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THE SHORT-DISTANCE BEHAVIOR OF A $g\phi^4$ THEORY,
AND CONFORMAL INVARIANT FOUR-POINT FUNCTIONS

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Abstract

The "zero-mass" limiting short-distance theory for the $g\phi^4$ interaction is explicitly constructed. The propagator is computed to third order and the vertex to fourth order in $\bar{g}$, the zero-mass coupling constant. The Gell-Mann and Low short-distance equations which result as exact consequences can be explicitly determined to all orders in perturbation theory. If the eigenvalue condition is imposed, one can construct a theory in terms of the massless free propagator and a conformal invariant four-point vertex function only. The energy-momentum tensor of this theory is the same as that proposed by Callan, Coleman and Jackiw.

Introduction

In their pioneering work, Gell-Mann and Low⁽¹⁾ (GML) studied the short-distance behavior of quantum electrodynamics by what is now known as the renormalization group method. Their results not only included some facts about the asymptotic behavior of the photon and electron Green's functions, but also put forth some very striking conditions that the bare, unrenormalized electron-photon coupling constant should obey. These were, in the short-distance limit, that it be either infinite, as perturbation theory seems to predict, or that it be a finite number independent of the physical coupling constant, and that it must be a small positive root α of some equation (GML equation)

$$\bar{\Psi}(x) = 0 \tag{I. 1}$$

now known as the eigenvalue equation. Unfortunately from the formalism leading to these results one is not capable of telling which of the two alternatives, if either, represent physical reality.

Even though much work and soul-searching is still to be done, some recent theoretical studies seem to indicate that the interesting eigenvalue situation might be of some significance to the understanding, in terms of field theory, of the basic dynamics underlying the high-energy behavior of the fundamental interactions and other physical phenomena. The first hint came from Johnson, Baker and collaborators^(2, 3, 4, 5) who in their model theory of QED, where photon propagators do not contain photon self-energy insertions, realized that the theory could be made finite if an eigenvalue equation is satisfied. In fact, they found that the photon renormalization constant

Z_3 takes the form ($Z_1 = Z_2$ can be made separately finite order by order by choosing an appropriate gauge)

$$Z_3 =_{\Lambda^2 \gg q^2} F_1(\alpha_0) \log \Lambda^2/q^2 + \text{FINITE TERMS} \quad (I.2)$$

where α_0 is the bare coupling (fine structure) constant. The function $F_1(x)$ has the property that if for some x it has a zero, then the same x is also a zero at $\Psi(x)$, the GML function. Later on, Wilson⁽⁶⁾ suggested that eigenvalue type equations might be of a wider dynamical interest, especially that they might be relevant in understanding strong interactions and phase transition phenomena⁽⁷⁾. More recently, Adler⁽⁸⁾, using the very elegant techniques developed by Symanzik⁽⁹⁾ and Callan⁽¹⁰⁾ was able to reproduce Johnson and Baker's results and went further on to state that⁽¹¹⁾ if a different approach is taken in summing up graphs contributing to Z_3 , then the J-B function $F_1(x)$ should also satisfy

$$F_1(\alpha) = 0 \quad (I.3)$$

where α is now the physical coupling constant. Furthermore, he points out that the solution to (I.3) must be essential zero.

In this work⁽¹²⁾, we will use the physically simple, but non-trivial, $g\phi^4$ field theory to carry out two studies. First, we will show that one can derive, in a simple and compact manner, the GML equations, exactly, once an asymptotic zero-mass theory has been properly defined. Secondly, we will try to add a modest contribution toward the understanding of the significance of eigenvalue type equations in field theories at short distances by analyzing their effects on our zero-mass theory.

The existence of a zero-mass limiting theory is the crucial stepping stone, in the GML analysis, to arrive at the short distance equations. Most renormalization group approaches take this step for granted⁽¹³⁾. An explicit construction of the limiting theory will not only put GML results on solid theoretic ground, but will also allow for a well-defined theoretical formalism which one can use to analyze its possible consequences. As it turns out, our approach is compact and straightforward and allows us to compute all of the short-distance functions in a simple way. The simplicity is an inheritance from the method we devised to carry out the renormalization of the theory in question⁽¹⁴⁾. Basically, we feel that the determination of a proper renormalization program is crucial if one wants to construct, simply, zero-mass (short-distance) theories explicitly. We recall that our renormalization procedure was characterized by avoiding the introduction of renormalization constants and a canonical Lagrangian formalism, which, we explained, "non-commute" with the short-distance analysis of relativistic field theories.

The zero-mass formalism, in effect, should allow us to calculate the form of the eigenvalue equation in a very direct way and so determine the eigenvalue, if it exists. Here, unfortunately, one is faced with the stumbling block which has, thus far, prevented any well-defined conclusion to the problem; namely one is in the position of trying to make statements which are beyond perturbation theory and yet one still has perturbation theory as the only calculational tool available. Only the full determination of the eigenvalue function and its eigenvalues would represent the first real breakthrough in understanding the short-distance behavior along these theoretical lines. We are able to tell, however, what happens to the short-distance theory if an eigenvalue condition is obeyed. For the $g\phi^4$ interaction we observe that the theory becomes what we shall call a "classical-like asymptotic" theory. By "classical-like" we mean that the theory at the

eigenvalue resembles, or can be abstracted to resemble, a classical theory. For instance, the massless $g\phi^4$ interaction is well known to be both canonically scale- and conformal-invariant. The renormalization procedure, by necessity, introduces large cut-off masses, thus destroying those invariances. The canonical field dimension d , defined for a boson field by

$$i [D, \varphi(x)] = (d + x \frac{\partial}{\partial x}) \varphi(x) \quad (I.4)$$

$$d = 1$$

where D is the generator of dilatations, will lose meaning through the perturbative quantum corrections procedure. Similarly, the vertex function will be neither scale- nor conformal-invariant, for the same reasons. The Callan-Coleman-Jackiw⁽¹⁵⁾ energy-momentum tensor, which is seen to be the relevant one in the study of those invariances of the theory, becomes ill-defined (i. e. it will not behave as a CCJ tensor) through the quantum corrections. If the eigenvalue equation is valid one observes that, for the short-distance theory, the propagator and vertex function become scale- and conformal-invariant, and the concept of dimensionality is re-established, the dimension being of course anomalous

$$d_{g_0} = 1 + F(g_0) \quad (I.5)$$

$F(g_0)$ is a constant greater than zero, its value being determined by g_0 , the eigenvalue. Furthermore, from the theory at the eigenvalue we can construct a theory which is in terms of the massless free propagator and an explicitly conformal-invariant four-point vertex function. The construction of our theory allows for the determination

of such a function (within a perturbative expansion). The energy-momentum tensor of such a theory is the same as the CCJ tensor.

This paper is divided into three parts: in the first the zero-mass theory is constructed; in the second the short-distance GML equations are derived; in the third the eigenvalue condition and its validity are discussed, and then, assuming the existence of an eigenvalue, the short-distance theory is analyzed accordingly and the conformal-invariant theory is constructed. One appendix will spell out some of the techniques used to calculate the zero-mass and GML functions.

Chapter 1: The "Zero-Mass" Theory

1.1 The Zero-Mass Limit

If we take the limit of the renormalized theory as the physical mass approaches zero, we see that we obtain infrared logarithmic divergences coming from subtraction constants divided at on-shell points. Effectively these divergences are, then, of the same nature as the original ultraviolet ones of the unrenormalized theory. This procedure, therefore, will not present us with a well-defined asymptotic theory, and so we must spell out what it is we mean exactly by a "zero-mass" theory. Our object is to find a "zero-mass" propagator function $\bar{\Delta}$, and a "zero-mass" vertex function $\bar{\Gamma}$ defined by

$$\bar{\Delta}^{-1} = \bar{\Delta}^{-1}(p^2, p_0^2; m^2; \bar{g}) \equiv p^2 [1 + \bar{\Theta}(p^2, p_0^2; m^2; \bar{g})] \quad (1.1)$$

and

$$\bar{\Gamma} = \bar{\Gamma}[p_1, p_2, p_3, p_4; p_0^2; m^2; \bar{g}; \bar{\Delta}] \quad (1.2)$$

such that

$$\begin{aligned} \text{A) } & \left(\lim_{m \rightarrow 0} \right) \bar{\Delta} \quad \text{and} \\ & \left(\lim_{m \rightarrow 0} \right) \bar{\Gamma} = \text{finite functions to all orders in } \bar{g} \end{aligned}$$

$$\text{B) } \lim_{p^2 \rightarrow 0} \left[\left(\lim_{m \rightarrow 0} \right) p^2 \bar{\Theta}(p^2, m^2; p_0^2; \bar{g}) \right] = 0 \quad (1.3)$$

By $\bar{g}$, we mean the "zero-mass" coupling constant which is defined by evaluating $\left(\lim_{m \rightarrow 0}\right) \bar{T}$ at the symmetry point, off the mass-shell, defined by

$$(p_i^2 + p_j^2)^2 = \frac{4}{3}(1 + 2\delta_{ij}) p_0^2 \quad (1.4a)$$

$$\left(\lim_{m \rightarrow 0}\right) \bar{T}\left[1, \frac{m^2}{p_0^2}; \bar{g}; \bar{\Delta}\right] = \bar{g} \quad (1.4b)$$

By $\left(\lim_{m \rightarrow 0}\right)$ we understand that all the momenta of the theory, including the off-shell momenta p_0^2 , must be taken to be much larger than m^2 . This is, essentially, GML's program. Here the off-shell momentum p_0^2 will set the scale and it will replace m^2 in the logarithmic expressions. The success of the theory will then depend on showing that remaining m^2 terms will vanish as m becomes small compared to all the momenta. Because of the simple analytic structure of the equations defining $\bar{T}$, in our interactive theory, the validity of condition B), equation (1.3), is crucial to the completion of our program.

Since the theory is renormalizable, we expect that the off-shell theory must be related to the on-shell theory. So there must be a function $S(p_0^2, m^2)$ such that

$$\bar{\Delta}^{-1}(p_i^2, p_0^2; m^2; \bar{g}) = S(p_0^2, m^2; g) \Delta^{-1}(p_i^2, m^2; g) \quad (1.5)$$

where Δ and g are the renormalized propagator and coupling constant respectively.

If $\left(\lim_{m \rightarrow 0}\right) \bar{\Delta}^{-1}$ is to be finite, then the expansion of $S(p_0^2, m^2; g)$ will contain logarithmic terms depending on the mass which will cancel those in Δ^{-1} when S and Δ^{-1} are multiplied. The residual terms must be of $\mathcal{O}(m/p^2, m^2/p_0^2, p^2/p_0^2)$

$$\left(\lim_{m \rightarrow 0}\right) S(p_0^2, m^2; g) \Delta^{-1}(p^2, m^2; g) = \text{FINITE TERM} + \mathcal{O}\left[\frac{m^2}{p_0^2}, \frac{m^2}{p_0^2}\right] \quad (1.6)$$

A simple and explicit determination of $S(p_0^2, m^2; g)$ is by no means obvious.

But next we will show that our particular construction of the renormalized theory allows us to specify it in a natural and simple way.

1.2 The "Zero-Mass" Functions and Their Equations

Assuming that we have found a connecting function $S(p_0^2, m^2; g)$ such that $\bar{\Delta}$, the zero-mass propagator defined by (1.5), will satisfy the condition (1.3), the zero-mass vertex function, and its differential equations, can be readily determined. Let us recall from paper I⁽¹⁴⁾ that if the propagator Δ is known, then the vertex function T obeys the formal graphical equation in Figure 1, which we rewrite, for completeness, in symbolic form as

$$\begin{aligned} T &= K_S + \frac{1}{2} K_S (\Delta \Delta)_S T \\ K_S &= V + \frac{1}{2} K_t (\Delta \Delta)_t T + \frac{1}{2} K_U (\Delta \Delta)_U T \\ K_t &= V + \frac{1}{2} K_U (\Delta \Delta)_U T + \frac{1}{2} K_S (\Delta \Delta)_S T \\ K_U &= V + \frac{1}{2} K_S (\Delta \Delta)_S T + \frac{1}{2} K_t (\Delta \Delta)_t T \end{aligned} \quad (1.7)$$

V and the kernels K_S , K_t , K_U have been properly defined in paper I. It was then also noticed that if we re-define a new vertex and propagators by

$$\bar{T}_R = R^2 T$$

$$\bar{\Delta}_R = \bar{v}^1 \Delta$$

(1.8)

where R is some arbitrary constant, then $\bar{T}_R$ and $\bar{\Delta}_R$ obey (1.7) except for an undefined constant term. Furthermore, $\bar{T}_R$ and $\bar{\Delta}_R$ obey, uniquely, the differentiated form of (1.7), which we will also rewrite for completeness

$$\dot{T} = \dot{K}_S + \dot{\tau}_S$$

$$\dot{K}_S = \dot{V} + \dot{\tau}_t + \dot{\tau}_u$$

$$\dot{K}_t = -\dot{V} + \dot{\tau}_u + \dot{\tau}_s$$

$$\dot{K}_u = \dot{V} + \dot{\tau}_s + \dot{\tau}_t$$

(1.9a)

where

$$\dot{\tau}_s = \frac{1}{3} \pi (\Delta \Delta)_s \dot{T} + \frac{1}{3} \dot{K}_s (\Delta \Delta)_s T +$$

$$+ \frac{1}{3} \pi (\Delta \Delta)_s \dot{K}_s + \frac{1}{4} \pi (\Delta \Delta)_s \dot{K}_s (\Delta \Delta)_s T$$

(1.9b)

All "primed" quantities are to be understood as being differentiated by an appropriate differential operator.

Specializing these results to our case, if $\bar{\Delta}$ is related to Δ via $S(P_0^2, m^2)$ as given by (1.5), then we can define

Therefore, using (1.13), (1.14) and the boundary conditions (1.4b), if the zero-mass propagator $\bar{\Delta}$ is known, one can determine the zero-mass vertex from (1.9) as follows

$$\bar{\Gamma}[P_i, P_f; \bar{g}] = \bar{g} + \int_0^1 \frac{d\lambda}{\lambda} \bar{\Gamma}[\lambda P_i + (1-\lambda)P_f; \bar{g}] \quad (1.15)$$

So, once $S(P_0^2, m^2)$ and $\bar{\Delta}(P^2)$ are found, the first part of our project will be concluded. Their determination will be made in analogy to the method introduced by Baker and Johnson⁽⁴⁾ in order to find the asymptotic form of the electron propagator in QED.

Let us recall, from paper I, the equations that are obeyed by the renormalized propagator

$$[P^2 \Delta(P^2, m^2)]^{-1} = - \int_{-m^2}^{q^2} \frac{dx}{x^2} M(x, m^2/x) \quad (1.16)$$

where

$$M(P^2, m^2) \equiv - P^4 \frac{\partial}{\partial P^2} \left[\frac{1}{P^2 \Delta(P^2, m^2)} \right] \quad (1.17)$$

$M(P^2, m^2)$ turned out to be the trace of the forward field energy-momentum tensor.

It can be explicitly computed by evaluating the integral-differential equation

$$\frac{\partial M}{\partial P^2} = - \frac{P^2}{6} \int_0^\infty dk^2 \frac{\partial}{\partial P^2} \left\langle \frac{K_S(P, k) G_0(k, P)}{P^2} \right\rangle_{\hat{k}} [k^2 \Delta(k^2)]^2 \frac{\partial}{\partial k^2} [\Delta(k^2)] \quad (1.18)$$

subject to the boundary condition

$$\bar{T}[P_i, P_0^0; m^2, \bar{\theta}] = S^2(P_0^0, m^2, g) T(P_i, m^2; g) \quad (1.10)$$

such that $(\lim_{m \rightarrow 0}) \bar{T}$ is finite, if (1.3) is satisfied,

$$(\lim_{m \rightarrow 0}) \bar{T} = \bar{T}\left[\frac{P_i^2}{P_0^0}, \frac{S}{\frac{4}{3}P_0^0}, \frac{t}{\frac{4}{3}P_0^0}; \bar{\theta}\right] \quad (1.11)$$

Then $\bar{T}$ and $\bar{\Delta}$ satisfy the differentiated equation (1.9). In order to have an explicit set of equations, we must define the appropriate differential operator that will make (1.9) a (overlapping) divergence-free equation. Because the mass term is not present in the propagator, the scaling derivative (on-shell)

$$\bar{\Theta}_S = \sum_{i=1}^4 P_i \frac{\partial}{\partial P_i} \quad (1.12)$$

will not give well-defined results upon integration. As suggested by Ward, one has to introduce the off-shell scaling derivative ⁽¹⁶⁾

$$\bar{\Theta}_S^0 = \sum_{i=1}^4 (P_i - P_i^0) \cdot \frac{\partial}{\partial P_i} \quad (1.13)$$

where the momenta P_i^0 obey condition (1.4a). If $A(P_i)$ is a function of the external momenta P_i , then

$$\begin{aligned} \bar{\Delta}'(P_i, P_i^0) &= \bar{\Theta}_S^0 A(P_i) = \\ &= \lambda \frac{d}{d\lambda} \left[A(\lambda P_i + (1-\lambda) P_i^0) \right] \end{aligned} \quad (1.14)$$

$$m(p^2) = m^2 \quad \text{at } p^2 = -m^2 \quad (1.19)$$

K_s is the same kernel defined by (1.7) and $C_2(\hat{p} \cdot \hat{k})$ is the D-wave Gegenbauer polynomial defined by

$$(1+t^2-2tz)^{-1} = \sum_{n=0}^{\infty} C_n(z) t^n \quad (1.20)$$

If we integrate (1.17) with respect to the off-shell point p_0^2 , we will get instead of (1.16)

$$[p^2 \Delta(p^2, m^2)]^{-1} = [p_0^2 \Delta(p_0^2, m^2)]^{-1} - \int_{p_0^2}^{p^2} \frac{dx}{x^2} M(x, m^2/x) \quad (1.21)$$

so that

$$\frac{p_0^2 \Delta(p_0^2, m^2)}{p^2 \Delta(p^2, m^2)} = 1 - p_0^2 \int_{p_0^2}^{p^2} \frac{dx}{x^2} \Delta(p_0^2, m^2) M(x, m^2/x) \quad (1.22)$$

So, if we define

$$p^2 \bar{\Delta}(p^2, p_0^2, m^2) = p^2 \Delta(p^2, m^2) S^{-1}(p_0^2, m^2) \quad (1.23)$$

where

$$S^{-1}(p_0^2, m^2) = p_0^2 \Delta(p_0^2, m^2) \quad (1.24)$$

then $(\lim_{m \rightarrow 0}) p^2 \bar{\Delta}(p^2, p_0^2, m^2)$ will be a finite, well-defined quantity if

$$\left(\lim_{m \rightarrow 0}\right) \left[-P_0^2 \int_{P_0^2}^{P^2} \frac{dx}{x^2} M(x) \Delta(P^2, m^2) \right] = \Theta(P^2, P_0^2) = \text{FINITE} \quad (1.25)$$

and

$$\lim_{P^2 \rightarrow 0} [\Theta(P^2, P_0^2) P^2] = 0 \quad (1.26)$$

The connecting function will then be $P_0^2 \Delta(P_0^2, m^2)$. To investigate the possibilities (1.25) and (1.26), let us define the function

$$F(P^2/P_0^2, g) = \lim_{m \rightarrow 0} [M(P^2) \Delta(P_0^2, m^2)] \quad (1.27)$$

If $F(P^2/P_0^2, \bar{g})$ exists as a finite, well-defined function, then for (1.26) to be true one must impose on it the condition

$$F(P^2) = 0 \quad \text{AT } P^2 = 0 \quad (1.28)$$

otherwise $P^2 \Theta(P^2, P_0^2)$ is either a constant or diverges as $P^2 \rightarrow 0$. Assuming that (1.27) is true, then one can calculate $F(P^2)$ from an equation analogous to (1.18) written in terms of the zero-mass function

$$\frac{\partial F}{\partial P^2} = -a \frac{P^2}{P_0^2} \frac{1}{6} \int_0^\infty dk^2 \frac{\partial}{\partial P^2} \left\langle \frac{\bar{E}_S(P, k) G(P, k)}{P^2} \right\rangle_{\hat{k}} (k^2 \Delta(k^2))^2 \frac{\partial}{\partial k^2} (\Delta^{-1}(k^2))$$

$$a = \pi^2 / (2\pi)^4 \quad (1.29)$$

and so the propagator is given by

$$[P^2 \Delta(P^2)]^{-1} = 1 - \int_1^{P^2/P_0^2} \frac{dz}{z^2} F(x) \quad (1.30)$$

It can be shown by induction that (1.9), (1.15), (1.27), (1.28), (1.29) and (1.30) constitute, order by order in perturbation theory, a completely consistent, unique, finite and exact representation of the short-distance limit: $(q^2, q_0^2) \gg m^2$, for the two- and four-point functions of our theory. The proof is straightforward and will not be given here with full details (it is given in reference (12)). Let us just remark that, because of the structure of the vertex function (see appendix) the $P^2 \rightarrow 0$ behavior of the kernel K_s is such that Eq. (1.29) will always be finite and well-defined⁽¹⁴⁾. We should point out how critical the procedure of the renormalization program is to carrying out the zero-mass limiting procedure. Had we used Eq. (3.35) of I instead of (3.43), which gives (1.16), we would have run into complications, for the simple constraint given by (1.28) would have to have been replaced by a sequence of constraints on $\theta_1(P^2)$ and all its derivatives, for the program to work.

The simplicity and compactness of the renormalized equation has been transmitted, then, to the equations for $\bar{\Delta}$ and $\bar{T}$. Furthermore, since now the propagators have a simpler structure with respect to the renormalized ones, easier explicit analytic iterative solutions can be carried out, if enough patience and interest are at hand. In the appendix we discuss our iterative method for solving equations (1.9), and we report the results for $\bar{\Delta}$ to third order in $\bar{g}$ and for $\bar{T}$ to fourth order in $\bar{g}$ in the tables.

Chapter 2: The Short Distance Behavior

Using only the fact that the theory in question is renormalizable, after a sequence of clever manipulations of equations expressing such a fact, as for example (1.8), one can derive the following relationship between propagators and vertex functions defined at various points⁽¹⁾

$$\begin{aligned} I\left(\frac{p^2}{m^2}; \frac{s}{\frac{4}{3}m^2}; \frac{t}{\frac{4}{3}m^2}; -\frac{m^2}{p^2}; g\right) &= I\left[\frac{p^2}{p^2}; \frac{s}{\frac{4}{3}p^2}; \frac{t}{\frac{4}{3}p^2}; \frac{p^2}{m^2}; \frac{p^2}{p^2}; g\right] \\ &= I\left[\frac{p^2}{Q^2}; \frac{s}{\frac{4}{3}Q^2}; \frac{t}{\frac{4}{3}Q^2}; \frac{Q^2}{m^2}; \frac{Q^2}{p^2}; g\right] = \dots \end{aligned}$$

(2.1)

where

$$\begin{aligned} I\left[\frac{p^2}{m^2}; \frac{s}{\frac{4}{3}m^2}; \frac{t}{\frac{4}{3}m^2}; -\frac{m^2}{p^2}; g\right] &= \left[d\left(\frac{p^2}{m^2}; g\right)\right]^2 \bar{\pi}\left[\frac{p^2}{m^2}; \frac{s}{\frac{4}{3}m^2}; \frac{t}{\frac{4}{3}m^2}; g\right] \\ &= \left[\bar{d}\left(\frac{p^2}{p^2}; \frac{m^2}{p^2}; g\right)\right]^2 \bar{\pi}\left[\frac{p^2}{p^2}; \frac{s}{\frac{4}{3}p^2}; \frac{t}{\frac{4}{3}p^2}; \frac{m^2}{p^2}; g\right] \\ &= \left[\bar{d}\left(\frac{p^2}{Q^2}; \frac{m^2}{Q^2}; g\right)\right]^2 \bar{\pi}\left[\frac{p^2}{Q^2}; \frac{s}{\frac{4}{3}Q^2}; \frac{t}{\frac{4}{3}Q^2}; \frac{m^2}{Q^2}; g\right] = \dots \end{aligned}$$

(2.2)

The functions π , $\bar{\pi}$, $\bar{\pi}$, etc. are the vertex functions defined at the various symmetry points, $\frac{4}{3}m^2$, $\frac{4}{3}p^2$, $\frac{4}{3}Q^2$, etc. The functions d , $\bar{d}$, $\bar{d}$, etc. represent the propagators defined at those points

$$\bar{d}\left[\frac{p^2}{P^2}; \frac{P_0^2}{m^2}; \bar{g}\right] = (q^2 + m^2) \bar{\Delta}\left(\frac{p^2}{P^2}; \frac{P_0^2}{m^2}; \bar{g}\right) \quad (2.3)$$

and obey functional equations of the type

$$\bar{d}\left[\frac{p^2}{P^2}; -\frac{m^2}{P^2}; \bar{g}\right] = \bar{d}\left[\frac{Q_0^2}{P^2}; -\frac{m^2}{P^2}; \bar{g}\right] \bar{d}\left[\frac{p^2}{Q_0^2}; -\frac{m^2}{Q^2}; \bar{g}\right] \quad (2.4)$$

The imposition of the boundary conditions defining the T's and d's implies that function I must satisfy:

$$\begin{aligned} I(1; g) &= g \\ I(1; \bar{g}) &= \bar{g} \\ I(1, \bar{\bar{g}}) &= \bar{\bar{g}} \quad \text{ETC.} \end{aligned} \quad (2.5)$$

Equations (2.1) and (2.5) imply that function I is independent of the different subtraction points. It is called the invariant charge. It should be stressed, again, that Eq. (2.1) as it stands, beyond saying that the theory is a renormalizable one, cannot be used to extract any physical prediction.⁽¹⁷⁾ As Gell-Mann and Low pointed out, only if $\left[\lim_{m \rightarrow 0}\right] I$ exists and gives a finite result, then one might draw some conclusions about the asymptotic behavior of the theory.

In the previous chapter we have shown that the $\left(\lim_{m \rightarrow 0}\right)$ does exist. In fact we have exact, explicit expressions for the $\left(\lim_{m \rightarrow 0}\right) \bar{\Delta}$ and $\bar{T}$ functions and, therefore, as a consequence of (1.10) and (1.24) we can construct the exact $\left(\lim_{m \rightarrow 0}\right)$ invariant

function I_0

$$\begin{aligned}
 I_0 &= \left(\lim_{m \rightarrow 0} \right) I \left[\frac{P^2}{m^2}; \frac{S}{\frac{4}{3}m^2}; \frac{t}{\frac{4}{3}m^2}; \frac{P^2}{m^2}; g \right] = \\
 &= \left(\lim_{m \rightarrow 0} \right) \left\{ \left[(P^2 + m^2) \Delta(P^2/m^2, g) \right]^2 \overline{\Pi} \left(\frac{P^2}{m^2}; \frac{S}{\frac{4}{3}m^2}; \frac{t}{\frac{4}{3}m^2}; g \right) \right\} = \\
 &= \left[P^2 \overline{\Delta}(P^2/P_0^2; \bar{g}) \right]^2 \overline{\Pi} \left[\frac{P^2}{P_0^2}; \frac{S}{\frac{4}{3}P_0^2}; \frac{t}{\frac{4}{3}P_0^2}; \bar{g} \right]
 \end{aligned}$$

(2.6)

If we now set

$$(P_i + P_j)^2 = \frac{4}{3}(1 + 2\delta_{ij})P^2 \quad (2.7)$$

the invariant charge is seen to obey the functional relationship

$$I_0(P^2) = \left[P^2 \overline{\Delta}(P^2/P_0^2; I_0(P_0^2)) \right]^2 \overline{\Pi} \left[P^2/P_0^2; I_0(P_0^2) \right] \quad (2.8)$$

where we have used the boundary condition (2.5) to express $\bar{g}$ in terms of I . This relationship can be made more explicit by differentiating (2.8)

$$P^2 \frac{\partial}{\partial P^2} I_0(P^2) \Big|_{P_0^2 = P^2} = \Phi \left[I_0(P^2) \right] \quad (2.9)$$

Furthermore, because of equations (1.17) and (1.27), we have that

$$F(P^2/P_0^2; I_0(P_0^2)) = - \frac{P^2}{P_0^2} \frac{\partial}{\partial P^2} \left[\frac{\bar{\Delta}^{-1}(P^2/P_0^2; I_0(P_0^2))}{P^2} \right] \quad (2.10)$$

therefore

$$\Phi(I_0(P^2)) = 2 F(1, I_0(P^2)) I_0(P^2) + \frac{P^2}{\partial P^2} \bar{T} \left[\frac{P^2}{P_0^2}; I_0(P_0^2) \right] \Big|_{P_0^2 = P^2} \quad (2.11)$$

Because of the connection between $\bar{\Delta}$ to the renormalized propagator via the $\left(\lim_{m \rightarrow 0} \right)$ process (2.10) determines, exactly, its asymptotic behavior

$$\frac{P^2}{\partial P^2} \left[\log P^2 \Delta(P^2/m^2) \right]_{P^2 \gg m^2} = F(1, I_0(P^2)) \quad (2.12)$$

Coupled equations (2.9), (2.11) and (2.10) are exact statements of the GML equations (also known by the awkward name of renormalization group equations) which specify the short-distance behavior of the theory. As they stand, their solution appears to involve quite a formidable task. One can calculate $\Phi(I)$ and $F(1, I)$ in perturbation theory as a power series in I using the methods of chapter 1 (the lowest order results are compiled in Table 2). Conversely, the knowledge of $\Phi(I)$ allows for the determination of all the log terms of $\bar{T}[Q^2/Q_0^2, I(Q_0^2)]$. Whether these terms can be summed up into a well-defined analytic function is not at all clear yet. As we mentioned in the introduction, Eq. (2.9) could allow for the solution

$$\begin{aligned} I_0(P^2) &= q_0 & \text{if} \\ \Phi(q_0) &= 0 \end{aligned} \quad (2.13)$$

that is, g_0 is a root of the function $\Phi(x)$. This "bare" asymptotic coupling, which will obviously be independent of $\bar{g}$ and g , the zero-mass and physical coupling constant, respectively, is called the eigenvalue and (2.13) the eigenvalue equation. The nature of the eigenvalue and its consequences with respect to our short-distance theory will be explored in the next section.

Chapter 3: The Theory of the "Eigenvalue"

3.1 Existence of the Eigenvalue

In what follows, we will assume that (2.13) is a valid statement and its impact on the theory will thus be analyzed. Whether this assumption is justified or not, we are not able to answer. An intensive, semi-quantitative study of (2.13) has been carried out within our only known and available means, perturbation theory⁽¹²⁾. Basically it was concluded that, because of the nature of relevant integrals leading to its computation to some high order of perturbation (say $N = 5$ to $N \approx 10$), $\Phi^{(N)}(g_0)$ is a polynomial with coefficients of alternating signs, the magnitude of the coefficients appearing to grow as N increased. Whether this behavior persists at higher and higher orders, we cannot tell with certainty. One is, unfortunately, faced with the problem of making predictions which are beyond perturbation theory and yet one has, seemingly, just perturbation theory as a mechanism to work with. It is not at all improbable that by studying the consequences of the eigenvalue condition on the theory, one could get, at least, an inkling of the general nature of these roots⁽¹¹⁾. We will try to report further on this matter in a future paper.

3.2 Green's Functions at the Eigenvalue and the Conformal Group

If (2.13) is satisfied by some $g = g_0$, then the other GML equation becomes

$$P^2 \frac{\partial}{\partial P^2} \left[\log P^2 \Delta(P^2 - m^2) \right] = F(1, g_0) = F(g_0)$$

(3.1)

which can be readily solved to give (in the $\left(\lim_{m \rightarrow 0}\right)$ region)

$$\begin{aligned}\Delta(P^2/m^2) &= \frac{1}{P^2} \left(\frac{P^2}{m^2}\right) F(g_0) \\ \bar{\Delta}(P^2/B^2) &= \frac{1}{P_0} \left(\frac{P^2}{B^2}\right) F(g_0)\end{aligned}\tag{3.2}$$

and thus expresses the exact asymptotic behavior of the propagator function. Some interesting consequences are attached to this result. First, we notice that

$$F(g_0) > 0\tag{3.3}$$

since otherwise the theory would have unwanted ghosts as $P^2 \rightarrow 0$. This is an additional constraint that g_0 , the eigenvalue, must obey. The third order in the g_0 result for $F(g_0)$ is given in the tables. The second consequence of this result is that the renormalization constant Z , defined by

$$\lim_{P^2 \rightarrow -m^2} \Delta^u(P^2) = \frac{Z}{P^2 + m^2}\tag{3.4}$$

where Δ^u is the unrenormalized propagator and m^2 is the physical mass, will be in the $\left(\lim_{m \rightarrow 0}\right)$

$$Z = 0\tag{3.5}$$

The theory at the eigenvalue is, thus, non-canonical. This behavior can also be understood if we look at the situation from a different angle, namely by introducing

the 15-parameter space-time transformation group, the conformal group⁽¹⁵⁾, given by the Poincaré group and

$$\text{Dilatations: } x_\mu \rightarrow \lambda x_\mu \quad (3.6)$$

$$\text{Inversions: } x_\mu \rightarrow x_\mu / x^2 \quad (3.7)$$

and the Special Conformal Transformation:

$$x_\mu \rightarrow \frac{x_\mu + Q_\mu x^2}{1 + 2Q \cdot x + Q^2 x^2} \quad (3.8)$$

The generators of dilatations, D , obey the canonical field commutation rule

$$i [D, \tilde{\varphi}(x)] = (x \cdot \partial_x + \frac{d}{2}) \tilde{\varphi}(x) \quad (3.9)$$

where d is the dimension of the field. For the free field case the canonical dimension is equal to the physical dimension of the field. Once interactions are introduced one realizes that, in perturbation theory, the concept of dimensionality starts losing meaning, as can be observed by analyzing the structure of the propagator⁽¹⁸⁾. This is, of course, due to the strong dilatation-breaking effects introduced by the regularizations which are necessarily brought in to make the theory finite. At the eigenvalue, we note from (3.2), all the breaking terms, coming in this theory as a sequence of logarithmic terms, sum up to a simple power behavior. The concept of dimensionality is thus again recovered, but now the dimension of the field has the non-canonical value.

$$d g_0 = 1 + F(g_0) \quad (3.10)$$

The eigenvalue propagator is thus a conformal covariant object like the free propagator. Some interesting conclusions, along these lines, can also be made for the eigenvalue vertex function. Let us recall Eq. (2.8) defining the invariant charge. At the eigenvalue it will be now written as

$$g_0 = \left(\frac{p^2}{P_0^2} \right)^{2F(g_0)} \bar{\Gamma}(P^2/P_0^2, g_0) \quad (3.11)$$

and, therefore, the vertex function evaluated at the invariant symmetry point defined by (2.7) is

$$\bar{\Gamma}(P^2/P_0^2, g_0) = g_0 \left(P^2/P_0^2 \right)^{-2F(g_0)} \quad (3.12)$$

In terms of the zero-mass coupling $\bar{g}$, $\bar{\Gamma}$ can be written, in perturbation theory

$$\bar{\Gamma}(P^2/P_0^2; \bar{g}) = \bar{g} + \sum_{n=1}^{\infty} \psi_n(\bar{g}) \log^n \frac{P^2}{P_0^2} \quad (3.13)$$

so, when $\bar{g}(g_0) = 0$ the coefficients ψ_n must be such that the power behavior (3.12) is obtained. The condition is that (at the eigenvalue only)

$$\psi_n(\bar{g}) = - \frac{(2F)^{n-1}}{n!} (-1)^n \quad (3.14)$$

Equation (3.12) is a clear indication that the zero-mass vertex, at the eigenvalue, is not just scale covariant but also that its coordinate space representation is conformal

covariant. That is, we can find a function $S(p_1^2, p_2^2, p_3^2, p_4^2; s, t)$ such that

$$\tau(p_1, p_2, p_3, p_4) = \frac{S(p_1^2, p_2^2, p_3^2, p_4^2; s, t)}{(p_0^2)^{2F(\phi)}} (2\pi)^4 i \delta\left(\sum_{i=1}^4 p_i\right) \bar{\tau}(p_1, p_2, p_3, p_4) \quad (3.15)$$

has a coordinate space transform

$$\tau(x_1, x_2, x_3, x_4) = \int \frac{d^4 p_1 d^4 p_2 d^4 p_3 d^4 p_4}{(2\pi)^{16}} e^{-i \sum_{i=1}^4 p_i \cdot x_i} \tau(p_1, p_2, p_3, p_4) \quad (3.16)$$

where τ is related to the connected piece of $\langle 0 | T^*(\varphi(x_1) \varphi(x_2) \varphi(x_3) \varphi(x_4)) | 0 \rangle$ which is invariant under the rules (3.6) - (3.9). These constrain τ to be of the form

$$\tau(x_1, x_2, x_3, x_4) = \Theta(X, Y) \quad (3.17)$$

where X and Y are the four-dimensional harmonic ratios

$$\begin{aligned} X &= \frac{x_{12} x_{34}}{x_{13} x_{24}} \\ Y &= \frac{x_{12} x_{34}}{x_{14} x_{23}} \end{aligned} \quad (3.18)$$

where $x_{ij} = (x_i - x_j)^2$ for $i \neq j$. Conformal invariance, per se, cannot specify more than (3.17). Since our theory must be invariant under the coordinate changes $x_i \leftrightarrow x_j$, $i \neq j$, Θ is further constrained to obey

$$\begin{aligned}\Theta(x, y) &= \Theta(y, x) = \Theta\left(\frac{1}{x}, \frac{y}{x}\right) = \\ &= \Theta\left(\frac{x}{y}, \frac{1}{y}\right)\end{aligned}\quad (3.19)$$

Once the obvious choice of $S(p_1^2, p_2^2, p_3^2, p_4^2; s, t)$ is specified we will show that, within our theoretical framework, $\tau(p_1, p_2, p_3, p_4)$ can be computed in principle.

3.3 Conformal Scattering Amplitude

As we have mentioned in paper I, the nature of the interaction allows us to identify the vertex function of the theory with a two-body, off-shell, scattering amplitude. In order to work with the vertex function itself, let us define

$$\begin{aligned}\tau(p_1, p_2, p_3, p_4) &= (2\pi)^4 i \delta^4\left(\sum_{i=1}^4 p_i\right) G(p_1, p_2, p_3, p_4) \\ G(p_1, p_2, p_3, p_4) &= \frac{S(p_i^2; s, t)}{(p_0^+)^{2F}} \bar{T}[p_1, p_2, p_3, p_4; q_0]\end{aligned}\quad (3.20)$$

so that, in coordinate space, C will be a conformal density, and from (3.17)

$$C(x_1, x_2, x_3, x_4) = \frac{\Theta(x, y)}{x_{12} x_{13} x_{14} x_{23} x_{24} x_{34}} \quad (3.21)$$

The simplest choice of S , in accordance with (3.11) and (3.12) is, obviously

$$S(p_i^2; s, t) = \prod_{i=1}^4 (p_i^2)^{F/2} \quad (3.22)$$

of the log terms is due only to renormalization procedure. The relevance of these characteristic terms will become more transparent at the eigenvalue. From (3.23) and (3.24) and the eigenvalue condition, we obtain, in fact

$$C(P_1 P_2 P_3 P_4; q) = \left\{ \left[q/3 + t(P_1 P_2; P_3 P_4; q) \right] \prod_{i=1}^4 \left(\frac{P_i^2}{S} \right)^{\frac{F(q)}{2}} + \right. \\ \left. + (2 \leftrightarrow 3; S \leftrightarrow U) + (2 \leftrightarrow 4; S \leftrightarrow t) \right\} \quad (3.25)$$

The third order contribution to t is given in the tables. One can further obtain a much simpler form for C if we exploit the invariant nature, Eq. (1.8), of the vertex equations. By applying (3.23) to the equation for $\bar{T}$ and defining (the notation should be obvious)

$$G_{S,t,U} = \prod_{i=1}^4 \left(\frac{P_i^2}{P_0^2} \right)^{F/2} \bar{K}_{S,t,U} \quad (3.26) \\ V_C = \prod_{i=1}^4 \left(\frac{P_i^2}{P_0^2} \right)^{F/2} \bar{V}$$

and using Eq. (3.2), the "conformal amplitude" C will satisfy the formal equation (symbolically)

$$C = C_S + \frac{1}{3} C_S (\Delta_0 \Delta_0)_S C \\ C_S = \bar{V}_C + \frac{1}{2} C_t (\Delta_0 \Delta_0)_t C + \frac{1}{2} C_U (\Delta_0 \Delta_0)_U C \\ C_t = \bar{V}_C + \frac{1}{2} C_U (\Delta_0 \Delta_0)_U C + \frac{1}{2} C_S (\Delta_0 \Delta_0)_S C \\ C_U = \bar{V}_C + \frac{1}{2} C_S (\Delta_0 \Delta_0)_S C + \frac{1}{2} C_t (\Delta_0 \Delta_0)_t C \quad (3.27)$$

Δ_0 is the free, massless propagator. Since C obeys the boundary condition at the symmetry point

$$C(P_i^S; g_0) = g_0 \quad (3.28)$$

$$(P_i^S + P_j^S)^2 = \frac{4}{3}(1 + 2\delta_{ij})P^2 \quad (3.29)$$

one can define differentiated equations analogous to (1.9). We should remark that the integrals in (3.27) are finite because of the good convergence of the functions t defined by (3.24). Furthermore, because of (3.5) there will be no ambiguous constant in V_c . From the result (3.25) the function C , once g_0 is known, is strictly dependent only on the ratio of invariants and external masses and one cannot properly talk about a "series expansion in terms of g_0 ". Equation (3.25) is just the inheritance of perturbation theory which is the only method we know to compute $F(g_0)$ and $t(Q_1 Q_2 Q_3 Q_4; g_0)$. Equation (3.27) and its differentiated form are a more precise, though less explicit (since their solution is quite hopeless) statement about C . With this in mind, let us now try to solve (3.27) via its appropriate differentiated form and boundary conditions with respect to some coupling parameter. The result would effectively be the same as solving the "zero-mass" theory with free propagators only. If we now force the expansion parameter to be g_0 , the eigenvalue, then because of (3.25) we would have as a result

$$C(P_1 P_2 P_3 P_4; g_0) = g_0 + \overset{F=0}{t}(P_1 P_2; P_3 P_4; g_0) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \quad (3.30)$$

and

$$\psi^{F=0}(\infty) = 0 \quad (3.31)$$

By the subscript, $F=0$, we mean that we are computing graphs without propagator self-energy insertions. Equation (3.31) is a somewhat simpler version of the eigenvalue condition (2.13). Effectively, it is a reflection of the fact that the "zero-mass" vertex function, without propagator self-energy insertions, is the relevant "invariant charge". The function $\Gamma^{F=0}$, at the eigenvalue, is then related to the function calculated with the full propagator by

$$\begin{aligned} \Gamma^{F \neq 0}(p_1, p_2, q) &= \frac{g_0}{g} \left(1 + \prod_{i=1}^4 \left(\frac{S}{4p_i^2} \right)^{F/2} \right) + \\ &+ \prod_{i=1}^4 \left(\frac{4p_i^2}{3S} \right)^{F/2} \Gamma^{F=0}(p_1, p_2, q) \end{aligned} \quad (3.32)$$

Its conformal invariance reflects the conformal invariant nature of the massless quartic boson self-interaction. Because, at the eigenvalue, the theory has some characteristics which are reminiscent of the unquantized, canonical theory, we could call it a "Classical-like" asymptotic theory. This "classical-like" nature is also observed if we analyze a relevant energy-momentum tensor in this theory.

3.4 The Energy-Momentum Tensor

The existence of the eigenvalue condition has some interesting consequences in regard to the short-distance behavior of the energy-momentum tensor. Some years ago Callan, Coleman and Jackiw⁽¹⁵⁾ pointed out that, canonically, the energy-momentum

tensor which is relevant to studying the breaking of scale and conformal transformations is given by

$$\Theta_{CCJ}^{\mu\nu} = \Theta_C^{\mu\nu} - \frac{1}{6}(\partial_\mu \partial_\nu - g^{\mu\nu} \square) \varphi^2 \quad (3.33)$$

where $\Theta_C^{\mu\nu}$ is the canonical tensor. $\Theta_{CCJ}^{\mu\nu}$ has the same Poincaré structure as $\Theta_C^{\mu\nu}$ but it has the additional property that

$$\text{Tr } \Theta_{CCJ}^{\mu\nu} = \partial_\mu D^\mu \quad (3.34)$$

where D^μ is the dilatation current. D^μ is conserved for the massless case and $\Theta_{CCJ}^{\mu\nu}$ is traceless. When quantization effects are taken into consideration, condition

(3.34) is never satisfied and renormalization effects leave a term of the form

$(\partial_\mu \partial_\nu - g^{\mu\nu} \square) \varphi^2$ unspecified¹⁴. At short distances, and assuming the existence of the eigenvalue, Shroer⁽¹⁹⁾ has shown that the trace of the tensor vanishes. Within

our theoretical framework this can be easily seen, as follows. In paper I, we defined a forward energy-momentum tensor by exploiting the invariance of the theory with respect to general coordinate transformations through the insertion of a zero-energy graviton into the propagator. The resulting Ward identity thus defined the tensor. If

we now, instead, make an insertion into the vertex function and go to our $(\lim_{m \rightarrow 0})$ region, the resulting trace identity will be proportional, at the symmetry point, to

$p^2 \frac{\partial}{\partial p^2}$ of the invariant charge, and thus proportional to $\Phi(g_0) = 0$ if there is

an eigenvalue. More interesting, yet, is the fact that within the "semi-classical

asymptotic" theory defined by Δ_0 and C , the relevant energy-momentum tensor is

the CCJ tensor. Let us recall from I that the traceless energy-momentum tensor obeys the equation

$$\begin{aligned} \bar{\Theta}^{\mu\nu}(P, P+Q) = & (P_\mu P_\nu - \frac{1}{4} g^{\mu\nu} P^2) A(P, Q) + \frac{1}{2} (P_\mu Q_\nu + P_\nu Q_\mu - \frac{1}{2} g^{\mu\nu} P \cdot Q) B(P, Q) + \\ & + (Q_\mu Q_\nu - \frac{1}{4} g^{\mu\nu} Q^2) C(P, Q) \end{aligned} \quad (3.35)$$

$$\bar{\Theta}^{\mu\nu}(P, P+Q) = \bar{\Theta}_0^{\mu\nu} + \int \frac{d^4 k}{(2\pi)^4} K_S \Delta \Delta \bar{\Theta}^{\mu\nu}(k, k+Q)$$

The form factors A and B are connected via Ward's identity

$$Q_\mu \bar{\Theta}^{\mu\nu} = -P^\nu \Delta^-(P+Q) - (P+Q)^\nu \Delta^-(P) \quad (3.36)$$

where the full tensor $\Theta^{\mu\nu}$ is given by

$$\Theta^{\mu\nu} = \bar{\Theta}^{\mu\nu} + \frac{1}{4} g^{\mu\nu} \text{Tr}[\Theta^{\mu\nu}] \quad (3.37)$$

The form factor $C(P, Q)$ specifies the trace of $\Theta^{\mu\nu}$, once A and B are known.

A scheme to compute to all orders finite A, B and C has been given to within

specifying $C(0, 0) = C_0 = \text{undetermined constant}$. We can now define a "zero-mass"

energy-momentum tensor by taking the $(\lim_{m \rightarrow 0})$

$$\widetilde{\bar{\Theta}}^{\mu\nu}(P, P+Q; P_0^2) = P_0^2 \Delta(P_0^2/m^2) \bar{\Theta}^{\mu\nu}(P, P+Q; m^2) \quad (3.38)$$

so that its traceless part obeys the formal equation

$$\tilde{\Theta}^{\mu\nu} = \tilde{\Theta}_0^{\mu\nu} + \frac{1}{2} \int \bar{K}_S \Delta \Delta \tilde{\Theta}^{\mu\nu} \quad (3.39)$$

In analogy to the analysis given in paper I, one can reduce the above equation into a set of differential-integral equations for the zero-mass form factors $\bar{A}$, $\bar{B}$ and $\bar{C}$, analogous to Eqns (4.51), (4.54) and (4.59) of that paper. If we are at the eigenvalue, since the form factors are proportional to Δ^{-1} , their P_0^2 dependence is $\sim (P_0^2)^F$. So we can define a traceless tensor which is scale covariant with respect to P and Q as follows

$$\tilde{\zeta}^{\mu\nu} = \left(\frac{P^2}{P_0^2} \right)^{\frac{F}{2}} \left(\frac{(P+Q)^2}{P_0^2} \right)^{\frac{F}{2}} \tilde{\Theta}^{\mu\nu}(P, P+Q) \quad (3.40)$$

Because of condition (3.5), the homogeneous term is not present, so that we obtain an equation

$$\tilde{\zeta}^{\mu\nu} = \frac{1}{2} \int C_S \Delta_0 \Delta_0 \tilde{\zeta}^{\mu\nu} \quad (3.41)$$

Since the form factors are scale-invariant, if we take a scaling derivative of the above, we obtain, obviously

$$\left(Q \cdot \frac{\partial}{\partial Q} + P \cdot \frac{\partial}{\partial P} \right) [\tilde{A}_c, \tilde{B}_c, \tilde{C}_c] = 0 \quad (3.42)$$

But, within the context of our theory, the Ward identity which specifies the boundary conditions on A and B is

$$Q^\mu \tilde{C}^{\mu\nu} = -P^\mu (Q+P)^2 - Q^\nu P^2 \quad (3.43)$$

and therefore $\tilde{A}_C = -1$, $\tilde{B}_C = -1$. At this stage nothing can be said about $\tilde{C}_C$ beyond recognizing that it must be some (arbitrary) scale invariant function of P , Q .

Since the existence of the eigenvalue is consistent with the vanishing of the trace of the asymptotic tensor $\tilde{C}^{\mu\nu}$, if we impose this condition, then $C_C = 1/6$ and therefore

$$\tilde{C}_{\mu\nu} = -\left(\frac{P_\mu P_\nu}{2} - \frac{1}{4} g^{\mu\nu} P^2\right) - \frac{1}{2} (P_\mu Q_\nu + P_\nu Q_\mu - \frac{1}{2} g^{\mu\nu} P \cdot Q) + \frac{1}{6} (Q_\mu Q_\nu - \frac{1}{4} g^{\mu\nu} Q^2)$$

which is the same as the CCJ tensor. We will then call $C^{\mu\nu}$ the "classical-like" asymptotic energy-momentum tensor.

Appendix A : Computational Techniques

A.1 Feynman Rules

The Feynman rules are defined, symbolically, by the graphical equation of Figure 1 and by the Eqns (1.7) and (1.9). We will summarize them here a little more explicitly:

1) Replace every non-differentiated vertex by $\bar{\Gamma}(Q_1, Q_2, Q_3, Q_4)$.

2) Replace every non-differentiated propagator line by

$$\bar{\Delta}(Q; \bar{g}) = Q^2 \left(1 + \sum_{n=1}^{\infty} \bar{g}^n \bar{\Delta}_n(Q^2/Q_0) \right)$$

3) Replace every differentiated kernel and propagator by

$$\bar{A}(Q_i) = \sum_{i=1}^4 (Q_i \cdot Q_i^0) \cdot \bar{Q}_i \cdot A(Q_i) = \lambda \frac{d}{d\lambda} A(\lambda Q_i + (1-\lambda) Q_i^0)$$

where $A = \{K_s, K_t, K_u, \bar{V}, \bar{\Delta}\}$ and $(Q_i^0 + Q_j^0) = 4/3 (1 + 2\delta_{ij}) Q_0^2$.

4) For every integration loop write a factor

$$i \frac{d^4 k}{(2\pi)^4}$$

A.2 Iteration of the Vertex Function Equations

A.2.1 Iteration Scheme

Assuming that the propagator is known, from (1.9a) and (1.9b), we note that we can write

$$\bar{\Gamma} = \bar{\Gamma}_s + \bar{\Gamma}_t + \bar{\Gamma}_u \quad (\text{A.1})$$

and integrating

$$\Gamma = \Gamma_s + \Gamma_t + \Gamma_u \quad (\text{A.2})$$

where

$$\begin{aligned} \tau'_t = & \frac{1}{3} \bar{V} + \frac{1}{3} K_t (\Delta \Delta)_t \bar{T} + \frac{1}{3} \bar{T} (\Delta \Delta)_t K_t + \\ & + \frac{1}{3} \bar{T} (\Delta \Delta)'_t \bar{T} + \frac{1}{4} \bar{T} (\Delta \Delta) K_t (\Delta \Delta) \bar{T} \end{aligned} \quad (A.3)$$

Computation of τ'_t and its subsequent integration are very straightforward to third order since they involve an integration over a single loop. Furthermore, we want now to show that, in an order by order sense, τ'_t can be written, for the order N

$$\tau^{(N)}_t = \sum_{i=1}^4 (P_i, P_i^0) \cdot \frac{\partial}{\partial P_i} \left\{ \bar{T}^{(N)}_t [\bar{T}^{(N_1)}, \bar{T}^{(N_2)}, \bar{\Delta}^{(N_3)}] \right\} \quad (A.4)$$

where N_1 , N_2 and N_3 are smaller than N , so that $\bar{T}^{(N_1)}$, $\bar{T}^{(N_2)}$ and $\bar{\Delta}^{(N_3)}$ are explicitly known from a previous iteration step; and $\bar{T}^{(N)}$ involves only the evaluation of a single loop integral. Result (A.4) follows essentially from the fact that if we iterate the formal equation repeatedly, we obtain a sum of terms that depends only on $\bar{T}$ and $\bar{\Delta}$ plus a single term that depends on the undefined kernels. More explicitly, one iteration of (1.7) will yield:

$$\bar{T}_t = \frac{1}{3} \bar{V} + \frac{1}{2} \bar{T} (\Delta \Delta)_t \bar{T} - \frac{1}{4} \bar{T} (\Delta \Delta)_t K_t (\Delta \Delta)_t \bar{T} \quad (A.5)$$

up to second order the term depending on K_s , K_t , K_u will not be present, so

$$\bar{T}^{(2)}_t = \frac{1}{3} \bar{T}^0 (\Delta \Delta)_t \bar{T}^0 - \{Q_i = Q_i^0\} \quad (A.6)$$

By repeating the iteration N times, we will have,

$$\bar{T} = \bar{T}_t + \bar{T}_s + \bar{T}_0$$

where

$$\begin{aligned} \bar{T}_t = & \frac{1}{3}\bar{V} + \frac{1}{3}\bar{T}(\Delta\Delta)_t\bar{T} - \frac{1}{4}\bar{T}(\Delta\Delta)_t\bar{T}(\Delta\Delta)_t\bar{T} + \\ & + \dots + \frac{(-1)^{N+1}}{2^N} \bar{T}(\Delta\Delta)_t\bar{T} \dots \bar{T}(\Delta\Delta)_t\bar{T} + \\ & + \frac{(-1)^N}{2^{N+1}} \bar{T}(\Delta\Delta)_t\bar{T} \dots \bar{T}(\Delta\Delta)_t\bar{T}(\Delta\Delta)_t\bar{T}(\Delta\Delta)_t\bar{T} \end{aligned} \quad (A.7)$$

$\leftarrow N \text{ times} \rightarrow$

The last term, depending on K_t , will not contribute to N^{th} order perturbation theory. We will define $\bar{T}_t^{(N)}$ to be given by the above formula without the $\bar{K}_t$ term. Furthermore, by using the analogous expressions for the lower orders we can re-express it, compactly, by the following one loop integral which is to be used to evaluate (A.4) order by order

$$\begin{aligned} & [\bar{T}^{(N)} - \frac{1}{3}\bar{V}^{(N)}] = \\ & = \frac{1}{3} \left\{ \sum_{n_1=0}^{N-2} \sum_{n_2=0}^{N-2-n_1} \sum_{n=1}^{N-n_1-n_2-1} \left[A_{n_1 n_2 n}^{(N)} \left(\bar{T}^{(n)} - b_n^{N-n_1-n_2} \bar{T}_t^{(n)} \right) (\Delta^{(n_1)} \Delta^{(n_2)})_t \bar{T}^{(N-n_1-n_2-n)} \right. \right. \\ & \quad \left. \left. + \bar{T}^{(n)} (\Delta^{(n_1)} \Delta^{(n_2)})_t \left(\bar{T}^{(N-n_1-n_2-n)} - b_{N-n_1-n_2-n}^{N-n_1-n_2} \bar{T}_t^{(N-n_1-n_2-n)} \right) \right] \right\} \\ & \quad - [\phi_i = \phi_i^0] \end{aligned} \quad (A.8)$$

where the coefficients $A_{n_1 n_2 n}^N$ have the values

$$A_{00n}^N = 1 - \frac{1}{2} \delta_{nN} \quad (\text{A.9})$$

and

$$A_{n_1 n_2 n}^N = 1 - \frac{1}{2} \delta_{n, n_1 - n_2 + N} \quad (\text{A.10})$$

The coefficients $b_n^{N-n_1-n_2}$ turn out to take the values

$$b_n^{N-n_1-n_2} = (1 - \delta_{1n}) \tilde{b}_n^{N-n_1-n} \quad (\text{A.11})$$

where

$$\tilde{b}_n^{N-n_1-n_2} = \frac{1}{2} (1 + \delta_{n, N-n_1-n_2-n}) \quad (\text{A.12})$$

Let us now analyze the basic structure of the one loop integrals that are involved in the above scheme.

A.2.2 Basic Integrals and Gegenbauer Polynomials

A typical graphical contribution to (A.8) is given in Figure . It is easy to realize that all these contributions are complicated generalizations of the following integrals

$$I_N(q_1, q_4) = i \int \frac{d^4 k}{(2\pi)^4} \log \frac{N k^2}{\frac{4}{3} q_4^2} \Delta^{(0)}(k-q_1) \Delta^{(0)}(k+q_4) \quad (\text{A.13})$$

which, in Euclidean form, becomes

$$I_n(Q_1, Q_4) = -Q J_n(Q_1, Q_4) \quad (A. 14)$$

where

$$J_n(Q_1, Q_4) = \int_0^\infty dk^2 k^2 \log \frac{k^2}{\frac{Q}{3} Q_3} \left\langle \frac{1}{(k-Q_1)^2} \frac{1}{(k+Q_4)^2} \right\rangle_{\{Q_i = Q_i^0\}} - \quad (A. 15)$$

and

$$Q = \pi^2 / (2\pi^4) \quad (A. 16)$$

The generalized integrals will contain log terms that depend on the external momenta and will thus involve much more complex angular dependences. Interestingly enough, it is easy to verify that one can compute, to all orders, the coefficients of the logarithmic powers by evaluating a sequence of J_N integrals. In fact, for this purpose it is sufficient to set either Q_1 or Q_4 equal to zero in (A. 16). Furthermore, explicit evaluation of J_n will determine the basic structure of the Q_0^2 -independent terms of the vertex function which are relevant to the discussion of conformal invariant amplitudes. Of course, explicit determination of these contributions becomes already quite cumbersome at the fourth order level. The function $J_n(Q_1, Q_4)$ defined by (A. 15) can be evaluated exactly by expanding the angular dependent quantities in terms of Gegenbauer polynomials. The properties of these have already been given, but for reasons of completeness we will summarize again. They obey:

1) the generating formula

$$(1+t^2-2tz)^{-1} = \sum_{n=0}^{\infty} C_n(z) t^n$$

(A.17)

2) the normalization condition

$$\int_{-1}^1 dz (1-z^2)^{1/2} C_n(z) C_m(z) \equiv \langle C_n(z) C_m(z) \rangle_z = \delta_{nm}$$

(A.18)

3) the addition property

$$\langle C_n(\hat{k} \cdot \hat{\Phi}_1) C_m(\hat{k} \cdot \hat{\Phi}_2) \rangle_k = \frac{\delta_{nm}}{n+1} C_m(\hat{\Phi}_1 \cdot \hat{\Phi}_2)$$

(A.19)

4) the reflection property

$$C_n(-z) = (-1)^n C_n(z)$$

(A.20)

5) the recursion formulae

$$C_1 C_n = C_{n+1} + C_{n-1}$$

$$C_1 = 2z$$

$$C_0 = 1$$

(A.21a)

$$C_m C_n = \sum_{s=0}^m C_{m+n-2s} \quad n > m$$

(A.21b)

The integrals are then expressed by

$$J_n(Q_1, Q_4) = \sum_{m=1}^{n+1} B_m \log^m \left(\frac{(Q_1 + Q_4)^2}{\frac{4}{3} Q_2^2} \right) + \Theta_n \left(\frac{Q_1^2}{Q_4^2}; \frac{Q_1 Q_4}{Q_2^2} \right)$$

(A.22)

where

$$B_{n+1} = \frac{1}{n+1}$$

$$B_m = (-1)^{n-m-1} \frac{n!}{m!} \quad m \neq n+1$$

(A.23)

Because of the definition of $J_n(Q_1, Q_4)$ we recall that

$$\Theta_n \left(\frac{Q_1^2}{Q_4^2}; \frac{Q_1 Q_4}{Q_2^2} \right) \Big|_{\text{symm. pt}} = 0$$

(A.24)

A.2.3 Determination of the 4th Order V-graph Contribution to $\Psi_1(g)$

To this order the graph is of the skeleton form and therefore there is only a single log term. Since it is symmetric with respect to $Q_1 Q_2 Q_3 Q_4$ (see Figure) it must be of the form

$$V^{(4)}(Q_1 Q_2 Q_3 Q_4; Q_0^2) = \frac{V_4}{3} \left[\log \frac{Q_0^2}{3} + (t) + (u) \right] + V_4(Q_1 Q_2 Q_3 Q_4)$$

and V_4 is independent of Q_0^2 and symmetric with respect to the external momenta.

Explicit determination of $V_4(Q_1 Q_2 Q_3 Q_4)$ is quite tedious and involved, and it will not be given. The constant V_4 can be easily determined if we go into coordinate space:

$$\begin{aligned} (2\pi)^4 i \delta^4(\sum Q_i) V^{(4)}(Q_1 Q_2 Q_3 Q_4) &= \\ &= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i \sum Q_i \cdot x_i} \frac{1}{x_{12} x_{13} x_{14} x_{23} x_{24} x_{34}} \end{aligned} \quad (\text{A.25})$$

where

$$x_{ij} = (x_i - x_j)^2 \quad i \neq j \quad (\text{A.26})$$

by translational invariance, we have

$$V(Q_1 Q_2 Q_3) = \int d^4x_1 d^4x_2 d^4x_3 e^{i \sum_{i=1}^3 Q_i \cdot x_i} \frac{1}{x_1^2 x_2^2 x_3^2 x_{12} x_{13} x_{23}} \quad (\text{A.27})$$

Now, suppose that we set $Q_1 = Q_2 = 0$, $Q_3 = Q$; then

$$V^{(4)}(Q) = \int d^4x_3 \frac{e^{iQ \cdot x_3}}{x_3^2} \int d^4x_1 d^4x_2 \frac{1}{x_1^2 x_2^2 x_{12} x_{13} x_{23}} \quad (\text{A.28})$$

The $x_1 - x_2$ integral is convergent and well-defined, and by dimensional arguments

$$\int d^4x_1 d^4x_2 \frac{1}{x_1^2 x_2^2 x_{12} x_{13} x_{23}} = \frac{A_4}{\mu^2} \quad (\text{A.29})$$

therefore

$$V^{(4)}(Q) = A_4 \log Q^2/\Lambda^2$$

and thus A is the coefficient we are looking for. Equation (A.29) can be evaluated in a very straightforward manner using the Gegenbauer polynomial expansion.

A.2.4 Structure of the Vertex Function in Perturbation Theory

Using the above facts, we have explicitly carried out computations to fourth order in $\bar{g}$. This is, in fact, the first non-trivial order since the V-graphs start contributing. The coefficients of the logs are determined, as we explained, by computing the relevant basic integrals. The contribution to the single log power from the V-graph has been discussed separately in the previous section. The results are summarized in Table 1. By a straightforward, but cumbersome and tedious induction analysis, one can show that to all orders, the vertex function takes the simple form

$$\begin{aligned} \bar{\Gamma} = & \bar{g} + \gamma(Q_1 Q_2 Q_3 Q_4) + \sum_{n=1}^{\infty} \frac{1}{3} \psi_n(\bar{g}) \left[\log^n \frac{s}{\frac{4}{3} Q^2} + \log^n \frac{t}{\frac{4}{3} Q^2} + \log^n \frac{u}{\frac{4}{3} Q^2} \right] + \\ & + \sum_{n=1}^{\infty} \left[C_n(Q_1 Q_2 Q_3 Q_4; \bar{g}) \log^n \frac{s}{\frac{4}{3} Q^2} + C_n(Q_1 Q_2 Q_3 Q_4 Q_5; \bar{g}) \log^n \frac{t}{\frac{4}{3} Q^2} + \right. \\ & \left. + C_n(Q_1 Q_2 Q_3 Q_4 Q_5; \bar{g}) \log^n \frac{u}{\frac{4}{3} Q^2} \right] \end{aligned} \quad (\text{A.30})$$

where

$$\tau(\phi_1\phi_2\phi_3\phi_4) = t(\phi_1\phi_2;\phi_3\phi_4;\bar{g}) + t(\phi_1\phi_3;\phi_2\phi_4;\bar{g}) + t(\phi_1\phi_4;\phi_2\phi_3;\bar{g}) \quad (\text{A.31})$$

is the cut-off independent part of τ . The functions C_n are also cut-off independent.

The polynomials $\psi_n(\bar{g})$ behave as

$$\psi_n(\bar{g}) = \bar{g}^2 \sum_{m=0}^{\infty} B_m^m \bar{g}^{n-1+m} \quad (\text{A.32})$$

The results up to fourth order are included in Table 1.

Acknowledgements

I would like to thank Professor Kenneth A. Johnson for his guidance and suggestions at the various stages of this work.

Tables of Computational Results

Table 1A : Zero-Mass Vertex Function to Fourth Order in $\bar{g}$

Definitions:

$$a\bar{\Gamma} = \sum_{n=1}^{\infty} (\bar{g}_a)^n \bar{\Gamma}^{(n)} = \sum_{n=1}^{\infty} (\bar{g}_a)^n [\bar{\Gamma}_s^{(n)} + \bar{\Gamma}_t^{(n)} + \bar{\Gamma}_u^{(n)}]$$

$$a = \pi^2 / (2\pi)^4$$

$$\bar{g}_a = a\bar{g}$$

$$\bar{V}^{(n)} = \text{contributions from V-graphs}$$

$$\bar{\Gamma}_{s,t,u}^{(n)} = \text{contributions from T.P.R. graphs.}$$

If

$$\bar{\Gamma}_s^{(n)} = \tau^{(n)}(q_1 q_2; q_3 q_4; q_5^2)$$

then

$$\bar{\Gamma}_t^{(n)} = \tau^{(n)}(q_1 q_4; q_3 q_2)$$

$$\bar{\Gamma}_u^{(n)} = \tau^{(n)}(q_1 q_3; q_4 q_2)$$

Results:

$$1) \quad \bar{\Gamma}^{(1)} = 1$$

$$2) \quad \bar{\Gamma}^{(2)} = \frac{1}{4} \log \frac{(q_1 + q_2)^2}{\frac{4}{3} q_5^2} + \frac{1}{4} \log \frac{(q_3 + q_4)^2}{\frac{4}{3} q_5^2}$$

$$3) \quad \overline{\Pi}_S^{(3)} = \frac{3}{8} \log^2 \frac{(\varphi_1 + \varphi_2)^2}{\frac{4}{3} \varphi_3^2} + \frac{3}{8} \log^2 \frac{(\varphi_3 + \varphi_4)^2}{\frac{4}{3} \varphi_3^2} - \\ - \frac{1}{2} \log^2 \frac{(\varphi_1 + \varphi_2)^2}{\frac{4}{3} \varphi_3^2} - \frac{1}{2} \log^2 \frac{(\varphi_3 + \varphi_4)^2}{\frac{4}{3} \varphi_3^2} + \Theta_3(\varphi_1 \varphi_2) + \Theta_3(\varphi_3 \varphi_4)$$

$$\Theta_3(\varphi_1 \varphi_2) = \frac{1}{2} \left(\log \frac{S}{\frac{4}{3} \varphi_1^2} - \log \frac{S}{\frac{4}{3} \varphi_2^2} \right) \sum_{n=0}^{\infty} \frac{C_n(-\hat{\varphi}_1, \hat{\varphi}_2)}{n+2} \left(\frac{\varphi_2}{\varphi_1} \right)^{n+2} \\ - \frac{1}{4} \log^2 \frac{S}{\varphi_1^2} - \frac{1}{4} \log^2 \frac{4}{3} - \frac{1}{8} \sum_{n=0}^{\infty} \frac{C_n(-\hat{\varphi}_1, \hat{\varphi}_2) C_n(-\hat{\varphi}_1, \hat{\varphi}_2)}{(n+1)^2} \left(\frac{\varphi_2}{\varphi_1} \right)^{n+1} \\ + \frac{1}{8} \sum_{n=0}^{\infty} \frac{2}{3} \frac{C_n(1/3)}{(n+1)^2} \quad \text{FOR } (\varphi_1 > \varphi_2) \\ = (\varphi_1 \leftrightarrow \varphi_2) \quad \text{FOR } (\varphi_1 < \varphi_2)$$

$$4) \quad \overline{\Pi}_S^{(4)} = \frac{9}{16} \log^3 \frac{(\varphi_1 + \varphi_2)^2}{\frac{4}{3} \varphi_3^2} + \frac{9}{16} \log^3 \frac{(\varphi_3 + \varphi_4)^2}{\frac{4}{3} \varphi_3^2} - \\ - \frac{89}{48} \log^2 \frac{(\varphi_1 + \varphi_2)^2}{\frac{4}{3} \varphi_3^2} - \frac{89}{48} \log^2 \frac{(\varphi_3 + \varphi_4)^2}{\frac{4}{3} \varphi_3^2} \\ + \gamma_4 \left[\log \frac{(\varphi_1 + \varphi_2)^2}{\frac{4}{3} \varphi_3^2} + \log \frac{(\varphi_3 + \varphi_4)^2}{\frac{4}{3} \varphi_3^2} \right] + \\ + C(\varphi_1 \varphi_2; \varphi_3 \varphi_4) \log^2 \frac{4}{3} \varphi_3^2 + \Theta_4(\varphi_1 \varphi_2) + \Theta_4(\varphi_3 \varphi_4)$$

$$\gamma_4 = \frac{37}{6} + \frac{1}{12} \log \frac{4}{3} - \frac{5}{4} \log^2 \frac{4}{3} - \frac{5}{8} \sum_{n=0}^{\infty} \frac{C_n(1/3)}{(n+1)^3}$$

$C_1(Q_1 Q_2; Q_3 Q_4; Q_0^2)$ and $\theta_4(Q_1 Q_2)$ have not been determined explicitly.

$$5) \quad \bar{V}^{(4)} = 2 \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} \log \frac{StU}{(\frac{4}{3}Q_0^2)^3} + V_4(Q_1 Q_2 Q_3 Q_4)$$

v_4 has not been determined explicitly.

Table 1B : Zero-Mass Propagator to Third Order in $\bar{g}$

$$\bar{D}^{-1}(Q^2/Q_0^2) = Q^2 \left(1 + \sum_{n=2}^{\infty} (\bar{g}_n)^n \bar{D}_n(Q^2/Q_0^2) \right)$$

$$\sum_{n=2}^{\infty} (\bar{g}_n)^n \bar{D}_n = - \int_1^{Q^2/Q_0^2} \frac{dx}{x^2} F(x, \bar{g}_n)$$

$$F(x, \bar{g}_n) = \sum_{n=2}^{\infty} (\bar{g}_n)^n f_n(x)$$

$$f_2(x) = \frac{1}{12}$$

$$\bar{D}_2 = -\frac{1}{12} \log Q^2/Q_0^2$$

$$f_3(x) = \frac{x}{6} \left[\log x - \left(\frac{3}{4} + \log \frac{4}{3} \right) \right]$$

$$\bar{D}_3 = -\frac{1}{12} \log^2 \frac{Q^2}{Q_0^2} + \frac{1}{6} \left(\frac{3}{4} + \log \frac{4}{3} \right) \log \frac{Q^2}{Q_0^2}$$

Table 2: Eigenvalue-Theory Functions

$$A) \quad F(q) = q^2 [F_2 + q_0 F_3 + \mathcal{O}(q^4)]$$

$$F_2 = \frac{1}{12}$$

$$F_3 = -\frac{1}{6} \left(1 + \frac{4}{3} \log \frac{4}{3} \right)$$

$$B) \quad \psi(q) = q^2 [B_1^0 + q_0 B_1^1 + q_0^2 B_1^2 + \mathcal{O}(q^3)]$$

$$B_1^0 = 3/2$$

$$B_1^1 = -3$$

$$B_1^2 = 37/2 + \frac{1}{4} \log \frac{4}{3} - \frac{15}{4} \log^2 \frac{4}{3} + \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} (6 - 5C_n(1/3))$$

$$C) \quad \Phi(q) = 2q_0 F(q) + \psi(q)$$

$$= q^2 [\varphi_1 - q_0 \varphi_2 + q_0^2 \varphi_3 + \dots]$$

$$\varphi_1 = 3/2$$

$$\varphi_2 = 17/6$$

$$\varphi_3 = \frac{73}{4} - \frac{1}{12} \log \frac{4}{3} - \frac{15}{4} \log^2 \frac{4}{3} + \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} (6 - 5C_n(1/3))$$

References

- 1) Gell-Mann, M.; Low, F. : Phys. Rev. 95, 1300 (1954).
- 2) Johnson, K.; Willey, R.; Baker, M. : Phys. Rev. 163, 1699 (1967).
- 3) Baker, M.; Johnson, K. : Phys. Rev. 183, 1292 (1969).
- 4) Baker, M.; Johnson, K. : Phys. Rev. D3, 2516 (1971).
- 5) Baker, M.; Johnson, K. : Phys. Rev. D3, 2541 (1971).
- 6) Wilson, K. : Phys. Rev. D3, 1818 (1971).
- 7) Wilson, K. : Phys. Rev. D4, 3174, 3184 (1971).
- 8) Adler, S. L.; Bardeen, W. A. : Phys. Rev. D4, 3045 (1971).
- 9) Symanzik, K. : Comm. Math. Phys. 18, 227 (1970).
- 10) Callan Jr., C. C. : Phys. Rev. D2, 1541 (1970).
- 11) Adler, S. L. : Phys. Rev. D5, 3021 (1972).
- 12) Poggio, E. C. : M. I. T. Ph. D. Thesis (unpublished).
- 13) Bogoliubov, N. N.; Shirkov, D. V. : Introduction to the Theory of Quantized Fields,
New York, Interscience Pub. (1957), Ch. VIII.
- 14) Poggio, E. C. : Columbia University preprint #CO-2271-6 (1973). ✓
- 15) Callan Jr., C. C.; Coleman, S.; Jackiw, R. : Ann. of Phys. (N. Y.) 59, 42 (1970).
- 16) Ward, J. C. : Phys. Rev. 84, 897 (1951).
- 17) Huang, K.; Low, F. E. : J. E. T. P. 46, 845 (1962).
- 18) Coleman, S.; Jackiw, R. : Ann. of Phys. (N. Y.) 67, 552 (1971).
- 19) Schroer, B. : Lett. al Nuovo Cimento 17, 867 (1971).

Figure Captions

- Figure 1 Graphical "Bethe-Salpeter-Dyson-Schwinger" equation for the zero-mass vertex function. Single lines represent the zero-mass propagator $\bar{\Delta}$.
- Figure 2 Graphical representation of the differentiated T.P.I. s-channel contribution to $\bar{T}$. These are analytically finite.
- Figure 3 Typical one-loop graphs occurring in the determination of the vertex function. Convenient choices of internal momenta are shown.
- Figure 4 The fourth order graph contributions to the vertex function.

FIGURE 1

$$\begin{aligned}
 & \begin{array}{c} 1 \\ \diagup \\ \textcircled{\bar{\tau}_S} \\ \diagdown \\ 2 \end{array} \begin{array}{c} 4 \\ \diagdown \\ \\ \diagup \\ 3 \end{array} = 1/2 \begin{array}{c} 1 \\ \diagup \\ \textcircled{\bar{S}} \end{array} \text{---} \begin{array}{c} 4 \\ \diagdown \\ \textcircled{\bar{T}} \\ \diagup \\ 3 \end{array} + 1/2 \begin{array}{c} 1 \\ \diagup \\ \textcircled{\bar{T}} \\ \diagdown \\ 2 \end{array} \text{---} \begin{array}{c} 4 \\ \diagdown \\ \textcircled{\bar{S}} \\ \diagup \\ 3 \end{array} + \\
 & + 1/2 \begin{array}{c} 1 \\ \diagup \\ \textcircled{\bar{T}} \\ \diagdown \\ 2 \end{array} \text{---} \begin{array}{c} 4 \\ \diagdown \\ \textcircled{\bar{T}} \\ \diagup \\ 3 \end{array} + 1/4 \begin{array}{c} 1 \\ \diagup \\ \textcircled{\bar{T}} \end{array} \text{---} \begin{array}{c} 4 \\ \diagdown \\ \textcircled{\bar{S}} \end{array} \text{---} \begin{array}{c} 4 \\ \diagdown \\ \textcircled{\bar{T}} \\ \diagup \\ 3 \end{array}
 \end{aligned}$$

FIGURE 2

FIGURE 3

FIGURE 4