FEB 28 1967

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BNL 11041

BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, L.I., N.Y.

AADD-128

RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

ACCELERATOR DEPARTMENT
Internal Report

THEORY OF LONGITUDINAL INSTABILITIES
IN SYNCHROTRONS, WITH APPLICATIONS

GESTI PRICES

H.C. \$ 3.00; MN .65

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December 16, 1966

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Summary

By combining the results of longitudinal instability theory and synchrotron oscillation theory, it is possible to derive two simple and useful expressions for longitudinal instability particle threshold. The first, which holds at injection, is given directly in terms of design parameters; in practical cases it turns out to depend only slightly on injection energy and to be independent of field index and vacuum chamber size, provided only that the vacuum chamber is large enough to contain particle oscillations. The second, which is based on adiabatic bunch behavior, displays the variation of threshold with energy; it decreases monotonically from injection to transition (if a real transition exists) and then increases monotonically. Different regions of this universal curve are relevant, depending upon whether the machine considered is a booster, a weak-focussing machine or a standard AG synchrotron in which transition is crossed. Practically, it appears that recently proposed boosters will be free from longitudinal instabilities if a little care is exercised in design. It is quite evident that proposed "improved" AG machines will have no problems at injection; the treatment of the transition region is still crude, and further analysis will be required for a definite answer - it appears that there might be trouble at least in Phase II (2×10^{13} protons) unless the capture at injection is extremely efficient.

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I. Introduction

When the number of particles in an accelerator beam exceeds a certain value, called the particle threshold, the accelerator becomes susceptible to longitudinal beam degeneration caused by the amplification of beam irregularities. Such a state is called a state of longitudinal instability, and the instability is described as resistive or negative-mass depending upon the mechanism producing it. (Below transition, only the former exists; above transition, the latter predominates.)

As is shown in Section II, it is possible to express the particle threshold for longitudinal instabilities in terms of accelerator constants, particle energy, a form factor depending on particle distribution across the vacuum chamber, a bunching factor, and some measure of the instability suppression which may be taken as the energy spread of the beam or the synchrotron (not betatron) width.

As will be shown in Section III, however, when the beam is bunched at injection, the beam parameters can be determined from synchrotron oscillation theory. So, by combining instability theory and synchrotron oscillation theory, one can predict the actual longitudinal instability threshold for a machine.

Hence it is possible to find, as is done in Section IV, an expression for longitudinal instability threshold at injection which is given directly in terms of design parameters. The formula turns out to be remarkably simple. It varies with energy only as γ^2 , is independent of γ_t and hence of the type of machine, and does not depend upon the dimensions of the vacuum chamber (except that the vacuum chamber must be large enough to contain synchrotron and betatron oscillations and various fluctuations).

It is further possible to construct, as is done in Section V, a curve which shows how the particle threshold varies with particle energy throughout the acceleration process. The energy dependence turns out to involve only (γ/γ_t) , so a universal curve can be constructed for all machines, and portions of it used depending on the relation between initial, transition and final energy. The curve decreases monotonically from injection to transition - where the threshold is zero* - then increases monotonically. Different portions

*On closer analysis finite but very small.

of the curve are shown to be relevant for different types of machines - weak focussing machines, standard AG machines and boosters.

Practical numerical applications to existing and proposed machines are given in Section VI, and results are discussed in Section VII.

II. Longitudinal Instability Theory^{1-5*}

A longitudinal instability is a state of the accelerator beam in which the amplitude of any collective longitudinal oscillations will increase with time. The state can be produced under two circumstances: (1) a field index and energy such that particles of higher kinetic energy circulate with a smaller angular velocity (giving rise to a so-called negative-mass instability), or (2) an in-phase component of electromagnetic feedback from the vacuum chamber walls (giving rise to a so-called resistive instability). A tendency to negative mass instability is present only above the transition energy (which may be taken as the definition of transition energy). A tendency to resistive instability is always present, but the resistive instability is generally swamped by the negative-mass instability when the latter is present.

Generally, not all the circulating particles have precisely the same frequency and energy, and there is a small radial spread across the tank; this is always the case after bunching has been completed. If all the circulation frequencies are close together, it is easy to impose a collective plasma oscillation, but if there is a wide spread in frequency it is not. Collective motion is also easier if there are more particles packed into the beam. For a given accelerator and given state of the beam, there is a certain number of particles in the beam below which the instability will not exist. This is known

1. A.A. Kolomensky and A.N. Lebedev, "Proceedings of the International Conference on High Energy Accelerators and Instrumentation," Geneva, 1959, pp 115-124 (CERN, Geneva, 1959).
2. C.E. Neilsen, A.M. Sessler and K.R. Symon, *ibid.*, pp 239-252.
3. V.K. Neil and A.M. Sessler, *Rev. Sci. Instr.* 36, 429-36 (1965).
4. R.L. Pease, *Trans. IEEE NS-12*, No. 3, 561-5 (1965).
5. R.L. Pease, "Longitudinal Instability Rise Rates," BNL Accelerator Dept. Int. Rpt. AADD-101, March 22, 1966.

* This section is taken mainly from References 3, 4 and 5.

as the particle threshold N_{bt} . It turns out that

$$N_{bt} \propto f^2 \left(\frac{df}{dE} \right) \left(\frac{\Delta E}{f} \right)^2 \sim (\Delta f) (\Delta E) \quad (1)$$

where we will follow the notation of conventional synchrotron oscillation theory and take Δf and ΔE as half-spreads of circulation frequency and energy respectively across the beam.

Equation (1) is important because it shows that a finite radial width per se does not prevent collective motion. Synchrotron widths contribute to stability, but betatron widths do not. For the particles in a bunch are moving (on the time scale of a revolution) in parallel paths, for which $\Delta f \sim \Delta r$ and $\Delta E \sim \Delta r$, whereas if a particle executes betatron oscillations about an equilibrium path its frequency and energy are not changed in first order. (As the crudest example, $\Delta E \sim m \omega_B^2 (\Delta r)^2 / 2$ for betatron oscillations.) Hence the threshold is proportional to the square of the synchrotron oscillation amplitude Δr_s , and we may if we wish write⁵

$$N_{bt} = \frac{(2\Delta r_s)^2}{r_o R} \frac{B \pi/2}{1 + (Z_o/30)} \frac{\gamma^2 - 1}{\gamma} \left| \gamma_t^2 - \gamma^2 \right| \gamma_t^2 \frac{1}{|U'| \left(1 + \frac{v^2}{u^2} \right)} \quad (2)^*$$

where $2\Delta r_s$ is the full synchrotron width, r_o is the classical radius

$$r_o = \frac{e^2}{m_o c^2} \quad [= 1.53 \times 10^{-16} \text{ cm for the proton}] , \quad (3a)$$

R is the average accelerator radius,

*

The one place where Eq. (2) breaks down is close to transition, because [from the discussion of Eq. (1)] df/dE changes sign at transition. Precisely at transition, Eq. (2) would predict an N_{bt} which went to zero with the power $|\gamma_t - \gamma|^2$, multiplied by any effect due to the $(2\Delta r_s)^2$ term. In actuality, however, since there is an energy spread, not all the particles are at transition. We would expect Eq. (2) to hold rather well if the total fractional energy spread $2\Delta\gamma \equiv 2\Delta E/m_o c^2$ obeys the relation $2\Delta\gamma \ll |\gamma_t - \gamma|$ and rather poorly if $2\Delta\gamma \gg |\gamma_t - \gamma|$. Other difficulties encountered at transition will be discussed in Section VII, and a close analysis will be made in a subsequent paper.

B is the bunching factor

$$B \equiv \frac{\text{actual number of particles in the beam}}{\text{number of particles in an equivalent uniform beam}}, \quad (3b)$$

Z_0 is the characteristic impedance in ohms between beam and tank considered as a transmission line [$= 60 \ln (b/a)$ for a coaxial cylindrical geometry in which b is the inner tank radius and a is the beam radius], γ and γ_t are the energy and transition energy* respectively in units of $m_0 c^2$. The constant U' is a quantity dependent on the density distribution across the tank and on whether the behavior is above or below transition. The ratio V/U is given by⁵ (for a coaxial geometry)

$$\frac{V}{U} = \frac{\sqrt{R}}{b \sqrt{n} [1 + 2 \ln (b/a)]} \sqrt{\frac{\mu c}{2\pi\sigma}} \frac{\beta^{3/2}}{1 - \beta^2}, \quad (4)$$

where b is the (coaxial) tank radius and n is the harmonic order of the instability, i.e., the approximate ratio of plasma frequency to particle circulation frequency; for relevant** values of n this ratio is very small:

$$\frac{V}{U} \ll 1. \quad (5)$$

If we further define, for convenience,

$$\eta \equiv \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \quad (6)$$

* The transition energy $\gamma_t m_0 c^2$ is related to compaction by

$$\gamma_t^2 = \frac{\Delta p/p}{\Delta r/R}$$

where R is the average, not instantaneous, radius. We can see this readily by noting that only with this definition is the quantity $\Delta f/f = \Delta v/v - \Delta r/R$ zero when $\gamma = \gamma_t$.

** See discussion of the value of n , below.

then we may rewrite (2) in the more compact form

$$N_{bt} = \frac{(2 \Delta r_s)^2}{r_o R} \frac{B \pi/2}{1 + (Z_o/30)} \frac{1}{|U'|} \gamma(\gamma^2 - 1) \gamma_t^4 |\eta| \quad (7)$$

The problem of finding the threshold as a function of energy becomes simple if we use Eq. (7), because we can find Δr_s and B as functions of energy from synchrotron oscillation theory.

But before we consider synchrotron oscillations we shall consider rise rates as well as thresholds, because even if a system is operating above particle threshold, there will be no problem if the rise rate is slow enough.

The standard rise rate for negative mass instabilities, which is an upper limit⁵ to the actual rise rate, is given by

$$\frac{1}{\tau_N} = n c \left\{ \frac{N_b}{2\pi B} \frac{r_o}{R^3} [1 + (Z_o/30)] \frac{\eta}{\gamma^3} \right\}^{1/2} \quad (8)$$

where N_b is the actual number of particles in the beam.

The standard rise rate for resistive instabilities is given by⁵

$$\frac{1}{\tau_R} = \frac{1}{2} \frac{V}{U} \frac{1}{\tau_N} \quad (9)$$

which, from (5), is somewhat less than the negative mass rise rate.

We must now try to find out how many time constants τ_N or τ_R a machine can tolerate. For this it is necessary to consider the nature of longitudinal instabilities more closely.

As mentioned briefly in the Introduction, a longitudinal instability is a state of an accelerator in which any longitudinal beam fluctuations will grow. It is not a disturbance in itself - but if there is a disturbance in the beam, it will be amplified. It is like the state of a man standing on one foot - he is unstable, but he must receive some sort of a push, or unbalance himself slightly, before he will actually fall.

It appears from some experiments carried out at Princeton⁶ that the beam fluctuations involved may be primarily those of ordinary shot noise, and that the beam degenerates if the shot noise is amplified so much that the fluctuations in beam current are of the order of magnitude of the current itself.* This process takes something of the order of five or ten time constants. Hence, since the τ 's are minimum times,⁵ there will probably be trouble in an accelerator if the duration of the state of instability is more than say fifteen or twenty times the value of the relevant minimum τ . **

There remains the question of n . One can see from Eq. (8) that the rise rate for negative mass instabilities is proportional to n , and from (8), (9) and (4) that the rise rate for resistive instabilities varies as $\sqrt{n}$. If the beam degeneration goes on for five or ten time constants, only the highest frequency fluctuations will remain as they are amplified so much faster. There is, however, a cutoff, and hence we can use the cutoff value as a good estimate of n . It appears from the experiments at Princeton⁶ that cutoff occurs at about

$$k b \sim 1 \quad (10)$$

where the wavenumber

$$k \equiv \frac{\omega}{v} \sim \frac{n}{R} \quad (11)$$

i.e., when

$$n \sim \frac{R}{b} \quad (12)$$

For this value of n , and reasonable values for the conductivity and dimensions, we find the ratio V/U of Eq. (4) quite small, and hence Eq. (7) is justified.

We now consider the problem of synchrotron oscillations.

6. M. V. Isalia, A. Passner and R.L. Pease, "Accelerator Beam Degeneration from Amplified Shot Noise," Princeton-Pennsylvania Accelerator Report PPAD-609D (in press).

* Similar fluctuations had been observed earlier at the Cosmotron.⁷

7. M.Q. Barton and C.E. Nielsen, Proc. Intl. Conf. on High Energy Accel. (September 1961), pp 163-174 (A.E.C., Washington, 1961).

** Note that the amplification of the beam degeneration would be cumulative if successive machines (e.g., a booster and a main synchrotron) happened to amplify any of the same frequency range; however, this possibility is unlikely.

III. Synchrotron Oscillation Theory^{8,9 *}

Let bunches of particles be travelling around the machine, accelerated by r.f. voltages across gaps. Let E represent the energy of the synchronous particle, which maintains the constant phase φ_s with respect to the r.f. voltage; it will circulate with an angular frequency ω_s which (in the absence of frequency error) is the h^{th} subharmonic of the r.f., i.e.

$$\omega_{\text{rf}} = h \omega_s .$$

Further let there be N gaps around the machine, each with maximum r.f. voltage V_m across it.

Consider some other particle, with energy $E + \Delta E$ and phase φ different from φ_s . Since it receives a kick $V_m \sin \varphi$ rather than $V_m \sin \varphi_s$ at each gap, its energy difference ΔE will change with time. Since in one revolution (a time $2\pi/\omega_s$), ΔE will change by $N e V_m (\sin \varphi - \sin \varphi_s)$, it follows that ΔE obeys the equation

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_s} \right) = \frac{N e V_m}{2\pi} (\sin \varphi - \sin \varphi_s) . \quad (13)$$

Conversely, however, a particle with energy different from E will travel with a different velocity from that of the synchronous particle, and hence φ will be changing as long as ΔE is not zero. This equation is

$$\frac{d\varphi}{dt} = h \omega_s \frac{\Delta t}{t}$$

8. E.D. Courant and H.S. Snyder, *Annals of Phys.* 3, 1-48 (1958).

9. J.J. Livingood, "Principles of Cyclic Particle Accelerators," (Princeton: D. Van Nostrand, 1961), Chapter 6.

*

This section is taken mainly from Reference 8; their V is my NV_m , i.e., the total peak r.f. voltage per turn. Reference 9 has an excellent discussion of phase plots, but the expression for adiabatic behavior of Δr does not agree with the results of Reference 8 and would lead to a quite different threshold curve.

where t is the period of revolution. After some kinematics this may be written

$$\frac{d\phi}{dt} = h \omega_s \frac{\eta}{\beta^2} \frac{\Delta E}{E} \quad (14)$$

where we remember that

$$\frac{1}{\beta^2} = \frac{\gamma^2}{\gamma^2 - 1} \quad (15)$$

We note that in (13), $d(\Delta E)/dt$ is given as a function of ϕ , whereas in (14), $d\phi/dt$ is given as a function of ΔE . Let us combine the two equations to obtain

$$\frac{1}{h\omega_s^2} \left[\frac{d}{dt} \left(\frac{E}{\eta} \frac{d\phi}{dt} \right) \right] = \frac{N e V_m}{2 \pi} (\sin \phi - \sin \phi_s) \quad (16)$$

where $\omega_o \equiv \omega_s/\beta = c/R$ is independent of velocity.

We shall do three things with Eq. (16).

A. Small Oscillations. Define

$$\xi \equiv \phi - \phi_s. \quad (17)$$

Then, if $\xi \ll 1$, and we assume E and η to be constants, Eq. (16) may be written

$$\frac{d^2 \xi}{dt^2} + \Omega^2 \xi = 0 \quad (18)$$

$$\Omega^2 \equiv - \frac{N e V_m h \omega_o^2 \eta \cos \phi_s}{2 \pi E} \quad (19)$$

This gives oscillations of angular frequency Ω .

B. The First Integral of Motion and the Radial Spread at Injection (large oscillation, constant energy).

To investigate further we find the analogue of the energy integral. Assuming E (and η) constant, and using the definition (19) for Ω^2 , we may

write the synchrotron oscillation Eq. (16) as

$$\frac{d^2\varphi}{dt^2} + \Omega^2 \left(\frac{\sin \varphi - \sin \varphi_s}{\cos \varphi_s} \right) = 0, \quad (20)$$

where Ω is now just a convenient parameter (though it is the limiting frequency as $\varphi \rightarrow \varphi_s$).

But with the usual identity

$$\frac{d^2\varphi}{dt^2} = \frac{1}{2} \frac{d}{d\varphi} \left(\frac{d\varphi}{dt} \right)^2 \quad (21)$$

we can integrate (10) to obtain

$$\frac{1}{2} \left(\frac{d\varphi}{dt} \right)^2 - \Omega^2 \left(\frac{\cos \varphi + \varphi \sin \varphi_s}{\cos \varphi_s} \right) = \text{const.} \quad (22)$$

where the first term is the analogue of the kinetic energy and the second term is the analogue of the potential energy. If we then assume that

$$\frac{d\varphi}{dt} = \left(\frac{d\varphi}{dt} \right)_i \quad \text{when } \varphi = \varphi_i \quad (23)$$

we can solve for the $d\varphi/dt$ term and get

$$\left(\frac{d\varphi}{dt} \right)^2 = \left(\frac{d\varphi}{dt} \right)_i^2 + 2 \Omega^2 \left(\frac{\cos \varphi + \varphi \sin \varphi_s - \cos \varphi_i - \varphi_i \sin \varphi_s}{\cos \varphi_s} \right). \quad (24)$$

Plots of $d\varphi/dt$ vs. φ , known as phase diagrams or fish diagrams, are given in Reference 9.

The oscillations will be stable until φ reaches $(\pi - \varphi_s)$, so the maximum stable value of $d\varphi/dt$ will correspond to $(d\varphi/dt)_i = 0$, $\varphi_i = \pi - \varphi_s$ and $\varphi = \varphi_s$, whence from (24)

$$\left(\frac{d\varphi}{dt} \right)_{\max} = \sqrt{-2 \Omega^2 \left[\frac{(\pi - 2 \varphi_s) \sin \varphi_s - 2 \cos \varphi_s}{\cos \varphi_s} \right]}. \quad (25)$$

Then, from Eq. (14), the maximum energy spread is

$$(\Delta E)_{\max} = \frac{\beta E}{\eta h \omega_o} \left(\frac{d\phi}{dt} \right)_{\max} \quad (26)$$

and, finally, the maximum radial spread, or half radial synchrotron width at injection is

$$\begin{aligned} \Delta r_s &= \frac{R}{\gamma_t^2 \beta^2} \left(\frac{\Delta E}{E} \right)_{\max} = \frac{R}{\gamma_t^2 \eta h \beta \omega_o} \left(\frac{d\phi}{dt} \right)_{\max} \\ &= \frac{R}{\gamma_t^2} \sqrt{\frac{N e V_m}{\pi E h \eta \beta^2} \left[(\pi - 2 \varphi_s) \sin \varphi_s - 2 \cos \varphi_s \right]} \quad (27) \\ &\quad \text{(at injection)} \end{aligned}$$

We must remember that (27) represents the radial spread only if there are actual particles available for capture at the limiting values of ϕ . This will be true at injection, when particles are being captured from a uniform beam (or a beam correctly pre-bunched) - but during the acceleration process the longitudinal size of the beam shrinks, there are simply no particles out as far as $\phi = \pi - \varphi_s$, and (27) does not give a true picture of the actual spread available to suppress longitudinal instabilities.

C. Adiabatic Behavior (small oscillations, variable energy).

To see how the amplitude of synchrotron oscillations will change with time, assume ξ small but do not assume E and η constant. Then Eq. (16) becomes

$$\frac{d}{dt} \left(\frac{E}{\eta} \frac{d\xi}{dt} \right) + \Omega^2 \frac{E}{\eta} \xi = 0 \quad (28)$$

If now E and η do not vary much in a synchrotron oscillation period - which is true except near transition - i.e., if

$$\frac{\frac{d}{dt} \left(\frac{E}{\eta} \right)}{\frac{E}{\eta}} \ll \Omega \quad , \quad (29)$$

we may use the WKB method¹⁰ and write

$$\xi \sim \frac{\text{const}}{\sqrt{\frac{E}{\eta}} \Omega} \exp \left(\pm j \int \Omega dt \right) \quad (\eta \neq 0) \quad (30)$$

or, since, from (19), Ω varies as $(\eta/E)^{1/2}$,

$$\varphi - \varphi_s \cong C \left(\frac{\eta}{\gamma} \right)^{1/4} \exp \left(\pm j \int \Omega dt \right) \quad (\eta \neq 0) \quad (31)$$

By differentiating (31) and using (29) we see that $d\varphi/dt$ varies as $\Omega(\eta/\gamma)^{1/4}$ and hence from (19) as $(\eta/\gamma)^{3/4}$; but from (27), Δr varies as $(1/\eta\beta)(d\varphi/dt)$, whence

$$\Delta r \cong \frac{D}{\eta^{1/4} \beta \gamma^{3/4}} \exp \left(\pm j \int \Omega dt \right) \quad (\eta \neq 0) \quad (32)$$

When η is near zero it is necessary to use one-third order Bessel functions.^{8*}

IV. Threshold at Injection

We can now calculate the longitudinal instability threshold at injection by simply substituting Eq. (27) for the radial half-width at injection into Eq. (7) for the threshold, to obtain

$$N_i \equiv N_{bt}^{(inj)} = \frac{2B}{|U'|} \frac{1}{1 + (Z_o/30)} \frac{R}{r_o} \frac{N e V_m}{E_o h} \gamma^2 \times [(\pi - 2\varphi_s) \sin \varphi_s - 2 \cos \varphi_s] \quad (33)$$

(injection)

where E_o is the rest energy $m_o c^2$.

10. See for instance J.C. Slater and N.H. Frank, "Mechanics" (New York: McGraw-Hill, 1947), Chapter X.

* See Section VII.

The interesting thing about (33) is how many parameters it does not depend on, or depends on only slightly.

In the first place, all the γ_t terms, including η , have cancelled out, so (33) is the same no matter what type of machine. In the second place, the β terms have cancelled out and the only energy dependence left is the γ^2 term, which does not vary very much within the usual range of injection energies, and which certainly varies less in this range than the space charge threshold.* In the third place - and again in contrast with the case of charge instabilities - the threshold is practically independent of beam or transverse tank dimensions, except that

- (1) there is a small contribution from the $\ln$ term, and
- (2) the tank must be large enough to contain the beam, including synchrotron and betatron oscillations, effects of fluctuations, and any spreading.

What then does the threshold ultimately depend on? There are practical limits to which a good designer can push B , U' , Z_0 and the term in square brackets, and for the machines we are considering, the γ^2 term is not strong. Hence the threshold is basically limited by

- (1) the energy gain per turn,
- (2) the harmonic number, and
- (3) the machine radius,

provided again the tank is large enough in cross section.

We shall consider numerical examples in Section VI, after we have gotten more perspective by investigating the variation in threshold during the acceleration process.

* This does not mean that the synchrotron width does not vary strongly with energy - it does. But at the same time there is an almost compensating change in the effectiveness of a given synchrotron width in suppressing longitudinal instabilities, so that, in terms of the factors of Eq. (33), the threshold is almost energy-independent at low energy. And it is the parameters at the right of Eq. (33) with which the designer will be working.

V. Variation of Threshold with Energy: the Universal Curve

Now we look at the variation of longitudinal instability threshold as a function of energy through the acceleration cycle.

We start with the threshold equation (7) as before. But now we assume that $(2 \Delta r_s)$ varies with energy adiabatically as described by Eq. (32), and - since except for extreme values the phase plots all have about the same shape - that the bunching factor B varies as $(\varphi - \varphi_s)$ as given by Eq. (31). We assume* that the terms not containing γ do not vary with energy. Then we can write

$$N_{bt} \propto \gamma^{5/4} |\eta|^{3/4} = \gamma^{5/4} \left| \frac{1}{\gamma_t} - \frac{1}{\gamma} \right|^{3/4} \quad (34)$$

We note specifically that Eq. (34) does not contain β either explicitly or as contained in $\gamma^2 - 1$, and hence in general N_{bt} varies rather slowly near injection. On the other hand, since N_{bt} does turn out to depend upon a positive power of η , the threshold (in this approximation**) goes to zero at transition.

It is simpler to plot Eq. (34) if it is written in terms of the ratio (γ/γ_t) , i.e. as

$$N_{bt} \propto \frac{|1 - (\gamma/\gamma_t)^2|^{3/4}}{(\gamma/\gamma_t)^{1/4}} \quad (35)$$

Note that the only energy dependence is upon the ratio (γ/γ_t) . Hence the curve, which is plotted in Figure 1, is universal in the sense that it applies to all accelerators.*** Different machines correspond merely to different portions

* Specifically we assume that neither Z_0 nor $|U'|$ vary with energy. Actually Z_0 does vary, but it depends logarithmically on the overall beam radius (due to synchrotron width, betatron width and fluctuations, as well as any possible beam spreading); its variation is small and not certain and we shall neglect it. The same comment applies to $|U'|$, which might have a small variation with energy as the beam profile changed.

** See Section VII.

*** For the behavior at transition see Section VII.

Figure 1.
Universal threshold curve: relative longitudinal instability threshold as a function of energy throughout the acceleration cycle.

of the curve. Note that the curve decreases monotonically to zero at transition, and then increases monotonically.*

To see what portions of Figure 1 are relevant for what types of accelerator, let us consider three machines in historical order.

A. The Cosmotron.** For a conventional-gradient machine like the Cosmotron, the factor γ_t^2 is replaced by $(1 - n_f) [1 + (s/m)]$, where n_f is the field index and s/m is the ratio of the length of straight sections to the length of magnet sections, i.e., $1 + (s/m) = R/r$; for the standard field index of 0.6 or so this is less than unity. (For example, in the Cosmotron,¹¹ $\gamma_t \sim 0.7$.) Hence a machine like the Cosmotron operates in the right hand portion of the universal curve of Figure 1. [For the Cosmotron itself, the limits are - or rather were - $(\gamma/\gamma_t) = 1.4$ and $(\gamma/\gamma_t) = 6$.] Note that the injection value of threshold is a lower limit (i.e., things get better), and the threshold increases during the acceleration process to about ten times its injection value.

B. The Brookhaven AGS. The AGS has a $\gamma_t \sim 8.75$ and operates at present between 50 MeV ($\gamma = 1.05$) and 33 BeV ($\gamma = 36.2$), whence the range of (γ/γ_t) is from 0.12 to 4.1, almost the entire range shown in Figure 1. The threshold decreases monotonically to about zero*** at transition and then increases again. Here the value of the threshold at injection is no guarantee of stability - the AGS will always be unstable near transition, and the problem is one of making the time of traverse of the unstable region so small that the beam degeneration will not have had enough time to build up to a dangerous value.

C. A Proposed Booster. The booster proposed by Maschke and Smith¹² for use with the Brookhaven AGS is an alternating-gradient synchrotron of moderate field gradient (resulting in this case in $\gamma_t \sim 2.5$) operating from 50 MeV ($\gamma = 1.05$) up to either 200 MeV ($\gamma = 1.21$) or about 630 MeV ($\gamma = 1.67$). Hence on Figure 1 it operates in a small range in the left hand portion of the curve, from $(\gamma/\gamma_t) = 0.42$ to $(\gamma/\gamma_t) = 0.48$ or 0.67. So during the restricted acceleration

* The first derivative $dN_{bt}/d(\gamma/\gamma_t)$ has no real zeroes for finite real $(\gamma/\gamma_t) \neq 1$.

** The Cosmotron is not operating, but it is the prototype of several existing machines. Note, however, that these remarks do not apply to the Argonne ZGS, which has $n_f = 0$ and hence $\gamma_t > 1$.

11. M.H. Blewett, Rev. Sci. Instr. 24, 725-37 (1952).

*** See Section VII.

12. A.W. Maschke and L.W. Smith, BNL informal draft.

cycle of the booster, the threshold does not vary very much, dropping to about 92% of its injection value for 200 MeV output, and to about 66% of its injection value for 630 MeV output. However, the injection value is an upper limit to the threshold, so if it is used as a measure of stability it should be treated conservatively.

VI. Numerical Applications

We first calculate a rough but useful formula for threshold at injection, bearing in mind that this may or may not be a relevant figure for the machine as a whole.

Consider Eq. (33). Let us try to make it much more workable. Assume limits on φ_s at one extreme to be 150° or 30° (which corresponds to a normal working value), and at the other extreme to be 175° or 5° (which corresponds to an attempt to get a very high capture efficiency at injection, perhaps by lowering $\dot{B}$ or distorting the r.f. wave). Then the bunching factor B would vary¹³ from 0.36 up to 0.56 and the absolute value of the quantity in square brackets would vary from 0.68 to 1.74. If we assume reasonable values of between 2 and 3 for $|U'|$ and say 2 for $(Z_o/30)$, we would have

$$\frac{2B}{|U'|} \frac{1}{1 + (Z_o/30)} \left| [(\pi - 2\varphi_s) \sin \varphi_s - 2 \cos \varphi_s] \right| \approx 0.054 \text{ to } 0.32 \quad (36)$$

Then, if we express the accelerator radius R in meters and the peak cavity voltage V_m in kilovolts, we can rewrite (33) as

$$N_{bt} \approx (0.38 \text{ to } 2.2) \times 10^{11} (R_{\text{meters}}) \frac{N}{h} (V_m \text{ kV}) \gamma^2 \quad (37)$$

(injection)

where the first figure corresponds to an unenhanced capture system with $\varphi_s = 150^\circ$ or 30° and to $|U'| = 3$, and the second figure to values of 175° or 5° and 2 respectively.

13. F.T. Cole and P.L. Morton, "Areas and Bunching Factors of Partially Filled Buckets," Lawrence Radiation Lab. Report UCID 10130 (September 21, 1964).

We apply Eq. (37) to our three archetypal machines, but this time treating the AGS last as it is by far the most complex.

The radius of the Cosmotron¹¹ - and for that matter the approximate radius of most 3 BeV proton machines - is about 11 meters. There is one accelerator gap ($N = 1$), the r.f. is at the fundamental ($h = 1$) and the peak voltage V_m is about 2 kV. The value of ϕ_s is about 30° and injection takes place at about 3.6 MeV. Hence the variable factor in Eq. (36) is of the order of 0.38 to 0.57 and

$$N_{bt} \approx (8 \text{ to } 13) \times 10^{11} \text{ (Cosmotron at injection)} \quad (38)$$

which is just about the operating value.

Hence we must consider the rise time. Since the Cosmotron always operates above transition, the instability will be of the negative mass type. If we use Eq. (8) with the estimates $B \sim 0.36$, $n \sim 40-45$, $N_b \sim 10^{12}$, we find that at injection the standard rise time is very fast, about 2 μsec , so that more than 10^{12} or so particles would be destroyed quickly before the threshold had a chance to rise.

Consider now the proposed booster,¹² which is to have a radius of 10.8 m and is to accelerate from 50 MeV (if necessary) to 1 BeV in 130 msec. We find $N V_m$ as follows: If $N V_m \sin \phi_s$ is the energy gain per turn, the energy gain per second is

$$\frac{dE}{dt} = N V_m \sin \phi_s \frac{\beta c}{2\pi R} \quad (39)$$

which converts to the differential equation in γ :

$$\frac{d\gamma}{dt} = \frac{N V_m \sin \phi_s}{E_0} \frac{c}{2\pi R} \frac{(\gamma^2 - 1)^{1/2}}{\gamma} \quad (40)$$

whose solution may be written most simply in terms of the dimensionless momentum

$$\zeta \equiv \frac{p}{m_0 c} \quad (41)$$

(often called η), as

$$\zeta_f - \zeta_i = \frac{N V_m \sin \varphi_s}{E_o} \frac{c}{2\pi R} (t_f - t_i) \quad (42)$$

whence the required energy gain per turn ($N V_m \sin \varphi_s$) is about 2.4 keV. Hence, depending on the value of φ_s , $N V_m$ will vary from 5 to 27.5 keV. Since (except possibly at the top extreme) this can be done with one gap, we assume $N = h = 1$, whence from Eq. (37), at injection,

$$N_{bt} \approx (2.3 \text{ to } 70) \times 10^{12} \text{ particles} \quad (43)$$

(booster at injection)

The very wide range arises because $N V_m \sin \varphi_s$, rather than $N V_m$, is stipulated to be fixed.

As discussed in the preceding section, the value of N_{bt} drops slightly as the energy increases. Nevertheless, we see that, with a very efficient bunching system, the machine will be stable not only in Phase I of the AGS Improvement Program (about 10^{13} particles) but in Phase II (2×10^{13} particles) as well. But one cannot use a nominal value of φ_s without any thought.

We might note incidentally the rise time; here the instability will always be resistive. From Eq. (8), with $n \sim 120$, $N_b \sim 2 \times 10^{13}$, we find $\tau_N \sim 0.2 \mu\text{sec}$; from Eq. (4), taking $\sigma \sim 10^{16} \text{ sec}^{-1}$ and hence $\sqrt{\mu c / 2\pi\sigma} \sim 10^{-3} \text{ cm}^{1/2}$, we find $V/U \sim 2 \times 10^{-5}$, whence from Eq. (9) $\tau_R \sim 20 \text{ msec}$ at injection. The value is about a third of this (three times as fast) at a 630 MeV extraction. These times are enough less than the boosting times (about 80 msec to get up to 630 MeV) that it is indeed necessary to take care to be below the particle threshold.

The situation with the Brookhaven AGS is very different. The threshold is very high, for the machine¹⁴ has a radius of 125 meters, and has $N = h$ and $V_m = 15 \text{ keV}$, whence

$$N_{bt} \sim \begin{bmatrix} (0.79 \text{ to } 4.6) \\ (1.05 \text{ to } 6.1) \\ (1.67 \text{ to } 9.7) \end{bmatrix} \times 10^{14} \text{ particles} \begin{bmatrix} (50 \text{ MeV}) \\ (200 \text{ MeV}) \\ (500 \text{ MeV}) \end{bmatrix} \quad (44)$$

(AGS at injection)

14. See for instance G.K. Green and E.D. Courant, Handbuch der Physik XLIV (Berlin: Springer, 1959), pp 300-335.

But the problem is not at injection but at transition. This is a complex calculation indeed. What we shall do here is make rough calculations of

- (A) how long the AGS is below threshold, and
- (B) the (minimum) rise time constant of the instability.

By comparing these two quantities we can get a rough idea of whether or not there will be longitudinal instability problems.

(A) Duration of the Instability. Let $\delta\gamma$ be the half width* of γ during which the machine will be unstable. If we assume $\gamma_i \sim 1 \ll \gamma_t$, $\delta\gamma \ll \gamma_t$, then from Eq. (35), at injection

$$N_{bt}^{(inj)} \equiv N_i \sim (\text{const}) \left(\frac{1}{\gamma_t} \right)^{-1/4} \quad (45a)$$

while at the point near transition where the system becomes unstable, i.e. where the actual number of particles N_b in the beam is equal to the threshold,

$$N_b \sim (\text{const}) \left(\frac{2\delta\gamma}{\gamma_t} \right)^{3/4} \quad (45b)$$

whence the half-width $\delta\gamma$ depends on the number of particles in the beam as

$$\delta\gamma = \frac{1}{2} \left(\gamma_t \frac{N_b}{N_i} \right)^{4/3} \quad (46)$$

Now the energy of the protons increases about 100 MeV in 3 msec,¹⁴ so that half of the total time during which the system is unstable is

$$\delta t = 0.25 \left(\frac{N_b}{N_i} \right)^{4/3} \text{ seconds} \quad (47)^{**}$$

* Not to be confused with $\Delta\gamma$, the half-spread of γ in the beam.

** The numerical factor is not precisely one-fourth, merely 0.25 to two decimal places.

We enter figures in the first row of Table I, where we consider three values of N_b : the present AGS (2×10^{12}), the improved AGS in Phase I (10^{13}) and in Phase II (2×10^{13}). In each case the two figures correspond to extremes in Eq. (44).

	Present $N = 2 \times 10^{12}$	Improved Phase I $N = 10^{13}$	Improved Phase II $N = 2 \times 10^{13}$
Half-time in unstable zone δt , msec.	1.9 - 0.18	11 - 1.0	15 - 1.4
Minimum rise time, τ_N , msec	0.51 - 1.5	0.12 - 0.35	0.074 - 0.22
Ratio $\delta t/\tau_N$	4 - 0.1	90 - 3	200 - 6
Remarks	OK	?	??

TABLE I
AGS Behavior near Transition

B. Minimum Rise Time. The instability will exist as a resistive one during the first half of the crossing time and as a negative mass one during the second half. If we assume $n \sim 2250$ and again $\sqrt{\mu c/2\pi\sigma} \sim 10^{-3} \text{ cm}^{1/2}$, we find from Eq. (4) the value $V/U \sim 0.01$. Hence we simply ignore what happens during the resistive part and just find τ_N from Eq. (8). We see from (8) that the rise rate is zero precisely at transition. Its value at an amount $\delta\gamma$ away is obtained by replacing $1/\gamma^3$ by $1/\gamma_t^3$ and η by $2\delta\gamma/\gamma_t^3$; if we replace the term in square brackets in Eq. (8) by 3, we get a minimum rise time at the threshold point (which corresponds to N_b particles) of

$$(\tau_N)_{\min} \approx 1.2 \left(\frac{10^{12}}{N_b} \frac{B_{th}}{\delta\gamma} \right)^{1/2} \text{ msec} \quad (48)$$

where B_{th} is the bunching factor at the threshold point, i.e., from Eq. (31),

$$\frac{B_{th}}{B_{inj}} \approx \left(\frac{\eta_t}{\gamma_t} \right)^{1/4} \left(\frac{\gamma_i}{\eta_i} \right)^{1/4} \approx \left(\frac{2\delta\gamma}{\gamma_t} \right)^{1/4} \quad (49)$$

whence

$$(\tau_N)_{min} \approx 0.44 \left(\frac{10^{12}}{N_b} \frac{B_{inj}}{(\delta\gamma)^{3/4}} \right)^{1/2} \text{ msec} \quad (50)$$

Then we can use Eq. (46) to get finally

$$(\tau_N)_{min} \approx 0.19 \left(\frac{10^{12}}{N_b} \frac{N_i}{N_b} B_{inj} \right)^{1/2} \text{ msec} \quad (51)$$

We calculate this for the cases we are considering and put the result in Table I, second row.

The relevant figure is the number of τ_N time constants in half the unstable zone crossing time; this is listed in the third row of Table I. In general, it takes 5 to 10 full time constants to raise shot noise to a dangerous level. But τ_N is the minimum time. The actual time constant is infinite (no stability) exactly at transition, and by the time the system has reached the threshold the reduction factor has made the process somewhat slower than would be represented by $(\tau_N)_{min}$. Hence we guess very crudely that if the number in the third row is as much as 15 or 20 there may be trouble. Unless there is extremely efficient bunching (φ_s close to 5° at injection), there might be difficulties in Phase I and even more likely in Phase II.

This entire AGS transition calculation is of course extremely crude and serves only to point out that there may be problems and that the choice of φ_s is crucial; the calculation must be done much more precisely.*

* See discussion, Section VII.

VII. Discussion

This report had two aims:

- (1) to recognize the connection between the spread required to prevent longitudinal instabilities and the spread resulting from synchrotron oscillations, and hence to derive some new more direct formulas,
- (2) to test certain current and proposed machine designs to see if there might be trouble with longitudinal instabilities.

As for (1), Eq. (33) gives the longitudinal instability threshold at injection as a function of design parameters, and Eq. (35) gives the variation of threshold with energy. In general, if transition is not crossed, the value at injection gives a good idea of whether or not there will be trouble with longitudinal instabilities - because the time constant for the negative mass instability of the weak-focusing machines is so fast that only the injection value of threshold counts anyway, while in a booster operating between fifty and a few hundred MeV the variation in threshold is slight. For AGS calculations - and only a crude approach is given here - the two calculations must be combined; an example is given in detail in Section VI.

As for (2), the same general conclusion would appear to follow for the AGS as for the proposed boosters - with some care, there will probably be no trouble in Phase I ($N_b = 10^{13}$), and with somewhat more care, there should be no trouble in Phase II ($N_b = 2 \times 10^{13}$). But the injection synchronous phase angle must be chosen closer to 0° or 180° than one normally would.

The choice of phase angle is so much more important than one would think - particularly in connection with the behavior of the AGS at transition - because of the many ways in which it affects the stability. A large bunching factor raises the threshold by making the beam more uniform. Further, the corresponding larger synchrotron oscillation amplitude means greater energy spread and hence greater protection against instabilities, and the larger bunching factor means a slower rise rate. More experimental work in this field would be very useful.

As has been mentioned several times, the calculations near transition (i.e., for the AGS) are extremely crude. This crudity comes from making five sorts of approximations:

(1) The WKB formulas are assumed to hold even at transition, whereas in fact they fail, as can be seen from looking at (29). The correct expressions are given by Courant and Snyder.⁸

(2) The energy spread is assumed to be so small that the system is mono-energetic even when the threshold is varying wildly with energy at transition. This was discussed in a footnote in Section II.

(3) The theory of Neil and Sessler³ is assumed here to hold at transition, and this will bear closer analysis.

(4) As discussed below Eq. (51), the "average" rise time is estimated to be about two or three times the minimum. This crude guessing must be replaced by an integration as soon as the form of the various factors is known more precisely.

(5) The estimate of five or ten time constants for beam breakup must be replaced by more accurate calculations for the AGS, preferably when more experimental work has been done on this phenomenon. The same applies to the estimate of n , the harmonic number of the most virulent instabilities.

So a more careful analysis of what happens at transition is the next step. It is important that stronger predictions be made about these longitudinal instabilities. For in contrast to transverse instabilities, they cannot be servoed away, and the only static scheme for suppressing them,¹⁵ which involves coating the inside of the vacuum chamber probably with an oxide, raises problems of vacuum maintenance and radiation damage, as well as breaking down theoretically at the high frequency limit, just where the instability is worst.

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Distr.: AADD External

15. R.J. Briggs and V.K. Neil, J. Nucl. Energy C8, 255-69 (1966).