TRANSCENDENTAL FUNCTIONS GENERALIZING THE
EXPONENTIAL INTEGRALS

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Transcendental Functions Generalizing the Exponential Integrals*

I. K. Abu-Shumays

ABSTRACT

A new set of generalized exponential integrals $E^{(m)}(x, v)$ which arise in linear transport theory are defined, and their asymptotic and series expansions are developed. The special cases $E^{(m)}(x, n)$, n positive integer, are shown to correspond to the functions $E_n^{(m)}(x)$ as defined by C. Kaplan, H. C. Van de Hulst and others. Considerable simplification results for the series development for $E_n^{(m)}(x)$. The connections with the incomplete gamma, the gamma, the psi, the polygamma and the Riemann zeta functions are noted.

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1. INTRODUCTION

The exponential integrals

$$E_n(x) = x^{n-1} \int_x^\infty e^{-t} \frac{dt}{t^n} = \int_1^\infty e^{-xt} \frac{dt}{t^n} = \int_0^1 e^{-x/\mu} \mu^{n-2} d\mu \quad (1.1)$$

and related functions have received considerable attention in the literature [1-20]. Computer codes are currently available [4, 5; also W. Gautschi, private communication] for their accurate numerical evaluation.

The generalized exponential integral $E_1^{(2)}(x)$ appears frequently in radiative transfer problems [13, 17]. More recently, C. Kaplan [14, 15] has encountered the generalized exponential integrals

$$E_n^{(m)}(x) = x^{n-1} \int_x^\infty E_n^{(m-1)}(t) \frac{dt}{t^n} = \int_1^\infty E_n^{(m-1)}(xt) \frac{dt}{t^n} \quad (1.2)$$

and related integrals in a problem involving the Oseen version of the Navier-Stokes equations. M. D. Chipman [3] encountered related functions in a problem in molecular quantum mechanics. Such functions and their generalizations below, also arise in reactor problems [1, also work in progress].

Kaplan has devoted considerable attention to developing the properties of the generalized exponential integrals $E_n^{(m)}(x)$ and their series expansions [14-16]. Through a lengthy procedure, he develops an algorithm for the derivation of the series expansion for $E_n^{(1)}(x)$, $E_n^{(2)}(x)$, $E_n^{(3)}(x)$, $E_n^{(4)}(x)$, etc. He notes that the derivation of these series becomes increasingly complex with increasing values of m . One step in the algorithm of Kaplan calls for replacing the Riemann zeta function $(-1)^t \zeta(t)$ in the expression for $\Gamma^{(m)}(1)$ by $\left[\sum_{t=1}^{n-1} t^{-t} + (-1)^t \zeta(t) \right]$. In an attempt to duplicate some of Kaplan's results the present author noted an ambiguity. Two equivalent representations for $\Gamma^{(m)}(1)$, $m > 3$, were derived, one involving

$\zeta^2(2) = \pi^4/36$, the other involving instead a linear combination of $\zeta^2(2)$ and $\zeta(4) = \pi^4/90$. Since $\zeta^2(2) = 2.5\zeta(4)$, it is essential that the user knows which representation for $\Gamma^{(m)}(1)$ to accept in order to correctly apply the algorithm.

Kaplan's work [14-16] provides the first and only extensive and self-contained treatment of the generalized exponential integrals. The present extension of the work of Kaplan leads to a simplified and unambiguous way to derive the desired series representations.

2. TRANSCENDENTAL FUNCTIONS AND THEIR PROPERTIES

The basis for the present paper is the following observation. The exponential integrals $E_n(x)$ are related to the incomplete gamma function [8, 9, 12, 19]

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt \quad (2.1)$$

by the expression

$$E_n(x) = E_n^{(1)}(x) = x^{n-1} \int_x^\infty e^{-t} \frac{dt}{t^n} = x^{n-1} \Gamma(1-n, x) \quad (2.2)$$

The fact that for non-integral α , $\Gamma(\alpha, x)$ is a multiple-valued function of x with a branch point at $x = 0$ causes no difficulty. In the applications envisioned here, x is restricted to the interval $0 < x < \infty$ (or to the x -plane cut along the negative real axis). For fixed x in this interval, $\Gamma(\alpha, x)$ is an entire function of α . It follows that replacing the integer n in (2.2) by a variable ν , the new function

$$E(x, \nu) \equiv E_\nu(x) \equiv x^{\nu-1} \int_x^\infty e^{-t} \frac{dt}{t^\nu} = x^{\nu-1} \Gamma(1-\nu, x) \quad (2.3a)$$

is also an entire function of ν for fixed $x > 0$. Furthermore all partial derivatives of $E(x, \nu)$ with respect to ν exist and are entire functions in ν for fixed x . In particular, it is convenient to introduce the notation

$$E^{(1)}(x, \nu) \equiv E(x, \nu); \quad E^{(m+1)}(x, \nu) \equiv \frac{(-1)^m}{m!} \frac{\partial^m}{\partial \nu^m} E(x, \nu) \quad (2.4)$$

Rather than differentiate (2.3a) to get $E^{(m+1)}(x, \nu)$, it is convenient to differentiate the equivalent definitions

$$E(x, \nu) = \int_1^\infty e^{-xt} \frac{dt}{t^\nu} = \int_0^1 e^{-x/\mu} \mu^{\nu-2} d\mu; \quad (2.3b)$$

this leads to the compact expressions

$$E^{(m+1)}(x, \nu) = \frac{1}{m!} \int_1^\infty e^{-xt} \log^m t \frac{dt}{t^\nu} = \frac{1}{m!} \int_0^1 e^{-x/\mu} \log^m \left(\frac{1}{\mu}\right) \mu^{\nu-2} d\mu = \frac{x^{\nu-1}}{m!} \int_x^\infty e^{-t} \log^m \left(\frac{t}{x}\right) \frac{dt}{t^\nu} \quad (2.5)$$

The functions $E^{(m)}(x, \nu)$ of (2.4) and (2.5) will henceforth be called "generalized exponential integrals of order m ". The following recurrence relations can be derived directly from (2.4) and (2.5)

$$\frac{\partial}{\partial \nu} E^{(m)}(x, \nu) = -m E^{(m+1)}(x, \nu), \quad (2.6)$$

$$\frac{\partial}{\partial x} E^{(m)}(x, \nu) = -E^{(m)}(x, \nu-1). \quad (2.7)$$

Furthermore, by integration by parts and admissible interchanges of integrations it can be shown from (2.5) that

$$\nu E^{(m+1)}(x, \nu+1) = E^{(m)}(x, \nu+1) - x E^{(m+1)}(x, \nu), \quad (2.8)$$

and

$$\int_1^\infty E^{(m)}(xt, \nu) \frac{dt}{t^\nu} = \frac{1}{(m-1)!} \int_1^\infty \frac{dt}{t^\nu} \int_1^\infty \frac{ds}{s^\nu} e^{-stx} \log^{m-1} s = \frac{1}{m!} \int_1^\infty \frac{ds}{s^\nu} e^{-sx} \log^m s,$$

or simply,

$$E^{(m+1)}(x, \nu) = \int_1^\infty E^{(m)}(xt, \nu) \frac{dt}{t^\nu}. \quad (2.9)$$

These recurrence relations (2.8)-(2.9) hold for $m > 0$ and can be extended to the case $m = 0$ by defining

$$E^{(0)}(x, \nu) \equiv e^{-x}, \quad \text{for all } \nu \quad (2.10)$$

The details in the derivation of (2.9) are given in order to complete the link between the present work and that of Kaplan [14-16]. In fact for $\nu = n$ a positive integer, (2.9) coincides with the definition of the exponential integral of order $(m+1)$ as given by Kaplan

$$E^{(m+1)}(x, n) = E_n^{(m+1)}(x) \equiv \int_1^\infty E_n^{(m)}(xt) \frac{dt}{t^n} \quad (2.11)$$

Thus while Kaplan defines the exponential integrals of order m in terms of integrals of lower order, the new expression (2.5) above has the advantage of further generalizing this definition and expressing it in terms of integrals of elementary functions. In fact our work on neutron transport theory (research in progress) led directly to integrals of the form (2.5) which then were identified as generalized exponential integrals..

Note that the recurrence relations (2.6)-(2.10) also hold for ν a positive integer. In addition, (2.8) and (2.10) imply that

$$E^{(m+1)}(x,0) = - \frac{E^{(m)}(x,1)}{x} \quad (2.12)$$

$$E^{(m+1)}(0,\nu+1) = \frac{E^{(m)}(0,\nu+1)}{\nu} = \frac{1}{\nu^{m+1}}, \quad \nu \neq 0 \quad (2.13)$$

These results (2.12)-(2.13) reduce to the results of Kaplan when $\nu = n$.

3. ASYMPTOTIC EXPANSIONS

Several inequalities for $E_n(x)$ are already available in the literature [9, p. 229] and these can be easily extended to cover the transcendental functions $E^{(m)}(x, \nu)$ with x, ν real and $x \geq 0$, $\nu \geq 0$. Of interest here are

$$E(x, \nu+1) \leq e^{-x}/\nu ; \quad E(x, \nu) \leq e^{-x}/x \quad (3.1)$$

and their generalization

$$E^{(m+1)}(x, \nu+1) \leq E^{(m)}(x, \nu+1)/\nu ; \quad E^{(m+1)}(x, \nu) \leq E^{(m)}(x, \nu+1)/x . \quad (3.2)$$

Strict equality in the first part of (3.1), (3.2) holds for $x = 0$, and in the second part holds for $\nu = 0$. These inequalities follow directly from (2.8), (2.10) and the fact that, by definition (2.5), $E^{(m)}(x, \nu) > 0$ for $x \geq 0$, $\nu \geq 0$.

Rearranging (2.8) and repeated application of this recurrence relation yields

$$\begin{aligned} E^{(m+1)}(x, \nu) &= \frac{1}{x} E^{(m)}(x, \nu+1) - \frac{\nu}{x} E^{(m+1)}(x, \nu+1) \\ &= \frac{1}{x} E^{(m)}(x, \nu+1) - \frac{\nu}{x} E^{(m)}(x, \nu+2) + \dots \\ &\quad + \frac{(-1)^k \nu(\nu+1) \dots (\nu+k-1)}{x^{k+1}} E^{(m)}(x, \nu+k+1) \\ &\quad - \frac{(-1)^k \nu(\nu+1) \dots (\nu+k)}{x^{k+1}} E^{(m+1)}(x, \nu+k+1) \quad (3.3) \end{aligned}$$

Note that the last term in (3.3) approaches zero as x approaches infinity.

In fact, it follows from the inequalities (3.2), that if the last term in (3.3) is neglected, the result on the right-hand side is an asymptotic expansion for $E^{(m+1)}(x, \nu)$ in terms of lower order exponential integrals $E^{(m)}(x, \nu+l)$. In particular, the asymptotic series for $E(x, \nu)$ is (see (3.2) and (2.13))

$$E(x, \nu) \sim \frac{e^{-x}}{x} \left[1 - \frac{\nu}{x} + \frac{\nu(\nu+1)}{x^2} + \dots + \frac{(-1)^k \nu(\nu+1) \dots (\nu+k-1)}{x^k} \right] \quad (3.4).$$

As expected from the theory of asymptotic expansions, the error in (3.4) is smaller in absolute value than the first term neglected. Thus the asymptotic series for $E^{(m+1)}(x, \nu)$ can be evaluated recursively starting from (3.4) and (3.3).

An alternate and more attractive procedure to derive the asymptotic series for $E^{(m)}(x, \nu)$ in terms of elementary functions, is to apply the definition (2.4) and differentiate (3.4) directly. Thus

$$E^{(2)}(x, \nu) = -\frac{\partial}{\partial \nu} E(x, \nu) \sim \frac{e^{-x}}{x^2} \left[1 - \frac{2\nu+1}{x} + \frac{3\nu^2+6\nu+2}{x^2} - \frac{4\nu^3+18\nu^2+22\nu+6}{x^3} + \dots \right] \quad (3.5)$$

$$E^{(3)}(x, \nu) = \frac{1}{2} \frac{\partial^2}{\partial \nu^2} E(x, \nu) \sim \frac{e^{-x}}{x^3} \left[1 - \frac{3\nu+3}{x} + \frac{6\nu^2+18\nu+11}{x^2} - \dots \right], \quad (3.6)$$

and clearly

$$E^{(m+1)}(x, \nu) = \frac{(-1)^m}{m!} \frac{\partial^m}{\partial \nu^m} E(x, \nu) \sim \frac{e^{-x}}{x^{m+1}} \left[1 - \frac{(m+1)\nu+m(m+1)/2}{x} + \dots \right]. \quad (3.7)$$

The expressions (3.5), (3.6) with ν a positive integer $\nu = n$, correspond to those given by Kaplan [16]. These expressions, together with (3.7) are useful only for sufficiently large x [16]; the larger the value of x , the more rapid the convergence. For small x , the complete series expansion would be more useful for numerical computations.

4. SERIES EXPANSIONS FOR $E(x, 1+\nu)$ AND $E^{(m)}(x, 1)$

Development of a series expansion for $E^{(m)}(x, 1)$, is simpler, though illustrative of the general case for $E^{(m)}(x, n)$. By definition (2.3a)

$$E(x, 1+\nu) = x^\nu \int_x^\infty e^{-t} \frac{dt}{t^{1+\nu}} = x^\nu \int_\epsilon^\infty e^{-t} \frac{dt}{t^{1+\nu}} - x^\nu \int_\epsilon^x e^{-t} \frac{dt}{t^{1+\nu}}. \quad (4.1)$$

Consider the restrictions $\epsilon > 0$ and arbitrarily small, and $-1 < \nu < 1$.

Integrating the first term on the right-hand side of (4.1) by parts, expanding e^{-t} in the second term in a power series in t and integrating term by term yields

$$E(x, 1+\nu) = \frac{1}{\nu} (1 - x^\nu \int_\epsilon^\infty e^{-t} \frac{dt}{t^\nu}) - \sum_{\ell=1}^{\infty} \frac{(-x)^\ell}{(\ell - \nu)\ell!} + O(\epsilon), \quad (4.2)$$

and in the limit $\epsilon \rightarrow 0$,

$$E(x, 1+\nu) = (1 - x^\nu \Gamma(1-\nu))/\nu - \sum_{\ell=1}^{\infty} \frac{(-x)^\ell}{(\ell - \nu)\ell!}, \quad -1 < \nu < 1. \quad (4.3)$$

An alternate expression can be deduced from (4.3) and the product of the series

$$x^\nu = e^{\nu \log x} = \sum_{k=0}^{\infty} (\nu \log x)^k \quad (\text{power series}) \quad (4.4)$$

$$\Gamma(1 - \nu) = \sum_{k=0}^{\infty} \frac{(-\nu)^k}{k!} \Gamma^{(k)}(1) \quad (\text{Taylor's series}), \quad (4.5)$$

and is

$$E(x, 1+\nu) = \sum_{m=0}^{\infty} \nu^m \sum_{\ell=0}^{m+1} \frac{(-1)^{\ell+1} \log^{m+1-\ell} x \Gamma^{(\ell)}(1)}{(m+1-\ell)\ell!} - \sum_{\ell=1}^{\infty} \frac{(-x)^\ell}{(\ell - \nu)\ell!}. \quad (4.6)$$

It is now straightforward to differentiate $E(x, 1+\nu)$ with respect to ν and set $\nu = 0$ to get

$$\begin{aligned}
 E^{(m+1)}(x, 1) &= \frac{(-1)^m}{m!} \frac{\partial^m}{\partial v^m} E(x, 1+v) \Big|_{v=0} \\
 &= \sum_{\ell=0}^{m+1} \frac{(-1)^{m+1-\ell} \log^{m+1-\ell} x \Gamma^{(\ell)}(1)}{(m+1-\ell)! \ell!} \\
 &= (-1)^m \sum_{\ell=1}^{\infty} \frac{(-x)^{\ell}}{\ell^{m+1} \cdot \ell!} .
 \end{aligned} \tag{4.7}$$

Equation (4.7) is the series expansion sought. However it is convenient for computational purposes to express $\Gamma^{(\ell)}(1)$, the derivatives of the gamma function, in terms of the psi ($\psi(x)$, digamma), the polygamma functions ($\psi^{(n)}(x)$) and the Riemann zeta function ($\zeta(x)$), which are extensively tabulated in the literature [6-9]. Contrary to what Kaplan asserts [16] the expressions for $\Gamma^{(m)}(1)$ do conform to a rule. Starting from the definition for the psi function $\psi(x) = \Gamma'(x)/\Gamma(x)$ and expanding in a Taylor series as in [19] we get

$$\Gamma^{(1)}(1+x) = \Gamma(1+x) \cdot \psi(1+x) \tag{4.8}$$

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{\Gamma^{(k+1)}(1)}{k!} x^k &= \sum_{\ell=0}^{\infty} \frac{\Gamma^{(\ell)}(1)}{\ell!} x^{\ell} \cdot \sum_{m=0}^{\infty} \frac{\Gamma^{(m)}(1)}{m!} x^m \\
 &= \sum_{k=0}^{\infty} x^k \left[\sum_{\ell=0}^k \frac{\Gamma^{(\ell)}(1)}{\ell!} \frac{\Gamma^{(k-\ell)}(1)}{(k-\ell)!} \right] .
 \end{aligned}$$

Equating the coefficients of the same power of x in the above series yields the recurrence relation

$$\Gamma^{(k+1)}(1) = \sum_{\ell=0}^k \frac{k!}{\ell! (k-\ell)!} \Gamma^{(\ell)}(1) \psi^{(k-\ell)}(1) . \tag{4.9a}$$

Further, it is known that

$$\Gamma^{(1)}(1) = \psi(1) = -\gamma ; \tag{4.9b}$$

where $\gamma = .57721 \dots$ is Euler's constant.

Equations (4.9) are useful for evaluating $\Gamma^{(m)}(1)$ recursively in terms of known values of the polygamma functions $\psi^{(\ell)}(1)$. On the other hand, these polygamma functions are related to the zeta functions by

$$\psi^{(\ell)}(1) = (-1)^{\ell+1} \ell! \zeta(\ell+1) \quad (4.10)$$

where

$$\zeta(1) \equiv \gamma; \quad \zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^n}, \quad n > 1. \quad (4.11)$$

Numerous computer installations have programs for the accurate evaluation of the zeta function.

While the above relations are useful for computing $\Gamma^{(m)}(1)$ recursively, the following direct expression in terms of the Riemann zeta functions can be deduced directly from Nielsen [19, I p. 43].

$$\Gamma^{(m)}(1) = (-1)^m \sum_{k=1}^m \frac{m!}{k!} \sum \frac{\zeta(\ell_1) \zeta(\ell_2) \cdots \zeta(\ell_k)}{\ell_1 \ell_2 \cdots \ell_k}. \quad (4.12a)$$

Here the second summation on the right-hand side of (4.12a) is over all combinations of positive integers

$$\ell_1 + \ell_2 + \cdots + \ell_k = m, \quad \ell_i \geq 1. \quad (4.12b)$$

A few examples are

$$\Gamma^{(1)}(1) = -\zeta(1) \quad (4.13a)$$

$$\Gamma^{(2)}(1) = \zeta(2) + \zeta^2(1) \quad (4.13b)$$

$$\Gamma^{(3)}(1) = -2\zeta(3) - 3\zeta(1)\zeta(2) - \zeta^3(1) \quad (4.13c)$$

The above results (4.9)-(4.13) make it possible to verify the series expansions for $E^{(m+1)}(x,1) = E_1^{(m+1)}(x)$ already available in the literature [14-16].

5. SERIES EXPANSIONS FOR $E(x, n+v)$ AND $E^{(m)}(x, n)$

Following steps similar to (4.1)-(4.3) it can be shown that

$$E(x, n+v) = \frac{(-x)^{n-1}}{(n-1)!} \left[\frac{1}{v} - \frac{x^v \Gamma(1-v) \Gamma(v) \Gamma(n)}{\Gamma(n+v)} \right] + \sum_{\substack{\ell=0 \\ \ell \neq n-1}}^{\infty} \frac{(-x)^\ell}{(n+v-\ell-1) \cdot \ell!} \quad (5.1)$$

Furthermore, it follows from the definition (2.4) that

$$E^{(m+1)}(x, n) = \frac{(-1)^m}{m!} \frac{\partial^m}{\partial v^m} E(x, n+v) \Big|_{v=0} \quad (5.2)$$

In order to facilitate the differentiation in (5.2), it is desirable to express the first term on the right-hand side of (5.1) in a power series in v . The series for x^v is given in (4.4). Another factor in (5.1) is [7, 8, 12]

$$\Gamma(1-v)\Gamma(v) = \frac{\pi}{\sin \pi v} = \frac{1}{v} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1)}{(2k)!} B_k \pi^{2k} v^{2k-1} \quad (5.3)$$

where B_k are the Bernoulli Numbers [8, I p. 52; 19, I p. 46]

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{r=1}^{\infty} \frac{1}{r^{2n}} = \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \quad (5.4a)$$

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66}, \text{ etc.} \quad (5.4b)$$

The third factor of interest in (5.1) can be expressed in the form

$$\frac{\Gamma(n)}{\Gamma(n+v)} = \sum_{k=0}^{\infty} \gamma_k v^k, \quad \gamma_0 = 1 \quad (5.5)$$

A recurrence relation for γ_k follows from

$$\frac{d}{dv} \frac{\Gamma(n)}{\Gamma(n+v)} = - \frac{\Gamma(n)}{\Gamma(n+v)} \cdot \psi(n+v) \quad (5.6)$$

by expanding $\psi(n+v)$ in a Taylor series in v , substituting (5.5) for the remaining terms and equating coefficients of the same power of v on both sides of the resulting expression, and is

$${}^{(m+1)}\gamma_{m+1} = - \sum_{k=0}^m \frac{\psi^{(k)}(n)}{k!} \gamma_{m-k}, \quad \gamma_0 = 1. \quad (5.7)$$

Again, the polygamma functions $\psi^{(k)}(n)$ are either available directly [6, 7] or can be evaluated in terms of the Riemann zeta function from the expression

$$\psi^{(m)}(n) = (-1)^{m+1} m! \left[\zeta(m+1) - \sum_{k=1}^{n-1} \frac{1}{k^{m+1}} \right], \quad n > 1. \quad (5.8)$$

Now multiplying the series (4.4), (5.3) and (5.5), substituting in (5.1) and simplifying yields

$$E(x, n+v) = \frac{(-x)^{n-1}}{(n-1)!} \left[-(\log x - \psi(n)) - \sum_{m=1}^{\infty} \gamma_m \left\{ A_{m+1}(n, x) + \sum_{k=1}^{\left[\frac{m+1}{2} \right]} \frac{2(2^{2k-1}-1)}{(2k)!} \pi^{2k} B_{k, m+1-2k}(n, x) \right\} \right] \\ - \sum_{\substack{\ell=0 \\ \ell \neq n-1}}^{\infty} \frac{(-x)^{\ell}}{(\ell+1-n-v) \cdot \ell!}, \quad \left[\frac{m+1}{2} \right] \text{ is largest integer } \leq \frac{m+1}{2}, \quad (5.9)$$

where

$$A_m(n, x) \equiv \sum_{k=0}^m (\gamma_{m-k} \log^k x) / k!. \quad (5.10)$$

It is straightforward at this stage to apply (5.2) to (5.9) to express the series for $E^{(m)}(x, n)$. Examples (simplifying and omitting the details) are

$$E(x, n) = - \frac{(-x)^{n-1}}{(n-1)!} [\log x - \psi(n)] - \sum_{\substack{\ell=0 \\ \ell \neq n-1}}^{\infty} \frac{(-x)^{\ell}}{(\ell+1-n) \cdot \ell!}, \quad (5.11)$$

$$E^{(2)}(x, n) = \frac{(-x)^{n-1}}{(n-1)!} \left\{ \frac{1}{2} [\log x - \psi(n)]^2 - \frac{1}{2} \psi^{(1)}(n) + \frac{\pi^2}{6} \right\} + \sum_{\substack{\ell=0 \\ \ell \neq n-1}}^{\infty} \frac{(-x)^{\ell}}{(\ell+1-n)^2 \cdot \ell!}, \quad (5.12)$$

$$E^{(3)}(x, n) = \frac{(-x)^{n-1}}{(n-1)!} \left\{ -\frac{1}{3!} [\log x - \psi(n)]^3 + \left(\frac{1}{2} \psi^{(1)}(n) - \frac{\pi^2}{6} \right) [\log x - \psi(n)] + \frac{1}{6} \psi^{(2)}(n) \right\}$$

$$- \sum_{\substack{\ell=0 \\ \ell \neq n-1}}^{\infty} \frac{(-x)^{\ell}}{(\ell+1-n)^3 \cdot \ell!}, \quad (5.13)$$

$$\begin{aligned}
E^{(4)}(x, n) = & \frac{(-x)^{n-1}}{(n-1)!} \left\{ \frac{1}{4!} [\log x - \psi(n)]^4 + \left(\frac{\pi^2}{12} - \frac{1}{4} \psi^{(1)}(n) \right) [\log x - \psi(n)]^2 \right. \\
& - \frac{1}{6} \psi^{(2)}(n) [\log x - \psi(n)] - \frac{1}{4!} \psi^{(3)}(n) + \frac{1}{8} [\psi^{(1)}(x)]^2 - \frac{\pi^2}{12} \psi^{(1)}(n) \\
& \left. + \frac{7\pi^4}{360} \right\} + \sum_{\substack{\ell=0 \\ \ell \neq n-1}}^{\infty} \frac{(-x)^{\ell}}{(\ell+1-n)^4 \cdot \ell!} \quad (5.14)
\end{aligned}$$

The above examples (5.11)-(5.14) are alternate and can be shown to be equivalent to the corresponding expressions for $E_n^m(x)$, $m = 1, 2, 3, 4$ given by Kaplan [16]. The main contribution of the present paper, however, is the generalization of the exponential integrals and the simplification resulting in the derivation of the asymptotic and series expansions for higher order terms.

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