**BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, New York**

**ACCELERATOR DEPARTMENT
Informal Report**

NONLINEAR TUNE SHIFT DUE TO THE LONG-RANGE BEAM-BEAM INTERACTION

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May 1, 1973

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Abstract

The long-range interaction of colliding particle beams in the weak-strong limit is considered. The linear and nonlinear ν -shifts are calculated using both uniform density beams and Gaussian beams. In particular, two limiting cases are detailed. In the case of large crossing angles, the beams are effectively separated in the long-range region. The relevant condition is $\alpha > \sigma_0/\beta_0$ (or, $\eta_0 > 1$), where α is the crossing angle, while β_0 and σ_0 are respectively the β value and the half-beam-size at the crossing point. The other limiting case is that of small η_0 ($\eta_0 < 1$), which corresponds physically to the situation where the test particle does not cross the "strong" beam at a crossing angle sufficiently large to escape the expanding envelope. The beams are therefore not properly separated in the long-range region. It is concluded that, except in the case η_0 much larger than one, the linear ν -shift is a true reflection of the nonlinear interaction in the long-range region. In rough terms, then, scaling applies. The precise meaning of scaling in the long-range region is discussed. However, for moderate $\eta_0 > 1$, the long-range linear ν -shift changes sign. This could lead to an underestimate of harmful nonlinear effects. It is found that for $\eta_0 < 1$ the nonlinear terms are sensitive to the density distribution in the long-range region. For low multipole order, n , they are generally smaller for uniform beam densities than for Gaussian densities. For $\eta_0 > 1$, there is no such sensitivity as a consequence of the fact that the beams are separated in the long-range region. It is finally pointed out that in attempting to limit the beam-beam interaction by separating the beams, then either the crossing angle must be sufficiently large ($\alpha > \sigma_0/\beta_0$), or the beams must be separated by externally imposed means, i.e., they may be bent apart by sufficiently strong bending magnets, or the interaction region may be limited by keeping the beams bunched.

1. Introduction

The luminosity that can be achieved when particle beams collide in storage ring devices is limited by the electromagnetic interaction of the two beams. A measure of this force is the linear tune shift induced in one beam (the "weak" beam) by the other beam (the "strong" beam), although it should be recognized that the mechanism which causes beam loss is the production of resonance stopbands.

A confusion has developed over the precise significance of the linear tune shift because in the past proper account of the variation of the betatron function, $\beta(s)$, as a function of longitudinal distance, s , was not taken. Correction of this neglect has given rise to the notion of a long-range linear beam-beam ν -shift.¹ The point is as follows: If, for example, two beams collide in a low- β region, then, due to the rapid, i.e., quadratic, increase of β with s , a test particle oscillating in the beam will remain within the expanding envelope unless the crossing angle of the beams, α , is greater than some fixed value, which is determined by the β function, β_0 , and the beam size (let σ_0 be the half size) at the crossing point. As we will see the required condition is

$$\alpha > \frac{\sigma_0}{\beta_0} . \quad (1.1)$$

If this condition is fulfilled the contribution to the beam-beam ν -shift will drop off for

$$s > \beta_0 , \quad (1.2)$$

with the drop-off increasing in magnitude as α is made larger. If it is not, however, the contribution to the ν -shift does not diminish unless there is physical intervention, such as the use of bending magnets to separate the beams.

The region $s > \beta_0$ has been referred to as the region of the long-range beam-beam interaction. If (1.1) is satisfied, its contribution to the ν -shift is small. If (1.1) is not satisfied, an automatic limitation on the interaction region is not obtained and other means must be sought to separate the beams, e.g., by the use of bending fields at the ends of the interaction region or by bunching the beams.

The linear tune shift by itself is not the cause of beam loss.² The highly nonlinear nature of the beam-beam interaction strongly suggests that resonance stopbands are mainly responsible. Thus, there is the notion of scaling. To what extent do the nonlinear forces scale with the linear force?

To answer this question, we will here consider scaling in terms of the parameter, r_ν , the ratio of the nonlinear ν -shift, $(\Delta\nu)_{NL}$, to the linear ν -shift, $(\Delta\nu)_L$:

$$r_\nu = \frac{(\Delta\nu)_{NL}}{(\Delta\nu)_L} . \quad (1.3)$$

In view of the point-like nature of the interaction, this is an adequate representation of scaling in that the tune shift in this situation is a direct measure of the stopband width.

Although developing the techniques in general, our results will be expressed for two simplified examples. Specifically we will treat the cases (1) of uniform density colliding beams, and (2) of Gaussian density beams. Further, we neglect all questions of self-consistency. This corresponds to the weak-strong limit, where one beam has a fixed density distribution (the strong beam) and a test particle of the other beam (the weak beam) is assumed to be moving in the fixed field of the strong beam. Finally, we neglect image effects due to the vacuum chamber wall and magnetic boundaries.

2. Beam-Beam ν -Shift

The force on a particle travelling through a beam can be conveniently written in terms of a quantity, $\Delta\nu$, which reduces to the real linear tune shift along the central orbit of the test particle. We will use this symbol, $\Delta\nu$, to refer to both tune shift as well as force.

Consider the case of two beams crossing vertically at an angle α . Since we neglect any dipole forces caused by the beam-beam interaction, we can write the horizontal force, F_x , and the vertical force, F_y , acting on a particle at (x,y) as

$$F_x = \frac{\partial F}{\partial x} x \quad , \quad (2.1a)$$

$$F_y = \frac{\partial F}{\partial y} y \quad . \quad (2.1b)$$

We define the quantity $\Delta\nu$ by its derivative with respect to s , the distance along the equilibrium orbit, measured from the crossing point. Writing

$$\Delta'\nu = \frac{d\Delta\nu}{ds} \quad , \quad (2.2)$$

we have for unbunched beams,¹

$$\Delta'_{v_x} = - \frac{Nr \beta_x(s)}{C \beta^2 \gamma} \left(\frac{\epsilon_0}{e\lambda} \frac{\partial F_x}{\partial x} \right)_{AV}, \quad (2.3)$$

$$\Delta'_{v_y} = - \frac{Nr \beta_y(s)}{C \beta^2 \gamma} \left(\frac{\epsilon_0}{e\lambda} \frac{\partial F_y}{\partial y} \right)_{AV}, \quad (2.4)$$

where

N is the total number of particles in each beam,

$\beta_{x,y}(s)$ are the betatron functions,

C is the circumference ($= 2\pi R$) of the equilibrium orbit,

r_p is the classical proton radius ($= 1.54 \times 10^{-18}$ m),

β is the velocity of light in units of c ,

γ is the particle energy in units of proton mass,

e is the electric charge,

ϵ_0 is the free space dielectric constant,

λ is the linear charge density, and

AV refers to averaging over phase in the $(x, \frac{dx}{ds})$ and $(y, \frac{dy}{ds})$ planes.

Let the test particle be defined by the coordinates (x,y,s) , and let the "strong" beam producing the force be defined with respect to coordinates (x',y',s') given by

$$\begin{aligned} x' &= x \\ y' &= y \cos \alpha + s \sin \alpha \\ s' &= s \cos \alpha - y \sin \alpha \end{aligned} \quad (2.5)$$

If we now assume that derivatives (with respect to y) of the beam profile can be neglected, and this is valid to lowest order in α , we can replace s' by s and we obtain for the forces (for $\alpha \ll 1$),

$$F_x = \frac{e\lambda}{\epsilon_0} (1 + \beta^2) \int_0^\infty \int_0^\infty dq_1 dq_2 \rho(q_1, q_2, s) \frac{q_1}{q_1^2 + q_2^2} \sin q_1 x \cos q_2 y', \quad (2.6)$$

$$F_y = \frac{e\lambda}{\epsilon_0} (1 + \beta^2) \int_0^\infty \int_0^\infty dq_1 dq_2 \rho(q_1, q_2, s) \frac{q_2}{q_1^2 + q_2^2} \cos q_1 x \sin q_2 y', \quad (2.7)$$

where ρ is the Fourier transform of the normalized charge density $\rho(x,y,s)$:

$$\rho(q_1, q_2, s) = \frac{1}{\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty dx dy \rho(x, y, s) \cos q_1 x \cos q_2 y. \quad (2.8)$$

The normalization is simply,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \rho(x, y, s) = 1 \quad (2.9)$$

The v -shift functions are, therefore, for small α ,

$$\Delta'_{v_x} = - \frac{Nr_p (1 + \beta^2) \beta_x(s)}{C\beta^2 \gamma} \int_0^{\infty} \int_0^{\infty} dq_1 dq_2 \rho(q_1, q_2, s) \frac{q_1^2}{q_1^2 + q_2^2} \cos q_1 x \cos q_2 y', \quad (2.10)$$

$$\Delta'_{v_y} = - \frac{Nr_p (1 + \beta^2) \beta_y(s)}{C\beta^2 \gamma} \int_0^{\infty} \int_0^{\infty} dq_1 dq_2 \rho(q_1, q_2, s) \frac{q_2^2}{q_1^2 + q_2^2} \cos q_1 x \cos q_2 y'. \quad (2.11)$$

Notice that we have replaced s' by s and that

$$y' \approx y + s\alpha. \quad (2.12)$$

Also, we implicitly assume that the averaging over x and y is to be done properly. This will be discussed in detail later.

3. Uniform Density Colliding Beams

Specializing to the case of uniform beams, the Fourier transform ρ [Eq. (2.8)] is given by

$$\rho(q_1, q_2, s) = \frac{\sin q_1 a \sin q_2 b}{\pi^2 ab q_1 q_2}, \quad (3.1)$$

where all the s -dependence is in the horizontal and vertical beam half-sizes, a and b . Let us further make the simplifying assumptions that

$$\begin{aligned} (1) \quad a &= b = \sigma(s), \\ \text{and } (2) \quad \beta_x &= \beta_y = \beta(s). \end{aligned} \quad (3.2)$$

Finally, we will restrict ourselves to calculations involving the vertical tune shift.

Using (3.1), we can evaluate the integrals in (2.11) to obtain

$$\Delta'_{v_y} = - \frac{Nr_p (1 + \beta^2) \beta_y(s)}{C\beta^2 \gamma (8\pi ab)} \left\{ \tan^{-1} \frac{a+x}{b+y} + \tan^{-1} \frac{a+x}{b-y} + \tan^{-1} \frac{a-x}{b+y} + \tan^{-1} \frac{a-x}{b-y} \right\}, \quad (3.3)$$

where the $\tan^{-1}$ function is defined in the range $-\pi/2$ to $+\pi/2$. With the assumptions in (3.2) and writing

$$\begin{aligned} u &= x/\sigma, \\ v' &= y'/\sigma, \end{aligned} \quad (3.4)$$

we have

$$\begin{aligned} \Delta' v_y &= \Delta' v_o \frac{1}{\pi} \\ &\left\{ \tan^{-1} \left(\frac{1+u}{1+v'} \right) + \tan^{-1} \left(\frac{1+u}{1-v'} \right) + \tan^{-1} \left(\frac{1-u}{1+v'} \right) + \tan^{-1} \left(\frac{1-u}{1-v'} \right) \right\}, \end{aligned} \quad (3.5)$$

where

$$\Delta' v_o = - \frac{N r_p (1 + \beta^2) \beta(s)}{8 \sigma^2 c \beta^2 \gamma} . \quad (3.6)$$

Since $\sigma(s)$ (2σ is the full beam size) is proportional to $\sqrt{\beta(s)}$, this quantity is independent of s and corresponds to the linear v -shift for head-on collisions ($\alpha = 0$).

We have obtained the vertical v -shift function, Eq. (3.5), as a function of the horizontal and vertical displacements from the equilibrium orbit, i.e., x and y respectively. It can be shown that this function when integrated over s is dependent on x^2 and y^2 . If we assume that there are no nearby resonances, then x^2 and y^2 can be separately averaged over phase in their respective phase planes, $(x, dx/ds)$ and $(y, dy/ds)$. Thus, the effect of the beam-beam interaction is simply to produce an amplitude dependent v -shift. This, of course, is not true. In actuality, it can cause beam growth; although, whether this develops because of the presence of isolated resonances or due to a condition of overlapping resonances inducing stochastic motion or both is as yet an unresolved issue.³ In any case, however, a measure of these nonlinear effects is, on the one hand, the magnitude of the stopband widths at their respective resonances, or, equivalently, the v -shift away from resonance. Here, we will adopt the latter approach, simplifying our analysis by treating only one-dimensional motion.

If we are sufficiently removed from any resonance condition, we can average over horizontal and vertical phases. Furthermore, for particles very close to the horizontal equilibrium orbit, we have for the squared horizontal amplitude,

$$\langle x^2 \rangle \cong 0 . \quad (3.7)$$

For the vertical motion, the squared amplitude can be written

$$\langle y^2 \rangle = \frac{\beta(s)}{\beta_0} \langle y_0^2 \rangle, \quad (3.8)$$

where β_0 is the value of $\beta(s)$ at the interaction point, $s = 0$, and $\langle y_0^2 \rangle$ is the value of the squared amplitude at $s = 0$.

For motion on the horizontal equilibrium orbit, the v -shift function is given by

$$\Delta'v_y = \Delta'v_0 \frac{2}{\pi} \left\{ \tan^{-1} \left(\frac{1}{1+v'} \right) + \tan^{-1} \left(\frac{1}{1-v'} \right) \right\}, \quad (3.9)$$

where

$$v' = \frac{y + s\alpha}{\sigma}. \quad (3.10)$$

The discontinuity in $\Delta'v_y$ at $|v'| = 1$ can be made explicit, that is,

$$\Delta'v_y = \begin{cases} \Delta'v_0 \left[1 + \frac{2}{\pi} \tan^{-1} \left(\frac{v'^2}{2} \right) \right] & \text{if } |v'| < 1 \\ \Delta'v_0 \left[-1 + \frac{2}{\pi} \tan^{-1} \left(\frac{v'^2}{2} \right) \right] & \text{if } |v'| > 1 \end{cases}. \quad (3.11)$$

The $\tan^{-1}$ function is now continuous across $|v'| = 1$ and we can expand $\Delta'v_y$ in powers of y . We can keep only powers of y^2 since the symmetric s integration causes the odd powers of y to vanish. Thus,

$$\Delta'v_y = \Delta'v_0 \left[f_L(s) + \sum_{n=1}^{\infty} \frac{1}{(2n)!} f_{2n}(s) y^{2n} \right]; \quad (3.12)$$

where $f_L(s)$ is an implicit function of y , defined by

$$f_L(s) = \begin{cases} 1 + \frac{2}{\pi} \tan^{-1} \left(\frac{s^2 \alpha^2}{2\sigma^2} \right) & \text{if } \left| \frac{y + s\alpha}{\sigma} \right| < 1 \\ -1 + \frac{2}{\pi} \tan^{-1} \left(\frac{s^2 \alpha^2}{2\sigma^2} \right) & \text{if } \left| \frac{y + s\alpha}{\sigma} \right| > 1 \end{cases} \quad (3.13a)$$

$$(3.13b)$$

and

$$f_{2n}(s) = \frac{2}{\pi} \frac{\partial^{2n}}{\partial y^{2n}} \left[\tan^{-1} \left(\frac{v'^2}{2} \right) \right]_{y=0}. \quad (3.14)$$

If we average the "nonlinear" terms over phases, then, using (3.8), there results

$$\Delta'v_y = \Delta'v_o \left\{ f_L(s) + \sum_{n=1}^{\infty} g_{2n} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} \right\} \quad (3.15)$$

In order to obtain the proper v -shift, the averaging over phase should be done by replacing y^{2n} by $[2y^{(2n+2)} / \langle y^2 \rangle]$ and then averaging. This can be seen if one uses the phase-amplitude form of solution, in which case, the v -shift is the rate of phase change, averaged over azimuth. Alternatively, a Hamiltonian approach gives the same results. Thus, writing

$$G_{2n} = \frac{(2)1.3. \dots (2n+1)}{2.4 \dots (2n+2)} \quad (3.15a)$$

we have

$$g_{2n}(s) = \frac{2G_{2n}}{(2n)! \pi} \sigma_o^{2n}(s) \frac{\partial^{2n}}{\partial y^{2n}} \left[\tan^{-1} \left(\frac{v'^2}{2} \right) \right]_{y=0} \quad (3.16)$$

which can be written

$$g_{2n}(s) = \frac{2G_{2n}}{(2n)! \pi} \frac{\partial^{2n}}{\partial q^{2n}} \left[\tan^{-1} \frac{1}{2} \left(q + \frac{s\alpha}{\sigma} \right)^2 \right]_{q=0} \quad (3.17)$$

The power series expansion in y will be valid for sufficiently small y . The discontinuity in $f_L(s)$ [see (3.13)] is determined by the condition

$$S(y_o, s) = \left| \frac{y_o}{\sigma_o} + \frac{s\alpha}{\sigma} \right| = 1 \quad (3.18)$$

where y_o is the vertical displacement at the crossing point, $s = 0$. This condition should be considered as a function of s . When the test particle reaches a distance s such that (3.18) is satisfied, the particle is at the beam edge. Thus, when $S(y_o, s) > 1$ the particle designated by (y_o) is outside the "strong" beam, while when $S(y_o, s) < 1$, the test particle is moving within the uniform density beam.

In the long-range region, $|s| > \beta_o$, and (3.18) becomes

$$\eta(y_o) = \left| \frac{y_o + \beta_o \alpha \text{sign}(s)}{\sigma_o} \right| = 1 \quad (3.19)$$

which is essentially s independent. Equation (3.19) means that if $\eta < 1$, the test particle does not reach the edge of the "strong" beam. In other words

the crossing angle is not large enough to overcome the effect of the expanding beam envelope. In this case, the long-range contribution to the linear ν -shift [Eq. (3.13a)] is large. If, on the other hand, $\eta > 1$, the test particle will reach the beam edge before the long-range region. Thus, the long-range contribution to the ν -shift is small, given by Eq. (3.13b).

The linear and nonlinear ν -shifts can now be found using Eqs. (3.13), (3.15) and (3.17). For an interaction length extending from $-\ell/2$ to $+\ell/2$, we have

$$\Delta \nu_L = \Delta' \nu_0 \int_{-\ell/2}^{\ell/2} f_L(s) ds, \quad (3.20)$$

$$\Delta \nu_{NL} = \Delta' \nu_0 \sum_{n=1}^{\infty} \frac{\langle y_0^{2n} \rangle}{\sigma_0^{2n}} \int_{-\ell/2}^{\ell/2} g_{2n}(s) ds. \quad (3.21)$$

In particular, for the octupole contribution to the ν -shift,

$$(\Delta \nu)_8 = \Delta' \nu_0 \frac{\langle y_0^2 \rangle}{\sigma_0^2} \int_{-\ell/2}^{\ell/2} g_2(s) ds, \quad (3.22)$$

and for the twelve-pole contribution ($n = 3$),

$$(\Delta \nu)_{12} = \Delta' \nu_0 \frac{\langle y_0^4 \rangle}{\sigma_0^4} \int_{-\ell/2}^{\ell/2} g_4(s) ds. \quad (3.23)$$

For the long-range region, denoted LR, and defined by $s > \beta_0$, we have for the ν -shift contributions,

$$\Delta \nu_L = \begin{cases} \Delta' \nu_0 \left[1 + \frac{2}{\pi} \tan^{-1} \frac{1}{2} \eta_0^2 \right] \int_{LR} ds, & \text{if } \eta < 1 \\ \Delta' \nu_0 \left[-1 + \frac{2}{\pi} \tan^{-1} \frac{1}{2} \eta_0^2 \right] \int_{LR} ds, & \text{if } \eta > 1 \end{cases} \quad (3.24a)$$

$$(3.24b)$$

$$\Delta \nu_{NL} = \Delta' \nu_0 \sum_{n=1}^{\infty} \frac{\langle y_0^{2n} \rangle}{\sigma_0^{2n}} g_{2n} \int_{LR} ds, \quad (3.25)$$

where g_{2n} is independent of s and is given by

$$g_{2n} = \frac{2G_{2n}}{(2n)! \pi} \frac{\partial^{2n}}{\partial q^{2n}} \left[\tan^{-1} \frac{1}{2} (q + \eta_0)^2 \right]_{q=0} ; \quad (3.26)$$

and

$$\eta_0 = \frac{\beta_0 \alpha}{\sigma_0} . \quad (3.27)$$

For each multipole, designated by n , the ratio of nonlinear to linear v -shifts is

$$r_v(n) = \frac{\Delta v_{NL}}{\Delta v_L} = \begin{cases} \frac{\langle y_0^{2n} \rangle g_{2n}}{\sigma_0^{2n} \left[1 + \frac{2}{\pi} \tan^{-1} \frac{1}{2} \eta_0^2 \right]} , & \text{if } \eta < 1 \\ \frac{\langle y_0^{2n} \rangle g_{2n}}{\sigma_0^{2n} \left[-1 + \frac{2}{\pi} \tan^{-1} \frac{1}{2} \eta_0^2 \right]} , & \text{if } \eta > 1 \end{cases} \quad (3.28a)$$

$$(3.28b)$$

Since

$$\eta = \left(\frac{y_0}{\sigma_0} + \frac{\beta_0 \alpha}{\sigma_0} \right)$$

is a function of y_0 (the displacement of the test particle from the center of the "strong" beam at the crossing point), the multipole nomenclature we have used is not, strictly speaking, valid. However, for two limiting cases, the y_0 dependence in $f_L(s)$ tends to disappear. Thus, if $\eta_0 \ll 1$, then it follows that $\eta(y_0) < 1$, essentially independent of y_0 . Thus, for the long-range v -shift contributions, (3.24a) and (3.28a) are applicable. On the other hand, if $\eta_0 \gg 1$, then in the long-range region, $\eta(y_0) > 1$, independent of y_0 . Thus, (3.24b) and (3.28b) are the proper expressions for computing the long-range v -shifts. In these cases, we can obtain relatively simple expressions for the long-range v -shifts.

The linear tune shifts are easily obtained from the first terms of the expansions of the $\tan^{-1}$ function for small and large arguments. If we let L_R be the length of the long-range region, we have,

$$\Delta v_L = \begin{cases} \Delta' v_0 L_R & \eta_0 < 1 \\ -\frac{4}{\pi \eta_0^2} \Delta' v_0 L_R & \eta_0 > 1 \end{cases} . \quad (3.29)$$

The nonlinear ν -shift can be obtained from (3.25). First we can evaluate g_{2n} for the two limiting cases. We write (3.26) in the form

$$g_{2n} = \frac{2G_{2n}}{(2n)! \pi} \frac{\partial^{2n}}{\partial \eta_0^{2n}} \tan^{-1} \left(\frac{1}{2} \eta_0^2 \right) . \quad (3.30)$$

For $\eta_0 > 1$, we require only two terms of the large η_0 expansion - i.e., for $\eta_0 > 1$,

$$g_{2n} \approx - \frac{4G_{2n}}{(2n)! \pi} \frac{\partial^{2n}}{\partial \eta_0^{2n}} \left(\frac{1}{\eta_0^2} \right) . \quad (3.31)$$

For $\eta_0 < 1$, we require the n^{th} term of the expansion; that is, for $\eta_0 < 1$, we have for g_{2n} ,

$$g_{2n} = \begin{cases} \frac{2G_{2n}}{(2n)! \pi} \frac{(-1)^{(n-1)/2}}{n 2^n} \frac{\partial^{2n}}{\partial \eta_0^{2n}} (\eta_0^{2n}) , & n \text{ odd} \\ 0 , & n \text{ even} . \end{cases} \quad (3.32)$$

Performing the derivatives in (3.31) and (3.32), we obtain,

$$g_{2n} \approx - \frac{4(2n+1) G_{2n}}{\pi} \frac{1}{\eta_0^{2n+2}} , \quad \eta_0 > 1 \quad (3.33)$$

$$g_{2n} \approx \begin{cases} \frac{2(-1)^{(n-1)/2} G_{2n}}{n 2^n} , & \eta_0 < 1 \text{ (n odd)} \\ 0 , & \eta_0 < 1 \text{ (n even)} . \end{cases} \quad (3.34)$$

With (3.25), the nonlinear tune shift of multipole order n is

$$\Delta \nu_{NL}(n) = g_{2n} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} \Delta' \nu_o L_R . \quad (3.35)$$

We thus obtain for the $r_\nu(n)$ parameter,

$$r_\nu(n) = \begin{cases} \frac{2(-1)^{(n-1)/2} G_{2n}}{n 2^n} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} , & \eta_0 < 1 \text{ (n odd)} \\ 0 , & \eta_0 < 1 \text{ (n even)} \end{cases} \quad (3.36)$$

and

$$r_v(n) = (2n + 1) G_{2n} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} \frac{1}{\eta_o^{2n}}, \quad \eta_o > 1. \quad (3.37)$$

From (3.36) and (3.37), we can conclude that for $\eta_o < 1$, the nonlinearities scale with the linear v -shift in the long-range region; while for $\eta_o > 1$, they drop off with η_o faster than the linear tune-shift. However, for $\eta_o > 1$, the long-range linear tune shift contribution is also falling with η_o . Therefore the nonlinearities, even in this case, still scale roughly with the linear tune shift — that is, as the long-range linear v -shift drops, so do the nonlinearities.

Note that we have given a precise significance to the notion of scaling in the long-range region — that is, although the nonlinear terms are said to be reflected by the linear tune shift, we can compute the way the higher multipoles behave as a function of n . In this sense, scaling does not mean that the higher multipole contributions are constant with n . In fact, for $\eta_o < 1$, the drop off is roughly $\frac{1}{n} 2^{-n}$, while for $\eta_o > 1$, it is like η_o^{-2n} (neglecting the n dependence arising from the phase averaging). Indeed, as we will see in the next section, the n dependence is, at least for $\eta_o < 1$, sensitive to the charge density distribution.

4. Colliding Beams with Gaussian Density Distributions

When using a uniform density beam, a difficulty arises due to the discontinuity at the edge of the beam. Specifically, the v -shift function (i.e., the rate of change of v -shift as a function of longitudinal distance, s) is discontinuous. When a test particle is travelling within the "strong" beam, it has one value; when travelling outside the "strong" beam, it has a different value. For example, see Eqs. (3.13). Thus, a particle with a value y_o at the crossing point might find itself within the beam for a certain s -range; but on succeeding revolutions, as y_o changes, this s -range may be considerably different. This type of amplitude dependence does not lend itself to a multipole expansion. However, we found that in certain limiting cases [see the discussion following (3.28)], such an expansion could be obtained in the long-range region ($s > \beta_o$). In this section, we will remove the difficulties encountered because of the discontinuous beam edge by treating the case of Gaussian beams, providing us with a continuous density distribution.

The normalized Gaussian density function is given by

$$\rho(x, y, s) = \frac{1}{2\pi ab} \exp\left(-\frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right), \quad (4.1)$$

where $a(s)$ and $b(s)$ are the horizontal and vertical rms beam sizes, respectively. Its Fourier transform is

$$\rho(q_1, q_2, s) = \frac{1}{\pi^2} e^{-(\frac{1}{2}a^2 q_1^2 + \frac{1}{2}b^2 q_2^2)} \quad (4.2)$$

Substituting this into (2.11), we can obtain the vertical v -shift function. If we assume (1) a circular beam [i.e., $a = b = \sigma(s)$], (2) equal vertical and horizontal β -functions, (3) small α , taking only the lowest order, and (4) zero horizontal amplitude (i.e., $\langle x \rangle = 0$), we obtain [see also Ref. 1],

$$\Delta'v_y = \Delta'v_o \frac{\partial}{\partial v} \left\{ \frac{2}{v} \left[1 - e^{-\frac{1}{2}v'^2} \right] \right\}, \quad (4.3)$$

where $\Delta'v_o$ is s -independent and is given by

$$\Delta'v_o = - \frac{Nr_p (1 + \beta^2) \beta(s)}{4\pi C \beta^2 \gamma \sigma^2}, \quad (4.4)$$

and

$$v' = y'/\sigma = (y + s\alpha)/\sigma \quad (4.5)$$

This can be expanded in powers of y . We keep only the even powers, the odd ones vanishing because of the symmetric s -integration. Properly averaging over phases in the $(y, dy/ds)$ plane, we obtain [see discussion following (3.15)],

$$\Delta'v_y = \Delta'v_o \sum_{n=0}^{\infty} g_{2n} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}}, \quad (4.6)$$

where

$$g_{2n} = \frac{2G_{2n}}{(2n)!} \frac{\partial^{2n+1}}{\partial q^{2n+1}} \left\{ \frac{1}{\left(q + \frac{s\alpha}{\sigma}\right)} \left[1 - e^{-\frac{1}{2}\left(q + \frac{s\alpha}{\sigma}\right)^2} \right] \right\}_{q=0}, \quad (4.7)$$

and

$$G_{2n} = \frac{(2) 1.3. \dots (2n+1)}{2.4. \dots (2n+2)} \quad (4.8)$$

In the long-range region ($s > \beta_0$), the v -shift function becomes independent of s . We have

$$s_{2n} = \frac{2G_{2n}}{(2n)!} \frac{\partial^{2n+1}}{\partial \eta_0^{2n+1}} \left\{ \frac{1}{\eta_0} (1 - e^{-\frac{1}{2}\eta_0^2}) \right\} , \quad (4.9)$$

where

$$\eta_0 = \frac{\beta_0 \alpha}{\sigma_0} . \quad (4.10)$$

For the limiting cases of large and small η_0 , we obtain

$$s_{2n} = \frac{(-1)^n (2n+1)}{(n+1)! 2^n} G_{2n} , \quad \eta_0 < 1 \quad (4.11)$$

and

$$s_{2n} = - \frac{2(2n+1)}{\eta_0^{2n+2}} G_{2n} , \quad \eta_0 > 1 . \quad (4.12)$$

The long-range linear and nonlinear contributions to the beam-beam v -shift are therefore

$$\Delta v_L = s_{20} \Delta' v_{0L} , \quad (4.13)$$

$$\Delta v_{NL}(n) = s_{2n} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} \Delta' v_{0L} . \quad (4.14)$$

Thus, the $r_v(n)$ parameter is given by

$$r_v(n) = \frac{s_{2n}}{s_{20}} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} . \quad (4.15)$$

In the two limiting cases, this becomes

$$r_v(n) = \frac{(-1)^n (2n+1) G_{2n}}{(n+1)! 2^n} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} , \quad \eta_0 < 1 \quad (4.16)$$

and

$$r_v(n) = \frac{(2n+1) G_{2n}}{\eta_0^{2n}} \frac{\langle y_o^{2n} \rangle}{\sigma_o^{2n}} , \quad \eta_0 > 1 . \quad (4.17)$$

The long-range nonlinear v -shift therefore falls off with increasing η_0 faster than the linear v -shift (if $\eta_0 > 1$). However, if $\eta_0 > 1$, the long-range linear contribution also is decreasing. Thus, if $\eta_0 < 1$, the long-range region contributes at a rate equal to that in the short-range region ($s \leq \beta_0$), i.e., for $\eta_0 < 1$

$$(\Delta v)_L = \Delta' v_0 L_R \quad ; \quad (4.18)$$

while if $\eta_0 > 1$, the long-range contribution diminishes with increasing η_0 :

$$(\Delta v)_L = - \frac{2}{\eta_0^2} \Delta' v_0 L_R \quad . \quad (4.19)$$

We notice that for $\eta_0 > 1$, the $r_v(n)$ parameter has the same n and η_0 dependence as in the case of uniform beams. This is simply a reflection of the fact that if the beams are "separated" in the long-range region, the tune shifts are insensitive to the charge distribution. However, for $\eta_0 < 1$, there is an n -dependence in $r_v(n)$ quite different from that in the uniform density case. The behavior for Gaussian beams is $\frac{1}{n!} 2^{-n}$ as compared to $\frac{1}{n} 2^{-n}$ for uniform density beams. Actually the nonlinear v -shifts are smaller in the uniform beam case for small n , but with Gaussian beams, the drop-off with n is faster. (Note, however, that the beam size σ_0 is defined differently in the two cases.)

5. Conclusions

We have considered the long-range interaction of colliding particle beams in the weak-strong limit. We have calculated the linear and nonlinear v -shifts for uniform density beams and Gaussian beams. Our results can be most readily expressed in terms of the parameter η_0 , which has the following physical significance: if $\eta_0 < 1$, the beams remain unseparated throughout the interaction region (the crossing angle is not large enough to overcome the expanding beam envelope); while if $\eta_0 > 1$, the beams will separate.

Generally, we conclude that the linear v -shift in the long-range region is a true reflection of the nonlinear interaction terms. For small η_0 as well as for η_0 greater than 1, but not too large, the nonlinear terms can be said to scale with the linear long-range v -shift. The dependence of the nonlinear v -shift on the multipole order, n , is calculated, and in this context, the

precise meaning of scaling in the long-range region is discussed. Some of our results can be readily seen in terms of the parameter $r_v(n)$, i.e., the ratio of nonlinear to linear v -shifts for a multipole of order n . The results for uniform density beams are given in (3.36) and (3.37), while those for Gaussian beams can be found in (4.16) and (4.17). If η_0 is much larger than 1, the drop-off of the multipole terms with n depends on η_0 , and will be somewhat faster than in the $\eta_0 < 1$ situation, going like $2^{-n} \ln \eta_0^2$. However, in this case the linear long-range v -shift is also reduced. Thus, scaling, in the sense we are using the term, still applies.

There is one point with regard to scaling that is worth emphasizing. What we have shown here is that scaling is applicable to the long-range interaction. From this, we are tempted to infer scaling for the complete beam-beam interaction. For the case $\eta_0 < 1$, this is probably valid since the linear v -shifts for short and long-range interactions have the same sign and simply add. However, for $\eta_0 > 1$, the long-range contribution to the vertical linear v -shift has the opposite sign to the short-range contribution. Thus, adding the long-range tune shift actually decreases the total linear beam-beam v -shift. For sufficiently large linear v -shifts compared to the long-range contribution, this will not lead to error; however, for moderate $\eta_0 > 1$, we could, by only observing the total linear tune shift, underestimate the harmful nonlinear influences of the beam-beam interaction. Of course, with vertical crossing, one should first look at the horizontal tune shift, where the change of sign does not occur.

Our calculations have shown that the nonlinear terms are sensitive to the density distribution. Of course, for $\eta_0 > 1$, there is no sensitivity in the long-range region. This can be seen from the expressions for the linear and nonlinear tune shifts in Sections 3 and 4. It is also clear from the fact that in this case, $\eta_0 > 1$, the beams are separated in the long-range region. We would then expect that a test particle moving away from the "strong" beam would be insensitive to the charge distribution of the "strong" beam. However, for $\eta_0 < 1$, the beams overlap in the long-range region. Thus, there is indeed a sensitivity of the nonlinear terms to the density distribution. As can be seen from (3.36) and (4.16), they are quite different for uniform and Gaussian density distributions. They even differ in sign for some multipoles. From these expressions, we see that for the lower multipoles the uniform density distribution gives smaller nonlinear contributions from the long-range region; in fact, every other term actually vanishes.

Finally, we make a comment about the physical significance of the condition $\eta_0 = 1$. This corresponds to the physical situation where a central particle in the "weak" beam remains in the expanding envelope of the "strong" beam. For $\eta_0 < 1$, the beams do not properly separate. From the point of view of design of an experimental interaction region, we must have a clean separation in a prescribed distance. To accomplish this with a nonzero crossing, there is a minimum crossing angle required, defined by the condition $\eta_0 > 1$, i.e., $\alpha > \sigma_0/\beta_0$. For crossing angles less than this minimum, proper separation must be obtained by physical intervention, e.g., by bending the beams away from each other at the ends of the prescribed interaction region, or by bunching the beams.

References

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