JAN 15 1965

LA-AC-6803

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MASTER

STABILITY OF SOLUTIONS OF THE POLYNOMIAL EQUATION*

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RELEASED FOR ANNOUNCEMENT
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ABSTRACT

A complete analysis is carried out of the stability of solutions of the differential equation

$$dx/dt = -x(x - a_1) \dots (x - a_n) ,$$

where $0, a_1, \dots, a_n$ are distinct roots of a real polynomial. It is shown that no periodic solutions exist. Every nonsingular solution of the equation approaches asymptotically one of the nodes either for $t \rightarrow \infty$ or for $t \rightarrow -\infty$. Moreover, every nonsingular solution which has one point between two nodes remains between them, approaching one asymptotically for $t \rightarrow \infty$, and the other for $t \rightarrow -\infty$.

1. INTRODUCTION

The differential equation to be considered in this note is an equation in a single dependent variable x , which may be written in the form

$$dx/dt = -x(x - a_1)(x - a_2) \cdots (x - a_n), \quad (1)$$

where $a_1 \cdots a_n$ are constants. The right side of the equation is thus a polynomial in x of degree $n + 1$. By an elementary change of variables, any first order equation in which the right side is a polynomial with at least one real root may be put in the form of Eq. (1).

Since it is intended to restrict x to real values, it will be supposed that the right side of Eq. (1) corresponds to a polynomial with real coefficients. Thus, any root, a_c , with a nonzero imaginary part, is accompanied by its complex conjugate, a_c^* . It will further be supposed that each of the roots $0, a_1 \cdots a_n$ is distinct, i.e. there are no multiple roots.

2. PROOF AND THEOREM

It is clear that $x = 0$ is a singular solution of Eq. (1). Moreover, $x = a_r$, where a_r , is any real root of the right side of Eq. (1), is also a singular solution. The singular points may be classified according to the infinitesimal stability of Liapunov's first method [1,2]. The coefficient of the linear term in Eq. (1) as it stands is

$$(-1)^{n+1} a_1 \cdots a_n. \quad (2)$$

If this product is negative, $x = 0$ will be called a right hand node; if the product is positive, $x = 0$ will be called a left hand node.

Each of the other points corresponding to real roots a_r may be classified by making the transformation

$$\xi = x - a_r, \quad (3)$$

under which the equation becomes

$$d\xi/dt = -\xi(\xi + a_r)(\xi + a_r - a_1) \cdots (\xi + a_r - a_{r-1}) \quad (4)$$

$$(\xi + a_r - a_{r+1}) \cdots (\xi + a_r - a_n) . \quad (4)$$

The coefficient of the linear term in Eq. (4) is

$$-a_r(a_r - a_1) \cdots (a_r - a_{r-1})(a_r - a_{r+1}) \cdots (a_r - a_n) . \quad (5)$$

If the product in Eq. (5) is negative, $x = a_r$ is a right-hand node; if the product in Eq. (5) is positive, it is a left-hand node.

The complex roots do not have significance as far as asymptotic stability is concerned, and need not be classified.

Now, designating the right side of Eq. (1) by $-F(x)$, and its derivative by $-F'(x)$, it is easily verified that Eq. (2) is simply $-F'(0)$ and Eq. (5) is $-F'(a_r)$.

Having completed the definitions, our proof is almost finished. For, under the hypothesis of no multiple roots the rational function

$$\{F(x)\}^{-1} \text{ may be decomposed } \left[3 \right] \text{ as} \quad (6)$$

$$\{F(x)\}^{-1} = \{xF'(0)\}^{-1} + \{(x - a_1) F'(a_1)\}^{-1} + \dots + \{(x - a_n) F'(a_n)\}^{-1} .$$

Solution of Eq. (1) is expressed by

$$\int \{F(x)\}^{-1} dx = -t + c, \quad (7)$$

where c is a constant. Integrating the part of Eq. (6) corresponding to each real root as

$$\{F'(a_r)\}^{-1} \log |x - a_r|, \quad (8)$$

it is clear that for a right hand node the coefficient of $\log |x - a_r|$ is positive, and Eq. (8) may be written as

$$\alpha_r \log |x - a_r|, \quad \alpha_r > 0. \quad (9)$$

For a left hand node the coefficient of $\log |x - a_r|$ is negative, and Eq. (8) has the form

$$-\alpha_r \log |x - a_r|, \quad \alpha_r > 0. \quad (10)$$

The complex roots may be treated as follows. For the pair of roots a_c, a_c^* the part of the decomposition (6) is

$$\{(x - a_c) F'(a_c)\}^{-1} + \{(x - a_c^*) F'(a_c^*)\}^{-1}. \quad (11)$$

It is easily verified that $F'(a_c^*)$ is the complex conjugate of $F'(a_c)$, so that Eq. (11) becomes

$$\frac{\left[2 \operatorname{Re}\{F'(a_c)\} \right] x - 2 \operatorname{Re}\{a_c F''(a_c)\}}{|F''(a_c)|^2 \left\{ (x - a_{cr})^2 + a_{ci}^2 \right\}},$$

$$\equiv \frac{\gamma(a_c) x + \delta(a_c)}{(x - a_{cr})^2 + a_{ci}^2}, \quad (12)$$

where $a_c = a_{cr} + (\sqrt{-1})a_{ci}$ and $\gamma(a_c)$ and $\delta(a_c)$ are real.

Summing over the integrals of the decomposition (6), we have

$$\begin{aligned} & \sum_{\text{Right hand nodes}} \alpha_r \log |x - a_r| - \sum_{\text{Left hand nodes}} \alpha_r \log |x - a_r| \\ & + \sum_{\text{Complex roots}} \left[\frac{1}{2} \gamma(a_c) \log \left\{ (x - a_{cr})^2 + a_{ci}^2 \right\} \right. \\ & \quad \left. + \frac{\delta(a_c) + \gamma(a_c) a_{cr}}{a_{ci}} \operatorname{Arctan} \left(\frac{x - a_{cr}}{a_{ci}} \right) \right] \\ & = -t + c. \end{aligned} \quad (13)$$

From (13),

$$\begin{aligned}
 & \frac{\prod_{\text{Right hand nodes}} |x - a_r|^{\alpha_r}}{\prod_{\text{Left hand nodes}} |x - a_r|^{\alpha_r}} \prod_{\text{Complex roots}} \left\{ (x - a_{cr})^2 + a_{ci}^2 \right\}^{\gamma(a_c)/2} \\
 & \prod_{\text{Complex roots}} \exp \left[\left\{ \frac{\delta(a_c) + \gamma(a_c)a_{cr}}{a_{ci}} \right\} \text{Arctan} \left(\frac{x - a_{cr}}{a_{ci}} \right) \right] \\
 & = \exp(c - t) . \tag{14}
 \end{aligned}$$

Eq. (14) represents an integral of Eq. (1). For a polynomial $F(x)$ of low degree, solution of the algebraic equation (14) may be convenient. For polynomials of higher degree, solution of the transcendental equation (14) may be no more convenient than direct numerical integration of Eq. (1), say by the Runge-Kutta method. In either case, however, Eq. (14) yields information on the asymptotic solutions. It cannot be used to specify the asymptotic character of all solutions of Eq. (1). However, it does specify the asymptotic character of those solutions of Eq. (1) which 1) are not singular solutions and 2) exist, i.e. remain bounded, as $t \rightarrow \infty$, or as $t \rightarrow -\infty$.

For, the right side of Eq. (14) vanishes as $t \rightarrow \infty$ and increases beyond all bounds as $t \rightarrow -\infty$. All of the coefficients α_r are positive. If x remains finite, the factors in Eq. (14) arising from complex roots can neither vanish nor become infinite. The left side of Eq. (14) can

vanish as $t \rightarrow \infty$ only if $x \rightarrow a_r$ where a_r is one of the right hand nodes.

Similarly the left side of Eq. (14) can become infinite as $t \rightarrow -\infty$ only if $x \rightarrow a_r$ where a_r is one of the left hand nodes. Thus any non-singular solution which remains finite for $t \rightarrow \pm \infty$ must approach one of the nodes. Stated fully, it has been shown that:

If the right side of Eq. (1) represents a polynomial in the real variable x , with real coefficients, in which all roots of the polynomial are distinct, then those non-singular solutions of Eq. (1) which remain bounded as $t \rightarrow \infty$ approach asymptotically one of the right hand nodes, while those non-singular solutions of Eq. (1) which remain bounded as $t \rightarrow -\infty$ approach asymptotically one of the left hand nodes.

3. PROPERTIES OF TRAJECTORIES

It is convenient to number the nodes in the order of their magnitude. Thus a transformation of the form of (3) will be made where a_r is the lowest node, and the remaining nodes are numbered so that

$$0 < a_1 < a_2 \cdots < a_m . \quad (15)$$

It is easily shown that the nodes are alternately right hand and left hand. For if a_r in Eq. (5) is a_j , then in virtue of (15), the sign of Eq. (5) is given by

$$(-1)^{m - j + 1} . \quad (16)$$

In obtaining the result of Eq. (16) use has been made of the fact that the complex roots may be ignored since $(a_r - a_c)(a_r - a_c^*)$ is always positive.

From (16), the highest node a_m is a right hand node; and the remaining nodes are alternately right and left hand.

The term trajectory will be used to refer to a non-singular solution of Eq. (1), which may be characterized by the fact that it has at least one point not on a node. In order for a trajectory to cross a node it must also have at least one point on the node. However, at the point on the node it cannot be true that $dx/dt = 0$, as for the usual crossing of a line of constant x , since the nodes are, by definition, values of x for which $dx/dt = 0$. If a solution of Eq. (1) has a point on a node, with $dx/dt = 0$, then since dx/dt changes sign at the node, the only possible crossing is by means of a cusp. But this would imply a function $x(t)$ which is double valued in the neighborhood of the cusp and would therefore contradict the uniqueness theorem for solutions¹ of Eq. (1). Consequently, it is not possible for a trajectory to cross a node.

Therefore, if a trajectory has one point between a pair of nodes, it is bounded by the nodes. Applying the result of the theorem of section 2, a trajectory with one point between a pair of nodes must approach the right hand node asymptotically for $t \rightarrow \infty$, and must approach the left hand node asymptotically for $t \rightarrow -\infty$.

At all points in the region between a pair of nodes the slope dx/dt has the same sign. It is easily verified that this sign is given by Eq. (16) where a_j is the node below the region. For the region above

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¹Since the derivative of the right side of Eq. (1) exists and is continuous, it satisfies a Lipschitz condition.

the highest node $dx/dt < 0$. If a trajectory has one point in this region, it cannot cross below the node. On the other hand, since $dx/dt < 0$, x decreases monotonically with t . If (t_0, x_0) is the point originally considered on the trajectory then points on the trajectory for $t > t_0$ are bound by the relation

$$a_m < x(t) < x_0 . \quad (17)$$

Applying the result of the theorem, the trajectory must approach asymptotically the node a_m as $t \rightarrow \infty$. It should be noted that no upper bound on this type of trajectory exists for the half-line $t < t_0$.

In the region below the lowest node, dx/dt has the sign of $(-1)^m$. For an odd number of nodes (m even), $dx/dt > 0$ in this region. A trajectory with one point (t_0, x_0) in the region increases monotonically with t , and for $t > t_0$ is bound by the relation

$$x_0 < x(t) < 0 . \quad (18)$$

Consequently it must approach asymptotically the node $x = 0$ for $t \rightarrow \infty$. For this type of trajectory there is no lower bound on $x(t)$ on the half-line $t < t_0$.

For an even number of nodes (m odd) $dx/dt < 0$ in the region $x < 0$,

and x increases monotonically with $-t$. In this case a trajectory passing through (t_0, x_0) is bound by the relation

$$x_0 < x(t) < 0, \quad (19)$$

for $t < t_0$. Consequently the trajectory must approach asymptotically the node $x = 0$ for $t \rightarrow -\infty$. In this case there is no lower bound on $x(t)$ on the half-line $t > t_0$.

It has now been shown that solutions of Eq. (1) which pass through each point of the t, x plane not on one of the nodes are asymptotic to one of the nodes either for $t \rightarrow \infty$ or for $t \rightarrow -\infty$, or both. The only other solutions are the nodes themselves. Therefore, there are no periodic solutions.²

Trajectories in the case of three nodes are sketched in Fig. 1. The complexity of the trajectory depends on the number of complex roots a_c . Because of the usual property [1] that Eq. (1) is unchanged by a change of the origin of t , trajectories in a given region have the same shape, being merely displaced from one another along the t -axis. For three, or any odd number of nodes, all trajectories are asymptotic to one

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2. It should be noted that neither the conclusion of the theorem alone, nor the constant sign of dx/dt in a region is adequate to assure this. The presence of nodes which the trajectory cannot cross is essential. For example, the equation

$$dx/dt = x^2 + 1,$$

violates the conditions of the theorem only by having no real roots, and possesses the periodic solution $x = \tan t$.

of the right-hand nodes for $t \rightarrow \infty$.

The theorem proved in section 2 may be regarded as a covering theorem for solutions of Eq. (1). The neighborhood of each node is seen to extend to the two nodes on each side. This is possible only because adjacent nodes are of the opposite character, right hand or left hand. The neighborhood of the highest node extends to $x = \infty$. The neighborhood of the lowest node extends to $x = -\infty$. The presence of a real root in Eq. (1), even as an apparently minor correction to a physical problem, implies asymptotic stability.

It should be pointed out that these results can also be obtained by an application of Liapunov's second method. For a right-hand node a_r , the Liapunov function

$$V = (x - a_r)^2, \quad (20)$$

may be used to show asymptotic stability over the region bounded by, but not including, the left-hand nodes on each side. The same function may be used for the left-hand nodes, after making the transformation $t = -\tau$ in Eq. (1).

The technique used above of seeking information on stability properties from an implicit integral of the differential equations is also applicable

to a limited class of multidimensional equations. For example, the equations

$$\begin{aligned} dx/dt &= f(x,y) , \\ dy/dt &= g(x,y) , \end{aligned} \tag{21}$$

yield a first integral if

$$\partial \left\{ f(x,y) \right\}^{-1} / \partial y = \partial \left\{ g(x,y) \right\}^{-1} / \partial x . \tag{22}$$

Extending this line of reasoning to more dimensions, an implicit first integral can be found if the equations can be formed into an exact differential.

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FIGURE CAPTION

Fig. 1. Trajectories in the case of three nodes.