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STABILITY OF THE INTERFACE BETWEEN
TWO IMMISCIBLE LIQUIDS OF UNEQUAL
VISCOSITY IN A POROUS SOLID

LIVERMORE SITE

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STABILITY OF THE INTERFACE BETWEEN TWO IMMISCIBLE LIQUIDS OF UNEQUAL VISCOSITY IN A POROUS SOLID

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ABSTRACT

The stability of the interface between two immiscible fluids moving at a uniform rate through a porous medium is investigated. The conditions for stability are derived and the rate of growth of instabilities of specified wavelength is obtained. The expression for the rate of growth of instability is similar in form to that obtained by Taylor for the case of uniformly accelerated incompressible liquids, but the growth rate is found to be inversely proportional to the wavelength rather than inversely proportional to the square root of the wavelength.

Introduction

It is generally known that when one liquid displaces a second immiscible liquid in a porous medium the interface between them is macroscopically unstable, at least at sufficiently high flow rates, if the viscosity of the displacing liquid is less than that of the displaced liquid. The instability is in many respects similar to that which exists in the accelerated motion of two liquids when the driving liquid has a lower density than the driven liquid. The initial rate of growth of instability in the latter case of accelerated motion was investigated theoretically by Sir Geoffrey Taylor in 1950.¹

¹ G. Taylor, Proc. Roy. Soc. A201, 192 (1950).

In the present paper the initial rate of growth of instability is determined for the case of immiscible liquid displacement in a porous medium. The theory closely parallels the treatment of a somewhat different problem given previously by the present author,² and the same notation will be used wherever possible. The theory is based on an appropriate linearization of the nonlinear boundary condition that must be satisfied at the interface between the liquids. The linearization is similar to that used in the linear theory of water waves and has been discussed in the author's previous paper.²

The Rate of Growth of Instability

Two immiscible liquids, designated as liquid 1 and liquid 2, are assumed to occupy a homogeneous isotropic porous medium which consists of an infinite strip of width L and of negligible thickness. The liquids are assumed to be incompressible, the flow of each liquid being governed by Darcy's law for homogeneous fluids.³ The configuration under consideration is illustrated in Fig. 1.

Fig. 1

² Ray E. Kidder, J. Appl. Phys. 27, 1546 (1956).

³ A discussion of Darcy's law and its limitations is given by M. Muskat, The Flow of Homogeneous Fluids in Porous Media (McGraw-Hill, New York, 1937).

At time $t = 0$, the interface between liquids 1 and 2 is any continuous smooth curve whose elevation above the horizontal x axis will be denoted by $s(x, t)$. The x axis will be located for convenience so that the mean value of s is zero when $t = 0$. Liquids 1 and 2 are moving vertically upward with a uniform velocity v at all points sufficiently distant from the interface.

Darcy's law, together with the assumption of incompressibility, implies the existence of velocity potentials Φ^1 and Φ^2 for liquids 1 and 2, respectively. These potentials will be defined as

$$\Phi^i = k(p_i + \rho_i g y) / f \mu_i + v y; \quad i = 1, 2 \quad (1)$$

and satisfy the Laplace equation

$$\Phi_{xx}^i + \Phi_{yy}^i = 0. \quad (2)$$

The components of fluid velocity are related to the velocity potential according to

$$u_i = -\Phi_x^i, \quad v_i = -\Phi_y^i + v. \quad (3)$$

The boundary conditions at the impermeable walls are

$$\Phi_x^i(0, y; t) = \Phi_x^i(L, y; t) = 0, \quad (4)$$

and the boundary conditions at infinity are

$$\Phi_y^1(x, -\infty; t) = \Phi_y^2(x, +\infty; t) = 0. \quad (5)$$

The dynamical condition to be satisfied at the interface between the two liquids is

$$P_c + (p_1 - p_2)_{y=s} = 0, \quad (6)$$

and the kinematical condition is

$$s_t - v = - \left[\frac{\Phi^i}{y} \right] y = s + s_x \left[\frac{\Phi^i}{x} \right] y = s \quad (7)$$

If we define

$$h = s - vt, \quad (8)$$

then Eq. (7) becomes:

$$h_t = - \left[\frac{\Phi^i}{y} \right] y = s + h_x \left[\frac{\Phi^i}{x} \right] y = s \quad (9)$$

In employing $h(x, t)$ to define the configuration of the interface we are simply choosing coordinates with respect to which the mean velocity of the interface is zero.

The symbols employed in the equations given above are defined as follows:

u_i, v_i : x and y components of the fluid velocity, respectively

f, k : porosity and permeability of the porous medium, respectively

ρ_i, μ_i : density and viscosity of the liquids, respectively

g : acceleration resulting from gravity

p_i : pressure

P_c : difference in pressure across interface due to capillary forces
(assumed constant).

The subscripts and superscripts 1 and 2 are used to denote the liquid being referred to, and the subscripts x, y , and t are used to denote partial differentiation with respect to the corresponding variables.

At this point we observe that the potentials satisfy the same boundary conditions (4) and (5), and interface relations (6) and (8), as did the potentials of the problem treated in reference 2. (The quantities H_1 and H_2 of reference 2 are here taken equal to infinity.)

Employing the results of reference 2 we may therefore immediately write the solution of the present linearized problem of interface motion as:

$$h(x, t) = \sum_{n=1}^{\infty} A_n e^{-\sigma_n t} \cos(n \pi x/L), \quad (10)$$

where

$$A_n = (2/L) \int_0^L h(x, 0) \cos(n \pi x/L) dx \quad (11)$$

$$\sigma_n = n \pi \left[kg(\rho_1 - \rho_2) + fv(\mu_1 - \mu_2) \right] / (\mu_1 + \mu_2) L . \quad (12)$$

It follows that any initial disturbance of the interface will tend to grow if

$$kg\rho_1 + fv\mu_1 < kg\rho_2 + fv\mu_2 . \quad (13)$$

Condition (13) is then the condition that the interface be unstable. If we denote the wavelength of a sinusoidal disturbance by λ ($= 2L/n$), the e-folding time τ for the growth of a disturbance of wavelength λ is

$$\tau = (\mu_1 + \mu_2) \lambda f / 2\pi \left[kg(\rho_2 - \rho_1) + fv(\mu_2 - \mu_1) \right] . \quad (14)$$

If the two liquids have the same density, or if the velocity v is sufficiently large that gravity effects can be neglected, the result (14) reduces to

$$\tau = (\lambda / 2\pi v) (\mu_2 + \mu_1) / (\mu_2 - \mu_1) . \quad (15)$$

The corresponding expression obtained by Taylor for the e-folding time τ in the case of an accelerated liquid interface is

$$\tau^2 = (\lambda / 2\pi a) (\rho_2 + \rho_1) / (\rho_2 - \rho_1) , \quad (16)$$

where a denotes the constant acceleration of the liquids.

Conclusions

There is a strong resemblance between our result (15) and that of Taylor (16). However, there is a basic difference in that in our case the e-folding time is directly proportional to the wavelength of the disturbance, whereas in Taylor's case it is proportional to the square root of the wavelength. The behavior of an unstable interface in a porous medium will be

qualitatively similar to that of an unstable free hydrodynamic surface, but will be quantitatively quite different due to the different wavelength dependence.

It is evident from the results (14) or (15) that short wavelength disturbances will tend to grow the most rapidly. The fact that the present analysis predicts that disturbances of vanishingly small wavelength will grow with infinite rapidity is of course a physical impossibility. The growth of very short wavelength disturbances will be limited by surface tension effects not taken into account in the present treatment. At sufficiently high flow rates, however, the effects of capillary forces will be negligible except at the microscopic or pore scale of the porous medium. Consequently it is to be expected that at high flow rates the most rapidly growing disturbances will be of the same order of size as the channels or pores of the porous medium, and the interface will become ill-defined in any macroscopic sense. On the other hand, if the interface is stable and the velocity of flow is again sufficiently high, the interface will tend to become macroscopically planar. All initial irregularities will decay with time, those of shortest wavelength decaying the most rapidly, so that an initially irregular interface will rapidly become smooth.

These qualitative predictions of the present linear theory of interface behavior have been confirmed by visual and photographic observation of experiments performed with unconsolidated porous media consisting of fine glass beads confined between parallel glass plates.^{4, 5} It should be stressed, however, that the quantitative treatment that has been given of interface behavior only applies to the initial behavior of an interface that is macroscopically relatively smooth to begin with. At later times, in the unstable case, the curvature of the interface will become large and nonlinear effects which have been neglected will become important.

⁴ Ray E. Kidder, Unpublished Report, California Research Corp., La Habra, California, January 1954.

⁵ P. Van Meurs, Trans. Am. Inst. Mining Met. Engrs. 210, 295 (1957).