^4 \frac{f_0}{\ell_e}) + \sum_h \left[\left(\frac{v'}{v}\right) v' \ell_0(v') / \ell_h(v') - v \ell_0(v) / \ell_h(v) \right] = 0$$

and

$$-a \frac{\partial f_0}{\partial v} - v f_1 / \ell_e = 0 \quad 3.1(5)$$

$$(I_e)^{-1} = 2\pi N \int_0^\pi (1 - \cos\psi) \sin\psi q_{e\psi}(\psi, v) d\psi$$

$$(I_h)^{-1} = 2\pi N \int_0^\pi \sin\psi q_h(\psi, v) d\psi$$

$$a = eE/m$$

We have followed here the standard practice to neglect in the second equation the terms

$$\frac{1}{3} \frac{m}{M} \frac{\partial}{\partial v} (v^4 f_1/I_e)$$

$$\Sigma_h \int \cos\theta \cos\theta' f_1(v')(v'/v) q_h(\psi', v') d\cos\theta d\cos\theta' dv - \frac{1}{3} \Sigma_h \int f_1(v) q_h(\psi', v') d\cos\psi dv$$

which are usually small (the first one because of the factor m/M , the second one since for moderate energies $q_h \ll q_{e\psi}$). If one works with the function $F(\epsilon, \cos\theta) = F_0(\epsilon) + F_1(\epsilon)\cos\theta$ such that

$$\int_0^{\epsilon \max} F_0(\epsilon) d\epsilon = 1$$

the foregoing equations become

$$\frac{eE}{3} \frac{\partial}{\partial \epsilon} (v F_1) - \frac{2m}{M} \frac{\partial}{\partial \epsilon} \left(\frac{e v F_0}{I_e} \right) - \Sigma_h \left[\frac{\sqrt{2/m} \sqrt{\epsilon + \epsilon_h}}{I_h(\epsilon + \epsilon_h)} F_0(\epsilon + \epsilon_h) - \frac{\sqrt{2\epsilon/m}}{I_h(\epsilon)} F_0(\epsilon) \right] = 0 \quad 3.1(6)$$

$$eE \frac{\partial}{\partial \epsilon} (v F_0) - \frac{2eE}{mv} F_0 + \frac{v F_1}{I_e} = 0 \quad 3.1(7)$$

or by eliminating F_1 ,

$$\begin{aligned} \frac{2}{3} \frac{(eE)^2}{m} \frac{\partial}{\partial \epsilon} \left[\epsilon I_e(\epsilon) \frac{\partial(F_0/v)}{\partial \epsilon} \right] + \frac{2m}{M} \frac{\partial}{\partial \epsilon} \left(\frac{e v F_0}{I_e(\epsilon)} \right) + \\ + \Sigma_h \left[\frac{\sqrt{2/m} \sqrt{\epsilon + \epsilon_h}}{I_h(\epsilon + \epsilon_h)} F_0(\epsilon + \epsilon_h) - \frac{\sqrt{2\epsilon/m}}{I_h(\epsilon)} F_0(\epsilon) \right] = 0 \end{aligned} \quad 3.1(8)$$

The various transport coefficients are

$$\begin{aligned} \eta_{||} &= \int v \cos\theta F(v, \cos\theta) d\cos\theta \frac{d\cos\theta}{2} = \int \frac{v F_1}{3} d\epsilon = -\frac{2}{3} \frac{eE}{m} \int \epsilon I_e \frac{\partial(F_0/v)}{\partial \epsilon} d\epsilon \\ &= \frac{2}{3} \frac{eE}{m} \int \frac{I_e}{v} F_0(\epsilon) d\epsilon + \frac{2}{3} \frac{eE}{m} \int \frac{\epsilon}{v} \frac{\partial I_e}{\partial \epsilon} v_0 d\epsilon = \frac{2}{3} \frac{eE}{m} \langle I_e/v \rangle + \frac{eE}{3m} \langle \frac{dI_e}{dv} \rangle, \end{aligned} \quad 3.1(9)$$

which is exactly the expression obtained in Section 2 (Eq. 2.1(5)). The diffusion coefficient D is from its definition

$$D = \int \frac{I_e v}{3} F_0(\epsilon) d\epsilon \quad 3.1(10)$$

One can show (see Section 3.4.2) that, if the magnetic field is small enough

$$\begin{aligned} w_{\perp} &= \int v \sin \theta F(v, \cos \theta) d\epsilon \frac{d \cos \theta}{2} = \frac{eB}{3m} \int l F_1 d\epsilon = \\ &= \frac{1}{3} \frac{eE}{m} \frac{eB}{m} \int \frac{l^2}{v} F_0(\epsilon) d\epsilon + \frac{2}{3} \frac{eE}{m} \frac{eB}{m} \int \frac{l}{v} \frac{dl}{dv} F_0(\epsilon) d\epsilon, \end{aligned} \quad 3.1(11)$$

which is again exactly the same result as in Section 2 (apart from the minus sign due to our axis convention).

The conservation of energy written in the same section has, of course, its exact analog. Performing on Eq. 3.1(8) the integration $\int \epsilon d\epsilon$, we get

$$eE \left[-\frac{2}{3} \frac{eE}{m} \int \epsilon l e(\epsilon) \frac{\partial(F_0/v)}{\partial \epsilon} d\epsilon \right] = \int \frac{2m}{M} \epsilon \frac{v F_0(\epsilon)}{I_e} d\epsilon + \sum_h \epsilon_h \int \frac{v}{I_h(\epsilon)} F_0(\epsilon) d\epsilon \quad 3.1(12)$$

or, $eEw =$ elastic losses per unit time + inelastic losses per unit time, which is Eq. 2.4(1).

Remarks:

We have assumed that f is independent of x . This is not right for an isolated pulse of electrons as we have seen in Section 2.4.3, since the distribution of velocity depends on the position in the pulse.

Relatively recently (1967) J. H. Parker and S. J. Lowke²¹ have developed a method of solution of the complete Boltzmann equation where the $\partial f / \partial x$ terms are not rejected. The result is quite complicated and we will not describe it here.

3.2 Druyvestein Distribution

If $I_e(\epsilon)$ is constant and there is no excitation, our Eq. 3.1(8) becomes

$$\frac{2}{3} \frac{(eE)^2}{m} \frac{\partial}{\partial \epsilon} \left(\epsilon l e \frac{\partial(F/v)}{\partial \epsilon} \right) + \frac{2m}{M} \frac{\partial}{\partial \epsilon} \left(\frac{\epsilon v F}{I_e} \right) = 0 \quad 3.2(1)$$

By integrating on $d\epsilon$, we get

$$\frac{2}{3m} \frac{(eE)^2}{\epsilon} \epsilon l e \frac{\partial(F/v)}{\partial \epsilon} + \frac{2m}{M} \frac{v\epsilon}{I_e} F = C$$

where C is a constant. From that we get easily, because of the constancy of the I_e

$$\frac{F(\epsilon)}{v} = k \exp \left[-\epsilon^2 / \frac{2\lambda}{3} (eEl_e)^2 \right] \left[1 + \int_0^\epsilon \frac{3mC}{2(I_e)^2 \epsilon l_e} \exp \left(\epsilon^2 / \frac{M}{3m} (eEl_e)^2 \right) d\epsilon \right]$$

The second term gives a non-convergent contribution ($\int F(\epsilon) d\epsilon = \infty$); so $C=0$ and, finally

$$F(\epsilon)d\epsilon = \frac{2\sqrt{\epsilon}}{\Gamma(3/4) \left[\sqrt{\frac{2}{3\Lambda}} eE l e \right]^{3/2}} \exp\left(\frac{-3\epsilon^2}{2\Lambda(eE l e)^2}\right) \quad 3.2(2)$$

where $\Lambda = 2m/M$ is the usual fraction of energy loss per collision. We can see that this distribution, found by Druyvestein⁷ in 1930, decreases faster at high energies than a Maxwellian one

$$F(\epsilon)d\epsilon = \frac{2}{\sqrt{\pi}} \frac{1}{kT} \sqrt{\frac{\epsilon}{kT}} e^{-\epsilon/kT} d\epsilon \quad 3.2(3)$$

With Druyvestein's distribution we can compute the transport parameters.

$$\begin{aligned} w &= \frac{2}{3} \frac{eE}{m} \left\langle \frac{l}{v} \right\rangle - \frac{1}{3} \frac{eE}{m} \left\langle \frac{dl}{dv} \right\rangle = \frac{1}{3} \frac{eE}{m} \int_0^\infty \left[\frac{2 l(\epsilon)}{v} - \frac{dl}{dv} \right] F(\epsilon) d\epsilon = \\ &= \frac{1}{3} \left(\frac{2}{m} \right)^{1/2} eE \int_0^\infty \epsilon^{-1/2} F(\epsilon) \left(l + \epsilon \frac{dl}{d\epsilon} \right) d\epsilon = \frac{2}{3} \frac{eE}{m} \int \epsilon l e \frac{\partial(F/v)}{\partial \epsilon} \\ &= \frac{2m}{eEM} \int \frac{v \epsilon F(\epsilon)}{l e} d\epsilon \end{aligned}$$

from where we can get (because $\int_0^\infty t^{\alpha-1} e^{-t} dt = \Gamma(\alpha)$)

$$w = \frac{2}{\Gamma(3/4)} \sqrt{\frac{\sqrt{2}}{3\sqrt{3}}} \sqrt{\Lambda} \frac{eE l e}{m} \quad 3.2(4)$$

Similarly we get

$$\begin{aligned} \langle \epsilon \rangle &= \int_0^\infty \epsilon F(\epsilon) d\epsilon = \frac{\Gamma(5/4)}{\Gamma(3/4)} \sqrt{2/3\Lambda} eE l e \\ D &= \sqrt{\frac{2}{m}} \frac{1}{3} \int_0^\infty \sqrt{\epsilon} l e(\epsilon) F(\epsilon) d\epsilon = \frac{1}{3} \frac{\sqrt{2}}{\Gamma(3/4)} \sqrt{\frac{2}{3\Lambda}} \frac{eE l^3}{m} \\ w_1 &= \frac{2}{3} \frac{eB}{m} eE \int_0^\infty \frac{l^2}{\epsilon} F(\epsilon) d\epsilon = \frac{eB}{m} \frac{\Gamma(5/4)}{\Gamma(3/4)} \sqrt{\frac{2\Lambda}{3}} l e \\ w_M &= \frac{w_1}{w} \frac{E}{B} = \frac{\Gamma(5/4) \Gamma(3/4)}{[\Gamma(3/2)]^2} = 1.067 w \quad 3.2(5) \end{aligned}$$

These expressions for the transport coefficients do not differ but for numerical factors of the order of 1 from those we got in Section 2.

3.3 General Case

We have already seen that in real life (Fig. 9 for argon) that cross sections are not constant. The solution of (Eq. 3.2(1)) is then

$$F(\epsilon) = C\sqrt{\epsilon} \exp\left(\int_0^\epsilon \frac{3\Lambda\epsilon}{[eE l_e(\epsilon)]^2} d\epsilon\right) \quad 3.3(1)$$

where $\Lambda = \frac{2m}{M}$ and C is fixed by $\int F d\epsilon = 1$.

However, we have neglected the excitation in the above equation. Its inclusion transforms the differential equation in a difference -- differential equation:

$$\begin{aligned} & \frac{\partial}{\partial \epsilon} \left[\frac{2}{3} \left(\frac{eE}{m} \right)^2 \epsilon l_e(\epsilon) \frac{\partial(F_0/v)}{\partial \epsilon} \right] + \frac{2m}{M} \frac{\partial}{\partial \epsilon} \left(\frac{\epsilon v F_0}{l_e(\epsilon)} \right) + \\ & + \sum_h^n \left[\frac{\sqrt{2/m} \sqrt{\epsilon + \epsilon_h}}{l_h(\epsilon + \epsilon_h)} F_0(\epsilon + \epsilon_h) - \frac{\sqrt{2/m} \sqrt{\epsilon}}{l_h(\epsilon)} F_0(\epsilon) \right] = 0 \end{aligned} \quad 3.3(2)$$

3.3.4 Approximate Methods

Most of the authors (Refs. 11, 22 for argon) have tried to get rid of the third term and to come back to an ordinary differential equation. Let us consider two extreme cases:

a) ϵ_h Large

In noble gas the excitation is large and one may safely assume that

$$\frac{\sqrt{2/m} \sqrt{\epsilon + \epsilon_h}}{l_h(\epsilon + \epsilon_h)} F_0(\epsilon + \epsilon_h) \ll \frac{\sqrt{2/m} \sqrt{\epsilon}}{l_h(\epsilon)} F_0(\epsilon)$$

due to the fact that

$$F(\epsilon + \epsilon_h) \ll F(\epsilon) ,$$

(except maybe in the neighborhood of $\epsilon=0$). We are then left with

$$\frac{2}{3} \left(\frac{eE}{m} \right)^2 \frac{\partial}{\partial \epsilon} \left[\epsilon l_e(\epsilon) \frac{\partial(F_0/v)}{\partial \epsilon} \right] + \frac{2m}{M} \frac{\partial}{\partial \epsilon} \left(\frac{\epsilon v F_0}{l_e(\epsilon)} \right) - \left(\sum_h \frac{1}{l_h} \right) \sqrt{\frac{2\epsilon}{m}} F_0(\epsilon) = 0 \quad 3.3(3)$$

Golant¹¹ has used this method very cleverly with approximations to l_e and l_h especially designed to allow analytical integration, and his result is not far from those obtained by Engerhardt and Phelps²³ or the present authors with rigorous methods.

b) ϵ_h Small

In the organic vapor, on the other hand, ϵ_h is usually small and one may replace the two last terms by

$$\sum_h \epsilon_h \frac{\partial}{\partial \epsilon} \left[\frac{v F(\epsilon)}{l_h(\epsilon)} \right] .$$

Inserting in the equation this gives

$$\frac{2}{3} \left(\frac{eE}{m} \right)^2 \epsilon l_e(\epsilon) \frac{\partial(F_0/v)}{\partial \epsilon} + \frac{\Lambda(\epsilon)}{l(\epsilon)} v F_0 = 0 \quad 3.3(4)$$

where we have replaced

$$\frac{2(m/M)\epsilon}{l_e} + \sum_n \epsilon h l_h^{-1} \quad \text{by} \quad \Lambda(\epsilon)/l(\epsilon)$$

where $l(\epsilon) = \left(\frac{1}{l_e} + \frac{1}{l_h} \right)^{-1}$ is the mean free path and $\Lambda(\epsilon)$ is the mean energy loss by collision.

We may note that the presence in the first term of l_e instead of l comes from the fact that we have neglected the angular dependence of excitation collision. If these are important they have to be taken into account and we would obtain approximately

$$\frac{2}{3} \left(\frac{eE}{m} \right)^2 \epsilon l \frac{\partial(F_0/v)}{\partial \epsilon} + \frac{\Lambda(\epsilon)}{l(\epsilon)} v F_0 = 0$$

The solution 3.3(1) is then replace

$$F(\epsilon) = C\sqrt{\epsilon} \exp \left(- \int_0^\epsilon \frac{3\Lambda(\epsilon') \epsilon' d\epsilon'}{[eE l(\epsilon')]^2} \right) \quad 3.3(5)$$

3.3.2 Rigorous Numerical Methods

These approximations were useful before the availability of large computers. Nowadays it is possible to solve Eq. 3.3(2) directly. The method follows directly from the observation first made by Sherman²⁴ that if you know F between ϵ and $\epsilon + \epsilon_h$ (ϵ_h being the last important excitation energy) and $F'(\epsilon)$, you can solve the equation between $\epsilon - \epsilon_h$ and ϵ ; it is just an ordinary differential equation. Sherman has developed such a method of backward prolongation of the elastic solution valid at infinity. Phelps and co-workers^{12,23} have used this method extensively for a large number of gases and mixtures.

The present authors have developed a similar method; instead of starting from equation 3.3.(2) we have started from a system of equations easily deduced from Eqs. 3.1(6) and 3.1(7):

$$\begin{aligned} \frac{\partial F_0/v}{\partial \epsilon} &= - \frac{F_1}{eE l_e v} \\ \frac{\partial F_1(\epsilon)}{\partial \epsilon} &= \frac{3}{eE} \left[\epsilon \frac{\partial}{\partial \epsilon} \frac{\Lambda}{l_e} + \frac{2\Lambda}{l_e} \right] F_0(\epsilon) + \sum_h \frac{3}{eE} \left[\frac{\sqrt{\epsilon + \epsilon_h}}{\sqrt{\epsilon}} \frac{F_0(\epsilon + \epsilon_h)}{l_h(\epsilon + \epsilon_h)} - \frac{F_0(\epsilon)}{l_h(\epsilon)} \right] - \\ &\quad - \left[\frac{1}{2\epsilon} + \frac{3\Lambda\epsilon}{(eE l_e)^2} \right] F_1(\epsilon) \end{aligned} \quad 3.3(6)$$

where $v = \sqrt{2\epsilon/m}$ and $\Lambda = 2m/M$. We take as starting value, a reasonably high energy ϵ_0 (about $4 \epsilon_k$), choose $F_0(\epsilon_0) = 0$, and $F_1(\epsilon_0) = 1$, since we expect the asymmetry to be large at high ϵ (in fact it is infinite in our approximation) and solve backward the system (with a modified version of the Lawrence Berkeley Laboratory library routine DIFSUB).^{25*}

* Our program is available upon request, from B. Sadoulet, Lawrence Berkeley Laboratory, Berkeley, California 94720 USA.

3.3.3 Application to Argon

On Fig. 9 we have plotted (dotted line) the momentum transfer cross section we have used:

$$\begin{array}{ll}
 \epsilon > 11.5 \text{ eV} & \sigma = 1.52 \times 10^{-15} \sqrt{\epsilon/11.5} \text{ cm}^2 \\
 1.15 < \epsilon < 11.5 & \sigma = 1.52 \times 10^{-15} \epsilon/11.5 \text{ cm}^2 \\
 .3 < \epsilon < 1.15 & \sigma = 1.46 \times 10^{-17} + 1.9 \cdot 10^{-16} (\epsilon - .3)^2 \text{ cm}^2 \\
 \epsilon < .3 & \sigma = 1.46 \times 10^{-17} + 1.24 \cdot 10^{-12} (.3 - \epsilon)^{6.5} \text{ cm}^2
 \end{array}$$

The excitation cross section has been taken as

$$\sigma = 9 \times 10^{-17} \sqrt{\epsilon/11.5} (\epsilon/11.5 - 1) \text{ cm}^2$$

and

$$\epsilon_h = 11.5 \text{ eV}$$

The resulting transport coefficient are plotted on Fig. 11. The agreement with the data is excellent over five orders of magnitude of E, a result already obtained by Golant¹¹ for the high energy region and by Engferardt and Phelps²³ in a more detailed analysis. The cross sections used by these authors are essentially identical to ours.

A remark may be useful for application of such calculation to drift for chambers operating in a magnetic field: The "magnetic velocity" w_M is very sensitive to the low energy part of the spectrum and therefore to the hypothesis made for the losses in excitation collision; the dashed curve in Fig. 11 shows the behavior of w_M when instead of assuming an energy loss equal to ϵ_h , we assume a total energy loss. In that context the deviation observed in our calculation (and the one of Engferardt and Phelps) between theory and experiment is not too worrying.

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Fig. 11. Results of rigorous approach to the argon transport coefficients compared to experimental results.

Another warning may be given. Figure 12 gives the distribution function and the asymmetry as functions of ϵ for various fields. At high fields (where we have taken into account the ionization)(see Section 5) the asymmetry is very large and our approximation of $F(\epsilon, \cos\theta)$ by first two terms of the Legendre expansion

$$F_0(\epsilon) + \cos\theta F_1(\epsilon)$$

breaks down.

The method described in the last section can easily be extended to a mixture of two gases 1 and 2 in proportion δ and $1-\delta$. It is only necessary to write

$$\begin{aligned} \frac{1}{I} &= \delta \frac{1}{I_1} + (1-\delta) \frac{1}{I_2} \\ \frac{\Lambda}{I} &= \delta \frac{\Lambda_1}{I_1} + (1-\delta) \frac{\Lambda_2}{I_2} \end{aligned} \quad 3.3(7)$$

Fig. 12. (a) Distribution function $F_0(\epsilon)$ for argon at different electric fields.

(b) Asymmetry, that is $F_1(\epsilon)/F_0(\epsilon)$ for argon in different electric fields. It is very large at very high fields where the approximation used in the computation (only two terms in the Legendre expansion) breaks down.

3.4 Special Cases: Low Electron Temperature or High Magnetic Fields

The above theory neglects several effects: the movement of gas molecules, the action of high magnetic fields and ionization. This last effect at high electric field is important and we will discuss it at length in Section 5. We will review here the modification to Eqs. 3.3(2) or 3.3(6) due to the first two effects.

3.4.1 Movement of Gas Molecules

If the electron temperature is low enough ($\epsilon \sim kT \sim 2.5 \cdot 10^{-2}$ eV at 18°C) we should take into account the movement of gas molecules. Moreover, if these are excitation energies of the order of kT , the thermally excited molecules can give back to electrons their energy (collision of the second kind). These two effects may be taken into account by the addition to Eq. 3.3(2) of the following terms:¹²

$$\frac{2\Lambda}{m} \frac{d}{d\epsilon} \left(\frac{\epsilon^2}{I_e} kT \frac{dF/v}{d\epsilon} \right) + \sum_h \left[\frac{\sqrt{2/m} \sqrt{\epsilon - \epsilon_h}}{I_h(\epsilon - \epsilon_h)} F(\epsilon - \epsilon_h) - \frac{\sqrt{2/m} \sqrt{\epsilon}}{I_h(\epsilon)} \frac{F(\epsilon)}{\epsilon} \right] \quad 3.4(1)$$

where I_h is the mean free path for collision of second kind and is given by detailed balance argument:

$$\frac{(\epsilon - \epsilon_h)}{I_h(\epsilon)} = \exp(-\epsilon_h/kT) \frac{\epsilon}{I_h(\epsilon)}$$

An a posteriori justification for the modification 3.4(1) comes from the fact that the energy loss due to elastic and inelastic collisions goes then to zero for a Maxwell distribution

$$F \sim \sqrt{\epsilon} e^{-\epsilon/kT}$$

In order to see the effect of the temperature let us assume that I_e is constant and that there is no inelastic collision. Equation 3.2(1) becomes now, after one integration

$$\left[\frac{(\epsilon E)^2}{3} I_e + \Lambda \frac{\epsilon}{I_e} kT \right] \frac{\partial F/v}{\partial \epsilon} = -\Lambda \epsilon F / I_e v$$

or writing

$$\epsilon_0^2 = (\epsilon E I_e)^2 / 3\Lambda$$

$$(\epsilon_0^2 + kT\epsilon) \frac{\partial(F/v)}{\partial \epsilon} = \epsilon F/v$$

Therefore

$$F/v = \epsilon \exp \left(- \int_0^\epsilon \frac{\epsilon}{\epsilon_0^2 + kT\epsilon} d\epsilon \right)$$

$$F = C \sqrt{\epsilon} \left(1 + \frac{kT\epsilon}{\epsilon_0^2} \right)^{\epsilon_0^2/(kT)^2} \exp(-\epsilon/kT)$$

which for large kT converges to the Maxwell distribution Eq. 3.2(3). Then from Eqs. 2.1(5) and 2.2(1)

$$W = \int \frac{2}{3} \frac{f}{v} \frac{eE}{m} F d\epsilon = \frac{2\sqrt{2} f e}{3\sqrt{\pi} \sqrt{m} kT} E$$

$$D = \int \frac{fv}{3} F d\epsilon = \frac{2\sqrt{2} f \sqrt{kT}}{3\sqrt{\pi} \sqrt{m}}$$

and

$$\epsilon_k = \frac{eD}{\mu} = kT.$$

It should be noted that ϵ_k is then fixed and equal to kT , and that the mobility is constant. It is for the same reason that at low field, the mobility is constant for ions: Λ is very big and therefore their temperature is close to kT .

It is difficult to incorporate collisions of the second kind in the backward prolongation method we have described in Section 3.3.2. Sherman and Gibson have developed other methods²⁴. In our program, in order to minimize modifications, we have used an approximation based on the following remarks:

a) For low ϵ of the order of kT , F converges towards a Maxwell distribution, and to first order

$$F \propto \sqrt{\epsilon} (1 + a\epsilon) e^{-\epsilon/kT}$$

with

$$a \approx - \frac{1}{kT} \frac{d}{d\epsilon} \left(\frac{F/\sqrt{\epsilon}}{e^{-\epsilon/kT}} \right)$$

b) For complex molecules $\epsilon_h \approx \epsilon$ for $\epsilon \leq kT$, since at that energy the vibrational cross sections are important. Therefore

$$\begin{aligned} \frac{\sqrt{\epsilon - \epsilon_h} F(\epsilon - \epsilon_h)}{f_h(\epsilon)} - \frac{\sqrt{\epsilon} F(\epsilon)}{f_h(\epsilon)} &= \frac{\epsilon}{f_h} e^{-\epsilon/kT} a \epsilon_h \approx \frac{\epsilon^2}{h} e^{-\epsilon/kT} a \approx \\ &\approx \frac{\sqrt{\epsilon}}{f_h(\epsilon)} \left(1 - e^{-\epsilon/kT} \right) \left(F + \sqrt{\epsilon} kT \frac{d(F/\sqrt{\epsilon})}{d\epsilon} \right) \end{aligned}$$

and we have used the last expression, which for large ϵ converges towards

$$\frac{\sqrt{\epsilon}}{f_h(\epsilon)} F(\epsilon)$$

These modifications have even for pure isobutane (see Section 3.5) very small effects and just prevent ϵ_k from decreasing below kT . The crudeness of our approximation has presumably very small effects on the final result.

3.4.2 High Magnetic Fields

We have already mentioned in Section 2 that in a large magnetic field, particles turn over an appreciable angle between collisions and that this increases the apparent density of the medium. The way to do it quantitatively with this phenomena has been given by Allis²⁶.

If the electric field is along Oz and the magnetic field is along Oy, the Boltzmann equation 3.1(1) is replaced by

$$v_x \frac{\partial f}{\partial x} + \frac{dv}{dt} \frac{\partial f}{\partial v} + \frac{d \cos \theta}{dt} \frac{\partial f}{\partial \cos \theta} + \frac{d\phi}{dt} \frac{\partial f}{\partial \phi} - \frac{\partial f}{\partial t} \Big|_{\text{coll.}} = 0$$

where θ and ϕ are the usual polar and azimuthal angles and

$$\begin{aligned} \frac{dv}{dt} &= \frac{eE}{m} \cos \theta \\ \frac{d \cos \theta}{dt} &= \frac{eE}{mv} \sin^2 \theta - \frac{eB}{m} \sin \theta \cos \phi \\ \frac{d\phi}{dt} &= -\frac{eB}{m} \frac{\cos \theta}{\sin \theta} \sin \phi \end{aligned}$$

The angles after a collision are given by Eq. 3.1(2) supplemented by

$$\begin{aligned} \sin \theta \cos \phi &= \cos \psi \sin \theta' \sin \phi' - \sin \psi \cos(\nu - \phi) \\ \sin \theta \sin \phi &= \cos \psi \sin \theta' \sin \phi' + \sin \psi \sin(\nu - \phi) \end{aligned}$$

Proceeding as before, but now expanding $f(v, \cos \theta, \phi)$ as

$$f(v, \cos \theta, \phi) = f_0(v) + \cos \theta f_1(v) + \sin \theta \cos \phi f_2(v) + \sin \theta \sin \phi f_3(v)$$

we get equations similar to 3.1(5):

$$\begin{aligned} & -\frac{eE}{3mv} \frac{\partial v^2 f_1}{\partial v} + \frac{1}{\sqrt{2}} \frac{m}{M} \frac{\partial}{\partial v} (v^4 f_0) + \sum_h \left[\left(\frac{v'}{v} \right) v' f_0(v') / I_h(v') - \right. \\ & \left. - v f_0(v) / I_h(v) \right] = 0 \\ & -\frac{eE}{m} \frac{\partial f_0}{\partial v} + \frac{eB}{m} f_2 - v \frac{f_1}{I_e} = 0 \\ & -\frac{eB}{m} f_1 = \frac{v f_2}{I_e} \\ & f_3 = 0 \end{aligned}$$

Therefore

$$f_2 = -\frac{eB I_e}{mv} f_1$$

from which one deduces transforming to F_0, F_1, F_2, F_3

$$\frac{eE}{3} \frac{\partial}{\partial \epsilon} (v F_1) - \frac{2m}{M} \frac{\partial}{\partial \epsilon} \left(\frac{\epsilon v F_0}{I_e} \right) - \sum_h \left[\frac{\sqrt{2/m} \sqrt{\epsilon + \epsilon_h}}{I_h(\epsilon + \epsilon_h)} F_0(\epsilon + \epsilon_h) - \sqrt{2/m} \frac{\sqrt{\epsilon}}{I_h(\epsilon)} F(\epsilon) \right] = 0$$

$$eE \frac{\partial}{\partial \epsilon} (v F_0) - 2 \frac{eE}{mv} F_0 + \left(1 + \frac{e^2 B^2 I_e^2}{2m\epsilon}\right) \frac{v F_1}{I_e} = 0$$

$$w_H = \int \frac{v}{3} F_1 d\epsilon = -\frac{2}{3} \frac{eE}{m} \int \frac{\epsilon I_e \frac{\partial F_0/v}{\partial \epsilon}}{1 + \frac{e^2 B^2 I_e^2}{2m\epsilon}} d\epsilon$$

$$w_L = \int \frac{v}{3} F_2 d\epsilon = \frac{eB}{3m} \int I_e F_1 d\epsilon$$

$$= -\frac{eB}{3m} eE \int \frac{I_e^2 v \frac{\partial F_0/v}{\partial \epsilon}}{1 + \frac{e^2 B^2 I_e^2}{2m\epsilon}} d\epsilon$$

where we find again the factor $\frac{1}{1 + \frac{e^2 B^2 I_e^2}{2m\epsilon}}$

which we have already encountered in Section 2.3. Similarly, the diffusion transverse to the magnetic field is

$$D_L = \frac{1}{3} \int \frac{I_e v F_0}{1 + \frac{e^2 B^2 I_e^2}{2m\epsilon}} d\epsilon$$

The effect on the transport coefficients for a field of 1 tesla is exemplified for pure argon in Fig. 13. The bumps observed for ϵ_k^* and w_H come from the fact that the diffusion D and w_L are less affected than w_H , which is dramatically reduced.

Fig. 13. Calculated variations of the transport coefficients for pure argon in a magnetic field of 1 Tesla (10 kG).

* Where we have kept the definition $\epsilon_k = eD/k$ where D is the diffusion parallel to the magnetic field.

3.5 Fit to Realistic Gas Mixtures

Unfortunately the gas used in multiwire proportional counters or drift chambers are more complex than pure argon: argon-CO₂, argon-isobutane, argon-methane or pure organic vapor such as methane or ethylene.

The fact that some of these gases are mixtures can be dealt with very easily: the various cross sections are just the appropriate weighted average of the cross sections for each component (Eq. 3.3(7)). On the other hand, the fact that we are using complex molecules introduces important modifications: in addition to electronic excitation, we now have excitation of the rotational and vibrational modes of the molecules. Figure 14 gives the excitation spectrum of CO₂ as deduced by Hake and Phelps²³ and shows two general effects which are very important for understanding the behavior of electrons in complex molecular gases:

- The vibrational excitation cross sections Q_v are very important compared to the momentum transfer elastic cross section Q_m in the energy range 0.1 to 1 eV. In other words, the mean fractional energy loss per collision Λ is large, and from Eq. 2.4(2) the drift velocities will be large and the mean energies low. So low in fact that they are of the order of the thermal energy kT and the movement of gas molecules should be taken into account as shown in Eq. 3.4(1).
- The important vibrational energies are limited to a few tenths of an electron volt. This leads for higher fields to an effective $\Lambda = \epsilon_{\max}/\epsilon$ where $\epsilon_{\max}$ is the highest important vibrational energy. This will generate as we will see, constant or even decreasing drift velocities.

Fig. 14. (a) Excitation spectrum of CO₂ (Q_m is the momentum transfer cross section, Q_v the vibrational cross section, Q_x the electronic cross section, and Q_i the ionization cross section). (b) Total cross sections for electrons with various complex molecules

Of course when speaking of realistic mixtures we encounter the usual problem that we have not enough data on cross sections to predict through the use of the above theory the various transport coefficients. We have therefore attempted to deduce these cross sections from a rough fit to the drift velocities in various mixtures of argon and the gas considered. We have done this for isobutane using data kindly given to us before publication by Charpak and Sauli² and for methane where drift velocity measurements²⁸ exist for the pure gas and one mixture with argon (10% CH₄, 90% Ar). In both cases, we can compare the predicted ϵ_k with either the ϵ_k which can be estimated from the measurement by Charpak, Sauli and Duinker² of the position accuracy in a drift chamber filled with 25% isobutane and 75% argon or the ϵ_k measured by Cochran and Forester²⁹ for pure methane.

3.5.1 Argon - Isobutane Mixtures

The total cross section for isobutane is only known above 1 eV and not very accurately³⁰ (Fig. 14b). The data on drift velocities seems to be incompatible with the simple extrapolation of the measured cross section to zero (over estimation of w at low field). As a first guess we took, therefore, a flat total cross section below 1 eV.^{*} We also took a flat excitation cross section and from infrared spectra³¹ we fixed the upper limit to vibrational energies at 0.36 eV.

We then fitted by hand the data at the lower and the upper concentrations (7% and 38% isobutane) varying only the value of the total cross section below 1 eV and the excitation cross section. Reasonable values seem to be

$$\begin{aligned} \sigma_{\text{tot}} &= \begin{cases} 1.1 \times 10^{-15} \text{ cm}^2 & \text{for } \epsilon \leq 1 \text{ eV} \\ [1.1 \times 10^{-15} (8. - \epsilon) + 4.8 \cdot 10^{-15} (\epsilon - 1)] / 7. \text{ cm}^2 & \text{for } 1 \leq \epsilon \leq 8 \text{ eV} \\ (4.8 \times 10^{-15} / \sqrt{\epsilon/8.}) \text{ cm}^2 & \text{for } \epsilon > 8 \text{ eV} \end{cases} \\ \sigma_{\text{exc}} &= 8 \times 10^{-17} \text{ cm}^2 \quad \text{for } \epsilon_h \leq 0.36 \text{ eV} \end{aligned}$$

Figures 15 and 16 show our results for w , w_M and ϵ_k . The agreement for drift velocities is fair, taking into account the crudeness of our model and the experimental difficulties of absolute normalization. As will be shown in Section 4.1, we predict rather well the position accuracy obtained by Charpak et al in their drift chamber.²

From our fit we can predict the drift velocity in pure isobutane. On the small range of E/p where it is measured,²⁷ we agree quite well with experiment (see Fig. 22).

3.5.2 Methane

In order to have more confidence in the mechanisms proposed to explain the data, we have played the same game for methane. With the same adjustment by hand of the total and excitation cross sections:

*Above we assume a straight line joining the point at 1 eV to the known maximum at 8 eV.

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Fig. 15. Computer w and w_m for different concentrations of argon-isobutane mixtures. Comparison is done with Sauli et al. measurements.

Fig. 16. Computed ϵ_K for different concentrations of argon-isobutane mixtures.

$$\sigma_{\text{tot}} = \begin{cases} 1.4 \times 10^{-16} \text{ cm}^2 & \text{for } \epsilon \leq 1 \\ [1.4 \times 10^{-16} (8. - \epsilon) + 2.8 \cdot 10^{-15} (\epsilon - 1)] / 7. \text{ cm}^2 & \text{for } 1 \leq \epsilon \leq 8 \text{ eV} \\ (2.8 \times 10^{-15} / \sqrt{\epsilon / 8.}) & \text{for } \epsilon > 8 \text{ eV} \end{cases}$$

$$\sigma_{\text{exc}} = 5 \times 10^{-17} \text{ cm}^2 \quad \text{for } \epsilon_h \leq 0.36 \text{ eV}$$

we obtained the drift velocities shown in Fig. 17. The data are from Borner et al.,²⁸ Cottrel Walker,²⁸ and Wagner et al.²⁸ The experimental points of Hurst et al.²⁸ for pure CH_4 seem too high and are usually not quoted in the literature.

Fig. 17. Computer w for pure methane and 10% methane in argon.

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The agreement is worse than in isobutane. In particular our assumption of flat cross section below 1 eV does not allow us to describe very well the sharp use of drift velocity for pure methane below 900 V/cm, and we predict too sharp a peak for the mixture 10% CH_4 , 90% argon. We may note that in this region of concentration the drift velocity predicted by our program depends very much on the exact percentage of methane as shown in Fig. 17 (dashed-dotted curve). There are also problems in our analysis due to the fact that F_1 is of the same order of magnitude as F_0 and that therefore the two first terms of the Legendre expansion of the distribution function $F(\epsilon, \cos\theta)$ are not sufficient.

We may however, compare our ϵ_k with the measured values of Cochran and Forester²⁹ (Fig. 18) which agree for pure CH_4 with Cottrell and Walker. The agreement is rather good.

We think therefore that in spite of its problems, our model is able to describe reasonably well the qualitative features of the data both for isobutane and methane mixtures with argon: the saturation of the drift velocity, the dramatic decrease of drift velocity at low concentration, and

Fig. 18. Computed ϵ_k for CH_4 . Experimental data for methane, CO_2 , and pure argon are also shown.

the small values of ϵ_k . We have not attempted to distinguish between the two coefficients of diffusion D and D_L (Section 2.4.3): experimentally (Wagner et al.²⁸) for CH_4 , D_L is not very different from D , presumably because of the small variation of the collision frequency.²¹

3.6 Intuitive Interpretation

The strange behavior of the drift velocities encountered in the preceding sections are somewhat difficult to understand intuitively. We therefore would like to conclude this section by the discussion of their origin.

3.6.1 Approximate Theory

Let us go back to our Eq. 3.3(5) which is an approximation to a general theory of electron in gases

$$F(\epsilon) = C\sqrt{\epsilon} \exp\left(-\int_0^\epsilon \frac{3\Lambda(\epsilon') \epsilon' d\epsilon'}{[eE l(\epsilon')]^2}\right).$$

The drift velocity is then

$$w = \int \frac{\sqrt{2/m}}{eE} \frac{\Lambda(\epsilon)}{l(\epsilon)} \epsilon^{3/2} F(\epsilon) d\epsilon.$$

Let us assume that

$$\begin{aligned}\Lambda(\epsilon) &= \lambda_0 \epsilon^m \\ l(\epsilon) &= l_0 \epsilon^{-n} \quad (n \sim \epsilon^n).\end{aligned}$$

We have then

$$F(\epsilon) d\epsilon = C\sqrt{\epsilon} \exp\left[-\frac{3\lambda_0}{l_0^2} \frac{\epsilon^{n+2m+2}}{(m+2n+2)(eE)^2}\right] d\epsilon$$

From dimensionality arguments ($F(\epsilon)d\epsilon$ has no dimension):

$$\begin{aligned}C &\approx E^{-3/(m+2n+2)} \\ \epsilon_k &\approx \langle \epsilon \rangle \approx E^{2/(m+2n+2)} \\ \text{and } w &\approx E^{m+1/(m+2n+2)}\end{aligned} \quad 4.1(3)$$

The same result may be deduced from the simplified theory of Section 2.4.

It is seen then ϵ_k is always a rising function of E , but may rise very slowly in agreement with intuition if the cross section or Λ are fast rising functions of ϵ (m and n large). On the other hand, w will rise slowly if the cross section rises rapidly or if Λ decreases with ϵ . Eventually for $m < -1$, w will decrease. We will now study how in practice we have a rising cross section or decreasing Λ .

3.6.2 Pure Gases

In order to exemplify these effects let us first look at argon. The cross section has been given in Fig. 9. The cross section below 0.3 eV is decreasing. Therefore, from 4.1(3) for

$\epsilon_k \lesssim 0.3$ eV we expect the strong rise of ϵ_k and w seen experimentally (Fig. 1). Then the cross section rises sharply and we expect w and ϵ_k to be nearly constant with ϵ_k rising twice as fast as w . This will continue until an appreciable proportion of the tail of the energy distribution is beyond the excitation potential of 11.5 eV. From the plots of Fig. 1 we see that this happens around $E/N = 3 \cdot 10^{-17}$ V/cm². Δ then is effectively increasing very much, producing a sharp rise in w and a leveling off of ϵ_k .

This qualitative discussion may also be applied to the other heavy noble gases where the Ramsauer dip in the cross section occurs (Fig. 19), producing basically the same structure (Fig. 1)

Fig. 19. Cross sections for argon and other noble gases showing the Ramsauer dip below 1 eV and cross sections for the other gases He and Ne. Units are $\pi a_0^2 \approx 10^{-15}$ cm².

or to CO_2 where the cross section (Fig. 14) is falling off rapidly at small ϵ , giving at relatively high fields ($E/N > 10^{-16}$ V/cm²) a very fast rising w and ϵ_k (Fig. 2). In the last case for small fields, ϵ_k is limited by ambient temperature and, as shown in Section 3.4.1, w is then proportional to the field (constant mobility).

However, the rising cross sections cannot alone explain the saturation of the drift velocity in organic vapors. The upper limit to the important vibrational energies is the responsible mechanism: it leads for high ϵ to an effective fractional energy loss

$$\Lambda = \epsilon_{\text{max}}/\epsilon$$

where ϵ_{max} is the highest vibrational energy. Therefore $m \approx -1$ and w is approximately constant.

3.6.3 Mixtures

Another mechanism may occur in mixtures especially with the heavy noble gases (argon, krypton, xenon) where there is a Ramsauer minimum in the cross section. Let us add to argon a hypothetical gas of constant cross section and constant Λ in such a proportion that the Ramsauer dip is filled partially. For small enough ϵ , the added gas will dominate, and l and Λ are constant. ϵ_k and w rise with the field until the cross section of argon becomes comparable to the cross section of the added gas. From that point on l and Λ begin to decrease. This results in the leveling off of the drift velocity, which then becomes constant or even decreases. If the proportion of the additional gas is increased, the Ramsauer dip is filled completely and such a behavior no longer occurs.

This is the mechanism responsible for the behavior of mixtures of argon- CO_2 (Fig. 3), argon-isobutane (Fig. 15) and argon-methane (Fig. 17).

SECTION 4

APPLICATION TO DRIFT CHAMBERS

We will attempt in this section to apply the theoretical results reviewed in the preceding sections to practical problems encountered in designing and operating drift chambers.

At least four properties are desirable for a gas to be used in a drift chamber:

- Minimum diffusion
- Low drift velocity
- Good behavior in magnetic field and especially low drift magnetic velocity.
- Constant drift velocities and stability

4.1 Minimum Diffusion

The accuracy of a drift chamber is limited by the diffusion in the gas. Let us consider the simplified case of a constant electric field E along Ox . After a drift length x the rms of the dispersion of electrons in x is (from Eqs. 1.1(3) and 1.1(4)):

$$\sigma_x = \sqrt{\frac{2Dx}{w}} = \sqrt{\frac{2\epsilon_k x}{eE}} \quad 4.1(1)$$

In order to decrease σ_x one should decrease ϵ_k and increase E in the drift region. However they are not independent as shown in Fig. 20 where we give experimental results for pure argon,²³ CO_2 ,³² CH_4 , C_2H_4 , and C_3H_6 .^{29*} We have also drawn (dotted lines) our theoretical estimate for pure isobutane, 25% isobutane - 75% argon mixture. We have chosen the field region of 1 kV/cm since the fields in the drift space of drift chambers are usually of this order of magnitude.

This figure shows how easy it is to decrease the effective temperature ϵ_k of the electrons by addition to argon of multiatomic molecules or using pure organic vapor:

Because of the excitation of a high number of rotational and vibrational modes, the effective fractional energy loss Λ is increased and from Eq. 2.4(2) ϵ_k is decreased. However many limits are encountered in that direction: ϵ_k is bounded any way from below by $kT \approx 0.025$ eV at ambient temperature; if the electrons are thermal then the drift velocity is proportional to the field as in CO_2 for $E < 1$ kV/cm and finally, the cooling should not be so efficient as to prevent any amplification before break down because of too high an electric field at the cathode.

It should be noted that the σ_x given in Eq. 4.1(1) will not be the spatial accuracy of a drift chamber. In addition to the possible imperfection of electronics, two other facts have to be taken into account.

a) The dispersion of the primary electrons due to δ rays (see Section 1.3). The theory of this effect is difficult since the knowledge of the stopping distance of the electrons taking into account their random walk (not their total range usually found in tables) would be needed in order to compute its contributions to the final accuracy. This may, in fact, be the ultimate limit on drift chamber accuracy.

*There are some questions about the measurements by Cochran and Forester for some gases (CO_2 for example, Fig. 18); their diffusion coefficients are systematically high. See Huxley and Crompton, Ref. 13.

Fig. 20. σ_x (diffusion after 1 cm path of the electron swarm) from the ϵ_k data for argon, CH₄, C₂H₄, C₃H₆ and theoretical estimation for argon-isobutane 25% mixture, pure isobutane, and argon-methane 10% mixture.

b) The fact the electronics will detect the time of arrival of the m^{th} fastest electron. Depending on the conditions, m may vary between 1 (sensitivity to a single electron pair, e. g., in magic gas³³) and 10 ($\sim 1/10$ of the total number of electrons (see Table I) if the threshold is set at $1/10$ of the mean pulse height and there is no saturation effect).

In order to estimate this effect let us consider the group of n electrons created at a distance x from the sense wire in a volume small compared to σ_x . The dispersion on the first electron,³⁴ for large n , is given by

$$\sigma_f \approx \frac{1.28 \sigma_x}{\sqrt{2 \text{Log } n}} \quad 4.1(2)$$

Let us consider a particle going through the chamber at a right angle at 2 cm from the sense wire (Fig. 21). The n electrons produced along 4 mm of the track will be at the same distance from the wire within 100μ which is small compared to the usual σ_x and may be considered to originate from the same point. In argon-isobutane mixtures or similar mixtures

$$\begin{aligned} n &\approx 0.4 \times 100 \approx 40 \\ \text{and } \sigma_f &\approx 0.5 \sigma_x \end{aligned} \quad 4.1(3)$$

Fig. 21. Illustration for the computation of σ_f in Section 4.1.

For a mixture of 25% isobutane and 75% argon at $E = 500$ V/cm, our theoretical estimate was $\epsilon_k = 1.1 \times 10^{-1}$ eV which leads for a drift of 2 cm to

$$\begin{aligned} \sigma_x &\approx \sqrt{\frac{2 \times .11 \times 2}{500}} \approx 0.03 \text{ cm} \\ \text{and } \sigma_f &\approx 150 \mu \end{aligned}$$

to be compared with the result of Charpak, Sauli and Duinker² of $175 \pm 15 \mu$. *

Equation 4.1(3) represents an upper limit for σ_f , since taking the m^{th} fastest electron, the dispersion will be decreased and taking the center of gravity

$$\sigma_f \approx \frac{\sigma_x}{\sqrt{n}} \approx 0.16 \sigma_x.$$

However these more elaborate methods based on constant fraction discriminators, may be limited in practice by the fact that the rise time of the pulses are not constant.

4.2 Low Drift Velocities

One potential advantage of drift chambers is the relatively low cost of electronics for a great accuracy. This advantage may be cancelled by too high a value of drift velocity. Unfortunately the cooling of electrons by complex molecules dramatically increases the drift velocity. The value obtained with argon-isobutane mixtures ($w \approx 4 \cdot 10^6$ cm/s²) for instance, leads to 500 MHz electronics if one wants to exploit the intrinsic accuracy of the chamber.

In the case where the drift velocity is not imposed by other considerations (e.g., dead time problems), it is interesting to cool the electrons down, not through an increase of Λ , but through an increase of the cross section at small ϵ . From Eq. 2.4(2) this will, for a given ϵ_k , decrease w . Experimentally higher cross sections and smaller Λ are achieved through the use of big molecules such as isobutane (Fig. 14b). It may be profitable, although to our knowledge no one has tried it yet, to use pure isobutane. Figure 22 gives our theoretical estimate for w and Fig. 16 gives our estimated ϵ_k ; although these predictions are presumably not very accurate, they show that low values of drift velocities and low diffusion might be achieved at the same time.

4.3 Behavior in Magnetic Field

Another disadvantage of large drift velocities is seen in the operation in magnetic fields. In a constant electric field E and transverse magnetic field B the transverse displacement of the electrons after a drift x is

$$\Delta y = \frac{w_{\parallel}}{w_M} x = w_M \frac{B}{E} x \quad 4.3(1)$$

and w_M for low field is usually close to w (see Figs. 1 and 15). For instance with $E = 5 \times 10^4$ V/m, $w_M \approx w \approx 4 \times 10^4$ m/sec

$$\begin{aligned} \Delta y &\approx 0.8 Bx; & (B \text{ in tesla}) \\ &\approx 0.3 x & \text{for } B = 4 \text{ kG} \\ &\approx 1.2 x & \text{for } B = 15 \text{ kG} \end{aligned}$$

* More recent results by the same authors² give for $E \sim 1400$ V/cm, 30% isobutane, $x = 2$ cm, $\sigma_f \approx 135 \pm 10 \mu$ when we predict 150μ . Note that our predictions are for gas mixtures without methylal. The presence of methylal may cool down the electrons.

Fig. 22. Experimental drift velocities for hydrocarbons. Data from Cottrell and Walker, private communication to Christophorou¹⁴, Bortner et al.,²⁸ Wagner et al.,²⁸ and L. Christophorou et al.²⁷

This dramatically distorts the trajectory of electrons as shown in Fig. 23 for 15 kG in a drift chamber of Walenta's type. The dimensions are given in the figure, the cathode plane is at -2150 V with respect to the field wire and the field wire at -2710 V (as in Fig. 7b). The gas used in this simulation is 38% isobutane-52% argon and the w_M have been taken from the theoretical computation described in Section 3.5.1. If the drift length is too large, such an effect will lead to the loss of electrons, thus requiring special methods for overcoming this problem (e.g., large gap or tilting of electric field²).

Fig. 23. Trajectories of electrons (dotted lines) in a drift chamber operating in a 1.5 Tesla magnetic field for the same conditions as in Fig. 7. Full lines are lines of equal drift time to the sense wire.

In addition to the distortion of the mean electron trajectory, high magnetic fields may modify the magnitude of the drift velocity through the factor

$$\frac{1}{1 + \frac{(eB\ell)^2}{2mc^2}}$$

encountered in Sections 2.3 and 3.4.2. This factor from the simplified theory is of the order of

$$\frac{1}{1 + \left(\frac{3}{2} \frac{wB}{E}\right)^2} \quad (\text{Section 2.4})$$

and for $w \approx 5 \times 10^4$ m/s, $B = 1$ tesla, and $E = 10^5$ V/m is of the order of 2/3. The effect is therefore not negligible, and it is interesting to investigate it more thoroughly. In Figs. 24a and b we give the values of w and w_M for pure isobutane and two mixtures of isobutane with argon for $B=0, 10$, and 20 kG as deduced from the numerical solution of the relevant form of Boltzmann's equation (Section 3.4.2). As could be guessed beforehand, w is reduced while w_M is increased and the effect is more pronounced for large concentrations of argon where the Ramsauer minimum plays a large role. Figure 24c gives the absolute value of the drift velocity.

Fig. 24. (a) Computer behavior of W in isobutane-argon mixtures in presence of different magnetic fields.

(b) Behavior of w_M .

(c) Behavior of the absolute value of the drift velocity

$$|w| = \sqrt{w_{||}^2 + w_{\perp}^2}.$$

$$|w| = \sqrt{w_{||}^2 + \frac{w_{||}^2 w_M^2 B^2}{E^2}}$$

For magnetic fields up to 20 kG and electric fields higher than 1000 V/cm the variation of $|w|$ is small.

4.4 Constancy and Stability of the Drift Velocity

Another property which may be useful in practice is a small dependence of drift velocity with respect to the electric field. This allows an easier correlation between time and position. We have seen in Sections 3.5 and 3.6 that this happens for some organic vapor such as methane, ethylene (Fig. 22) etc., and for argon-isobutane mixtures²⁷ (Fig. 15).

However, one should not exaggerate the importance of the linearity of the relation between position and time. There are many other factors than the drift velocity that will distort this curve, for example:

- a) Geometrical effects: In particular, even if the dependence of time on position is linear for particles normal to the drift chamber, it is no more linear for particles at an angle. Fig. 25 gives the simulated time to position function for the drift chamber of Walenta's type described in Sections 1.3.2 and 4.3 for tracks at 0° and at 45°. Chrapak and Sauli (private communication) have similar dependence for their chamber. Software connections have therefore had to be done anyway.
- b) Variation of the shape of the signal: Depending on the position of the primary particle, the number of electrons arriving in the avalanche region and their time interval vary. This leads to a variation in the shape of the signal. If the threshold is not low enough, there will be, in the response of the discriminator, time delays dependent on the position and this will induce threshold and

Fig. 25. Simulated time-position dependence (for a Walenta-type drift chamber) for tracks at 0° and 45°.

chamber-dependent non-linear effects.

We think that a better way of looking at this question is to speak of stability. A smooth dependence of w on the field decreases the dependence of the position accuracy on the mechanical inaccuracies of the chamber construction where imperfections may slightly change the field in the drift region. It will also decrease the dependence on the temperature: Fig. 26 gives the estimated variation of drift velocity for isobutane-argon mixtures between 0°C and 18°C . In first approximation the shape is unchanged and there is only a slight displacement to higher field at lower temperature since the transport coefficient is a function of the ratio of field to density E/ρ and ρ has increased slightly. Clearly it is then desirable to have dependence of w as small as possible with respect to E/ρ .

A method to achieve this result is to optimize the mixture composition: however, this method for achieving stability with respect to electric field and temperature, may introduce more

Fig. 26. Simulated temperature dependence (0 and 18°C) of the drift velocity in different argon-isobutane mixtures.

serious instabilities due to too large a dependence of the drift velocity on the exact composition of the mixture.

Note that if the variation of the drift velocity due to various factors is not too important, it can be calibrated away by the use of two chambers displaced by half a wire spacing. In addition to the solution of the left-right ambiguity, this method gives the sum of the two drift times which is, for a properly designed system, a smooth function of wire position; this enables computation of the scale factor necessary for the connection of possible small variations of w .

SECTION 5 MULTIPLICATION

5.1 Definitions and Experimental Behavior

In the preceding sections we have neglected the ionizing collisions which gives rise to a multiplication of the number of electrons for high enough fields. This phenomenon is characterized by another transport coefficient α usually called the first Townsend coefficient. It is defined as the fractional increase of the number of electrons per unit drift length in a constant field E .

$$\frac{dn}{n} = \alpha dx$$

α is then a function of E only. If E is strongly varying, no equilibrium is reached and α is a function not only of the local field but of the fields encountered before by the electrons. This may be absorbed in the function $\alpha(x)$ and after a drift from x_0 to x_1 the original number n_0 of electrons has been transformed into

$$n_1 = n_0 \exp \left[\int_{x_0}^{x_1} \alpha(x) dx \right] \quad 5.1(1)$$

One also defines

$$\eta = \alpha/E$$

Figure 27 shows the experimental behavior of $\alpha(E)$, measured in uniform fields for the noble gases together with the corresponding ionization cross sections. Larger ionization cross sections give

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Fig. 27. Townsend coefficients and ionization cross sections for noble gases (experimental).

larger α . It is seen also that α increases very fast with E in the region of low fields, then begins to saturate. The addition of organic vapor decreases α at low field but increases it at high field as shown in Fig. 28 where $\alpha(E)$ is plotted for argon and Ar-CH₄ mixture³⁵ (10% CH₄). Note that, from simple arguments, α/P or η should be in constant field universal function E/P . (P being the pressure).

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Fig. 28. Comparison of Townsend coefficient for argon-CH₄ mixture and pure argon. It is smaller for the mixture (cooling of the electrons) until the electric field is high enough to start ionizing collisions with CH₄ molecules (lower ionization potential than pure argon).

5.2 Multiplication Theory in a Constant Field

5.2.1 Holstein's Theory

Let us now describe how Holstein⁹ modified in 1946 the theory of Morse et al.⁸ to take into account the ionizing collision. As above we will start from the Boltzmann equation 3.1(1). But now we should introduce the $\partial f/\partial t$ term due to ionization and since the number of electrons is no more constant, f_0 and f_1 will depend on x . Equation 3.1(5) then becomes

$$-\frac{v}{3} \frac{\partial f_1}{\partial x} - \left(\frac{a}{3v^2} \right) \frac{\partial}{\partial v} (v^2 f_1) + v^{-2} \left(\frac{m}{M} \right) \frac{v}{\partial v} (v^4 \frac{f_0}{\epsilon_e}) + \Sigma_{i=1}^{\infty} \left[\left(\frac{v'}{v} \right) \frac{v' f_0(v')}{\epsilon_h(v')} - v \frac{f_0(v')}{\epsilon_h(v)} \right] + \frac{\partial f_0}{\partial t} \Big|_{\text{ionization}} = 0 \quad 5.2(1)$$

$$-v \frac{\partial f_0}{\partial x} - a \frac{\partial f_0}{\partial v} - \frac{v f_1}{\epsilon_e} = 0 \quad 5.2(2)$$

where $a = eE/m$, $1/2 mv^2 = 1/2 mv^2 + \epsilon_h$ and $\int f_0 dx v^2 dv = N$ the total number of electrons. As for excitation, we only considered the effect of ionization on f_0 . $\partial f_0 / \partial t$ can be computed in the following way:

$$\begin{aligned} \left. \frac{\partial f_0}{\partial t} \right|_{\text{ionization}} &= N \int_{\Omega'} f(\theta', \phi', v', x) v' q_1(v') [\chi_1(v', \psi, v) + \chi_2(v', \psi, v)] d\Omega' v'^2 dv' \\ &- N f(\theta, \phi, v, x) v q_1(v) \end{aligned} \quad 5.2(3)$$

where v' is the initial velocity of an electron before an ionizing collision, $q_1(v')$ is the ionization cross section at velocity v' , ψ the angle between the direction θ, ϕ after the impact and the direction θ', ϕ' before the impact, $d\Omega'$ is the solid angle of diffusion (Fig. 8d) ($d\Omega' = d\cos\psi dv$), $\chi_1(v', \psi, v) d\Omega' v^2 dv$ is the probability of extracting in an ionizing collision an electron of velocity v at an angle ψ and $\chi_2(v', \psi, v) d\Omega' v^2 dv$ is the probability that the initial electron scatters at an angle ψ with a final velocity v . Through integration we get

$$\left. \frac{\partial f_0}{\partial t} \right|_{\text{ionization}} = N \int f_0(v') v'^3 q_1(v') \left\{ \int [\chi_1(v', \psi, v) + \chi_2(v', \psi, v)] d\Omega' \right\} dv' - N f_0 v q_1(v) \quad 5.2(4)$$

Working with $F(x, \epsilon, \cos\theta) = F_0(x, \epsilon) + F_0(x, \epsilon) \cos\theta$, where the total number of electrons is

$$n = \int F_0 d\epsilon dx$$

$$\left. \frac{\partial F_0}{\partial t} \right|_{\text{ionization}} = N \int_{\epsilon+\epsilon_i}^{\infty} F_0(\epsilon') v' q_1(\epsilon') [\phi(\epsilon', \epsilon) + \phi(\epsilon', \epsilon' - \epsilon - \epsilon_i)] d\epsilon' - N F_0(\epsilon) v q_1(\epsilon) \quad 5.2(5)$$

where $\epsilon' = 1/2 mv'^2$, $\epsilon = 1/2 mv^2$, ϵ_i is the ionization energy and $\phi(\epsilon', \epsilon) d\epsilon = v^2 dv \int \chi_1(v', \psi, v) d\Omega'$ is the probability of extracting an electron with energy ϵ when the initial electron has an energy ϵ' . We have used the fact that from the energy conservation

$$v \int \chi_2(v', \psi, v) d\Omega' = \phi(\epsilon', \epsilon' - \epsilon - \epsilon_i)$$

Finally the Eqs. 3.1(6) and 3.1(7) become

$$\begin{aligned} \frac{v}{3} \frac{\partial F_1}{\partial x} + \frac{eE}{3} \frac{\partial}{\partial \epsilon} (v F_1) - \frac{2m}{M} \frac{\partial}{\partial \epsilon} \left(\frac{\epsilon v}{I_e} F_0 \right) &= \Sigma_h \left[v' \frac{F_0(\epsilon + \epsilon_h)}{I_h(\epsilon + \epsilon_h)} - v \frac{F_0(\epsilon)}{I_h(\epsilon)} \right] + \\ + \int_{\epsilon+\epsilon_i}^{\infty} d\epsilon' v' \frac{F_0(\epsilon')}{I_i(\epsilon')} [\phi(\epsilon', \epsilon) + \phi(\epsilon', \epsilon' - \epsilon - \epsilon_i)] &- \frac{F_0(\epsilon) v}{I_i(\epsilon)} \end{aligned} \quad 5.2(6)$$

and

$$v \frac{\partial F_0}{\partial x} + eE \frac{\partial}{\partial \epsilon} (v F_0) - \frac{2eE}{mv} F_0 + \frac{v F_1}{I_e} = 0 \quad 5.2(7)$$

where we have introduced $l_1(\epsilon) = 1/Nq_1(\epsilon)$. The drift velocity is now equal to

$$w = \frac{1}{n} \int v \frac{F_1 d\epsilon}{3} = -\frac{2}{3} \frac{1}{n} \frac{eE}{m} \int_{\epsilon_0}^{\infty} l_e \frac{\partial F_0 / v}{\partial \epsilon} d\epsilon - \frac{1}{n} \int_0^{\infty} \frac{v l_e}{3} \frac{\partial F_0}{\partial x} d\epsilon$$

where the second term accounts for the effect on the drift velocity of the onset of ionization collisions and $n = \int F_0 d\epsilon dx$

5.2.2 Numerical Evaluation

Let us first note that since we are in a constant electric field, a stationary state will be reached such that

$$\frac{\partial F(x, \epsilon, \cos\theta)}{\partial x} = \alpha F(x, \epsilon, \cos\theta)$$

where α is the first Townsend coefficient, which is given by

$$\alpha = \frac{1}{w} \int v \frac{F_0(x, \epsilon) d\epsilon}{l_1(\epsilon)} \times \frac{1}{\int F_0 d\epsilon} \quad 5.2(8)$$

This form of α may be deduced either from its definition or by integration of Eq. 5.2(6) with respect to ϵ . One wants therefore, to solve the system similar to 3.3(6)

$$\left. \begin{aligned} \frac{\partial F_1 / v}{\partial \epsilon} &= -\frac{F_1(\epsilon)}{eE l_e v} - \frac{\alpha}{eE} \left(\frac{F_0}{v} \right) \\ \frac{\partial F_1(\epsilon)}{\partial \epsilon} &= \frac{3}{eE} \left[\epsilon \frac{\partial}{\partial \epsilon} \frac{\Lambda}{l_e} + \frac{2\Lambda}{l_e} \right] F_0(\epsilon) + \sum_h \frac{3}{eE} \left[\frac{\sqrt{\epsilon + \epsilon_h}}{\sqrt{\epsilon}} \frac{F_0(\epsilon + \epsilon_h)}{l_h(\epsilon + \epsilon_h)} - \frac{F_0(\epsilon)}{l_h(\epsilon)} \right] - \\ &\quad - \left[\frac{1}{2\epsilon} + \frac{3\Lambda\epsilon}{(eE l_e)^2} \right] F_1(\epsilon) - \frac{\alpha}{eE} F_1(\epsilon) + \frac{3}{eE} \left[\frac{\sqrt{\epsilon + \epsilon_i}}{\sqrt{\epsilon}} \frac{F_0(\epsilon + \epsilon_i)}{l_i(\epsilon + \epsilon_i)} - \frac{F_0(\epsilon)}{l_i(\epsilon)} \right] \end{aligned} \right\} \quad 5.2(9)$$

For the resolution of this system we have used an iterative method: we first take $\alpha=0$, solve the system and calculate α from Eq. 5.2(8). This value is then used as input to solve once again system 5.2(9) giving then a new estimate of α and so on. In practice such a procedure converges in few iterations.

Figure 29 gives the result we obtained for argon, with the cross sections given in Section 3.3.3 and

$$\sigma_i = 3.10^{-16} \log \frac{\epsilon}{15.7} \text{ cm}^2 \text{ for } \epsilon > \epsilon_i = 15.7 \text{ eV}$$

The agreement with the experimental data of Kruithoff et al.³⁶ is excellent until $E = 5 \cdot 10^6$ V/cm where the asymmetry $F_1(\epsilon)/F_0(\epsilon)$ gets so large that our approximation of $F(\epsilon, \cos\theta)$ by the first two terms of the Legendre expansion breaks down (see Fig. 12b). α flattens off at high field because v/w decreases at high field and because most of the electrons have an energy much

Fig. 29. Theoretical and experimental dependence of α for pure argon.

greater than ϵ_i in a region where the cross section is increasing less rapidly.

5.3 Multiplication in a Multiwire Proportional Counter

As seen from Fig. 28, α for argon-CH₄ mixtures becomes appreciable (larger than 10^3 cm^{-1}) only for fields greater than $0.5 \times 10^5 \text{ V/cm}$, which, according to Fig. 6 occurs at about 60μ from the wire. In other words the avalanche effect only begins quite close to the wire in a region where the field has a $1/r$ dependence, and the theory outlined in the last section may not be correct since it relies on the fact that the equilibrium state has been reached.

5.3.1 Validity of the Equilibrium Assumption

We will first show that this equilibrium assumption is valid. In order to do so, we have written a Monte Carlo program following the development of an avalanche in the field shown in Fig. 6.

For easier use we have rewritten the Eq. 5.2(6) in a slightly different fashion:

$$F(x + \langle v_x \rangle_\theta dt, \epsilon + \Delta\epsilon) \cdot J = \int d\epsilon' v' \frac{F(x, \epsilon')}{I_1(\epsilon')} \left[\phi(\epsilon', \epsilon + \Delta\epsilon) + \phi(\epsilon', \epsilon' - \epsilon_i - \Delta\epsilon) \right] + \int_{\Sigma_h} \frac{F(\epsilon + \epsilon_h + \Delta\epsilon) \sqrt{2/m(\epsilon + \epsilon_h + \Delta\epsilon)}}{I_h(\epsilon + \epsilon_h + \Delta\epsilon)} + \bar{F}(x, \epsilon) \left[1 - \frac{v}{I_1(\epsilon)} - \frac{v}{I_h(\epsilon)} \right] \quad 5.3(1)$$

where

$$\langle v_x \rangle_\theta = \frac{v}{3} \frac{F_1(x, \epsilon)}{F_0(x, \epsilon)}$$

is the average velocity along $O\vec{x}$ of a particle of energy ϵ and random velocity v ,

$$\Delta\epsilon = \left(eE v_x - \frac{2m}{M} \epsilon \frac{v}{\tau_e} \right) dt$$

is the net energy gain after the time dt , and

$$J = 1 + \frac{\partial \langle v_x \rangle_\theta}{\partial x} dt + \frac{\partial \Delta\epsilon}{\partial \epsilon}$$

is a Jacobian necessary for normalization conservation with respect to x and ϵ in the absence of multiplication. In our numerical method where we look at the evolution of a histogram approximation to $F(\epsilon)$ as a function of time, this term is taken into account automatically.

For further simplicity we have neglected the diffusion along the three directions, and assumed that the average velocity along $O\vec{x}$ was for all ϵ equal to the drift velocity

$$\langle v_x \rangle_\theta = \langle v_x \rangle_{\theta, \epsilon} = w \quad 5.3(2)$$

This gives a functional dependence between the position of the avalanche and the time:

$$dx = w dt$$

and allows to define a first Townsend coefficient $\alpha = \alpha(E(x))$. The justification for such an approximation is the a posteriori observed weak dependence of $\alpha(E)$ on the exact form of $\langle v_x \rangle_\theta$. For the same reason, instead of computing w from the combination of Eqs. 5.3(1) and 5.2(7), for instance, we have assumed that $w(E)$ was the same as in a constant field E .

Figure 30 gives our results for argon with the same cross sections as in Sections 3.3 and 5.2. Points (a) were obtained for $w(E)$ directly from experimental data of Fig. 1 to higher fields,

Fig. 30. Computed dependence of α for $1/2$ varying electric fields compared with Kruthoff data for constant field configuration.

points (b) were obtained with the extra assumption that w was constant above 7.5×10^4 V/cm. The resulting α 's in both cases, have a slightly more rapid dependence on $\mathcal{E}$ (a fact which may be related to our approximation 5.3(2)), but agree in magnitude with the experimental results for constant fields. Therefore we may conclude that the non-homogeneity of the field does not have any big effect on α , and that the equilibrium assumption is approximately true.

5.3.2 The Empirical Formulae for Gain in Proportional Counters

With this result in mind, we may try to interpret the various empirical formulae proposed for the description of the gain in proportional counters.

They have been critically reviewed a few years ago by Charles³⁵ who pointed out that the possibility of errors in gas gain measurements made with non-modern amplifiers because of amplifier pulse shaping effects. This casts some doubt on the agreement of some of these measurements with data.

Table II gives the various formulae for α as a function of E and the corresponding gain resulting from Eq. 5.1(1) and the fact that the field at a radius r (if r is sufficiently small) is

$$E(r) = \frac{1}{2\pi\epsilon_0} \frac{CV}{r}$$

for a modern multiwire proportional counter, where C is the capacity per unit length of wire and V the potential applied to the chamber. These formulae, except for the last two, have, even for their authors, a limited range of applicability, and one must introduce as an additional parameter a field E_s at which amplification is said to start. We define r_s such that $E(r_s) = E_s$ and r_0 as the radius of the wire. V_s is the tension for which

$$r_s = r_0 \left(\frac{CV}{r_0} = \frac{CV}{r_s} = E_s \right).$$

In practice, because of the two parameters entering in these formulae and the resulting flexibility, it is difficult to distinguish between them without careful measurements for many different radii and large ranges of V , and any of them may be used for rough prediction of operating voltage of a chamber.

As regards theoretical justifications of these formulae, they are non-existent or not convincing. In Appendix A we give the derivation of the classical Rose Korff model,³⁷ which tries to isolate the unstated assumption that allows one to derive the formula (and which is false, as can be checked explicitly with the numerical results of the complete theory presented above in Section 5.2). In our opinion, their only merit is that they describe empirically the behavior α with E . For instance, if we take pure argon (Fig. 27), the best fit between 10^4 V/cm and 10^6 V/cm is provided by the Ward's formula with $\alpha = 4 \times 10^2 V^{+4/2} \text{ cm}^{-4/2}$; the Williams-Sara one with $\alpha = 0.7 \times 10^{-5} V \text{ cm}^{-1}$ gives too big a flattening at high fields and the formulae of Diethorn and Rose-Korff are valid only in a limited range of E around 10^5 V/cm and 5×10^5 V/cm respectively. In Section 5.2 we have given the reason for such a saturation of α ,

TABLE II

Authors	$\alpha(r)$	Amplification
Power behavior of α : Rose-Korff ($n = 1/2$) Khristov ($n = 0$) Diethorn ($n = 1$)	$\begin{cases} 0 & \text{if } E < E_s \\ A \cdot E^n & \text{if } E > E_s \end{cases}$	for $n \neq 1$ $\exp \left[\frac{A}{n+1} \left(\frac{CV}{2\pi\epsilon_0} \right)^n \left(r_s^{1-n} - r_0^{1-n} \right) \right] =$ $= \exp \left[\frac{A}{n+1} \left(\frac{CV}{2\pi\epsilon_0} \right)^n r_0^{1-n} \left(\frac{V}{V_s} \right)^{1-n} - 1 \right] =$ for $n = 1$ $\exp \left[A \frac{CV}{2\pi\epsilon_0} \log \left(\frac{r_s}{r_0} \right) \right] =$ $= \exp \left[A \frac{CV}{2\pi\epsilon_0} \log \left(\frac{V}{V_s} \right) \right]$
Zastanny	0 if $E < E_s$ $A(E - E_s)$	$\exp \left[A \frac{CV}{2\pi\epsilon_0} \log \left(\frac{r_s}{r_0} \right) + \frac{r_0}{r_s} - 1 \right] =$ $= \exp \left\{ A \frac{CV}{2\pi\epsilon_0} \left[\log \left(\frac{V}{V_s} \right) + \frac{V_s}{V} - 1 \right] \right\}$
Williams-Sara	$A e^{-a/E}$	$\exp \left[\frac{A}{a} \frac{CV}{2\pi\epsilon_0} \exp \left(\frac{-2\pi\epsilon_0 a r_0}{CV} \right) \right]$
Ward	$A e^{-a/\sqrt{E}}$	$\exp \left[\frac{2A}{a} \exp \left(-\frac{a\sqrt{2\pi\epsilon_0} r}{\sqrt{CV}} \right) \left(\sqrt{\frac{CV}{2\pi\epsilon_0} r_0} + \frac{CV}{2\pi\epsilon_0 a} \right) \right]$

5.4 Parasitic Phenomena

As a warning relative to straight application of the results of the last section, we summarize in this section the parasitic phenomena important in a multiwire proportional counter.

5.4.1 Loss of Electrons

First, there are several phenomena that may lead to loss of electrons; among which are recombination between ions and electrons, and back diffusion of electrons to the cathode where they may be trapped, and electron capture. The first two effects play very little role in common proportional counters as shown, for instance, by Rossi³⁸ or Wilkinson.²⁰

On the other hand, electron capture is a property that is used in proportional counters (see Section 5.4.4) in order to limit the sensitive region of the chamber.¹ The elementary process is easy to understand: given a mean free path of capture $\ell_c(\epsilon)$ for an electron of energy ϵ , the number of electrons after a drift x in a uniform electric field E is

$$N = N_0 e^{-x/L_c}$$

where

$$L_c = \frac{1}{w} \left\langle \frac{v}{\ell_c(\epsilon)} \right\rangle$$

is the effective free path. Unfortunately, the cross sections are not constant and L_c will have a strong dependence on the electric field. Moreover, if the electronegative component is in high enough proportion to modify the energy distribution of electrons, L_c will not be inversely proportional to the concentration. Figure 34 gives some electron capture cross section as a function of ϵ for O_2 and some halogen compounds.

5.4.2 Space Charge Effects

Not only some primary electrons may be lost, but the avalanche may be reduced by space charge effects. One should distinguish between local effects which affect only one avalanche and flux-dependent effects.

5.4.2.1 Local Effects⁴⁰ At high amplification the charge density of the avalanche may be comparable to the charge density on the sense wire. In that case the later density is decreased locally and in order to go on, the avalanche has presumably to spread along the wire. This is seen in high gain proportional counters as a knee in the curve of the dependence of the pulse height on the high voltage (e.g., Fig. 32) and a saturation of the pulses.^{33,41}

In order to give an order of magnitude for the amplification at which such a phenomenon takes place, let us look at a mixture mainly composed of argon. As we have seen the characteristic energy ϵ_k flattens out at about 10-15 eV. In the region of the avalanche the field is typically 10^5 V/cm (Fig. 6) and the avalanche starts at some $x=60\mu$ from the wire. The lateral spread of

Fig. 31. Cross sections for electron capture for O_2 and halogen compounds (from Frauzen-Cochran¹⁷ and I. S. Buchel'nikova³⁹).

Fig. 32. Pulse height vs high voltage dependence in multiwire proportional counter.⁴¹

the avalanche is then *

$$\sigma_z = \sqrt{\frac{2\epsilon_k x}{E}} \approx 10 \mu,$$

which is about the radius of the wire and explains why the avalanche slightly surrounds the wire.¹ The density of the charge on the wire necessary to give the field $E = 10^5$ V/cm at the radius 60μ is

$$q = 2 \epsilon_0 r E \approx 3 \cdot 10^{-14} \text{ C}/\mu \approx 210^5 \text{ electrons per micron.}$$

The density of electrons in the avalanche is comparable as soon as the amplification reaches 5×10^6 or 10^7 .

5.4.2.2 Rate-Dependent Effects When a large flux of particles crosses a proportional counter working at high amplification, the ions of the successive avalanches drifting in the gap build up a stationary density of charges sufficient to significantly decrease the field around the wires and therefore the pulse height as well. This has recently been seen⁴² for amplification around 10^7 and for a flux of 10^4 particles/s/mm².

5.4.3 Breakdown Phenomena

However the above high amplification regimes cannot be reached in any mixture because of breakdown phenomena. In the following discussion we follow closely Loeb¹³ with some modification however because modern experimental data (see Massey II¹⁴) seems to disagree, especially with some orders of magnitude he quotes for photo ionization. We have to distinguish between

* This formula is not strictly true. We assume here that everything behaves as if all the created electrons were originating from the same point.

phenomena happening at the cathode and those happening close to the anode. The gas in the gap itself is quite inactive usually because of the low field in that region.

5.4.3.1 Cathode Phenomena These are essentially the liberation of electrons by ionic impact, the photoelectric effect, the impact of metastable atoms, and field emission, especially by layers of insulating materials (dusts).

The first three effects will produce delayed pulses related to the crossing of ionizing particles and eventually to a breakdown of the chamber. In order to quantify this process, let us for simplicity consider a parallel plane chamber with gap x and assume that n_0 electrons are initially produced at the cathode. It will be related to the total number n of electrons after multiplication by

$$n'_0 = \gamma_i (n - n'_0 - n_0)$$

and

$$n = (n_0 + n'_0) e^{\alpha x}$$

where γ_i is the probability of extraction of an electron by an ion, for instance. Finally

$$n = \frac{n_0 e^{\alpha x}}{1 - \gamma_i (e^{\alpha x} - 1)} \quad 5.4(1)$$

and if γ_i is high enough or $e^{\alpha x}$ high enough, the denominator will be zero and the chamber will diverge. This breakdown point will depend through α on the field in the gap, through x on the gap width (a smaller gap is safer) and through γ on the field at the surface of the cathode since most of the mechanisms mentioned above may be enhanced by a high field.

5.4.3.2 Photo-Ionization Near the Anode The mechanisms active in the gas are essentially the photo-ionization by photons produced in the primary avalanche and ionization by collision of the second kind in which excited atoms or molecules ionize other ones. The energy of positive ions is usually not high enough to ionize the gas. (Loeb¹³).

We will first treat the first mechanism. Figure 33a shows the absorption coefficient k (i.e., the inverse of the mean free path) of ultraviolet light in the heavy noble gases. Since they are formed of monoatomic molecules, the absorption of light with quantum energy greater than the threshold for primary-ionization can only take place through photon-ionization* so that the ratio ξ between the photo-ionization cross section and the photo-absorption cross section is 1 (Massey II¹⁴, p. 1078). Let us assume for simplicity that in a (multiwire) proportional counter the avalanche starts at a distance x_0 from the wire and that the photons are emitted perpendicularly to the wire. If n ions are produced in the avalanche, $f \cdot n$ photons will be produced (f being the mean number of photons emitted in an ionizing collision) and $g \cdot f \cdot n \cdot k \cdot e^{-kx} dx$ will be absorbed between x and $x+dx$. (g is a geometrical factor.) A proportion ξ will ionize the gas giving rise finally to $e^{\alpha x}$ electrons if $x < x_0$ and $e^{\alpha x_0}$ electrons if $x \geq x_0$. Finally, the total number of secondary electrons produced is

* In excitation the light is reemitted.

Fig. 33. Absorption coefficients K of uv light (a) in the heavy noble gases, ¹⁴ and (b) in CO_2 . Fig. 33(c) on following page gives absorption coefficients for organic vapors.

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Fig. 33(c). Absorption coefficients K of uv light in various organic vapors.

$$\begin{aligned}
 P &= \int_0^{x_0} f g \xi k e^{(\alpha-k)x} dx + \int_{x_0}^{\infty} f g \xi k e^{(\alpha x_0 - kx)} dx \\
 &= f g \xi \left[\frac{\alpha}{\alpha-k} e^{(\alpha-k)x_0} - \frac{k}{\alpha-k} \right]
 \end{aligned}$$

and finally the total number of electrons is

$$n = \frac{n_0 e^{\alpha x_0}}{1 - g f \xi \left[\frac{\alpha}{\alpha-k} e^{(\alpha-k)x_0} - \frac{k}{\alpha-k} \right]}$$

which for $\alpha \ll k$ becomes

$$n \approx \frac{n_0 e^{\alpha x_0}}{1 - g f \xi} \quad 5.4(2)$$

and for $\alpha \geq k$ gives

$$n \approx \frac{n_0 e^{\alpha x_0}}{\Lambda - g f \xi e^{\alpha x_0}} \quad 5.4(3)$$

For argon, for instance, $\xi=1$, g is 0.5,¹³ and f is a few percent because the cross section for inelastic collision giving rise through excitation to photons of energy greater than $\epsilon_1 = 15.7$ eV is much smaller than the ionization¹⁴ cross section. For small α ($\ll k = 10^3$ cm⁻¹ for argon) and small amplification, Eq. 5.4(2) is appropriate and the amplification is only slightly increased. For large amplification on the contrary, Eq. 5.4(3) should be applied and breakdown occurs very rapidly (as soon as $\alpha \geq k = 10^3$ cm⁻¹ giving gain of the order of 10^4 if $x_0 \approx 100\mu$ or 10^2 if $x_0 \approx 50\mu$). Therefore, if one wants to obtain high gain, avoiding the breakdown, it is necessary to capture photons without photoionization. Complex molecules will have such a quenching effect since in most cases the photon will break the molecules. Figure 33 shows this effect for CO₂ and light organic vapor as compiled by Massey and Christophoron respectively.¹⁴ Large molecules, increasing k , and decreasing ξ allow much higher amplification to be reached.

The particular form taken by the breakdown depends on the relative values of $1/k$ and the spread σ_x of the avalanche along the wire (Fig. 34).

Fig. 34. Spread of the avalanche along the wire.

Longitudinal propagation (Geiger mode). If the mean absorption path is greater than the dimension of the avalanche, the photons may initiate other avalanches along the wire giving rise to Geiger discharges with slow rising time, large amplitudes and long dead time.

Transversal propagation (Streamer mode). On the contrary, if the mean absorption path is shorter than the dimension of the avalanche, the photons can only increase the size of the avalanche. At very high amplification the large number of ions will increase the field locally and the avalanche may start at a greater distance from the wire. This leads to a transversal breakdown (streamer) and eventually to a spark between the cathode and the anode wire.

Both mechanisms may be generated locally by irregularity of the wire (spikes) or small layer of insulator (grease, dust) and lead to self-sustaining avalanches often called corona effects because of the light they produce.

5.4.3.3 Collision of Second Kind The noble gases unfortunately have metastable excited states, which may produce electrons when the atoms de-excite on the wall of the chamber several micro-seconds after deposition. This may be prevented however by complex molecules to which the excitation energy may be transferred through collision.

It may happen that the excitation energy of the noble gas is greater than the ionization energy of the "quencher". This gives rise the Penning effect,¹³ namely an increase of α . We may note that this is a prompt effect since the mean collision time between molecules is 10^{-10} s, and therefore may have some importance in the gain of modern multiwire proportional counter.

5.4.4 An Example: The "Magic Gas"

As an example of the foregoing discussion, we may take the "magic gas" of Charpak and coworkers:³³ 24% isobutane, 0.5% CF_3Br , (Freon 13B1), 4% methylal $((\text{OCH}_3)_2\text{CH}_2)$, and argon. The action of the different components may be characterized as follows:

a) The isobutane action is mainly at the anode. By dissociative absorption of the photons, it prevents Geiger discharges: experimentally isobutane suppression results effectively into high instability of the chamber and low rising time pulses.^{33,41} In addition to these quenching properties, the Penning phenomenon may be effective ($\epsilon_0 = 11.5$ eV for argon, $\epsilon_1 = 11.3$ eV for isobutane).

b) The methylal is mainly added in order to prevent the ionized dissociation products of isobutane to polymerize.³³ Its ionization potential (9.7 eV) is smaller than those of the dissociation products of isobutane and by charge transfer collision it neutralizes them rapidly. It seems also to have quenching properties.³³

c) Finally, the freon 13B1 is strongly electronegative and captures low energy electrons* especially those originating from secondary effects at the cathode. This "quenching" effect at the cathode allows reduction in the wire spacing and maintenance of a high amplification, even though the cathode field is higher. As a secondary effect, it captures primary electrons produced far from the wire and therefore increases the time resolution and decreases the number of double hits on adjacent wires. It should be noted that, according to the discussion in Section 5.3.1, the effective mean absorption path depends strongly on the temperature of the electrons, and any variation in the proportion of isobutane should be matched by a variation in the amount of Freon.

The net result is the possibility of reaching for small wire spacing and high fields and therefore high amplification; the saturation of the pulse height by space charge effects is then effective and pulse height becomes independent of the initial ionization.³³ Some parasitic phenomena may still exist for the low Freon concentration suitable for 2-mm wire spacing as shown by the double peaks observed at certain gains.⁴¹

* CF_3Br_2 has presumably, a behavior very similar to CF_3I plotted in Fig. 34

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APPENDIX

THE KORFF MODEL FOR PROPORTIONAL COUNTERS

Rose and Korff³¹ with their pioneering work in 1941 tried to tackle the problem of multiplication. They made drastic hypotheses because they were looking for purely analytical solutions

These hypotheses, furthermore, are not all quite explicit in the original paper; for example, the interpretation of the Korff model that we are going to present is different from the one presented by other authors.¹⁵ In our interpretation the physical hypotheses of the model are

- $\sigma_1(\epsilon) = a N(\epsilon - \epsilon_1) Y(\epsilon - \epsilon_1)$, where N is the number of atoms in unit volume, and Y is the step function.
- The created electron has zero energy, i. e., $\Phi_1(\epsilon, \epsilon') = \delta(\epsilon)$.
- There is no excitative process (very crude approximation).
- $d\epsilon/dr \approx eE \approx q/2\pi\epsilon_0 r$ for a proportional counter (that is, no elastic energy loss!)
- There is a constant ratio v/w . (In fact they seem to forget about the difference between the random velocity and the drift velocity.)

It is then straightforward to see that Eq. 5.3(1) becomes

$$F(r+dr, \epsilon) = dr \delta(\epsilon) n(r) \langle (\epsilon - \epsilon_1) Y(\epsilon - \epsilon_1) \rangle aN + F(r, \epsilon + \epsilon_1) aN \epsilon Y(\epsilon) - F(r, \epsilon - \frac{d\epsilon}{dr} dr) \cdot (1 - dr a N(\epsilon - \epsilon_1) Y(\epsilon - \epsilon_1)) \quad (A1)$$

where $n(r) = \int F(r, \epsilon) d\epsilon$ is the number of electrons at the radius r , and the symbol $\langle \rangle$ is used for the average over ϵ . Computing this expression at $\epsilon=0$ we have

$$F(r+dr, 0) d\epsilon = dr n(r) \langle (\epsilon - \epsilon_1) Y(\epsilon - \epsilon_1) \rangle aN \quad (A2)$$

- The last and main hypothesis of the model is a scale invariance hypothesis

$$F(r, \epsilon) = \frac{n(r)}{\langle (\epsilon - \epsilon_1) Y(\epsilon - \epsilon_1) \rangle} \phi \left(\frac{\epsilon}{\langle (\epsilon - \epsilon_1) Y(\epsilon - \epsilon_1) \rangle} \right) \quad (A3)$$

which implies $F(r, 0) = \frac{n(r) \phi(0)}{\langle (\epsilon - \epsilon_1) Y(\epsilon - \epsilon_1) \rangle}$

where $\phi(0)$ is r independent. (The mean $\langle \rangle$ depends on r). Such form arises, for instance, from a triangular distribution peaked at $\epsilon=0$

$$F(r, \epsilon) = n(r) \frac{2}{\epsilon_{\max}(2)} \left(1 - \frac{\epsilon}{\epsilon_{\max}(2)} \right)$$

In that case

$$\langle (\epsilon - \epsilon_i) Y(\epsilon - \epsilon_i) \rangle = \frac{\epsilon_{\max}(r)}{3}$$

and we get

$$F(r, 0) = n(r) \frac{2}{3} \frac{1}{\langle (\epsilon - \epsilon_i) Y(\epsilon - \epsilon_i) \rangle}$$

In this hypothesis, comparing Eqs. (A2) and (A3) we get

$$\langle (\epsilon - \epsilon_i) Y(\epsilon - \epsilon_i) \rangle = \sqrt{\frac{\phi(0)}{aN} \frac{d\epsilon}{dr}} \quad .$$

If we integrate the starting equation (A1) over $d\epsilon$ disregarding $(dr)^2$ contributions, we get

$$\begin{aligned} a(r) &= \frac{n(r+dr) - n(r)}{n(r) dr} = aN \langle (\epsilon - \epsilon_i) Y(\epsilon - \epsilon_i) \rangle \\ &= \sqrt{\phi(0) \frac{d\epsilon}{dr} aN} = \sqrt{\phi(0) aN} eE \sim \frac{1}{\sqrt{r}} \quad . \end{aligned}$$