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TOPICS IN THE INCLUSIVE PHENOMENOLOGY *

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I. Exclusive and Inclusive cross-sections

I.1 Let me first recall the definition of the exclusive cross-section for the process

$$a + b \rightarrow c_1(p_1) + c_2(p_2) + \dots + c_n(p_n)$$

where c_i 's are identical and spinless particles.

Define the S-matrix by

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^4(p_f - p_i) M_{fi} / \sqrt{2E_a 2E_b \prod_{j=1}^n 2E_j} V^{n+2} \quad (\text{I.1})$$

where

$$p_i = p_a + p_b$$

and

$$p_f = p_1 + p_2 + \dots + p_n.$$

The transition rate is

$$\Gamma_{fi} = \frac{1}{T} \left| (2\pi)^4 \delta^4(p_f - p_i) M_{fi} / \sqrt{2E_a 2E_b \prod_{j=1}^n 2E_j} V^{n+2} \right|^2 \quad (\text{I.2})$$

which requires the interpretation of the square of the δ -function.

This can be done by first replacing

$$\begin{aligned} \delta^4(p_f - p_i) &= \frac{1}{(2\pi)^4} \int d^4\xi e^{i(p_f - p_i)\xi} = \frac{1}{(2\pi)^4} \int_V d^3\xi e^{-i(\vec{p}_f - \vec{p}_i) \cdot \vec{\xi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\xi_0 e^{i(E_f - E_i)\xi_0} \\ &= \frac{1}{2\pi} \delta^3(\vec{p}_f - \vec{p}_i) \frac{2}{(E_f - E_i)} \sin \left[\frac{T}{2} (E_f - E_i) \right] \end{aligned} \quad (\text{I.3})$$

and by interpreting

$$\begin{aligned} |\delta^3(\vec{p}_f - \vec{p}_i)|^2 &= \frac{1}{(2\pi)^3} \int_V d^3\xi e^{-i(\vec{p}_f - \vec{p}_i) \cdot \vec{\xi}} \cdot \delta^3(\vec{p}_f - \vec{p}_i) \\ &= \frac{V}{(2\pi)^3} \delta^3(\vec{p}_f - \vec{p}_i) \end{aligned}$$

Then (2) gives

$$\Gamma_{fi} = (2\pi)^4 \delta^4(p_f - p_i) \frac{|M_{fi}|^2 V}{2E_a 2E_b \prod_{j=1}^n 2E_j V^{n+2}} \quad (\text{I.4})$$

for

$$\lim_{T \rightarrow \infty} \frac{\sin^2 \frac{T}{2} x}{\frac{T}{2} x^2} = \pi \delta(x).$$

This procedure is equivalent to the formal replacement
(see Källén) $|(2\pi)^4 \delta^4(p_f - p_i)|^2 \longrightarrow VT (2\pi)^4 \delta^4(p_f - p_i)$.

The transition probability into the n-particle final state f in the momentum intervals $\prod_{j=1}^n d^3p_j$ is given by

$$\begin{aligned} d\omega_{fi} &= \prod_{j=1}^n \left(\frac{V}{(2\pi)^3} d^3p_j \right) \Gamma_{fi} \\ &= (2\pi)^4 \delta^4(p_f - p_i) \frac{d^3p_1}{(2\pi)^3 2E_1} \dots \frac{d^3p_n}{(2\pi)^3 2E_n} \frac{|M_{fi}|^2}{2E_a 2E_b V} \end{aligned} \quad (I.5)$$

This is equal to

$$d\sigma_{fi} = \rho(\text{target}) \cdot (\text{Incoming flux}) \quad (I.6)$$

where $\rho(\text{target})$ is the number density of the target, assumed here to be 1, and where the incoming flux is defined as the product of the number density of the beam ($\rho_{\text{beam}} = 1/V$) by the incident velocity v_{in} .

In the center of mass system of (a,b) particles

$$v_{\text{in}} = |\vec{v}_a - \vec{v}_b|_{\text{C.M.}} = \frac{p}{E_a} + \frac{p}{E_b} = \frac{p\sqrt{s}}{E_a E_b}$$

s is the invariant

$$s = (p_a + p_b)^2_{\text{C.M.}} = (E_a + E_b)^2$$

The incident momentum p is related to s by

$$p = \frac{1}{2\sqrt{s}} \sqrt{[s - (m_a + m_b)^2][s - (m_a - m_b)^2]} \equiv \frac{\lambda(s, m_a^2, m_b^2)}{2\sqrt{s}} \quad (I.7)$$

Note

$$\lambda^2(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

Then

$$d\sigma_{fi} = \prod_{j=1}^n \left(\frac{d^3p_j}{(2\pi)^3 2E_j} \right) \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) (2\pi)^4 \delta^4(p_f - p_i) |M_{fi}|^2 \quad (I.8)$$

Finally, the exclusive differential cross-section for

$$\frac{d\sigma(a+b \rightarrow c_1 + \dots + c_n)}{dp_1 dp_2 \dots dp_n} = \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) (2\pi)^4 \delta^4(p_f - p_i) \langle p_a p_b | M^\dagger | p_1 \dots p_n \rangle \langle p_1 \dots p_n | M | p_a p_b \rangle \quad (I.9)$$

where we denote

$$dp_j \equiv \frac{d^3 p_j}{(2\pi)^3 2E_j} \quad (I.10)$$

the Lorentz invariant phase-space volume element. The invariance property is immediately established by remarking that

$$\begin{aligned} \frac{d^3 p}{2E} &= \int dp_0 \frac{d^3 p}{2E} \left[\delta(p_0 - E) + \delta(p_0 + E) \right] \theta(p_0) \\ &= \int d^4 p \delta(p^2 - m^2) \theta(p_0) \end{aligned} \quad (I.11)$$

with

$$E = + \sqrt{\vec{p}^2 + m^2}$$

The four dimensional momentum space volume is of course Lorentz invariant, and so are the δ - and the θ - functions. (Note that the Lorentz-Group is $L^\uparrow$).

With (10), the Lorentz invariant normalization of the single particle state is

$$\langle p' | p \rangle = (2\pi)^3 2E \delta^3(\vec{p}' - \vec{p}) \quad (I.12)$$

since

$$\int d^3 p \langle p' | p \rangle = \int \frac{d^3 p}{(2\pi)^3 2E} \langle p' | p \rangle = 1. \quad (I.13)$$

In general, the normalization is specified by the completeness relation

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int dp_1 \dots dp_n |p_1 \dots p_n \rangle \langle p_1 \dots p_n| = I. \quad (I.14)$$

Since the n particles are identical, the partial cross-section is given by

$$\sigma^{(n)} = \frac{1}{n!} \int dp_1 \dots dp_n \frac{d\sigma(a+b \rightarrow c_1 + \dots + c_n)}{dp_1 dp_2 \dots dp_n} \quad (\text{I.15})$$

so that the total cross-section is

$$\sigma_{\text{tot}} = \sum_n \sigma^{(n)} \quad (\text{I.16})$$

which is a first trivial example of inclusive cross-section.

Let's consider the simple cases offered by the 2- or 3-particle cross-sections.

Eg.1) The two particle phase-space volume can be combined with the energy-momentum conservation, when the initial total momentum is given :

$$\begin{aligned} \sigma(a+b \rightarrow c_1+c_2) &= \int dp_1 dp_2 (2\pi)^4 \delta^4(p_1+p_2-p_a-p_b) \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) |M_{fi}|^2 \\ &= \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) \int \frac{d^3 p_1}{4E_1 \bar{E}_2} \frac{1}{(2\pi)^2} \delta(E_1 + \bar{E}_2 - \sqrt{s}) |M_{fi}|^2 \\ &= \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) \frac{1}{(2\pi)^2} \int \frac{p_1^2 dp_1 d\Omega_1}{4E_1 \bar{E}_2} \delta(E_1 + \bar{E}_2 - \sqrt{s}) |M_{fi}|^2 \end{aligned} \quad (\text{I.17})$$

we have introduced the notation

$$\bar{E}_2 = \sqrt{\vec{p}_1^2 + m_2^2} = \sqrt{E_1^2 - m_1^2 + m_2^2}$$

Then, using the property

$$p_1 dp_1 = E_1 dE_1 \quad (\text{I.18})$$

the integration over E_1 can be performed, which leads for $m_1=m_2$ to the angular distribution

$$\begin{aligned} \frac{d\sigma}{d\Omega_1} &= \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) \frac{p_1}{4\sqrt{s}} \frac{1}{(2\pi)^2} |M_{fi}|^2 \\ &= \frac{1}{4s} \frac{p_1}{p_a} \left| \frac{M_{fi}}{4\pi} \right|^2 \end{aligned} \quad (\text{I.19})$$

or equivalently

$$\frac{d\sigma}{d\cos\theta_1} = \frac{\pi}{2s} \frac{p_1}{p_a} \left| \frac{\langle p_1 p_2 | M | p_a p_b \rangle}{4\pi} \right|^2 \quad (\text{I.20})$$

and also

$$\frac{d\sigma}{dt} = \frac{\pi}{\lambda^2(s, m_a^2, m_b^2)} \left| \frac{M_{fi}}{4\pi} \right|^2 \quad (\text{I.21})$$

where

$$t = (p_a - p_1)^2 = m_a^2 + m_1^2 - 2(E_a E_1 - p_a p_1 \cos \theta_1)$$

Eg.2) The 3-particle phase-space together with the 4-momentum conservation can be worked out similarly :

$$\begin{aligned} \sigma(a+b \rightarrow c_1+c_2+c_3) &= \int dp_1 dp_2 dp_3 \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) (2\pi)^4 \delta^4(p_1+p_2+p_3 - p_a - p_b) |M_{fi}|^2 \\ &= \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) (2\pi)^{-5} \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \frac{d^3 p_3}{2E_3} \delta^4(p_1+p_2+p_3 - p_a - p_b) |M_{fi}|^2 \end{aligned} \quad (\text{I.22})$$

Introduce the variables

$$p = (p_a + p_b), \quad p^2 = s, \quad q = (p_1 + p_2), \quad q^2 = s'$$

Then, with the aid of

$$\int d^4 q \delta^4(q - p_1 - p_2) = 1 \quad (\text{I.23})$$

expression (22) becomes

$$\frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) (2\pi)^{-5} \int d^4 q \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta^4(q - p_1 - p_2) \int \frac{d^3 p_3}{2E_3} \delta^4(q + p_3 - p) |M_{fi}|^2 \quad (\text{I.24})$$

Now, integration over q_0 can be performed using

$$\int d^4 q \int ds' \delta(s' - q^2) = \int \frac{d^3 q}{2E'} ds' \quad (\text{I.25})$$

with $E' = \sqrt{\vec{q}^2 + s'} = q_0$, which leads finally to

$$\begin{aligned} \sigma &= \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) \int \frac{ds'}{2\pi} \left\{ \frac{1}{(2\pi)^2} \int \frac{d^3 q}{2E'} \frac{d^3 p_3}{2E_3} \delta^4(q - p - p_3) \right\} \\ &\quad \cdot \left\{ \frac{1}{(2\pi)^2} \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta^4(q - p_1 - p_2) \right\} \cdot |M_{fi}|^2 \end{aligned} \quad (\text{I.26})$$

or, as in the 2-particle case,

$$\sigma = \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) \int \frac{ds'}{2\pi} \int \frac{p_3 d\Omega_3}{(2\pi)^2 4\sqrt{s}} \int \frac{p_1 d\Omega_1}{(2\pi)^2 4\sqrt{s'}} |M_{fi}|^2 \quad (I.27)$$

where

$$p_3 = \frac{1}{2\sqrt{s}} \lambda(s, m_3^2, s') \quad , \quad p_1 = \frac{1}{2\sqrt{s'}} \lambda(s', m_1^2, m_2^2) \quad (I.28)$$

The 3-particle production differential cross-section reads.

$$\frac{d\sigma}{ds' d\Omega_1 d\Omega_3} = \frac{p_1 p_3}{32 (2\pi)^5 (ss')^{1/2}} \lambda^{-1}(s, m_a^2, m_b^2) |M_{fi}|^2 \quad (I.29)$$

I.2 In the inclusive process, one sums over all final particles that are not explicitly detected.

If $C_1(p_1)$ is the one detected, then from (15) the partial cross-section is given by

$$\frac{d\sigma^{(n)}}{dp} = \frac{1}{(n-1)!} \int dp_2 \dots dp_n \frac{d\sigma}{dp dp_2 \dots dp_n} \quad (I.30)$$

so that the one-particle inclusive cross-section is given by

$$\frac{d\sigma}{dp} = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int dp_2 \dots dp_n \frac{d\sigma}{dp dp_2 \dots dp_n} \quad (I.31)$$

The 2-particle inclusive cross-section is

$$\frac{d\sigma}{dp dq} = \sum_{n=3}^{\infty} \frac{1}{(n-2)!} \int dp_3 \dots dp_n \frac{d\sigma}{dp dq dp_3 \dots dp_n} \quad (I.32)$$

and in general, the n-particle inclusive cross-section is

$$\frac{d\sigma}{dp_1 \dots dp_n} = \sum_{m=1}^{\infty} \frac{1}{m!} \int dq_1 \dots dq_m \frac{d\sigma}{dp_1 \dots dp_n dq_1 \dots dq_m} \quad (I.33)$$

General References to §I

1. G.Källén, Elementary Particle Physics - Addison Wesley Publ.Cy
2. J.Benecke, T.T. Chou, C.N. Yang and E. Yen - Phys. Rev. 188 (1969) 2159.

II. Inclusive Sum Rules and some Elementary Consequences

II.1 Elementary Sum Rules

Introduce the following short-hand notation

$$\tilde{\sigma}^{(n)} \equiv \frac{d\sigma}{dp_1 dp_2 \dots dp_n} \quad (\text{II.1})$$

then, the expressions already obtained for the inclusive cross-sections simply read

$$\sigma_{\text{tot}} = \sum_n \sigma^{(n)} = \sum_n \frac{1}{n!} \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} \quad (\text{I.16})$$

$$\frac{d\sigma}{dp_1} = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int dp_2 \dots dp_n \tilde{\sigma}^{(n)} \quad (\text{I.31})$$

$$\frac{d\sigma}{dp_1 \dots dp_n} = \sum_{m=1}^{\infty} \frac{1}{m!} \int dq_1 \dots dq_m \tilde{\sigma}^{(n+m)} \quad (\text{I.33})$$

These formulae can be obviously generalized to the production of several species of stable hadrons :

$$a + b \rightarrow c_1(p_1) + \dots + c_n(p_n) + d_1(q_1) + \dots + d_m(q_m) + e_1(r_1) + \dots + e_\ell(r_\ell)$$

$$\frac{d\sigma}{dp_1 dq_1} = \sum_{n,m,\ell} \frac{1}{(n-1)!(m-1)!\ell!} \int dp_2 \dots dp_n dq_2 \dots dq_m dr_1 \dots dr_\ell \tilde{\sigma}^{(n+m+\ell)} \quad (\text{II.2})$$

$$\frac{d\sigma}{dp_{c_1} \dots dp_{c_i} dp_{d_1} \dots dp_{d_j}} = \sum_{n,\ell,m} \frac{1}{(n-i)!(m-j)!\ell!} \int dp_{c_{i+1}} \dots dp_{c_n} dp_{d_{j+1}} \dots dp_{d_m} dr_1 \dots dr_\ell \tilde{\sigma}^{(n+m+\ell)} \quad (\text{II.3})$$

One notices immediately that

$$\begin{aligned} \int \frac{d\sigma}{dp} dp &= \int dp \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int dp_2 \dots dp_n \tilde{\sigma}^{(n)} = \sum_n n \frac{1}{n!} \int dp dp_2 \dots dp_n \tilde{\sigma}^{(n)} \\ &= \sum_n n \sigma^{(n)} = \langle n \rangle \sigma_{\text{tot}} \end{aligned} \quad (\text{II.4})$$

and

$$\begin{aligned}
 \int \frac{d\sigma}{dp} p_\mu dp &= \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int dp_1 dp_2 \dots dp_n \tilde{\sigma}^{(n)} p_\mu \\
 &= \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \frac{1}{n} \left\{ \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} p_{1\mu} + \dots + \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} p_{n\mu} \right\} \\
 &= \sum_n \frac{1}{n!} \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} (p_1 + \dots + p_n)_\mu = (p_a + p_b)_\mu \sigma_{tot} \quad (\text{I.5})
 \end{aligned}$$

where the identity of the produced particles has been used, as well as 4-momentum conservation.

The generalization of (II 4 and 5) to the world of several kinds of stable hadrons is

$$\int \frac{d\sigma}{dp_c} dp_c = \langle n_c \rangle \sigma_{tot} \quad (\text{I.6})$$

and

$$\sum_c \int \frac{d\sigma}{dp_c} p_{c\mu} dp_c = (p_a + p_b)_\mu \sigma_{tot} \quad (\text{I.7})$$

From the two-particle inclusive cross-section

$$\frac{d\sigma}{dp_1 dp_2} = \sum_{n=3}^{\infty} \frac{1}{(n-2)!} \int dp_3 \dots dp_n \tilde{\sigma}^{(n)}$$

one gets

$$\begin{aligned}
 \int \frac{d\sigma}{dp_1 dp_2} dp_1 dp_2 &= \sum_{n=3}^{\infty} \frac{1}{(n-2)!} \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} = \sum_n n(n-1) \frac{1}{n!} \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} \\
 &= \sum_n n(n-1) \sigma^{(n)} = \langle n(n-1) \rangle \sigma_{tot} \quad (\text{I.8})
 \end{aligned}$$

which generalizes to

$$\int \frac{d\sigma}{dp_c dp_d} dp_c dp_d = \langle n_c n_d - n_c \delta_{cd} \rangle \sigma_{tot} \quad (\text{I.9})$$

Also

$$\begin{aligned}
 \int \frac{d\sigma}{dp_1 dp_2} p_{1\mu} dp_1 &= \sum_{n=3} \frac{1}{(n-2)!} \int dp_1 p_{1\mu} \int dp_3 \dots dp_n \tilde{\sigma}^{(n)} \\
 &= \sum_{n=3} \frac{1}{(n-2)!} \frac{1}{(n-1)} \left\{ \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} p_{1\mu} + \dots + \int dp_1 \dots dp_n \tilde{\sigma}^{(n)} p_{n\mu} \right\} \\
 &= \sum_{n=2} \frac{1}{(n-1)!} \int dp_1 dp_3 \dots dp_n (p_1 + p_3 + \dots + p_n)_\mu \tilde{\sigma}^{(n)} \\
 &= (p_a + p_b - p_2)_\mu \cdot \frac{d\sigma}{dp_2} \quad (\text{II.10})
 \end{aligned}$$

again we used the fact that the final particles are identical and the energy-momentum conservation. Note that there is no integration on p_2 .

From (II.5) and (II.10) one derives

$$\begin{aligned}
 \int dp dq p_\mu q_\nu \frac{d\sigma}{dp dq} &= \int dp p_\mu \int dq q_\nu \frac{d\sigma}{dp dq} \\
 &= \int dp p_\mu (p_a + p_b - p)_\nu \frac{d\sigma}{dp} \\
 &= (p_a + p_b)_\mu (p_a + p_b)_\nu \sigma_{tot} - \int dp p_\mu p_\nu \frac{d\sigma}{dp} \quad (\text{II.11})
 \end{aligned}$$

Consider this formula with $\mu=\nu$ and sum over index μ , then using

$$s = (p_a + p_b)^2, \quad p^2 = m^2,$$

one gets

$$\int dp dq (p \cdot q) \frac{d\sigma}{dp dq} + m^2 \langle n \rangle \sigma_{tot} = s \sigma_{tot} \quad (\text{II.12})$$

Note that the dominant contribution at high energies comes from the first term integrated only over the region of the two-particle phase space where $(p \cdot q) \sim s$, the double-fragmentation region as it will be called later.

Generalizations of (II 10-12) to the case of several kinds of hadrons are given by

$$\sum_c \int \frac{d\sigma}{dp_c dp_d} p_{c\mu} dp_c = (p_a + p_b - p_d)_\mu \frac{d\sigma}{dp_d} \quad (\text{II.13})$$

$$\sum_{c,d} \int \frac{d\sigma}{dp_c dp_d} p_{c\mu} dp_c dp_d = (p_a + p_b)_\mu \left(\sum_d \langle n_d \rangle - 1 \right) \sigma_{tot} \quad (\text{II.14})$$

$$\begin{aligned} \sum_{c,d} \int \frac{d\sigma}{dp_c dp_d} p_{c\mu} p_{d\nu} dp_c dp_d &= (p_a + p_b)_\mu (p_a + p_b)_\nu \sigma_{tot} \\ &\quad - \sum_d \int dp_d p_{d\mu} p_{d\nu} \frac{d\sigma}{dp_d} \end{aligned} \quad (\text{II.15})$$

and

$$\sum_{c,d} \int \frac{d\sigma}{dp_c dp_d} (p_c \cdot p_d) dp_c dp_d + \sum_d m_d^2 \langle n_d \rangle \sigma_{tot} = s \sigma_{tot} \quad (\text{II.16})$$

Consider the conservation of any discrete additive quantum number Q , such as charge, Baryon number,...

$$\sum_c n_c Q_c = Q = Q_a + Q_b$$

Then

$$\int Q_c \frac{d\sigma}{dp_c} dp_c = Q_c \langle n_c \rangle \sigma_{tot}$$

so that

$$\sum_c \int Q_c \frac{d\sigma}{dp_c} dp_c = \sum_c Q_c \langle n_c \rangle \sigma_{tot} \quad (\text{II.17})$$

On the other hand, one has

$$\begin{aligned} \int \frac{d\sigma}{dp_c} dp_c &= \sum_{n_c n_d} \frac{1}{(n_c-1)! n_d!} \int dp_c dp_{c_2} \dots dp_{c n_c} dp_{d_1} \dots dp_{d n_d} \tilde{\sigma}^{(n_c+n_d)} \\ &= \sum_{n_c n_d} n_c \frac{1}{n_c! n_d!} \int dp_{c_1} \dots dp_{c n_c} dp_{d_1} \dots dp_{d n_d} \tilde{\sigma}^{(n_c+n_d)} \\ &= \sum_{n_c n_d} n_c \sigma^{(n_c+n_d)} \end{aligned}$$

which gives

$$\begin{aligned} \sum_c Q_c \int \frac{d\sigma}{dp_c} dp_c &= \sum_c \sum_{n_c n_d} Q_c n_c \sigma^{(n_c+n_d)} = \sum_{n_c n_d} \left(\sum_c Q_c n_c \right) \sigma^{(n_c+n_d)} \\ &= \sum_{n_c n_d} Q \sigma^{(n_c+n_d)} = Q \sigma_{tot} \end{aligned} \quad (\text{II.18})$$

Finally, (II.17) and (II.18) together furnish

$$\sum_c Q_c \langle n_c \rangle = Q \quad (\text{II.19})$$

Similarly, one derives

$$\sum_c Q_c \langle n_c n_d \rangle = Q \langle n_d \rangle \quad (\text{II.20})$$

II.2 Elementary Consequences

Eg.1) Let's turn to the elementary consequences of these sum rules.

First, from

$$\sum_c \int \frac{d\sigma}{dp_c} p_{c\mu} dp_c = (p_a + p_b)_\mu \sigma_{tot}$$

taken for $\mu=0$, one gets

$$\sum_c \int \frac{d\sigma}{dp_c} \frac{d^3 p_c}{2(2\pi)^3} = \sqrt{s} \sigma_{tot} \quad (\text{II.21})$$

or, using the following decomposition of $\vec{p}_c$ relatively to the initial momentum

$$\vec{p}_c = p_c^\parallel + \vec{p}_c^\perp$$

and the Feynman variable x defined as

$$x_c \equiv \frac{p_c^\parallel}{p_{max}} \simeq \frac{2p_c^\parallel}{\sqrt{s}} \quad (\text{II.22})$$

formula (21) reads

$$\sum_c \int \frac{d\sigma}{dp_c} dx_c d^2 p_c^\perp \frac{1}{4(2\pi)^3} = \sigma_{tot} \quad (\text{II.23})$$

Remark that the kinematics implies the constraint

$$- \sqrt{1 - \left(\frac{2p_{\perp}}{\sqrt{s}}\right)^2} \leq x_c \leq + \sqrt{1 - \left(\frac{2p_{\perp}}{\sqrt{s}}\right)^2} \quad (\text{II.24})$$

Now, formula (II.21) is also

$$\sum_c \int \frac{d\sigma}{dp_c} \frac{E_c}{E_{tot}} dp_c = \sigma_{tot}$$

and

$$E_c = \sqrt{\vec{p}_c^2 + m_c^2} = \sqrt{p_{\parallel}^2 + \vec{p}_c^{\perp 2} + m_c^2} = \sqrt{p_{\parallel}^2 + \kappa^2} = \frac{1}{2} \sqrt{s x_c^2 + 4\kappa^2}$$

where

$$\kappa^2 \equiv \vec{p}_{\perp}^2 + m_c^2 \quad (\text{II.25})$$

The ratio

$$\frac{E_c}{E_{tot}} = \frac{1}{2} \sqrt{x_c^2 + \frac{4\kappa^2}{s}}$$

has always a lower bound in kinematical regions : e.g; in the fragmentation region

$$\frac{E_c}{E_{tot}} \geq \frac{1}{2}$$

Since the quantity which is integrated is positive definite, the restriction of the integral to one kinematical region can never be cancelled by the contributions of the other regions of phase-space.

By restricting to that region where

$$E_c > d \cdot E_{tot} \quad (\text{II.26})$$

and assuming the Froissart bound $\sigma_{tot} < c (\log s)^2$, one gets

$$\sigma_{tot} \simeq \int_{E_c > d E_{tot}} \frac{d\sigma}{dp_c} \frac{E_c}{E_{tot}} dp_c > d \int_{E_c > d \cdot E_{tot}} \frac{d\sigma}{dp_c} dp_c$$

or

$$\int_{E_c > d \cdot E_{tot}} \frac{d\sigma}{dp_c} dp_c < d^{-1} \sigma_{tot} < d^{-1} c (\log s)^2 \quad (\text{II.27})$$

Eg.2) Estimate of the Triple-Pomeron coupling

Consider the one-particle inclusive process

$$a + b \rightarrow c + \text{anything}$$

The Triple-Regge region is the region near the phase-space boundary which is defined by the following limits

$$S_{a\bar{c}} \text{ is fixed}$$

$$S = S_{ab} \rightarrow \infty$$

$$M^2 = (p_a + p_b - p_c)^2 \rightarrow \infty$$

} $\Rightarrow$

such that s/M^2 is large.

In terms of the Feynman variable x_c , the limit corresponds to $x_c \rightarrow 1$, since

$$S_{a\bar{c}} = (p_a - p_c)^2 \approx -\frac{\kappa^2}{x_c} = -\frac{\bar{p}_{1c}^2 + m_c^2}{x_c}$$

$$S_{b\bar{c}} = (p_b - p_c)^2 \approx -s x_c$$

so that $s/M^2 \sim 1/(1-x_c)$.

The energy dependence of the one-particle inclusive cross-section is given by

$$\begin{aligned} \frac{d\sigma}{dp_c} &\sim \frac{1}{s} \beta_{a\bar{c}} \beta_{\bar{a}c} s^{\alpha_{a\bar{c}} + \alpha_{\bar{a}c}} \text{Disc}_{M^2} A(M^2, \alpha_{a\bar{c}}, \alpha_{\bar{a}c}, \alpha_{b\bar{b}}(0)) \\ &\sim \beta_{a\bar{c}} \beta_{\bar{a}c} \beta_{b\bar{b}}(0) g(\alpha_{a\bar{c}}, \alpha_{\bar{a}c}, \alpha_{b\bar{b}}(0)) s^{\alpha_{a\bar{c}} + \alpha_{\bar{a}c} - 1} (M^2)^{\alpha_{b\bar{b}}(0) - \alpha_{a\bar{c}} - \alpha_{\bar{a}c}} \end{aligned} \quad (\text{II.29})$$

where β 's are the usual particle-particle-Reggeon coupling and $g(\alpha_1, \alpha_2, \alpha_3)$ is the triple Regge coupling constants.

Suppose that the trajectories have the unit intercept and in particular

$$\alpha_{a\bar{c}} = \alpha_{\bar{a}c} = 1 + s_{a\bar{c}} \quad ; \quad \alpha_{b\bar{b}}(0) = 1 \quad (\text{II.30})$$

then, roughly, the most singular part of (II.29) is given by

$$\begin{aligned} \frac{d\sigma}{dp_c} &\sim |\beta(0)|^2 \beta(0) g_{TR}(0,0,0) \left(\frac{M^2}{s}\right)^{-1-2s_{a\bar{c}}} \\ &\sim |\beta(0)|^2 \beta(0) g_{TR}(0,0,0) (1-x_c)^{-1-2s_{a\bar{c}}} \end{aligned} \quad (\text{II.31})$$

while the total cross-section approaches a constant.

On the other hand the total cross-section follows from (II.31) to be

$$\sigma_{\text{tot}} \sim \frac{\pi}{4(2\pi)^3} \int d^2 p_c^\perp \lim_{\epsilon \rightarrow 0} \int_{1-\delta}^{1-\epsilon} dx_c \frac{d\sigma}{dp_c} \quad (\text{II.32})$$

so that

$$\sigma_{\text{tot}} \sim \frac{\pi^2}{4(2\pi)^3} |\beta(0)|^2 \beta(0) g_{TR}(0,0,0) \int_0^\infty d(\bar{p}_{c\perp}^2) \lim_{\epsilon \rightarrow 0} \int_{1-\delta}^{1-\epsilon} dx_c (1-x_c)^{-1-2s_{a\bar{c}}} \quad (\text{II.33})$$

where

$$\epsilon = c \left(\frac{s_0}{s} \right)$$

Formula (II.33) then gives

$$\begin{aligned} \sigma_{\text{tot}} &\sim \frac{\pi^2}{4(2\pi)^3} |\beta(0)|^2 \beta(0) g_{TR}(0,0,0) \left\{ \frac{1}{2} \lim_{\epsilon \rightarrow 0} \ln \frac{2m_c^2 \ln \epsilon^{-1}}{2m_c^2 \ln \delta^{-1}} \right. \\ &\quad \left. + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \sum_{n=1}^\infty \frac{(-)^n}{n n!} \left[(2m_c^2 \ln \epsilon^{-1})^n - (2m_c^2 \ln \delta^{-1})^n \right] \right\} \end{aligned} \quad (\text{II.34})$$

where we used

$$\lim_{\epsilon \rightarrow 0} \int_0^\infty d(\bar{p}_{c\perp}^2) \int_{1-\delta}^{1-\epsilon} dx_c (1-x_c)^{-1-2s_{a\bar{c}}} = \lim_{\epsilon \rightarrow 0} -\frac{1}{2} \int_{m_c^2}^\infty \frac{dx}{x} \left[e^{-2x \ln \epsilon^{-1}} - e^{-2x \ln \delta^{-1}} \right] \quad (\text{II.35})$$

and

$$\int_a^\infty \frac{dx}{x} e^{-bx} = \int_{ab}^\infty \frac{dz}{z} e^{-z} = E_1(ab) = -\gamma - \ln(ab) - \sum_{n=1}^\infty \frac{(-)^n (ab)^n}{n n!} \quad (\text{II.36})$$

(II.34) shows that σ_{tot} can grow at least like $\ln \ln \epsilon^{-1} \sim \ln \ln s$.

(For $m_c \neq 0$, the degree of divergence is even worse). Thus if

$\alpha_p(0) = 1$, we must have

$$\lim_{t \rightarrow 0} g_{TR}(t, t, 0) = 0.$$

Eg.3) Implications of inclusive sum rules to the correlations.

Let me first define the two-particle correlation function by

$$C(p, q) = \frac{d\sigma}{dp dq} - \frac{1}{\sigma_{tot}} \frac{d\sigma}{dp} \cdot \frac{d\sigma}{dq} \quad (\text{II.37})$$

There is no unique prescription for the two-particle correlations; other definitions are used by other authors, e.g.

$$1) \quad \frac{1}{\sigma} \frac{d\sigma}{dp dq} - \left(\frac{1}{\sigma} \frac{d\sigma}{dp} \right) \left(\frac{1}{\sigma} \frac{d\sigma}{dq} \right)$$

$$2) \quad \frac{1}{\langle n_c n_d - n_c \delta_{cd} \rangle} \frac{d\sigma}{dp_c dp_d} - \left(\frac{1}{\langle n_c \rangle} \frac{d\sigma}{dp_c} \right) \left(\frac{1}{\langle n_d \rangle} \frac{d\sigma}{dp_d} \right)$$

All of them include some amount of kinematical correlation.

From (II.15), one can easily see that

$$\int dp dq C(p, q) p_\mu q_\nu = - \int dq q_\mu q_\nu \frac{d\sigma}{dq} \quad (\text{II.38})$$

For $\mu = \nu = 0$

$$\int dp dq E_p E_q C(p, q) = - \int dq E_q^2 \frac{d\sigma}{dq} \quad (\text{II.39})$$

and for $\mu = \nu = 3$ (the z-axis is // to the incident momentum)

$$\int dp dq \vec{p}_\parallel \cdot \vec{q}_\parallel C(p, q) = - \int dq q_\parallel^2 \frac{d\sigma}{dq}$$

so that

$$\begin{aligned} \int dp dq (E_p E_q - \vec{p}_\parallel \cdot \vec{q}_\parallel) C(p, q) &= - \int dq (E_q^2 - q_\parallel^2) \frac{d\sigma}{dq} \\ &= - \int dq (\vec{q}_\perp^2 + m^2) \frac{d\sigma}{dq} \end{aligned} \quad (\text{II.40})$$

The right-hand side of (II.40) is nearly zero (small transverse momenta and negligible masses).

Then, we have, for the kinematic arrangement in which

$$\vec{p}_\parallel \parallel \vec{q}_\parallel, \quad \vec{p}_\parallel \cdot \vec{q}_\parallel = -p_\parallel q_\parallel < 0, \quad (\text{II.41})$$

$$\int_{\vec{p}_\perp \cdot \vec{q}_\perp < 0} dp dq (E_p E_q + p_\parallel q_\parallel) C(p, q) = 2 \int_{\vec{p}_\perp \cdot \vec{q}_\perp < 0} dp dq (E_p E_q) C(p, q) = - \int dq (\vec{q}_\perp^2 + m^2) \frac{d\sigma}{dq}$$

or

$$\int_{\vec{p}_\perp \cdot \vec{q}_\perp < 0} dp dq E_p E_q C(p, q) = - \frac{1}{2} \int dq (\vec{q}_\perp^2 + m^2) \frac{d\sigma}{dq} \sim 0$$

while, from (II.39)

$$\int_{\vec{p}_\perp \cdot \vec{q}_\perp > 0} dp dq E_p E_q C(p, q) = - \int dq E_q^2 \frac{d\sigma}{dq}$$

In the double fragmentation region, the correlation is very weak but in the single fragmentation region, the correlation must be negative. Indeed, the momentum correlations must exist.

Take $\mu = \nu = 1, 2$ and sum ; then

$$\int dp dq \vec{p}_\perp \cdot \vec{q}_\perp \frac{d\sigma}{dp dq} + \int dq \vec{q}_\perp^2 \frac{d\sigma}{dq} = 0$$

or

$$\int dp dq \vec{p}_\perp \cdot \vec{q}_\perp C(p, q) + \int dq \vec{q}_\perp^2 \frac{d\sigma}{dq} = 0$$

so that $d\sigma/dp dq$ must have correlation in the azimuthal angle ($\cos \varphi = \hat{p}_\perp \cdot \hat{q}_\perp$). Indeed, it has been argued that the presence of the azimuthal angle dependence implies a cut component to the Pomeron singularity or some singularity more complicated than a simple pole by Freedman et al.

They write the 2-particle inclusive cross-section in the double fragmentation region obtained by taking the leading high-energy limit implied by the exchange of the Toller-Lorentz poles having the quantum numbers λ and M of the $O(3,1)$ group.

$$\frac{d\sigma}{dp dq} = \frac{1}{s} \sum_{M=0}^{\infty} (m^2)^{\alpha_V(0)} \cos M \varphi F_M(t_1, t_2, \frac{m_1^2}{m^2}, \frac{m_2^2}{m^2})$$

where

$\alpha_V(0)$ is the leading singularity in the Lorentz quantum number λ ,

$$m^2 = (p_a + p_b - p - q)^2, \quad m_1^2 = (p_a + p_b - q)^2, \quad m_2^2 = (p_a + p_b - p)^2.$$

The leading singularity must have $M=0$ component. If $\alpha_v(0)$ is a simple pole, $\sum_M$ must reduce to a single term with $M=0$; if $\alpha_v(0)$ is a cut, all terms of the $\sum_M$ contribute.

Eg.4) Consequences of the sum rules involving additive quantum numbers.

In (II.19) we had already obtained

$$\sum \langle n_c \rangle Q_c = Q.$$

Dealing with the 2-particle inclusive cross-section, we derive

$$\sum_c \int Q_c \frac{d\sigma}{dp_c dp_d} dp_c = (Q - \langle n_d \rangle Q_d) \frac{d\sigma}{dp_d} \quad (\text{II.42})$$

which reads, in term of the correlation function

$$- \langle n_d \rangle Q_d \frac{d\sigma}{dp_d} = \sum_c \int Q_c C_{c,d}(p_c, p_d) dp_c \quad (\text{II.43})$$

Imagine a world consisting only of pions ; Q is the electric charge.

We get

$$\begin{aligned} - \langle n_{\pi^+} \rangle \frac{d\sigma}{dq_{\pi^+}} &= \int dp \left(- C_{\pi^-\pi^+}(p, q) + C_{\pi^+\pi^+}(p, q) \right) \\ + \langle n_{\pi^-} \rangle \frac{d\sigma}{dq_{\pi^-}} &= \int dp \left(- C_{\pi^-\pi^-}(p, q) + C_{\pi^-\pi^+}(p, q) \right) \\ C_{\pi^-\pi^+} &= C_{\pi^+\pi^-} \end{aligned}$$

Integrate both sides with respect to q and sum up the two relations, then

$$\left(\langle n_{\pi^+} \rangle^2 - \langle n_{\pi^-} \rangle^2 \right) \sigma_{tot} = \int dp dq \left(C_{\pi^-\pi^-}(p, q) - C_{\pi^+\pi^+}(p, q) \right) \quad (\text{II.44})$$

It seems reasonable to put this difference equal to zero, so that

$$\langle n_{\pi^+} \rangle = \langle n_{\pi^-} \rangle .$$

From (II.9), one derives, using the correlation function,

$$\langle (n_c - \langle n_c \rangle)(n_d - \langle n_d \rangle) \rangle \sigma_{tot} = \langle n_c \rangle \delta_{cd} \sigma_{tot} + \int dp_c dp_d C(p_c, p_d).$$

For the pion-world, this gives

$$\begin{aligned} \langle (n_{\pi^+} - \langle n_{\pi^+} \rangle)(n_{\pi^-} - \langle n_{\pi^-} \rangle) \rangle \sigma_{tot} &= \int C_{\pi^+\pi^-}(p, q) dp dq \\ \langle (n_{\pi^+} - \langle n_{\pi^+} \rangle)^2 \rangle \sigma_{tot} &= \langle n_{\pi^+} \rangle \sigma_{tot} + \int C_{\pi^+\pi^+}(p, q) dp dq \\ \langle (n_{\pi^-} - \langle n_{\pi^-} \rangle)^2 \rangle \sigma_{tot} &= \langle n_{\pi^-} \rangle \sigma_{tot} + \int C_{\pi^-\pi^-}(p, q) dp dq \end{aligned} \quad (\text{II.45})$$

From (II.44) and (II.45)

$$\langle n_{\pi^+}^2 \rangle - \langle n_{\pi^-}^2 \rangle = \langle n_{\pi^+} \rangle - \langle n_{\pi^-} \rangle.$$

$$\langle (n_{ch} - \langle n_{ch} \rangle)^2 \rangle = 2 \langle n_{ch} \rangle + \langle n_{\pi^+} \rangle^2 + \langle n_{\pi^-} \rangle^2 + 2 \int (C_{\pi^+\pi^+} + C_{\pi^-\pi^-}) dp dq$$

which shows that the averaged deviation of the charged pions distribution can be larger than the averaged multiplicity itself.

General References to SII

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III. Inclusive and Exclusive Functionals.

Let me now introduce functionals for the cross-sections, from which we'll derive sum rules and the correlation functions.

Remind

$$\frac{d\sigma^{excl.}}{dp_1 \dots dp_n} = \frac{1}{2} \lambda^{-1}(s, m_a^2, m_b^2) (2\pi)^4 \delta^4\left(\sum_{i=1}^n p_i - P\right) |M_{ab \rightarrow c_1 + \dots + c_n}|^2 \quad (\text{I.9})$$

$$P = (p_a + p_b)$$

We rewrite the right-hand side in the form

$$= \delta^4\left(\sum_{i=1}^n p_i - P\right) |T_{fi}|^2 \quad (\text{III.1})$$

showing explicitly the presence of the 4-momentum conservation δ -function.

Define the exclusive functional

$$\Sigma(\eta) \equiv \sum_n \frac{1}{n!} \int dp_1 \dots dp_n \delta^4(\sum p_i - P) |T_{fi}|^2 \eta(p_1) \eta(p_2) \dots \eta(p_n) \quad (\text{III.2})$$

depending on the scalar parameter η .

Taking the limit $\eta \rightarrow 1$, one gets immediately (see I.15)

$$\Sigma(\eta)|_{\eta \rightarrow 1} = \sigma_{tot} \quad (\text{III.3})$$

For the functional derivative, we admit the following convention

$$\frac{\delta \eta(p_\alpha)}{\delta \eta(p_i)} = (2E_i) (2\pi)^3 \delta^3(\vec{p}_\alpha - \vec{p}_i) \quad (\text{III.4})$$

in order to preserve our normalization.

Then

$$\begin{aligned} \frac{\delta \Sigma(\eta)}{\delta \eta(p_\alpha)} &= \sum_n \frac{1}{n!} \int dp_1 \dots dp_n \delta^4(\sum p_i - P) |T|^2 \left\{ \frac{\delta \eta(p_1)}{\delta \eta(p_\alpha)} \eta(p_2) \dots \eta(p_n) + \dots \right. \\ &\quad \left. + \eta(p_1) \eta(p_2) \dots \frac{\delta \eta(p_n)}{\delta \eta(p_\alpha)} \right\} \\ &= \sum_n \frac{n}{n!} \int dp_1 \dots dp_n \delta^4(\sum p_i - P) |T|^2 (2E_j) (2\pi)^3 \delta^3(\vec{p}_j - \vec{p}_\alpha) \eta(p_1) \dots \eta(p_j) \dots \eta(p_n) \end{aligned}$$

j is one of the indices 1, 2, ..., n ; so that

$$\frac{\delta \Sigma(\eta)}{\delta \eta(p_\alpha)} = \sum_n \frac{1}{(n-1)!} \int dp_1 \dots dp_{\alpha-1} dp_{\alpha+1} \dots dp_n \delta^4(p_\alpha + \sum_{i \neq \alpha} p_i - P) |T|^2 \eta(p_1) \dots \eta(p_{\alpha-1}) \eta(p_{\alpha+1}) \dots \eta(p_n) \quad (\text{III.5})$$

whose limit for $\eta \rightarrow 1$ gives the one-particle inclusive cross-section

$$\left. \frac{\delta \Sigma(\eta)}{\delta \eta(p_\alpha)} \right|_{\eta \rightarrow 1} = \frac{d\sigma}{dp_\alpha} \quad (\text{III.6})$$

The same functional derivative procedure leads to

$$\left. \frac{\delta \Sigma(\eta)}{\delta \eta(p_\alpha) \dots \delta \eta(p_\gamma)} \right|_{\eta \rightarrow 1} = \frac{d\sigma}{dp_\alpha \dots dp_\gamma} \quad (\text{III.7})$$

when repeated several times.

Note that this functional $\Sigma(\eta)$ gives the exclusive cross section in the limit $\eta \rightarrow 0$

$$\left. \frac{\delta \Sigma(\eta)}{\delta \eta(p_\alpha) \dots \delta \eta(p_\gamma)} \right|_{\eta \rightarrow 0} = \frac{d\sigma^{\text{excl.}}}{dp_\alpha \dots dp_\gamma} \quad (\text{III.8})$$

In the same sence, one introduces the inclusive functional

$$I(\eta) = \Sigma(1+\eta) \quad (\text{III.9})$$

$$= \sigma_{\text{tot}} + \sum \frac{1}{n!} \int dp_1 \dots dp_n \frac{d\sigma^{\text{incl.}}}{dp_1 \dots dp_n} \eta(p_1) \dots \eta(p_n) \quad (\text{III.10})$$

which gives

$$\frac{\delta I(\eta)}{\delta \eta(p_\alpha) \dots \delta \eta(p_\gamma)} \Big|_{\eta \rightarrow 0} = \frac{d\sigma^{inel.}}{dp_\alpha \dots dp_\gamma} \quad (\text{III.11})$$

With the use of simple functional analysis, the sum rules already obtained in (II) can be derived.

One verifies immediately the following

$$\left[\int dp_\alpha p_{\alpha\mu} \eta(p_\alpha) \frac{\delta}{\delta \eta(p_\alpha)} \right] \Sigma(\eta) = P_\mu \Sigma(\eta) \quad (\text{III.12})$$

which remains after the substitutions

$$\eta \rightarrow 1 + \eta, \quad \Sigma(\eta) \rightarrow \Sigma(1 + \eta) = I(\eta)$$

so that

$$\left[\int dp_\alpha p_{\alpha\mu} (1 + \eta(p_\alpha)) \frac{\delta}{\delta \eta(p_\alpha)} \right] I(\eta) = P_\mu I(\eta) \quad (\text{III.13})$$

The limit $\eta \rightarrow 0$ gives

$$\int dp_\alpha p_{\alpha\mu} \frac{d\sigma}{dp_\alpha} = P_\mu \sigma_{tot} \quad (\text{II.5})$$

Differentiate (III.13) with respect of $\eta(p_\beta)$ and take the limit $\eta \rightarrow 0$ afterwards, then one recovers

$$\int dp_\alpha p_{\alpha\mu} \frac{d\sigma}{dp_\alpha dp_\beta} = (P - p_\beta)_\mu \frac{d\sigma}{dp_\beta} \quad (\text{II.10})$$

and n differentiations lead to

$$\int dp_\alpha p_{\alpha\mu} \frac{d\sigma}{dp_1 \dots dp_n dp_\alpha} = \left(P - \sum_{i=1}^n p_i \right)_\mu \frac{d\sigma}{dp_1 \dots dp_n} \quad (\text{III.14})$$

Furthermore, the iteration of (III.13) gives

$$\left[\int dp_\beta p_{\beta\nu} \eta(p_\beta) \frac{\delta}{\delta \eta(p_\beta)} \right] \left[\int dp_\alpha p_{\alpha\mu} \eta(p_\alpha) \frac{\delta}{\delta \eta(p_\alpha)} \right] \Sigma(\eta) = P_\mu P_\nu \Sigma(\eta) \quad (\text{III.15})$$

Substitute $\eta \rightarrow 1 + \eta$, $\Sigma(\eta) \rightarrow I(\eta)$, then, using (III.4)

$$\int d p_{\beta} p_{\beta \nu} p_{\alpha \mu} (1 + \eta(p_{\beta})) \frac{\delta I(\eta)}{\delta \eta(p_{\beta})} + \int d p_{\beta} d p_{\alpha} p_{\alpha \mu} p_{\beta \nu} (1 + \eta(p_{\alpha}))(1 + \eta(p_{\beta})) \frac{\delta I(\eta)}{\delta \eta(p_{\alpha}) \delta \eta(p_{\beta})} = P_{\mu} P_{\nu} I(\eta) \quad (\text{III.16})$$

whose limit for $\eta \rightarrow 0$ is just (II.11).

Again one more differentiation brings the result

$$\begin{aligned} (P_{\mu} P_{\nu} - p_{\gamma \mu} p_{\gamma \nu}) \frac{d\sigma}{dp_{\gamma}} &= \int d p_{\beta} d p_{\alpha} p_{\alpha \mu} p_{\beta \nu} \frac{d\sigma}{d p_{\alpha} d p_{\beta} d p_{\gamma}} \\ &+ \int d p_{\alpha} p_{\alpha \mu} p_{\gamma \nu} \frac{d\sigma}{d p_{\alpha} d p_{\gamma}} + \int d p_{\beta} p_{\beta \nu} (p_{\beta} + p_{\gamma})_{\mu} \frac{d\sigma}{d p_{\beta} d p_{\gamma}} \end{aligned} \quad (\text{III.17})$$

Summing up equations (III.13), (III.15) and the further iterations contracted with an arbitrary four-vector x^{μ} , one obtains the formal operator equation

$$\left[\exp. \int d p_{\alpha} p_{\alpha \mu} x^{\mu} \eta(p_{\alpha}) \frac{\delta}{\delta \eta(p_{\alpha})} \right] \Sigma(\eta) = \left[\exp. P_{\mu} x^{\mu} \right] \Sigma(\eta). \quad (\text{III.18})$$

Let's now reexpress those relations in terms of correlation functions. We have already defined the two-particle correlation function

$$C_2(p_{\alpha}, p_{\beta}) \equiv \frac{d\sigma}{d p_{\alpha} d p_{\beta}} - \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{d p_{\alpha}} \frac{d\sigma}{d p_{\beta}} \quad (\text{II.37})$$

which has the dimension of a cross-section when integrated over the momenta. One can easily build up from this the generalized definition of n-particle correlation function :

$$C_1(p_{\alpha}) \equiv \frac{d\sigma}{d p_{\alpha}} \quad (\text{III.19})$$

$$C_2(p_{\alpha}, p_{\beta}) = \frac{d\sigma}{d p_{\alpha} d p_{\beta}} - \frac{1}{\sigma_{\text{tot}}} C_1(p_{\alpha}) C_1(p_{\beta}) \quad (\text{III.20})$$

$$C_3(p_\alpha p_\beta p_\gamma) \equiv \frac{d\sigma}{dp_\alpha dp_\beta dp_\gamma} - \frac{1}{\sigma_{tot}} \left\{ C_1(p_\alpha) C_2(p_\beta p_\gamma) + C_1(p_\beta) C_2(p_\alpha p_\gamma) + C_1(p_\gamma) C_2(p_\alpha p_\beta) \right\} \\ - \frac{1}{\sigma_{tot}^2} C_1(p_\alpha) C_1(p_\beta) C_1(p_\gamma) \quad (\text{III.21})$$

etc....

Our sum rules then read, in terms of correlation functions

$$(\text{II.5}) \rightarrow P_\mu \sigma_{tot} = \int dp_\alpha p_{\alpha\mu} C_1(p_\alpha) \quad (\text{III.22})$$

$$(\text{II.10}) \rightarrow -p_{\beta\mu} C_1(p_\beta) = \int dp_\alpha p_{\alpha\mu} C_2(p_\alpha p_\beta) \quad (\text{III.23})$$

$$(\text{III.17}) \rightarrow -(p_\beta + p_\gamma)_\mu C_2(p_\beta p_\gamma) = \int dp_\alpha p_{\alpha\mu} C_3(p_\alpha p_\beta p_\gamma) \quad (\text{III.24})$$

and in general

$$-(p_\beta + p_\gamma + \dots + p_n)_\mu C_{n-1}(p_\beta \dots p_n) = \int dp_\alpha p_{\alpha\mu} C_n(p_\alpha p_\beta \dots p_n). \quad (\text{III.25})$$

Now, (III.23) substituted into (III.22) gives

$$P_\mu \sigma_{tot} = - \int dp_\alpha dp_\beta p_{\beta\mu} C_2(p_\alpha p_\beta) \quad (\text{III.26})$$

and (III.24) substituted into (III.26)

$$P_\mu \sigma_{tot} = \frac{1}{2} \int dp_\alpha dp_\beta dp_\gamma p_{\gamma\mu} C_3(p_\alpha p_\beta p_\gamma) \quad (\text{III.27})$$

finally

$$P_\mu \sigma_{tot} = (-)^{n-1} \frac{1}{(n-1)!} \int dp_\alpha dp_\beta \dots dp_n p_{n\mu} C_n(p_\alpha p_\beta \dots p_n). \quad (\text{III.28})$$

The relations (III.26) to (III.28) show explicitly that the correlation functions will never be identically zero, since, if $C_n \equiv 0$ then all C_i 's ($i < n$) will also be zero.

To close this chapter, let me make some remarks about the points I didn't mention.

1- The generalization of the functionals to the case of several species of stable hadrons is straightforward : instead of one kind of η parameter, one introduces η_c parameters for each type of hadrons.

2- What are the implications of these sum rules to the 1-plane analyticity ? These sum rules are so far only used with simple factorizable Regge poles (no cuts) which are related to the power behaviour of the cross-sections and lead to scaling.

3- Non-forward generalization of the sum rules : instead of cross-sections, the generalized sum rules for $t \neq 0$ use the discontinuity of the amplitudes. (for $t=0$ these are equivalent due to the generalized optical theorem). There are some technical points involved.

General References to §III

- 1. L.S. Brown, Phys. Rev. D5(1972)748
- 2. K.J. Biebl and J. Wolf - Phys. Lett. 37B(1971)197
- 3. Z. Koba, Lecture given at the Third International Colloquium on Multiparticle reactions. June 20-24, 1972, Zakopane, Poland.

IV. Known Properties in Experiments.

These are well summarized by various people ; for example Horn¹, Deutschmann² and Jacob³ among others. The details can be found in these reviews and we will just sketch some key features of production processes here.

IV.1 Most of the produced particles are pions, i.e., about 90% at 20~30 GeV and 80% at ISR energies. Thus the situation is far from being near the SU(3) limit and one may consider mostly the pion distribution when looking for the general properties.

IV.2 Limit on the transverse momenta ; the distribution is very different from what the phase space alone would suggest and shows a severe limit on the transverse momenta of the produced particles. The average value is of the order of 300 MeV and at accelerator energies (10~50 GeV) this appears to be independent of the type of reactions studied, the observed multiplicity and the incident energies. The distribution appears to be of the exponential form in $q_{\perp}$ for the range of 0.2~1.5 GeV/c. Although the abundance of pions at small transverse momenta are confirmed at the ISR, the observed distribution at larger values of $q_{\perp}$ shows wider and larger by several orders of magnitude than the one expected from the extrapolation of the exponential behavior below $q_{\perp} \sim 1.5$ GeV/c. (See Fig. IV-1). From Fig. IV-2, we note that this effect is energy dependent. Thus the production of high-mass objects is not as much suppressed as one may conclude from the exponential $q_{\perp}$ dependence of the production cross section. Also it is interesting because we know that the elastic differential p-p cross section shows a break from a single exponential $q_{\perp}$ behavior at large angles. Some say* that the spectrum may be fitted by $q_{\perp}^{-6}$.

* See for example D. Amati, L. Caneschi and M. Testa, CERN preprint TH 1597.

IV.3 Leading particle Effect ; When one considers the distribution of a final particle whose quantum numbers are the same as those of the target (beam), such particle is often found to be produced with a large fraction of the available energy. For example, the proton distribution is often concentrated near the location of the incoming (target) proton in the C.M. system and the π^- distribution has a strong tail in the direction of the incoming π^- . This trend of following the incoming particle is referred as "the leading particle effect". Thus only part of the C.M. energy is shared among the actual secondary particles. For example, the proton distribution is remarkably flat and very different from that of pion which is concentrated near the low momentum region. Although this effect is partly kinematical due to the mass difference between pion and proton ; on the average a final proton is found with the half or more of the C.M. energy of the incident particle whereas most of the pions are produced with relatively low C.M. energy. The existence of a cloud of pions with low C.M. moment that stays finite as the incident energy increases is sometimes referred to as pionization.

IV.4 Slow increase of the average multiplicity ; the average multiplicity of secondaries increases slowly with energy. This is partly due to the severe limits on $q_{\perp}$ and the leading particle effect. It is generally parametrized by $\langle n \rangle = a + b \log s$ and gives a fair approximation to the observed behavior at present energies. (Fig.IV-3) with $b = 1.2 \sim 1.5$.

General References to SIV

1. D. Horn, Physics Report 4C, 1 (1972).
2. M. Deutschmann, "Proceedings of the Amsterdam International Conference on Elementary Particles", (1971).
3. M. Jacob, "XVIth International Conference on High Energy Physics" Batavia, Ill. (1972).

FIGURE CAPTIONS

- Fig. IV-1 Transverse momentum distributions for π^+ and π^- at $x = 0$ and large $q_{\perp}$.
M. Banner et al., Saclay-Strasbourg Collaboration-Batavia (1972)
- Fig. IV-2 Large transverse momentum distribution of π^0 at $x = 0$ for different ISR energies.
- Fig. IV-3 Mean charged multiplicity in p-p collisions at accelerator and ISR energies.
The solid line gives a logarithmic fit through the accelerator data points.

Fig. IV-1.

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Fig. IV-2

Fig. 4v.3

V. Kinematics and choice of variables

We shall primarily focuss on the single-particle inclusive reaction

$$a + b \rightarrow c + \text{anything}$$

A) This quasi-two body process can be described by 3 independent invariant variables of energy, momentum transfer and missing mass

$$\left. \begin{aligned} S &= s_{ab} = (p_a + p_b)^2 && \text{Energy} \\ s_{a\bar{c}} &= (p_a - p_c)^2 && \text{Momentum transfer} \\ \text{or } s_{b\bar{c}} &= (p_b - p_c)^2 && \\ \text{and } M^2 &= (p_a + p_b - p_c)^2 && \text{Missing mass} \end{aligned} \right\} \quad (V-1)$$

Then

$$s_{ab} + s_{a\bar{c}} + s_{b\bar{c}} = m_a^2 + m_b^2 + m_c^2 + M^2 \quad (V-2)$$

Suppose particle a is the projectile, particle b is the target.

The rest frame of b is the Laboratory frame (lab), where

$$p_b = (m_b, 0, 0, 0)$$

$$p_a = (\omega_a, 0, 0, p)$$

with

$$\omega_a = (s - m_a^2 - m_b^2) / 2m_b, \quad p = \lambda(s, m_a^2, m_b^2) / 2m_b.$$

On the other hand, in the projectile frame, in which a is at rest

$$p_a = (m_a, 0, 0, 0)$$

$$p_b = (\omega_b, 0, 0, -p)$$

with

$$\omega_b = (s - m_a^2 - m_b^2) / 2m_a, \quad p = \lambda(s, m_a^2, m_b^2) / 2m_a.$$

In the center of mass frame (C.M.) of particles a and b

$$p_a = (E_a, 0, 0, p)$$

$$p_b = (E_b, 0, 0, -p)$$

with

$$E_a = (s + m_a^2 - m_b^2)/2\sqrt{s}, \quad E_b = (s - m_a^2 + m_b^2)/2\sqrt{s}, \quad p = \lambda(s, m_a^2, m_b^2)/2\sqrt{s}$$

B) In any frame in which the momenta p_a and p_b are collinear, we can parametrize the momenta as

$$p_a = (m_a \cosh \xi_a, 0, 0, m_a \sinh \xi_a)$$

$$p_b = (m_b \cosh \xi_b, 0, 0, m_b \sinh \xi_b)$$

(V-3)

$$p_c = (k \cosh \xi, p_x, p_y, k \sinh \xi)$$

where

$$k^2 = p_x^2 + p_y^2 + m_c^2 = \bar{p}_\perp^2 + m_c^2$$

The variable ξ is called the rapidity.

In the Lab. frame, $\xi_b^l = 0$ *

and $(p_a + p_b)^l = (m_a \cosh \xi_a^l + m_b, 0, 0, m_a \sinh \xi_a^l)$

so that $s = m_a^2 + m_b^2 + 2m_a m_b \cosh \xi_a^l$

from which we get

$$\cosh \xi_a^l = (s - m_a^2 - m_b^2)/2m_a m_b$$

and

$$\xi_a^l \simeq \ln \frac{s}{m_a m_b} \quad (V-4)$$

In the C.M. frame

$$\cosh \xi_a^c = \frac{E_a}{m_a} = (s + m_a^2 - m_b^2)/2m_a \sqrt{s}$$

$$\sinh \xi_a^c = \frac{p}{m_a} = \lambda(s, m_a^2, m_b^2)/2m_a \sqrt{s}$$

In the following, superscripts l, c, p will indicate quantities measured respectively in the lab., C.M. or projectile frame.

so that

$$e^{\pm \xi_a^c} = \frac{s + m_a^2 - m_b^2 \pm \lambda(s, m_a^2, m_b^2)}{2 m_a \sqrt{s}} \sim \begin{cases} \sqrt{s}/m_a \\ 0 \end{cases}$$

or

$$\xi_a^c \simeq \ln(\sqrt{s}/m_a) \quad (V-5)$$

Also

$$e^{\pm \xi_b^c} = \frac{s - m_a^2 + m_b^2 \pm \lambda(s, m_a^2, m_b^2)}{2 m_b \sqrt{s}} \sim \begin{cases} 0 \\ \sqrt{s}/m_b \end{cases}$$

$$\xi_b^c \simeq -\ln(\sqrt{s}/m_b). \quad (V-6)$$

From the invariant scalar products

$$p_a \cdot p_c = m_a \kappa \cosh(\xi_a - \xi), \quad p_b \cdot p_c = m_b \kappa \cosh(\xi_b - \xi),$$

one notices that

$$\xi_a - \xi \quad \text{and} \quad \xi_b - \xi$$

are invariant quantities in collinear frames ; since

$$p_{\perp} = p_{\perp}^c = p_{\perp}^l = p_{\perp}^p$$

and since $\kappa^2 = p_{\perp}^2 + m^2$ is invariant for such longitudinal boost ;

$$(\xi^c - \xi_a^c) = (\xi^p - \xi_a^p) = \xi^p \quad (V-7)$$

$$(\xi^c - \xi_b^c) = (\xi^l - \xi_b^l) = \xi^l$$

Thus at constant energy s , the rapidity variable ξ is invariant up to an additive constant.

If we define

$$p^{\pm} = E \pm p_{\parallel} \quad \text{with} \quad p^+ p^- = \kappa^2$$

then

$$p^{\pm} = \kappa \cosh \xi \pm \kappa \sinh \xi = \kappa e^{\pm \xi} \quad (V-8)$$

and, from (V-7)

$$\begin{aligned} (p^{\pm})^p &= \kappa e^{\pm \xi^p} = \kappa e^{\pm (\xi^c - \xi_a^c)} = e^{\mp \xi_a^c} (p^{\pm})^c \\ (p^{\pm})^l &= \kappa e^{\pm \xi^l} = \kappa e^{\pm (\xi^c - \xi_b^c)} = e^{\mp \xi_b^c} (p^{\pm})^c \end{aligned} \quad (V-9)$$

From (V-8),

$$\xi = \frac{1}{2} \ln \frac{p^+}{p^-} = \frac{1}{2} \ln \frac{E + p_{\parallel}}{E - p_{\parallel}} \quad (V-10)$$

and

$$\frac{p_{\parallel}}{E} = \tanh \xi = \beta_{\parallel} \quad (V-11)$$

Thus the longitudinal velocity $\beta_{\parallel}$ is independent of the mass of the particle and is given by the rapidity. For a given same value of rapidity (or velocity), the longitudinal momenta can be quite different from each other for the particles with different masses, because of the difference of the transverse mass $\kappa = \sqrt{p_{\perp}^2 + m^2}$. For example, $\kappa \sim 0.3$ for pion while $\kappa \sim 1$ for proton so that the proton longitudinal momenta can be larger by a factor 3 than that of pion for ξ fixed. This is the kinematical reason for the slow pion concentration around the C.M. which was mentioned before.

The rapidity plots tend to expand the low $p_{\parallel}^{C.M.}$ region, thus it is convenient to use the rapidity variable to study the concentration of low C.M. momenta particles whereas a $p_{\parallel}^{C.M.}$ plot is better in the region of negative $p_{\parallel}^l$.

C) Consider the variables $(p_{\parallel}^l, \theta^l = \tan^{-1}(p_{\perp}/p_{\parallel}^l))$.

Then

$$p_{\parallel}^l = p_{\perp} e^{-r} \quad \text{where} \quad r = \ln \tan \theta^l \quad (V-12)$$

which is related to the previous ones by

$$p_{\parallel}^l = \kappa \sinh \xi^l$$

so that, if $\xi^l \gg 1$, $p_{\perp} e^{-r} \approx \frac{\kappa}{2} e^{\xi^l}$

or $-r \approx \xi^l + \ln \frac{\kappa}{2p_{\perp}}$

and, if $p_{\perp} \gg m$, $-r \approx \xi^l + \ln \frac{1}{2}$

or $\xi^l \approx -\ln \tan \theta^l + \ln 2 = \ln \left(\frac{2}{\tan \theta^l} \right) \quad (V-13)$

This variable has been used in the analysis of cosmic ray data¹. A measurement of the cross-section at fixed lab angle and large momenta is equivalent to looking at the large $p_{\perp}$ structure for fixed rapidity.

D) Finally, consider the Feynman scaling variable

$$x = \frac{p_{\parallel}^c}{p_{\max}} \approx \frac{2p_{\parallel}^c}{\sqrt{s}}$$

Then the physical region is

$$p_{\perp} \leq \sqrt{s}/2 \quad \text{and} \quad -\sqrt{1 - \left(\frac{2p_{\perp}}{\sqrt{s}}\right)^2} \leq x \leq +\sqrt{1 - \left(\frac{2p_{\perp}}{\sqrt{s}}\right)^2}$$

From (IV, 5,6 and 11), the target and projectile fragments have the C.M. momenta

$$(p^{\pm})^c = e^{\pm \xi_b^c} (p^{\pm})^l \sim \left\{ \begin{matrix} 0 \\ \sqrt{s}/m_b \end{matrix} \right\} (p^{\pm})^l$$

$$(p^{\pm})^c = e^{\pm \xi_a^c} (p^{\pm})^p \sim \left\{ \begin{matrix} \sqrt{s}/m_a \\ 0 \end{matrix} \right\} (p^{\pm})^p$$

respectively.

Since

$$p_{\parallel} = \frac{1}{2}(p^+ - p^-)$$

the target fragment has $p_{\parallel}^c \sim -\frac{\sqrt{s}}{2m_b} (p^-)^l$ so that $x \approx -\frac{(p^-)^l}{m_b} < 0$ (V-14)

and

the projectile fragment has $p_{\parallel}^c \sim \frac{\sqrt{s}}{2m_a} (p^+)^p$ so that $x \approx \frac{(p^+)^p}{m_a} > 0$ (V-15)

To see the relation between x and ξ , use

$$x = \frac{2p_{11}}{\sqrt{s}} = \frac{2\kappa}{\sqrt{s}} \sinh \xi^c \approx \pm \frac{\kappa}{\sqrt{s}} e^{\pm \xi^c} \quad \text{as } s \rightarrow \infty$$

where $\pm$ correspond to the positive and negative ξ^c respectively.

For the projectile fragment, i.e. $x > 0$, $\sqrt{s} \approx m_a e^{\xi_a^c}$ so that

$$x \approx \frac{\kappa}{m_a} e^{\xi^c - \xi_a^c} = \frac{\kappa}{m_a} e^{\xi^p} = \frac{\kappa}{m_a} e^{-|\xi^p|} \quad (\text{V-16})$$

For the target fragment, i.e. $x < 0$, $\sqrt{s} \approx m_b e^{-\xi_b^c}$ so that

$$x \approx -\frac{\kappa}{m_b} e^{-\xi^c + \xi_b^c} = -\frac{\kappa}{m_b} e^{-\xi^t} \quad (\text{V-17})$$

E) Now calculate the invariant variables in terms of x ;

$$\begin{aligned}
 s_{a\bar{c}} &= (p_a - p_c)^2 = m_a^2 + m_c^2 - 2m_a K \cosh(\xi - \xi_a) = m_a^2 + m_c^2 - 2m_a K \cosh \xi^P \\
 &= m_a^2 + m_c^2 - m_a K \left(\frac{m_a x}{K} + \frac{K}{m_a x} \right) \\
 &= m_a^2 (1-x) + m_c^2 \left(1 - \frac{1}{x}\right) - \frac{P_\perp^2}{x} \quad (V-18)
 \end{aligned}$$

$$\begin{aligned}
 s_{b\bar{c}} &= (p_b - p_c)^2 = m_b^2 + m_c^2 - 2(E_b E_c - \vec{p}_b \cdot \vec{p}_c) = m_b^2 + m_c^2 - 2(\omega_b E_c + p p_\parallel)^2 \\
 &\cong m_b^2 + m_c^2 - 2 \frac{s}{2m_a} (E_c + p_\parallel)^2 = m_b^2 + m_c^2 - \frac{s}{m_a} (p^+)^2 \\
 &= m_b^2 + m_c^2 - xs \quad (V-19)
 \end{aligned}$$

Thus the projectile fragment has

$$\begin{cases} s_{a\bar{c}} \cong - \frac{P_\perp^2}{x} \\ s_{b\bar{c}} \cong - xs \end{cases}$$

which can be obtained from the C.M. expressions

$$\begin{aligned}
 s_{a\bar{c}} &\cong - \frac{s}{2} (\bar{x} - x) \\
 s_{b\bar{c}} &\cong - \frac{s}{2} (\bar{x} + x) \quad (V-20)
 \end{aligned}$$

where

$$\bar{x} = \sqrt{x^2 + \frac{4K^2}{s}}$$

Note from (V-20) that

$$s_{a\bar{c}} \cong s_{b\bar{c}} \rightarrow -\sqrt{s} K \quad \text{as } x \rightarrow 0. \quad (V-21)$$

Note also that

$$\begin{aligned}
 s_{a\bar{c}} s_{b\bar{c}} &\cong \left(m_a^2 (1-x) + m_c^2 - \frac{K^2}{x} \right) (m_b^2 + m_c^2 - xs) \\
 &\cong s \left(K^2 - (m_a^2 + m_c^2)x + m_a^2 x^2 \right)
 \end{aligned}$$

so that

$$s_{a\bar{c}} s_{b\bar{c}} / s$$

is independent of s and in the $x \rightarrow 0$ limit

$$s_{a\bar{c}} s_{b\bar{c}} / s \rightarrow K^2 \quad (V-22)$$

We remark that the central region, x small, can be reached either for s large or for $p_{\parallel}$ small. But the "Wee region" for which $p_{\parallel} < \kappa$ is different from the rest of dominant central region; this region becomes narrower and narrower as $s \rightarrow \infty$.

From (V-2, 18 and 19),

$$\begin{aligned} s + s_{a\bar{c}} + s_{b\bar{c}} &= (m_a^2 + m_b^2 + m_c^2) + M^2 \\ &\simeq s + m_b^2(1-x) + m_c^2 - \frac{\kappa^2}{x} + m_b^2 + m_c^2 - xs \\ &= s(1-x) + m_a^2(1-x) + m_b^2 + 2m_c^2 - \frac{\kappa^2}{x} \end{aligned}$$

or

$$M^2 = s(1-x) + m_c^2 - m_a^2 x - \frac{\kappa^2}{x}$$

so that

$$M^2/s = (1-x) + \frac{1}{s} (m_c^2 - m_a^2 x - \frac{\kappa^2}{x}) \quad (V-23)$$

Finally, we list the invariant phase-volume in a variety of variables

$$\frac{d^3 p}{E} = d^2 p_{\perp} d\xi = \pi d(p_{\perp}^2) d\xi = \pi d(p_{\perp}^2) \frac{dx}{x} \simeq -\pi dx ds_{a\bar{c}} \simeq \pi \frac{dM^2}{s} ds_{a\bar{c}} \quad (V-24)$$

Reference to SV.

1. The earlier reference is L. von Lindern, Nuovo Cimento 5, (1957) 491.

VI. Hypothesis of Scaling and Limiting Fragmentation

As the energy increases, the total cross-section approaches constant while the quasi two-body production cross-section decreases. Thus one is necessarily dealing with the process of many particle final states at high energies. For such multibody systems, the usual scheme of the analysis through the Mandelstam invariants is prohibitively complicated.

In a way to salvage this problem, Feynman, Yang and others have suggested that the characteristics of the single-particle spectra obtained after summing up all other final particles can yield the dynamical information governing the strong interactions. Of course this process of the single-particle inclusive reaction is only the next level of sophistication above a total cross-section measurement. As we have seen it already this is the next simplest situation also kinematically over the two-body $\rightarrow$ two-body reactions requiring only 3 independent variables.

In particular, Yang et al.¹ suggested the following conjecture for $a + b \rightarrow c + \text{anything}$, based on the results of the earlier work by Wu and Yang ;

$$\lim_{s \rightarrow \infty} \sigma_{ab}^c(s; \vec{p}_c^{\text{lab}}) = \sigma_{ab}^c(\vec{p}_c^{\text{lab}}) > 0 \quad (\text{VI-1})$$

where

$$\sigma_{ab}^c(s; \vec{p}_c^{\text{lab}}) = E_c \frac{d\sigma_{ab}^c}{d^3p_c} \quad (\text{VI-2})$$

is the single-particle cross-section and $\vec{p}_c^{\text{lab}}$ is the momentum of the particle c measured in the laboratory system.

On the other hand, Feynman² regarded no special preference on the lab. system and concentrated to the C.M. system of a and b and proposed the following scaling hypothesis based on the institution coming from the field theoretic Bremsstrahlung model :

for

$$x = p_{\parallel} / p_{\text{max}} = 2p_{\parallel} / \sqrt{s} \quad \text{and} \quad p_{\perp} \text{ fixed,}$$

$$\lim_{s \rightarrow \infty} \sigma_{ab}^c(s; x, \vec{p}_{\perp}) = \sigma_{ab}^c(x, \vec{p}_{\perp}) > 0 \quad (\text{VI-3})$$

whereas

$$\lim_{\substack{s \rightarrow \infty \\ x \rightarrow 0}} \mathcal{S}_{ab}^c(s; x, \vec{p}_\perp) = \mathcal{S}^c(\vec{p}_\perp) \neq 0 \text{ and finite.} \quad (\text{VI-4})$$

The generalization of these hypotheses to the n-particle inclusive process is straight-forward, and (VI-1) and (VI-3) are obviously the same but expressed in the different reference frames. We notice then from the discussion in §V, they are referring to the particles produced in the fragmentation region of target or projectile. The second part of the scaling hypothesis (VI-4) is for the pionization region.

As we will see later, this scaling property is intimately related to the Regge (power) behavior of the amplitudes and for the first time we come close to having a unified description of the two-body elastic reactions and many-particle production processes.

As it is well known by now, the concept of scaling in hadronic reactions was already known in the early work of Amati, Fubini and Stanghellini³ in 1962. In an attempt to generalize the techniques for the two-particle production through the single particle exchange mechanism to the multiple production, they introduced the multiperipheral configuration and asked for the most probable configuration consistent with the kinematical constraints. The conditions were :

1. all $p_{i\perp} = 0$
2. all p_i^+ / p_{i+1}^+ are roughly equal
3. $\langle n \rangle$ increases logarithmically with energy s.

Under such conditions, they have found that the multiperipheral model exhibits simple scaling laws at high energy and that these laws are independent of the details of the model ; one can have other particle or Reggeon exchanges besides pions⁴.

There are other ways to understand the scaling laws and we will discuss two of them in the following.

References to § VI

- 1) J. Benecke, T.T. Chou, C.N. Yang and E. Yen - Phys. Rev. 188 (1969) 2159
- 2) R.P. Feynman Phys. Rev. Letters 23 (1969) 1415
- 3) D. Amati, S. Fubini and A. Stanghellini, Nuovo Cimento 26 (1962) 896, S. Fubini, "Strong Interactions and High Energy Physics", Scottish Universities' Summer School (1963) Editor R.G. Moorhouse (Oliver and Boyd, 1964)
- 4) For example G.H. Chew and A. Pignotti, Phys. Rev. 176 (1968) 2112,
C.E. De Tar, Phys. Rev. D3 (1971) 128

VII. Feynman-Wilson gas analog

Here the multiparticle production cross-sections are compared to the multiparticle distribution functions of a classical gas and the total cross-section to the partition function of a gas.

We express the total cross-section (I.16) in the rapidity space using (V.24)

$$\begin{aligned}\sigma_{tot}(s) &= \sigma_{tot}(\bar{r}_a, \bar{r}_b) = \sum \frac{1}{n!} \int \prod_{i=1}^n d^3 r_i \frac{d\sigma(\bar{r}_a, \bar{r}_b; \bar{r}_1 \dots \bar{r}_n)}{d^3 r_1 \dots d^3 r_n} \delta^4(p_a + p_b - p_1 - \dots - p_n) \\ &= \sum \frac{1}{n!} \int \prod_{i=1}^n d^3 r_i \sigma^{(n)}(\bar{r}_a, \bar{r}_b; \bar{r}_1 \dots \bar{r}_n) \delta^4(p_a + p_b - \sum_{i=1}^n p_i)\end{aligned}\quad (VII-1)$$

where we used the notation

$$\bar{r}_i = (\bar{p}_{i\perp}, \xi_i), \quad d^3 r_i = d^2 p_{i\perp} d\xi_i \quad (VII-2)$$

Note that

$$\delta^4(P) = \delta(E) \delta^3(\bar{P}) = 2 \delta(E + P_{\parallel}) \delta(E - P_{\parallel}) \delta^2(\bar{P}_{\perp}) \quad (VII-3)$$

where

$$\begin{aligned}P &= p_a + p_b - \sum_{i=1}^n p_i \\ E + P_{\parallel} &= P^+ = m_a e^{\xi_a} + m_b e^{\xi_b} - \kappa_1 e^{\xi_1} - \dots - \kappa_n e^{\xi_n} \\ E - P_{\parallel} &= P^- = m_a e^{-\xi_a} + m_b e^{-\xi_b} - \kappa_1 e^{-\xi_1} - \dots - \kappa_n e^{-\xi_n} \\ \bar{P}_{\perp} &= p_{a\perp} + p_{b\perp} - \sum_i \bar{p}_{i\perp}\end{aligned}\quad (VII-4)$$

In this way, $\sigma_{tot}(\bar{r}_a, \bar{r}_b)$ with $\bar{r}_a, \bar{r}_b$ fixed is a partition function of a gas and the functions $\sigma^{(n)}(\bar{r}_a, \bar{r}_b; \bar{r}_1 \dots \bar{r}_n)$ are the n-particle distribution functions for the gas.

At large energy, $\xi_a - \xi_b \simeq \ln(s/m_a m_b)$ in C.M., or $\xi_a \simeq \ln(s/m_a m_b)$, $\xi_b = 0$ in the lab system which puts an upper bound on the ξ_i , i.e. ξ_i can not be larger than ξ_a through $\delta(P^+)$. Also $\delta(P^-)$ puts a lower bound on ξ_i , i.e. ξ_i can not be less than ξ_b .

So, one can think of a gas distributed inside a cylindrical volume.

Similarly the invariant single-particle inclusive cross-section can be written as

$$\frac{d\sigma}{dp}(\bar{r}_a, \bar{r}_b; \bar{r}) = \sum \frac{1}{(n-1)!} \int d^3r_1 \dots d^3r_{n-1} \sigma^{(n)}(\bar{r}_a, \bar{r}_b; \bar{r}_1 \dots \bar{r}_{n-1} \bar{r}) \delta^4(P) \quad (\text{VII-5})$$

In the real gas theory, one measures the quantity of statistical average instead of the distribution functions of the gas particles. In analogy, we define the density function by

$$f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r}) = \frac{1}{\sigma_{tot}(\bar{r}_a, \bar{r}_b)} \frac{d\sigma}{dp}(\bar{r}_a, \bar{r}_b; \bar{r}) \quad (\text{VII-6})$$

since $f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r})$ is the average number of particles to be found in the volume d^3r .

Then the two-particle density function can be written as

$$f^{(2)}(\bar{r}_a, \bar{r}_b; \bar{r}_1, \bar{r}_2) = \frac{1}{\sigma_{tot}(\bar{r}_a, \bar{r}_b)} \sum_{n=2} \int d^3r'_1 \dots d^3r'_{n-2} \sigma^{(n)}(\bar{r}_a, \bar{r}_b; \bar{r}_1' \dots \bar{r}_{n-2}' \bar{r}_1 \bar{r}_2) \delta^4(P) \quad (\text{VII-7})$$

and the two-particle correlation function as

$$C^{(2)}(\bar{r}_a, \bar{r}_b; \bar{r}_1, \bar{r}_2) = f^{(2)}(\bar{r}_a, \bar{r}_b; \bar{r}_1, \bar{r}_2) - f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r}_1) f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r}_2). \quad (\text{VII-8})$$

The n-particle density and correlation functions can be generalized in the straightforward way.

We introduce now a dynamical assumption : assume that the Feynman gas interact only through short range forces whose range stays fixed as s increases.

Then the part of the gas that is well away from the walls has no direct interaction with the walls, being many interaction lengths away from them, i.e. as

$$\xi_a - \xi \quad \text{and} \quad \xi - \xi_b \quad \text{become large then} \quad f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r}) \rightarrow f^{(1)}(\bar{r}).$$

By Lorentz invariance in the collision plane, which is translational invariance along the ξ -axis, $f^{(1)}(\bar{r})$ cannot depend on ξ , and so $f^{(1)}(\bar{r})$ depends only on $\vec{p}_\perp$

$$\lim_{\xi_a - \xi, \xi - \xi_b \gg 1} f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r}) \rightarrow f^{(1)}(\vec{p}_\perp) \quad (\text{VII-9})$$

This should be compared with Feynman scaling hypothesis in the pionization region ($s \rightarrow \infty, x \rightarrow 0$), see (VI.4).

The part of the gas near the projectile wall, i.e.

$$\begin{cases} \xi - \xi_a & \text{comparable to the range of the force} \\ \xi_b - \xi \gg 1 \end{cases}$$

will not feel the effects of the target wall. By combining this with the translational invariance, then

$$\lim_{\xi_b - \xi \gg 1} f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r}) \rightarrow f^{(1)}(\vec{p}_\perp, \xi - \xi_a) \quad (\text{VII-10})$$

This is just the scaling law in the projectile fragmentation region.

Similarly for $f^{(2)}$, we can see, for example that
1. if the two particles are independently produced, i.e.

$$|\xi_1 - \xi_2| \gg 1$$

a) with $|\xi_a - \xi_2|$ and $|\xi_b - \xi_1| \gg 1$ for fixed $|\xi_a - \xi_1|$ and $|\xi_b - \xi_2|$
then

$$f^{(2)}(\bar{r}_a, \bar{r}_b; \bar{r}_1, \bar{r}_2) \longrightarrow f^{(2)}(\xi_a - \xi_1, \xi_b - \xi_2; \bar{p}_{1\perp}, \bar{p}_{2\perp}) = f^{(1)}(\xi_a - \xi_1, \bar{p}_{1\perp}) f^{(1)}(\xi_b - \xi_2, \bar{p}_{2\perp})$$

(double fragmentation) (VII-11)

b) with $|\xi_i - \xi_b|$ and $|\xi_i - \xi_a| \gg 1$, ($i=1,2$), then the limit should only depend on transverse momenta

$$f^{(2)}(\bar{r}_a, \bar{r}_b; \bar{r}_1, \bar{r}_2) \longrightarrow f^{(1)}(\bar{p}_{1\perp}) \cdot f^{(1)}(\bar{p}_{2\perp})$$

(uncorrelated pionization) (VII-12)

2. if $|\xi_1 - \xi_2|$ remains finite and fixed

a) for $|\xi_1 - \xi_b| \gg 1$ and $|\xi_1 - \xi_a|$ fixed

$$f^{(2)}(\bar{r}_a, \bar{r}_b; \bar{r}_1, \bar{r}_2) \longrightarrow f^{(2)}(\xi_a - \xi_1, \xi_1 - \xi_2, \bar{p}_{1\perp}, \bar{p}_{2\perp})$$

(single fragmentation) (VII-13)

b) for $|\xi_1 - \xi_a|, |\xi_1 - \xi_b| \gg 1$

$$f^{(2)}(\bar{r}_a, \bar{r}_b; \bar{r}_1, \bar{r}_2) \longrightarrow f^{(2)}(\xi_1 - \xi_2, \bar{p}_{1\perp}, \bar{p}_{2\perp})$$

(correlated pionization) (VII-14)

Note that $(\bar{p}_{1\perp}$ and $\bar{p}_{2\perp})$ mean only 3 independent variables, $|\bar{p}_{1\perp}|, |\bar{p}_{2\perp}|$ and the azimuthal angle $\cos \varphi = \hat{p}_{1\perp} \cdot \hat{p}_{2\perp}$.

In this kind of short-range picture, one can conveniently introduce the correlation length ζ as the measure of the maximum separation $|\xi_1 - \xi_2|$ for which $C^{(2)}$ is appreciably different from zero, i.e.

$$C^{(2)} \sim e^{-|\xi_1 - \xi_2| / \zeta} \quad (\text{VII-15})$$

If the Feynman gas has the density independent of ξ except near the walls, then

$$\langle n \rangle = \int d^3r f^{(1)}(\bar{r}_a, \bar{r}_b; \bar{r}) \propto \xi_a - \xi_b \approx \ln(s/m_a m_b) \quad (\text{VII-16})$$

Because of this, it is clear that only the multiperipheral part of the inelastic cross-section behaves like the distribution functions of a gas with short-range forces. On the other hand, we know that the diffractive processes involve low multiplicity. In principle, these two-processes should clearly separate at sufficiently large s from each other. In this connection, the correct normalization of the density functions of the Feynman gas should be to the multiperipheral part of the inelastic cross-section rather than to the total cross-section.

References to § VII.

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We have seen that for the projectile fragment

$$s_{a\bar{c}} \approx -\kappa^2/x, \quad s_{b\bar{c}} \approx -xs \quad \text{and} \quad M^2 \approx (1-x)s.$$

Then the 3-3 amplitude can be regarded as a two-body amplitude for a particle of mass $\sqrt{s_{a\bar{c}}}$ (negative mass-squared) ^{scattered} by the particle b. Together with the assumption b), then one can write

$$s \frac{d\sigma}{d^3p/2E} \sim \frac{1}{2i} \text{---} \text{Disc}_{M^2} \text{---} \sim \frac{1}{2i} \text{---} \alpha_V, \alpha_R (R = \rho, \omega, A_2)$$

which means

$$s \frac{d\sigma}{d^3p/2E} \sim \beta_{b\bar{b}}^V(0) \beta^V(s_{a\bar{c}}, t=0) (M^2)^{\alpha_V(0)} + \beta_{b\bar{b}}^R(0) \beta^R(s_{a\bar{c}}, t=0) (M^2)^{\alpha_R(0)} \quad (\text{VIII-1})$$

or, if $\alpha_V(0) = 1$ and $\alpha_R(0) = \frac{1}{2}$,

$$\frac{d\sigma}{d^3p/2E} \sim f_1(\kappa^2, x) + f_2(\kappa^2, x) / \sqrt{s} \quad (\text{VIII-2})$$

since $M^2 \approx (1-x)s$.

Similar things can be obtained for the target fragment.

In the central region, we know that

$$s_{a\bar{c}} \sim s_{b\bar{c}} \sim -\sqrt{\kappa^2 s} \quad \text{while} \quad M^2 \sim s.$$

Then the 3-3 amplitude has a double Regge expansion

$$s \frac{d\sigma}{d^3p/2E} \sim \frac{1}{2i} \text{---} \text{Disc}_{M^2} \text{---} \sim \frac{1}{2i} \text{---} \alpha_V, \alpha_R (R = \rho, \omega, A_2)$$

$$\sim (s_{a\bar{c}})^{\alpha_{a\bar{a}}(0)} (s_{b\bar{c}})^{\alpha_{b\bar{b}}(0)} \beta_{a\bar{a}}(0) \beta_{b\bar{b}}(0) \beta \left(\frac{s_{a\bar{c}} s_{b\bar{c}}}{M^2} \right) \quad (\text{VIII-3})$$

↑
(two-particles - two Reggeons vertex)

Again by keeping the vacuum and the meson trajectories only, we obtain

$$\frac{d\sigma}{d^3p/2E} \sim \kappa^2 f(\kappa^2) + g(\kappa^2) s^{-1/4} \quad (\text{VIII-4})$$

Thus one can expect that the central region would be reached much slower than the fragmentation regions.

Near the phase-space boundary, one can think of the triple Regge limit, i.e. $x \sim 1$, $s_{a\bar{c}} \simeq -\kappa^2$, $s_{b\bar{c}} \simeq -s$ and $s/M^2 \simeq 1/(1-x) \rightarrow \infty$. In this limit the 3-3 amplitude can be regarded as the one for a Reggeon $\alpha_{a\bar{c}}$ scattered by the particle b , so that

$$s \frac{d\sigma}{d^3p/2E} \sim \frac{1}{2i} \left[\text{Diagram 1} \right] = \frac{1}{2i} \left[\text{Diagram 2} \right] \\ \sim s^{\alpha_{a\bar{c}} + \alpha_{\bar{a}c}} \beta_{a\bar{c}}(s_{a\bar{c}}) \beta_{\bar{a}c}(s_{\bar{a}c}) \frac{1}{2i} \text{Disc}_{M^2} \tilde{T}_{Rb \rightarrow Rb} \quad (\text{VIII-5})$$

where $\tilde{T}$ is the Reggeon-particle b scattering amplitude.

This gives the triple-Regge formula explained before

$$s \frac{d\sigma}{d^3p/2E} \sim |\beta_{a\bar{c}}(s_{a\bar{c}})|^2 \beta_{b\bar{c}}(0) g_{TR}(s_{a\bar{c}}, s_{\bar{a}c}, t=0) \left(\frac{M^2}{s}\right)^{\alpha_{b\bar{c}}(0) - 2\alpha_{a\bar{c}}} s^{\alpha_{b\bar{c}}(0)} \quad (\text{VIII-6})$$

Due to the vanishing of the triple Pomeron coupling $g_{TR}(0,0,0)$, the combinations allowed in (VIII.6) are PPR, RRP and RRR couplings, P and R being the Pomeron and meson trajectories respectively ;

$$s \frac{d\sigma}{d^3p/2E} \sim f(s_{a\bar{c}}) \left(\frac{M^2}{s}\right)^{\alpha_P(0) - 2\alpha_R(s_{a\bar{c}})} s^{\alpha_P(0)} + f'(s_{a\bar{c}}) \left(\frac{M^2}{s}\right)^{\alpha_R(0) - 2\alpha_P(s_{a\bar{c}})} s^{\alpha_R(0)} \\ + f''(s_{a\bar{c}}) \left(\frac{M^2}{s}\right)^{\alpha_R(0) - 2\alpha_R(s_{a\bar{c}})} s^{\alpha_R(0)} \quad (\text{VIII-7})$$

Because of the smallness of $(1-x)$, all three terms are in principle important - Phenomenologically, however, the p - p and π - p data taken at BNL and Serpukov respectively seen to be consistent with the first two terms.²

References to § VIII

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IX. Energy Dependence of the Inclusive Cross-sections

a) The hypothesis of Mueller opens a new Regge phenomenology, though it should be regarded as only a qualitative scheme as we know that cuts and other complicated singularities are playing also an important role in the analysis of two-body data. On the other hand, it seems almost essential that cuts and other singularities should not be present in order to explain the scaling property.

The new phenomenology is based on the Eq.(VIII-2) in the fragmentation region¹ ;

$$\frac{d\sigma}{d^3p_c/2E_c} = f_1(\kappa, x_c) + f_2(\kappa, x_c) s^{(\alpha_R(0) - \alpha_V(0))} = -\frac{1}{2} \quad (IX-1)$$

where α_R and α_V denote the particle and leading vacuum trajectories having the intercepts 1/2 and 1 respectively. α_R represents the approximately exchange degenerate meson trajectories ϱ, f, ω, A_2 .

Consider the reactions $\pi^+ p \rightarrow c + \text{anything}$, $\pi^- p \rightarrow c + \text{anything}$ for $c = (\pi, K, N \text{ or } \bar{N})$ in the proton fragmentation region. Then from (IX-1), f_2 will have two parts

$$f_2 = f_2(\varrho) + f_2(f)$$

Due to charge conjugation at the pion vertex, we have

$$f_1^{\pi^+ p} = f_1^{\pi^- p}, \quad f_2^{\pi^+ p}(\varrho) = -f_2^{\pi^- p}(\varrho), \quad f_2^{\pi^+ p}(f) = f_2^{\pi^- p}(f)$$

so that

$$\left(\frac{d\sigma}{d^3p/2E} \right)^{\pi^+ p} - \left(\frac{d\sigma}{d^3p/2E} \right)^{\pi^- p} = 2 f_2(\varrho)^{\pi^+ p} \cdot s^{-1/2} \quad (IX-2)$$

Thus the distributions for these two reactions will approach each other as $\sqrt{s}$. Similar conclusions can be drawn for the pairs of reactions initiated by (K^+, K^-) or $(p, \bar{p})$ as projectiles.

(1) Consider the situation when the missing mass channel $(a\bar{c}b)$ has exotic quantum numbers. Then the straight-forward extension of the duality argument in the two-body scattering implies that the imaginary parts of meson exchanges in the crossed channel should cancel each other through exchange degeneracy so that $f_2(\kappa, x_c) = 0$. This suggests that the invariant distributions for such case are practically constant as functions of the incoming energy¹. Examples of this sort are the reactions $K^+p \rightarrow \pi^\pm + \text{anything}$, $\pi^+p \rightarrow \pi^- + \text{anything}$.

(2) This is followed by another criterion for early scaling : Ellis et al² suggest that (ab) should be exotic in addition to $(a\bar{c}b)$ as the two-body channels should be taken into account, the direct channel energy being well above their threshold. Without this extra condition, they argued that factorization leads to inconsistency for an example. These two criteria are followed by a series of alternative proposals. Some of them are listed in the following:

(3) $f_2(\kappa, x_c) = 0$ in the fragmentation of b if (ab) and $(a\bar{c})$ are exotic³. This is obtained by studying a class of duality diagrams with quark lines. This criteria allows different s dependence for $x > 0$ and $x < 0$ in a given reaction.

(4) $f_2(\kappa, x_c) = 0$ if $(ab\bar{c})$, (ab) , $(a\bar{c})$ are exotic⁴ (for c fragment of b).

(5) One can show that $f_2(\kappa, x_c) = 0$ for c fragment of a whenever $(a\bar{c})$ is exotic (even if $(ab\bar{c})$ is not), through factorization and symmetry, provided $f_2(\kappa, x_c) = 0$ when $(ab\bar{c})$ and $(a\bar{c})$ are exotic⁵.

For example, $\pi^+p \rightarrow \pi^- + \text{anything}$ has $(ab\bar{c}) = (\pi^+p\pi^+)$ exotic
 $(a\bar{c}) = (\pi^+\pi^+)$ exotic

The related reaction $\pi^-p \rightarrow \pi^+ + \text{anything}$ has

$(a\bar{c}) = (\pi^-\pi^-)$ exotic
 though $(ab\bar{c}) = (\pi^-p\pi^-)$ non exotic.

If $\pi^+p \rightarrow \pi^- + \text{anything}$ is independent of s , then via factorization $\pi^-p \rightarrow \pi^+ + \text{anything}$ is also independent of s for $x > 0$. Further-

more, because only Pomeron exchange remains, $f_1^{\pi^-p \rightarrow \pi^+X} = f_1^{\pi^+p \rightarrow \pi^-X}$ for $x > 0$, which seems to get some experimental support⁶.

Another example $K^+p \rightarrow \pi^-X$ has $(ab\bar{c})$, (ab) , $(a\bar{c})$ exotic. The criteria (1) and (2) say the independence on s for both fragmentation regions. The criteria (3) and (4) imply such thing for $x < 0$. The related process $K^-p \rightarrow \pi^+X$ has only $(a\bar{c})$ exotic. Then according to (5), the criteria (1) and (2) must also predict, via factorization, an s -independence for this reaction in $x > 0$. Furthermore, because only Pomeron couples, $(\frac{d\sigma}{dp})_{K^+p \rightarrow \pi^-X} = (\frac{d\sigma}{dp})_{K^-p \rightarrow \pi^+X}$, in the K fragmentation region.

Experimentally it appears $(K^-p) \approx 1.5 (K^+p)$; for $x > 0$

These show that all these criteria offer an incomplete solution. The most elaborate study made along this line is the one by Tye and Veneziano⁷ by making use of the planar duality diagrams. This I will discuss now as a warm up to the dual-resonance model.

b) In an attempt to generalize the two component duality picture in two-body scattering to the single-particle inclusive process, these authors make use of the planar dual resonance model to describe the production amplitudes in the unitarity sum. Then for $a + b \rightarrow c +$ anything, there are seven different configurations for the production amplitudes :

Fig (IX-1)

where (α, β, γ) denotes the three cyclic permutation of $(a, b, \bar{c})$. Obviously, the second and third diagrams bring in the loop representation and correspond to the real background terms in contradiction to the resonance contribution of the first group.

When one incorporates the diffractive background term, which has pure vacuum quantum numbers in the t-channel and is assumed to be imaginary, there are six more configurations for the production amplitudes :

Here $(\alpha, \beta) = (a, b)$ or (b, a) . These six contributions should match with the diffractive total cross-section through the inclusive sum rule.

On the other hand, if one just considers the quantum number structure of the contributions to the inclusive sum rules, only the first diagram leads to two new structures, the other two diagrams match with those of the second and third diagrams of the seven components with $(\alpha, \beta) = (a, b)$ or (b, a) . Note also that the second and third diagrams are analogous to the "Nova excitations" and their non-exotic and exotic decays.

With these nine components, one can check the consistency of the inclusive sum rules with the conjecture that

$$\sigma_{tot} = \sum_{i=1}^9 \sigma_{tot}^i = c_1 + c_2/\sqrt{s}$$

$$\frac{d\sigma}{d^3p/2E} = \sum_{i=1}^9 \left(\frac{d\sigma}{d^3p/2E} \right)^i = f_1(\kappa, x_c) + f_2(\kappa, x_c)/\sqrt{s},$$

and give criteria for the scaling behavior. In this way, Tye and Veneziano conclude that scaling is achieved when all four channels (a, b) , $(a\bar{b})$, $(a\bar{c})$, $(b\bar{c})$ have the exotic quantum number as in the reactions

like $NN \rightarrow (\bar{N}, \bar{K}) + \text{anything}$ or $KN \rightarrow (\bar{K}, \bar{N}) + \text{anything}$.

This lengthy summary of the scaling criteria does not offer complete solution; the central plateau, while it seems to have been developed for the pion distributions at the ISR energies, is not yet reached for the antiproton distributions. The distribution at $x = 0$ increases by a factor of 2 between 500 GeV and 1500 GeV. A satisfactory solution perhaps requires a better understanding of the production mechanism. Other than pions, threshold effects are apparently important, as the inclusive Λ distribution in pp collisions indicates (the total Λ production cross-section at 205 GeV is 2.7 times larger than the one measured at 28 GeV) since this will bring in the nature of the particles produced. On the other hand, we have already seen that the approach to scaling, even in Mueller's framework, depends on the value of x so that, for example, for relatively fast Λ 's one can expect the asymptotic limit relatively sooner than for slower ones. This sort of consideration should be incorporated with the criterion obtained from the duality and factorization considerations.

With these remarks, we now turn to the dual-resonance model for the studies of inclusive spectra.

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X- Single particle Production in Dual-Resonance Model

Even within the framework of dual resonance model, there are two ways of constructing the inclusive cross sections, i.e., either one uses such model for the elastic multiparticle amplitudes and takes the appropriate discontinuities in the spirit of Mueller's analysis or one uses the model to construct the production amplitudes and sums up the resulting exclusive cross-sections through the unitarity relation. Since the dual resonance model is only an approximate S-matrix representation, the two ways can result different cross-sections. While the second approach appears more general in the sense of including the background terms, it brings about the difficulty of writing down the proper representations for the loop to express such background terms. The first approach cannot include the non-resonant background contributions, but it is computable. In fact, it is this approach which produced many interesting predictions of the distribution functions¹.

We have seen from the Mueller's framework that a factorizable multi-Regge assumption for the $\text{Disc}_{\mu_2} T_{3 \rightarrow 3}$ is needed to explain scaling. However, factorization of the inclusive cross-section is not necessarily a consequence of the Regge assumption for the multiparticle amplitude. In order for the cross-sections to have desired factorization, the amplitude must have special analyticity property. The dual model is the only known representation for multiparticle scattering which has non-trivial analyticity and multi-Regge behavior. The fact that the dual model used for the elastic multiparticle amplitude yields the inclusive cross-sections with many properties observed in experiments may be an indication that the dual model has the singularity structure of general validity and is reasonable for production processes. This is so despite the obvious drawback of the model, namely, the lack of unitarity. In fact, Di Giacomo and others² have argued that even if the dual model without loops gives the exclusive cross-sections inversely proportional to the widths parameters ($1/\Gamma_R$), so that in

the zero width limit they diverge, whenever one sums up exclusive cross-sections, such dependence on the widths is washed out in the inclusive cross-sections as if these are computed from the discontinuities of the dual amplitude.

The fact that the loops are not included may mean another drawback of this approach, namely the absence of Pomeron. But in so far as we do not have a clear understanding as to whether or not the twisted loops is the proper representation for the Pomeron, it is fair to say that this difficulty is of more fundamental level. In fact, the nature of Pomeron itself is not understood well yet. On the other hand, if we consider the number density, i.e., the single particle inclusive distribution normalized to the total cross-section, we do not have to make any specific assumptions on the leading vacuum trajectory intercept. With an arbitrary value for the intercept, $\alpha_v(0)$, we can examine the limiting distributions and then try to mock up the Pomeron if needed, by simply taking $\alpha_v(0) = 1$.

Even if one takes the attitude that the six-point dual model is to represent the elastic 3-3 amplitude for $a+b+\bar{c} \rightarrow a+b+\bar{c}$, there are $\frac{1}{2}(6-1)! = 60$ different diagrams corresponding to the different permutations of the external lines. However, only $\frac{1}{2}(3! \times 3!) = 18$ of them have a, b and $\bar{c}$ adjacent and contribute to the discontinuity in M^2 . Each of these can be expressed by a multiperipheral configuration (1,2,...,6) of Fig (X-1). (See appendix A)

Fig (X-1)

$$\begin{aligned}
 B(1,2,\dots,6) = & \int_0^1 \int_0^1 \int_0^1 du_{12} du_{13} du_{14} u_{12}^{-\alpha_{12}-1} u_{13}^{-\alpha_{13}-1} u_{14}^{-\alpha_{14}-1} \\
 & (1-u_{12})^{-\alpha_{23}-1} (1-u_{13})^{-\alpha_{34}-1} (1-u_{14})^{-\alpha_{45}-1} \\
 & (1-u_{12}u_{13})^{\alpha_{23}-\alpha_{24}+\alpha_{34}} (1-u_{13}u_{14})^{\alpha_{34}-\alpha_{35}+\alpha_{45}} \\
 & (1-u_{12}u_{13}u_{14})^{\alpha_{24}-\alpha_{25}-\alpha_{34}+\alpha_{35}} \quad (X-1)
 \end{aligned}$$

where $\alpha_{i,i+n} = \alpha_{i,i+n}(0) + (p_i + p_{i+1} + \dots + p_{i+n})^2$
 and u_{ij} 's are the Chan variables³.

Then using the symmetry relation discovered by Plahte,

$$B(1,2,\dots,6) + e_{12} B(2,1,\dots,6) + e_{12} e_{13} B(2,3,1,4,5,6) + e_{12} e_{13} e_{14} B(2,3,4,1,5,6) + e_{12} e_{13} e_{14} e_{15} B(2,3,4,5,1,6) = 0 \quad (X-2)$$

where $e_{12} = e^{-i\pi(\alpha_{12}+1)}$, etc

we obtain relationships between five different diagrams. Taking discontinuity in M^2 , then (X-2) reduces to relationships between three diagrams, which in turn can be reduced to relationships between pairs of diagrams by taking imaginary parts. For example, one can get

$$\text{Disc}_{M^2} B(a\bar{c}b\bar{a}c\bar{b}) = \frac{\sin \pi(\alpha - \alpha_{a\bar{c}} - \alpha_{b\bar{c}}) \sin \pi(-\alpha_{a\bar{c}}) \sin \pi(-\alpha_{b\bar{c}})}{\sin \pi(\alpha - \alpha_{a\bar{c}}) \sin \pi(\alpha - \alpha_{b\bar{c}}) \sin \pi(-\alpha_{a\bar{c}} - \alpha_{b\bar{c}})} \text{Disc}_{M^2} B(b\bar{c}a\bar{a}c\bar{b}) \quad (X-3)$$

where $\alpha = \alpha(0) + M^2$

and $\alpha_{ij} = \alpha_{ij}(0) + S_{ij}$

As they stand, these sine factors are infinitely oscillating and do not have a well-defined asymptotic limit. To give meaning to the asymptotic limit along rays in complex S , $S_{a\bar{c}}$, $S_{b\bar{c}}$ we adopt a familiar phase convention that the phases of S , $-S_{a\bar{c}}$, $-S_{b\bar{c}}$ have the same sign when they are asymptotic. With this convention, we can show that contributions from nine diagrams vanish exponentially due to the sine factors. Thus we are left with only a total of seven distinct diagrams contributing to the distributions.

They are as following, using the notation

$$B(\bar{b}c\bar{a}a\bar{c}b) = \begin{pmatrix} a & \bar{a} \\ \bar{c} & c \\ b & \bar{b} \end{pmatrix}$$

in the fragmentation region of a

$$\begin{pmatrix} a & \bar{a} \\ \bar{c} & c \\ b & \bar{b} \end{pmatrix} \quad \begin{pmatrix} \bar{c} & \bar{a} \\ a & c \\ b & \bar{b} \end{pmatrix} \quad \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \\ b & \bar{b} \end{pmatrix} \quad \begin{pmatrix} \bar{c} & c \\ a & \bar{a} \\ b & \bar{b} \end{pmatrix}$$

in the fragmentation region of b

$$\begin{pmatrix} a & \bar{a} \\ \bar{c} & c \\ b & \bar{b} \end{pmatrix} \quad \begin{pmatrix} a & \bar{a} \\ b & c \\ \bar{c} & \bar{b} \end{pmatrix} \quad \begin{pmatrix} a & \bar{a} \\ \bar{c} & \bar{b} \\ b & c \end{pmatrix} \quad \begin{pmatrix} a & \bar{a} \\ b & \bar{b} \\ \bar{c} & c \end{pmatrix}$$

and in the central region

$$\begin{pmatrix} a & \bar{a} \\ \bar{c} & c \\ b & \bar{b} \end{pmatrix}$$

This is in a way very satisfying, because, for example, the four diagrams contributing to the fragmentation of a are the only four diagrams with Regge trajectories in the $b\bar{b}$ channel, which are needed to produce limiting distributions according to Mueller's analysis. Also these diagrams provide signature factors naturally for the Regge trajectories in the $a\bar{c}$ and $\bar{a}c$ channels entering the triple Regge vertex. We note further that only one diagram has the double Regge behavior needed to contribute in the central region. In fact, this is the only diagram contributing to all regions of fragmentation and pionization. Therefore while all seven diagrams are important for the consideration of the limiting distributions in each kinematic region, if one wishes to examine the distributions for all values of x , we may elect to study the particular diagram $\begin{pmatrix} a & \bar{a} \\ \bar{c} & c \\ b & \bar{b} \end{pmatrix}$.

In doing so, we may obtain the informations about the approach to scaling for different regions of x . Although one can use the multiperipheral configuration of Fig.(X-1) for this diagram everywhere, we find it convenient to use the representation (X-1) in the fragmentation region, i.e.

$$\mathcal{B}_6 \equiv \mathcal{B}(1,2,\dots,6) = \mathcal{B}(b\bar{c}a\bar{a}c\bar{b}) \quad (X-4)$$

whereas in the central region, the representation (A-5) corresponding to Fig.(X-2)

Fig (X-2)

with

$$B_b \equiv B(1, 2, \dots, b) = B(\bar{b} b \bar{c} a \bar{a} c) \quad (X-5)$$

Starting with (X-4), we take the asymptotic limit $\alpha \rightarrow -\infty$ with $\alpha_{ac}, \alpha_{a\bar{c}}$, etc fixed and negative, where the integral representation is defined, and then continue analytically to the desired physical region i.e., $\alpha \rightarrow \infty$ with $\alpha_{b\bar{c}}, \alpha_{bc}, \alpha_{a\bar{c}}, \alpha_{ac}$, negative to take the discontinuity.

In this limit, the dominant contribution comes from u_{13} near 1. Substituting $u_{13} = 1 + \frac{z}{\alpha}$ and taking the leading order term,

$$\begin{aligned} B_b &= \int_0^1 du_{12} du_{14} \int_{-\alpha}^0 \frac{dz}{\alpha} e^{-z} \left(-\frac{z}{\alpha}\right)^{-\alpha_{a\bar{a}}-1} \left(1 - u_{12} - u_{12} \frac{z}{\alpha}\right)^{\alpha_{a\bar{c}} - \alpha_{a\bar{a}\bar{c}} + \alpha_{a\bar{a}}} \\ &\quad \left(1 - u_{14} - u_{14} \frac{z}{\alpha}\right)^{\alpha_{\bar{a}c} - \alpha_{a\bar{a}c} + \alpha_{a\bar{a}}} \\ &\quad \left(1 - u_{12}u_{14} - u_{12}u_{14} \frac{z}{\alpha}\right)^{\alpha_{a\bar{a}\bar{c}} - \alpha_{b\bar{b}} - \alpha_{a\bar{a}} + \alpha_{a\bar{a}c}} \\ &\quad u_{12}^{-\alpha_{b\bar{c}}-1} u_{14}^{-\alpha_{bc}-1} (1-u_{12})^{-\alpha_{a\bar{c}}-1} (1-u_{14})^{-\alpha_{\bar{a}c}-1} \\ &= \int_0^1 du_{12} du_{14} u_{12}^{-\alpha_{b\bar{c}}-1} u_{14}^{-\alpha_{bc}-1} (1-u_{12})^{-1-\alpha_{a\bar{a}\bar{c}}+\alpha_{a\bar{a}}} (1-u_{14})^{-1-\alpha_{a\bar{a}c}+\alpha_{a\bar{a}}} \\ &\quad (1-u_{12}u_{14})^{\alpha_{a\bar{a}\bar{c}} - \alpha_{b\bar{b}} - \alpha_{a\bar{a}} + \alpha_{a\bar{a}c}} \\ &\quad \alpha^{\alpha_{a\bar{a}}} \int_{-\alpha}^0 dz \left(-\frac{z}{\alpha}\right)^{-\alpha_{a\bar{a}}-1} \exp \left\{ -1 - \frac{u_{12}}{1-u_{12}} \frac{\alpha_{a\bar{c}} - \alpha_{a\bar{a}\bar{c}} + \alpha_{a\bar{a}}}{\alpha} \right. \\ &\quad \left. - \frac{u_{14}}{1-u_{14}} \frac{\alpha_{\bar{a}c} - \alpha_{a\bar{a}c} + \alpha_{a\bar{a}}}{\alpha} \right. \\ &\quad \left. - \frac{u_{12}u_{14}}{1-u_{12}u_{14}} \frac{\alpha_{a\bar{a}\bar{c}} - \alpha_{b\bar{b}} - \alpha_{a\bar{a}} + \alpha_{a\bar{a}c}}{\alpha} \right\} z \end{aligned}$$

$$B_6 = \int_0^1 du_{12} du_{14} u_{12}^{-\alpha_{b\bar{c}}-1} u_{14}^{-\alpha_{b\bar{c}}-1} (1-u_{12})^{-1-\alpha_{a\bar{a}\bar{c}}+\alpha_{a\bar{a}}} (1-u_{14})^{-1-\alpha_{a\bar{a}\bar{c}}+\alpha_{a\bar{a}}} \\ (1-u_{12}u_{14})^{\alpha_{a\bar{a}\bar{c}}-\alpha_{b\bar{b}}-\alpha_{a\bar{a}}+\alpha_{a\bar{a}\bar{c}}} \\ \cdot \alpha^{\alpha_{a\bar{a}}} \Gamma(-\alpha_{a\bar{a}}) \left[-1 - \frac{u_{12}}{1-u_{12}} \frac{\alpha_{a\bar{c}}-\alpha_{a\bar{a}\bar{c}}+\alpha_{a\bar{a}}}{\alpha} - \frac{u_{14}}{1-u_{14}} \frac{\alpha_{a\bar{c}}-\alpha_{a\bar{a}\bar{c}}+\alpha_{a\bar{a}}}{\alpha} \right. \\ \left. - \frac{u_{12}u_{14}}{1-u_{12}u_{14}} \frac{\alpha_{a\bar{a}\bar{c}}-\alpha_{b\bar{b}}-\alpha_{a\bar{a}}+\alpha_{a\bar{a}\bar{c}}}{\alpha} \right]^{\alpha_{a\bar{a}}}$$

$$B_6 = \int_0^\infty dz_1 dz_2 z_1^{-\alpha_{b\bar{c}}-1} z_2^{-\alpha_{b\bar{c}}-1} (1+z_1)^{\alpha_{b\bar{c}}+\alpha_{b\bar{b}}-\alpha_{a\bar{a}\bar{c}}} (1+z_2)^{\alpha_{b\bar{c}}+\alpha_{b\bar{b}}-\alpha_{a\bar{a}\bar{c}}} \\ (1+z_1+z_2)^{\alpha_{a\bar{a}\bar{c}}-\alpha_{b\bar{b}}-\alpha_{a\bar{a}}+\alpha_{a\bar{a}\bar{c}}} \\ \Gamma(-\alpha_{a\bar{a}}) \left[-\alpha - z_1(\alpha_{a\bar{c}}-\alpha_{a\bar{a}\bar{c}}+\alpha_{a\bar{a}}) - z_2(\alpha_{a\bar{c}}-\alpha_{a\bar{a}\bar{c}}+\alpha_{a\bar{a}}) \right. \\ \left. - \frac{z_1 z_2}{1+z_1+z_2} (\alpha_{a\bar{a}\bar{c}}-\alpha_{b\bar{b}}-\alpha_{a\bar{a}}+\alpha_{a\bar{a}\bar{c}}) \right]^{\alpha_{a\bar{a}}} \quad (X-6)$$

$$\text{where } z_1 = \frac{u_{12}}{1-u_{12}}, \quad z_2 = \frac{u_{14}}{1-u_{14}}$$

Note that as α is continued to the positive region, an imaginary part develops due to the last factor in the integrand.

$$\frac{1}{2i} \text{Disc}_{M^2} B_6 = \Gamma(-\alpha_{a\bar{a}}) \sin \pi \alpha_{a\bar{a}} \int_0^\infty dz_1 dz_2 z_1^{-\alpha_{b\bar{c}}-1} z_2^{-\alpha_{b\bar{c}}-1} (1+z_1)^{\alpha_{b\bar{c}}+\alpha_{b\bar{b}}-\alpha_{a\bar{a}\bar{c}}} \\ (1+z_2)^{\alpha_{b\bar{c}}+\alpha_{b\bar{b}}-\alpha_{a\bar{a}\bar{c}}} (1+z_1+z_2)^{\alpha_{a\bar{a}\bar{c}}-\alpha_{b\bar{b}}-\alpha_{a\bar{a}}+\alpha_{a\bar{a}\bar{c}}} H^{\alpha_{a\bar{a}}} \theta(H) \quad (X-7)$$

where

$$H = \alpha + z_1 (\alpha_{a\bar{c}} - \alpha_{a\bar{a}\bar{c}} + \alpha_{a\bar{a}}) + z_2 (\alpha_{\bar{a}c} - \alpha_{a\bar{a}c} + \alpha_{a\bar{a}}) + \frac{z_1 z_2}{1+z_1+z_2} (\alpha_{a\bar{a}\bar{c}} - \alpha_{b\bar{c}} - \alpha_{a\bar{a}} + \alpha_{a\bar{a}c}) \quad (X-8)$$

Put

$$\omega = 1 + z_1 + z_2$$

then

$$H = \alpha \left[1 + \frac{z_1 \alpha_{a\bar{c}}}{\alpha} \left(1 + \frac{\alpha_{a\bar{a}} - \alpha_{a\bar{a}\bar{c}}}{\alpha_{a\bar{c}}} \frac{1+z_1}{\omega} \right) + \frac{z_2 \alpha_{\bar{a}c}}{\alpha} \left(1 + \frac{\alpha_{a\bar{a}} - \alpha_{a\bar{a}c}}{\alpha_{\bar{a}c}} \frac{1+z_2}{\omega} \right) \right]$$

Making further change of variables

$$y_1 = -\frac{\alpha_{a\bar{c}}}{\alpha} z_1, \quad y_2 = -\frac{\alpha_{\bar{a}c}}{\alpha} z_2, \quad u = 1 + z_1, \quad v = (1 + z_2)$$

$$H = \alpha \left[1 - y_1 \left(1 + \frac{\alpha_{a\bar{a}} - \alpha_{a\bar{a}\bar{c}}}{\alpha_{a\bar{c}}} \frac{u}{\omega} \right) - y_2 \left(1 + \frac{\alpha_{a\bar{a}} - \alpha_{a\bar{a}c}}{\alpha_{\bar{a}c}} \frac{v}{\omega} \right) \right] \equiv \alpha h$$

so that

$$\begin{aligned} \frac{1}{2i} \text{Disc}_{M^2} B_6 &= \frac{\pi}{\Gamma(1+\alpha_{a\bar{a}})} \alpha^{\alpha_{a\bar{a}}} \left(-\frac{\alpha_{a\bar{c}}}{\alpha} \right)^{\alpha_{b\bar{c}}} \left(-\frac{\alpha_{\bar{a}c}}{\alpha} \right)^{\alpha_{b\bar{c}}} \\ &\int_0^\infty dy_1 dy_2 y_1^{-\alpha_{b\bar{c}}-1} y_2^{-\alpha_{b\bar{c}}-1} u^{\alpha_{b\bar{c}} - \alpha_{a\bar{a}\bar{c}} + \alpha_{b\bar{c}}} \\ &v^{\alpha_{b\bar{c}} - \alpha_{a\bar{a}\bar{c}} + \alpha_{b\bar{c}}} \omega^{\alpha_{a\bar{a}\bar{c}} + \alpha_{a\bar{a}c} - \alpha_{b\bar{c}} - \alpha_{a\bar{a}}} h^{\alpha_{a\bar{a}}} \theta(h) \quad (X-9) \end{aligned}$$

Finally, the single-particle density function is

$$\begin{aligned} f_{ab}^c &= \frac{1}{\sigma_{\text{tot}}} \frac{1}{2i} \text{Disc}_{M^2} B_6 \\ &= \left(\frac{\alpha}{\alpha_{ab}} \right)^{\alpha_{a\bar{a}}} \left(-\frac{\alpha_{a\bar{c}}}{\alpha} \right)^{\alpha_{b\bar{c}}} \left(-\frac{\alpha_{\bar{a}c}}{\alpha} \right)^{\alpha_{b\bar{c}}} \int_0^\infty dy_1 dy_2 y_1^{-\alpha_{b\bar{c}}-1} y_2^{-\alpha_{b\bar{c}}-1} \\ &u^{\alpha_{b\bar{c}} - \alpha_{a\bar{a}\bar{c}} + \alpha_{b\bar{c}}} v^{\alpha_{b\bar{c}} - \alpha_{a\bar{a}\bar{c}} + \alpha_{b\bar{c}}} \omega^{\alpha_{a\bar{a}\bar{c}} + \alpha_{a\bar{a}c} - \alpha_{b\bar{c}} - \alpha_{a\bar{a}}} h^{\alpha_{a\bar{a}}} \theta(h) \quad (X-10) \end{aligned}$$

where we used

$$\sigma_{\text{tot}} = \frac{\pi}{\Gamma(1+\alpha_{a\bar{a}})} (\alpha_{ab})^{\alpha_{a\bar{a}}-1}$$

as implied by the four-point amplitude. Note that (X-10) is explicitly depending on s , x and K at a given energy. To see the limiting distributions, we take further kinematical limit of each region:

Fragmentation of b :

Since $\alpha_{a\bar{c}} = \alpha_{\bar{a}c} \simeq -xs$, $\alpha_{b\bar{c}} = \alpha_{\bar{b}c} \simeq -\kappa^2/x$ and $\alpha \simeq (1-x)s$

and $\alpha_{b\bar{b}} = \alpha_{a\bar{a}} = \alpha_v$, $\alpha_{ab} \simeq s$

$$\begin{aligned} f_{ab}^c \longrightarrow f_b^c &= (1-x)^{\alpha_v + 2\kappa^2/x} x^{-2\kappa^2/x} \int_0^\infty dy_1 dy_2 (y_1 y_2)^{\kappa^2/x} \\ &\cdot \left(1 + \frac{1-x}{x} y_1\right)^{-\kappa^2/x + \alpha_v - \alpha_{a\bar{a}c}} \left(1 + \frac{1-x}{x} y_2\right)^{-\kappa^2/x + \alpha_v - \alpha_{a\bar{a}\bar{c}}} \\ &\cdot \left(1 + \frac{1-x}{x} (y_1 + y_2)\right)^{-2\alpha_v + \alpha_{a\bar{a}c} + \alpha_{a\bar{a}\bar{c}}} (1-y_1-y_2)^{\alpha_v} \theta(1-y_1-y_2) \end{aligned} \quad (X-11)$$

which is manifestly a function of x and K . Contributions to f_b^c of the other three terms can be obtained by permuting $b \leftrightarrow \bar{c}$ and/or $\bar{b} \leftrightarrow c$ in the integral of (X-10).

Triple Regge region : We take the limit $x \rightarrow 1$ in (X-11),

$$\begin{aligned} f_b^c \underset{x \rightarrow 1}{\sim} (1-x)^{\alpha_v - 2\alpha_{b\bar{c}}} \int_0^\infty dy_1 dy_2 \theta(1-y_1-y_2) y_1^{-\alpha_{b\bar{c}}-1} y_2^{-\alpha_{\bar{b}c}-1} (1-y_1-y_2)^{\alpha_v} \\ = (1-x)^{\alpha_v - 2\alpha_{b\bar{c}}} \frac{\Gamma(-\alpha_{b\bar{c}}) \Gamma(-\alpha_{\bar{b}c}) \Gamma(1+\alpha_v)}{\Gamma(-\alpha_{b\bar{c}} - \alpha_{\bar{b}c} + \alpha_v + 1)} \end{aligned} \quad (X-12)$$

One can easily see from this that the triple Regge vertex function is

$$g(\alpha_{b\bar{c}}, \alpha_{\bar{b}c}; \alpha_v) = \frac{\pi \Gamma(1+\alpha_v)}{\{\Gamma(1+\alpha_{b\bar{c}}) \Gamma(1+\alpha_{\bar{b}c})\}^{1/2} \Gamma(1+\alpha_v - \alpha_{b\bar{c}} - \alpha_{\bar{b}c})} \quad (X-13)$$

which vanishes for $\alpha_v = 1$ and $\alpha_{b\bar{c}} = \alpha_{\bar{b}c}$ integer ≥ 1 .

When the other three diagrams are summed to (X-12), they give the signature factors for the Regge trajectories $\alpha_{b\bar{c}}$ and $\alpha_{\bar{b}c}$ entering the triple Regge region.

The triple Regge distribution (X-12) can be obtained directly from B_6 taking $\alpha_{a\bar{c}}, \alpha_{\bar{a}c} \rightarrow \infty$ limits. This leads to

$$B_6 = \frac{\beta(\alpha_{b\bar{c}}) (-\alpha_{a\bar{c}})^{\alpha_{b\bar{c}}}}{\sin \pi \alpha_{b\bar{c}}} \cdot \frac{\beta(\alpha_{\bar{b}c}) (-\alpha_{\bar{a}c})^{\alpha_{\bar{b}c}}}{\sin \pi \alpha_{\bar{b}c}} \cdot R \quad (X-14)$$

in which R is the Reggeon-particle amplitude,

$$R = \beta(\alpha_{b\bar{c}}) \beta(\alpha_{\bar{b}c}) \int_0^1 du u^{-\alpha_v-1} (1-u)^{-\alpha_v-1+\alpha_{\bar{b}c}+\alpha_{b\bar{c}}} \quad (X-15)$$

$\beta(\alpha_{ij})$ being the Regge residue function. Besides, we see from (X-15) that the Reggeon-particle cross-section at high energy is given by

$$\sigma_{tot}^{R.P.} = \frac{\pi \beta(\alpha_{b\bar{c}}) \beta(\alpha_{\bar{b}c}) \alpha^{\alpha_v - \alpha_{b\bar{c}} - \alpha_{\bar{b}c} - 1}}{\Gamma(1 + \alpha_v - \alpha_{b\bar{c}} - \alpha_{\bar{b}c})} \quad (X-16)$$

from which (X-13) can also be obtained.

Central region :

Here, the distribution gets a contribution only from the diagram $B(b\bar{c}a\bar{a}c\bar{b})$. As $s_{a\bar{c}} = s_{b\bar{c}} = s_{\bar{a}c} = s_{\bar{b}c} \simeq -K\sqrt{s}$, the most contribution to the integral of (X-10) comes from the upper end of the integration range. For this reason, it is convenient to use the equivalent configuration (X-5), which treats the $(a\bar{c})$ and $(b\bar{c})$ channels in symmetric ways although the same result follows from (X-10). The limiting distribution turns out to be

$$f_c(\kappa^2) = \frac{\Gamma(1+\alpha_v)}{\pi} (\kappa^2)^{\alpha_v} \int_0^1 du u^{-\alpha_{a\bar{c}}-1} (1-u)^{-\alpha_{a\bar{c}}-1} \text{Disc}_\lambda V(\alpha_v, \alpha_v; \lambda = \frac{\kappa^2}{u(1-u)}) \quad (X-17)$$

where V is the two-Reggeon-one-particle vertex function

$$V(\alpha_1, \alpha_2; \lambda) = \int_0^\infty \int_0^\infty du dv u^{-\alpha_1-1} v^{-\alpha_2-1} \exp(-u-v + \frac{uv}{\lambda}). \quad (X-18)$$

We note from (X-17) that the distribution has an exponential cutoff in transverse momenta

$$f^c(k^2) \underset{k \rightarrow \infty}{\sim} k^{-2\alpha_v-3} e^{-4k^2} \quad (X-19)$$

in qualitative agreement with experiment.

Threshold for scaling :

The limiting distributions shown above are correct only for s infinite. At given finite values of s , one has to resort to the numerical study of (X-10). In this respect, it is interesting to establish the threshold for scaling. Obviously, such threshold will depend on the value of x as well as s . We have suggested a criteria based on how the respective limiting distributions are obtained from (X-10). It is crucial to have the Regge behavior in order to obtain scaling and the Regge behavior is achieved from the dual amplitude via the usual exponentiation.

$$\int_0^N dz \left(1 - \frac{z}{N}\right)^N \longrightarrow \int_0^N dz e^{-z} \quad (X-20)$$

Our criteria makes use of the reliability of this exponentiation procedure. The largeness of N determines the confidence of the approximation (X-20) and it turns out for a large class of test functions relevant to the distribution that $N \geq 20$ gives in general better than 10% accuracy. The exponentiation procedure (X-20) has another role in connection with duality. Namely, the distribution functions can also be obtained by summing over missing mass poles but one can show the equivalence of the two approaches, the resonance structure in missing mass and the distribution function (X-10), so long as N is large enough to make the exponentiation procedure (X-20) meaningful.

One then notes that in the fragmentation of b , $S_{a\bar{c}} \simeq -xs$ and $M^2 \simeq (1-x)s$ have to be large, i.e., $s \simeq N/x$ or $s \simeq N/(1-x)$ so that the fragmentation limit, say, for x around 0.5 is practically reached at the energies above $s \simeq 40 \text{ GeV}^2$. This seems to be supported by the the experiments. On the other hand, this criteria implies that the pionization limit is obtained at much higher energies, i.e., $s \simeq N^2/\langle K^2 \rangle \gg 4.000 \text{ GeV}^2$. This may mean that the scaling limit is not yet reached even at the ISR energies. In addition, the triple-Regge behavior is expected at higher energies than the fragmentation limit according to our criteria. This criteria is different from the many proposed ones discussed in SIX, that make use of the exoticity concept, and shows an explicit dependence on x for the approach to scaling.

Numerical Results :

Here, we list briefly the qualitative results obtained from (X-10) by Pu Shen and myself.

1) The distribution function $f_{ab}^c(x, K^2, s)$ shows the x and K^2 dependence in qualitative agreement with π^- spectrum seen in experiments, in particular by Ko and Lauder⁴. The decrease of the distribution with K^2 is faster for smaller x , while it approaches e^{-4K^2} for large K^2 independent of x .

The distribution is peaked near $x=0$ for both finite and infinite s indicating the slow pion concentration.

2) The smaller the value of x is, the smaller the average transverse momentum $\langle p_\perp \rangle$ is. Moreover, $\langle p_\perp \rangle$ is smaller for lower values of s . This is again in accord with the experimental trends.

3) The distribution can not be separated into a product of functions of $p_\perp$ and x only.

4) The average transverse momentum $\langle p_\perp \rangle$ is larger for heavier mass and the mass dependence of $\langle p_\perp \rangle$ is stronger near $x=0$. In particular, the maximum $\langle p_\perp \rangle$ is occurring in the fragmentation region for fixed s and

approaches asymptotically 340 and 350 MeV for pion and kaon respectively.

5) When the absolute normalization of the distribution (X-10) is fixed by matching the theoretical area $\int_{-0.4}^{+0.4} G(x) dx$ to that of experimental area⁵ for the π^+p reaction at $s=42 \text{ GeV}^2$ where

$$G(x,s) = \frac{\pi}{2} \int_0^\infty dp_\perp^2 f(x, \kappa, s) \quad (\text{X-21})$$

it reproduces the experimental spectrum for π^-, π^+ and K_S^0 productions surprisingly well, except the asymmetry of the spectrum around $x=0$ and the leading particle effect.

6) The average multiplicity $\langle n \rangle$ can be approximated by $\langle n \rangle = c \ln s + d$ (X-22) around $s = 10^3 \text{ GeV}^2$ where $c=1.28$ for pion and $c=0.1$ for kaon. With the normalization constants obtained above, this implies the slope of 0.42, 0.64 and 0.13 in $\ln s$ for π^-, π^+ and K_S^0 respectively which are certainly consistent with the available data.

However, the distribution (X-10) gives rise rigorously to the form of (X-22) but with the slope 0.26 after $s=4 \times 10^3 \text{ GeV}^2$, thus indicating that the pion production will be slowed down further, once the pionization limit is reached. Incidentally, one gets in this limit from (X-10) that

$$C = \frac{1}{2} \int_0^1 \int_0^1 du dv u^{-\alpha_{a\bar{a}c} + \alpha_v} (1-u)^{-\alpha_{a\bar{a}c} + \alpha_v} v^{\alpha_v} \exp \left[-m_c^2 / u(1-u)(1-v) \right] \quad (\text{X-23})$$

7) The differential multiplicity $d\langle n \rangle / dx = \sigma_{tot}^{-1} d\sigma / dx$ at $s=42 \text{ GeV}^2$ shows a peak at $x=0$ and is roughly exponential, i.e., e^{-bx} with $b=9.6$ and 6 respectively for pion and kaon. Experimentally $b=7 \sim 10$ for pions.

In short the B_0 analysis gives surprisingly good results for the single-particle distributions. Not only has one the desired limiting behavior in all kinematical regions, but also one obtains the experimentally observed cutoff in transverse momentum as well as the shape in x and $p_\perp$.

References to § X

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XI- Further Analysis of Dual Resonance Model

Encouraged by the success of the B_6 analysis, we have proceeded to the B_8 analysis for the two-particle production¹. It is cautioned in our work about the procedure of taking the discontinuity in the missing mass variable and the use of the kinematic relations which may exist in particular kinematic regions of interest. The distribution has the desired limiting behavior and factorization as well as non-trivial correlation.

1) While the single fragmentation distribution shows the desired scaling behavior, the double fragmentation distribution shows the desired factorization property

$$f_{ab}^{12} = f_a^1(x_1, x_1) \cdot f_b^2(x_2, x_2) \quad (X1.1)$$

where f_a^1 is the fragmentation distribution for $a + \text{target} \rightarrow c + \text{anything}$ and is given by the form of Eq.(X-11). In obtaining this the kinematic relation $s_{a\bar{1}\bar{2}} \cdot s_{b\bar{1}\bar{2}} / M^2 = s_{\bar{1}\bar{2}}$ plays an important role.

2) In the ditriple limit, i.e. $x_1 \approx 1$ and $x_2 \approx -1$ in (XI-1), one obtains

$$f_{ab}^{12} \approx (1-x_1)^{\alpha_v - \alpha_{\bar{a}1} - \alpha_{a\bar{1}}} (1+x_2)^{\alpha_v - \alpha_{b2} - \alpha_{b\bar{2}}} h_a^1 h_b^2 \quad (X1.2)$$

where h_a^1 is given by the coefficient of $(1-x)^{\alpha_v - 2\alpha_{b\bar{c}}}$ in Eq.(X-12) with c replaced by C_1 .

3) In the pionization limit but with relative rapidity y and arbitrary azimuthal angle, $0 \leq \phi \leq 2\pi$, defined by

$$s_{12} = m_1^2 + m_2^2 + 2x_1 x_2 \cosh y - 2p_{1\perp} p_{2\perp} \cos \phi \quad (X1.3)$$

the distribution shows again a cutoff in transverse momenta,

$$f^{12} \sim e^{-4(p_{1\perp}^2 + p_{2\perp}^2 + 2p_{1\perp} p_{2\perp} \cos \beta)} \quad (X1.4)$$

where β depends on y and ϕ , so long as $p_{1\perp} \gg p_{2\perp} \gg m_1, m_2$ or

$p_{1\perp} = p_{2\perp} \gg m_1, m_2$. In particular $\cos \beta(y, \phi) \geq \cos \beta(y, \pi)$ so that the two particles are more likely produced to move opposite to each other in the transverse momentum plane.

4) In the pionization region, the two-particle correlation function defined by

$$C^{(2)} = f^{12}(y, p_{1\perp}, p_{2\perp}, \cos \phi) / f^1(p_{1\perp}) \cdot f^2(p_{2\perp}) - 1 \quad (XI.5)$$

turns out to be, to the leading order²

$$C^{(2)} \simeq - \frac{32}{\alpha_V} (p_{1\perp} p_{2\perp})^2 \left\{ \frac{p_{1\perp}^2 + p_{2\perp}^2}{2 p_{1\perp} p_{2\perp}} + \cos \phi \right\} e^{-y} \quad (XI.6)$$

so that the correlation length in y is of the order 1. This is due to the fact that the second leading trajectory in B_8 is its first daughter. However, if there exists an additional vacuum trajectory $\alpha_{V'}$ whose intercept lies between those of α_V and $\alpha_V - 1$, then the correlation length would be $(\alpha_V - \alpha_{V'})^{-1}$ instead. Yet the minimal correlation in $\cos \phi$ would be given by (XI-6) as $\alpha_{V'}$ would factorize. In the $y \rightarrow \infty$ limit, the correlation ceases to exist and the distribution has the factorized form.

5) By restricting to the region of finite S_{12} or y where the two particles can form a resonance, the B_8 analysis can be used to examine the distribution of a single outgoing resonance which decays subsequently into particles 1 and 2. The case of the vector meson productions has been studied³. In general, the results obtained in this way contain the decay correlation effects and it is necessary to factor out the vertex function in order to isolate the single-resonance distribution. It is found that the vector meson vertex is factorizable only for large values of transverse mass K of the vector meson both in the fragmentation and central regions. In the fragmentation region, the helicity state $\lambda=0$ is favored over $\lambda=\pm 1$ for

the vector meson production and the angular correlation for the decay takes the $\cos^2\beta$ form along the preferred beam direction in the rest frame of the vector meson. On the other hand, in the central region, the helicity $\lambda = \pm 1$ states are favored over the zero state and the angular correlation contains a $\cos^2\beta$ term in addition to a $\cos\beta$ term which comes from the daughter contributions but vanishes if the intercept of the vacuum trajectory is taken to be unity. In addition, the distribution has the familiar exponential cutoff in the transverse mass K as in the spinless particle distributions. But because of the massiveness of the vector meson, the distribution is less sharp for low K than that of spinless meson and the vector meson (ρ) production is suppressed by a factor $\exp[-4(m_\rho^2 - m_\pi^2)]$ compared to the pion production.

Also the large K limit removes the correlation coming from decay of the daughter particles whose distribution is identical to the spinless meson.

There are other topics of interest such as the finite energy sum rules for the Reggeon-particle and Reggeon-Reggeon amplitudes based on the dual amplitude analysis⁴. But we will not go into the description here.

On the whole, the dual amplitude analysis have helped us to gain substantial insight into the inclusive phenomenology. I believe further that the dual resonance model will continue to play an important role for the study of multiparticle dynamics.

References to § XI

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APPENDIX A :

The six-point amplitude for Fig. (X-I) can be derived following the Chan convention. First we write down

$$B(1,2,\dots,6) = \int_0^1 du_{12} du_{13} du_{14} u_{12}^{x_{12}} u_{13}^{x_{13}} u_{14}^{x_{14}} u_{23}^{x_{23}} u_{24}^{x_{24}-1} u_{25}^{x_{25}-2} u_{34}^{x_{34}} u_{35}^{x_{35}-1} u_{45}^{x_{45}} \quad (A.1)$$

with $x_{ij} = -\alpha_{ij} - 1$. Then note that only three of the u_{ij} 's are independent. The remaining six variables can be expressed in terms of the three (chosen) u_{ij} 's through the condition that dual partitions should not have coincident poles, which can be expressed as

$$u_{ij} = 1 - \prod_{\text{all } (\overline{ij})} u_{\overline{ij}} \quad (A.2)$$

where $(\overline{ij})$ denotes the channel dual to (ij) . In fact, one can easily see that

$$\begin{aligned} u_{12} &= 1 - u_{23} u_{24} u_{25}, & u_{13} &= 1 - u_{24} u_{25} u_{34} u_{35} u_{45}, & u_{14} &= 1 - u_{25} u_{35} u_{45} \\ u_{23} &= 1 - u_{12} u_{34} u_{35}, & u_{24} &= 1 - u_{12} u_{13} u_{35} u_{45}, & u_{25} &= 1 - u_{12} u_{13} u_{14} \\ u_{34} &= 1 - u_{13} u_{23} u_{45}, & u_{35} &= 1 - u_{13} u_{14} u_{23} u_{24}, & u_{45} &= 1 - u_{14} u_{24} u_{34} \end{aligned} \quad (A.3)$$

Thus if we take U_{12} , U_{13} and U_{14} to be independent, then from (A.3) one finds

$$\begin{aligned} u_{23} &= \frac{1 - u_{12}}{1 - u_{12} u_{13}}, & u_{24} &= \frac{1 - u_{12} u_{13}}{1 - u_{12} u_{13} u_{14}}, & u_{25} &= \frac{1 - u_{12} u_{13} u_{14}}{1 - u_{12} u_{13} u_{14}}, \\ u_{34} &= \frac{1 - u_{13}}{1 - u_{13} u_{14}} \frac{1 - u_{12} u_{13} u_{14}}{1 - u_{12} u_{13}}, & u_{35} &= \frac{1 - u_{13} u_{14}}{1 - u_{12} u_{13} u_{14}}, & u_{45} &= \frac{1 - u_{14}}{1 - u_{13} u_{14}} \end{aligned} \quad (A.4)$$

When (A.4) is inserted into (A.1), we obtain the multiperipheral configuration (X-1)

On the other hand, if U_{12} , U_{13} and U_{45} are taken to be independent, then we express

$$u_{14} = \frac{1 - u_{45}}{1 - u_{13} u_{45}}$$

and insert to the rest of u_{ij} 's in (A.5). We then obtain from (A.1) that

$$B(1,2,\dots,6) = \int_0^1 du_{12} du_{13} du_{45} u_{12}^{-\alpha_{12}-1} u_{13}^{-\alpha_{13}-1} u_{45}^{-\alpha_{45}-1} (1-u_{12})^{-\alpha_{23}-1} (1-u_{13})^{-\alpha_{35}-1} (1-u_{45})^{-\alpha_{14}-1} \\ (1-u_{12} u_{13})^{\alpha_{23}-\alpha_{24}+\alpha_{34}} (1-u_{13} u_{45})^{\alpha_{25}-\alpha_{24}+\alpha_{14}} (1-u_{12} u_{13} - u_{13} u_{45} + u_{12} u_{13} u_{45})^{\alpha_{24}-\alpha_{25}-\alpha_{34}+\alpha_{35}} \quad (A-5)$$

This corresponds to the configuration of Fig. (X-2).

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