Relaxation Rate of a Two-Dimensional
Magnetized Plasma Model *

by

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ABSTRACT

The thermal relaxation of a two-dimensional single-species plasma model is investigated. The numerical results in the unmagnetized case are consistent with those of Montgomery and Nielson¹ when the mesh size of the computational grids is equal to a Debye length. It is shown that a smaller mesh size makes the relaxation faster. The secular term² in the kinetic equation arising from the calculation of the energy exchange of a two-particle system along the unperturbed orbit is well interpreted and overcome by the introduction of the drift of the guiding center. For weak magnetic fields, the pair correlation is enhanced and makes the relaxation time shorter than that in the absence of a magnetic field. Calculating the electric field fluctuation based upon the uncorrelated finite gyroradius model, we show the first cyclotron mode³ has maximum at $\Omega \cong 2.5 \omega_{pe}$. When the cyclotron frequency $\Omega \cong 2 \omega_{pe}$, the relaxation is fastest. We conjecture the first cyclotron mode is the mechanism which causes particles to exchange energy most efficiently. In the strongly-magnetized regime, the relaxation time is proportional to $\sqrt{n_o \lambda_D^2}$ and B. In the case of weaker magnetic fields, there is a transition to the unmagnetized regime in which the relaxation time varies as $n_o \lambda_D^2$.

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I. INTRODUCTION

The two-dimensional plasma model ^{2,4} consists of charged rods aligned parallel and interacting through the logarithmic Coulomb potential

$$\varphi_{ij} = - \frac{2e_i e_j}{L} \ln x_{ij} \quad (I-1)$$

Here, the charges e_i , e_j and separation $x_{ij} = |\vec{x}_i - \vec{x}_j|$ in the x-y plane, are referred to rods i, j, respectively. Every rod has length L with the property that e and L approach ∞ , but e/L stays finite. The kinetic theory which starts from the BBGKY hierarchy is most tractable in the limit of low density or weak coupling. For unmagnetized plasmas,⁵ the theories have been partially verified by numerical experiments. A thermal relaxation time defined by computing Boltzmann's H-function is one of the products of the simulations. In a two-dimensional unmagnetized plasma, with a uniform positive background, the relaxation time is linearly proportional to the inverse of the plasma parameter, $\epsilon^{-1} = n_o \lambda_D^2$; n_o is the electron number density; $\lambda_D^2 = LK_B T / 4 \pi n_o e^2$; $K_B T$ is the temperature in energy units.

The same system in the presence of an applied dc magnetic field $\vec{B} = B \hat{z}$ along the charged rods has been studied recently. In

the very strong magnetic field case, the gyroradii become extremely small, and the guiding center model introduced by Taylor and McNamara^{4,5} simplifies the analytical work. It predicts no thermal relaxation, however. Finite gyroradius effects must be included if thermal relaxation is to be obtained. Vahala and Montgomery² derived a single-species Fokker-Planck equation which predicts a relaxation time proportional to the gyrofrequency $\Omega (= eB/mc)$. The derivation was based upon an integration along the unperturbed orbits, which causes a secular behavior. The divergent integral was cut off by a physical argument. Our understanding now is that this cutoff has a dependence on the strength of magnetic field and can be shown to be related to the drift of the guiding center.

Computer simulation by particle-in-cell (abbreviated PIC) techniques was developed several years ago. The details are reviewed elsewhere.⁶ The main idea is to adopt a weighting procedure which assigns the fractions of charges inside an Eulerian cell onto the corners. From the grid charge density, the electric field $\vec{E}$ is determined. The particle positions and velocities can then be advanced. New Lagrangian coordinates of charges are obtained and the computation is repeated.

In Section II we discuss the relevant kinetic theory. In Section III, the energy relaxation and H-theorem are discussed. The details of the computer simulations are described in Section IV, with results in Section V, and interpretations in Section VI. We draw the conclusions in Section VII.

II. KINETIC THEORY OF A TWO-DIMENSIONAL, SINGLE-SPECIES PLASMA

Consider a collection of N charged rods in a volume $V = L\ell^2$ with uniform immobile positive background. The statistical description of the dynamical variables can be made in a $4N$ dimensional space. For the i th charge, let $x_i \equiv (\vec{x}_i, \vec{v}_i)$ be a four-dimensional vector which gives the position and velocity. The constancy of the probability $D_N(x, t)$ with its domain on the $4N$ dimensional phase space is given by Liouville's equation. With magnetic field $B\hat{e}_z$, the trajectory of the i th particle is given as follows:

$$\frac{d\vec{x}_i}{dt} = \vec{v}_i \quad (\text{II-1a})$$

$$\frac{d\vec{v}_i}{dt} = \frac{e}{m} \vec{E}(\vec{x}_i, t) + \vec{v}_i \times \vec{\Omega} \quad (\text{II-1b})$$

where $\vec{\Omega}$ is the vector gyrofrequency, and

$$\vec{E}(\vec{x}_i, t) = - \frac{\partial}{\partial \vec{x}_i} \sum_{j \neq i} \tilde{\varphi}_{ij}, \text{ with } \tilde{\varphi}_{ij} = \varphi_{ij}/e_i \quad (\text{II-2})$$

The Liouville equation becomes

$$\frac{dD_N}{dt} = \left[\frac{\partial}{\partial t} + \sum_i \frac{dx_i}{dt} \frac{\partial}{\partial x_i} \right] D_N = 0$$

or alternatively,

$$\left\{ \frac{\partial}{\partial t} + \sum_i \vec{v}_i \cdot \frac{\partial}{\partial \vec{x}_i} + \sum_i \left[\frac{e_i}{m_i} \vec{E}(\vec{x}_i, t) + \vec{v}_i \times \vec{\Omega} \right] \cdot \frac{\partial}{\partial \vec{v}_i} \right\} D_N = 0 \quad (II-3)$$

It is convenient to define the reduced probability distribution for s particles in the phase space regardless of where the other $N-s$ particles are. It can be written as

$$\frac{f_s(x_1, \dots, x_s, t)}{V^s} \equiv \int D_N dx_{s+1} \dots dx_N \quad (II-4)$$

Applying $V^s \int dx_{s+1} \dots dx_N$ to Eq. (II-3) gives

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \sum_{j=1}^s \left(\vec{v}_j \cdot \frac{\partial}{\partial \vec{x}_j} + \vec{v}_j \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_j} \right) - \frac{1}{m} \sum_{j<i}^s \frac{\partial \varphi_{ij}}{\partial \vec{x}_i} \cdot \left(\frac{\partial}{\partial \vec{v}_i} - \frac{\partial}{\partial \vec{v}_j} \right) \right] f_s \\ &= \frac{N-s}{mV} \sum_{i=1}^s \int dx_{s+1} \frac{\partial \varphi_{i,s+1}}{\partial \vec{x}_i} \cdot \frac{\partial f_{s+1}}{\partial \vec{v}_i} \quad (II-5) \end{aligned}$$

Equations (II-5), for $s = 1, 2, \dots$, are called the BBGKY hierarchy^{7,8} of equations. One may pass to the limit of large N and V such that $\frac{N-s}{V} \rightarrow n_0 = \text{constant}$.

Suppose there exists only one species, subject to (II-5), the hierarchy equations for $s = 1, 2$ can be written as

$$\left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_1 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_1} \right) f_1(\vec{x}_1, \vec{v}_1, t) = \frac{n_0}{m} \int d\vec{x}_2 \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial f_2}{\partial \vec{v}_1} \quad (\text{II-6})$$

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{x}_2} + \vec{v}_1 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_1} + \vec{v}_2 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_2} \right. \\ & \quad \left. - \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} - \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{v}_2} \right] f_2 \\ & = \frac{n_0}{m} \int d\vec{x}_3 \frac{\partial \varphi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial f_3}{\partial \vec{v}_1} + \frac{n_0}{m} \int d\vec{x}_2 \frac{\partial \varphi_{23}}{\partial \vec{x}_2} \cdot \frac{\partial f_3}{\partial \vec{v}_2} \quad (\text{II-7}) \end{aligned}$$

It is useful to define a hierarchy of correlations by the following:

$$\begin{aligned}
 f_1(1) &= f_1(1) \\
 f_2(1,2) &= f_1(1)f_1(2) + p(1,2) \\
 f_3(1,2,3) &= f_1(1)f_1(2)f_1(3) + p(2,3)f_1(1) + p(1,3)f_1(2) \\
 &\quad + p(1,2)f_1(3) + T(1,2,3) \quad . \quad (II-8)
 \end{aligned}$$

The hope is that the shielding of the Coulomb force weakens the interaction between particles so that the correlation functions $p, T, \dots$ are in some sense "small" or "unimportant".

Substituting (II-8) into (II-6) and (II-7), we obtain

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_1 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_1} \right) f_1 &= \frac{n_o}{m} \int d\vec{x}_2 \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial f_1(1)}{\partial \vec{v}_1} f_1(2) \\
 &\quad + \frac{n_o}{m} \int d\vec{x}_2 \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} p(1,2) \quad (II-9)
 \end{aligned}$$

$$\begin{aligned}
& \left[\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{x}_2} + \vec{v}_1 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_1} + \vec{v}_2 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_2} \right. \\
& \quad \left. - \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) \right] p(1,2) \\
& = \frac{n_o}{m} \int d\vec{x}_3 \frac{\partial \varphi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} [f_1(1)p(2,3) + f_1(2)p(1,3) + f_1(3)p(1,2)] \\
& \quad + \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_1(1)f_1(2) + \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{v}_2} f_1(1)f_1(2) \quad (II-10)
\end{aligned}$$

In obtaining Eq. (II-10), we neglect the contribution from the ternary correlation and we make use of Eq. (II-9).

A. Urmagnetized Case

If no magnetic field exists and the system is uniform in space, the key equations are

$$\frac{\partial}{\partial t} f_1(\vec{v}_1, t) = \frac{n_o}{m} \int d\vec{x}_2 \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} p(1,2) \quad (II-11)$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{x}_2} \right) p(1,2) &= \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_1} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) f_1(1) f_1(2) \\ &+ \frac{n_0}{m} \int d\vec{x}_3 \frac{\partial \phi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_1(1) p(2,3) + \frac{n_0}{m} \int d\vec{x}_3 \frac{\partial \phi_{23}}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{v}_2} f_1(2) p(1,3) \end{aligned} \quad (\text{II-12})$$

where we assume one body distribution function is spatially uniform. The last two terms of (II-12) are negligible for weak, short range forces. If we neglect them, we get the Fokker-Planck equation. If we included these terms, and treated the force as weak and long range, we would obtain the Balescu-Lenard equation. The standard way of solving Eq. (II-12) is described in many places (e.g., reference 7).

Here, we derive the Fokker-Planck equation from (II-12), neglecting the last two terms on the RHS. Assume

$$\begin{aligned} \vec{x}_1(\tau) &= \vec{x}_1 + \vec{v}_1(\tau - t), \quad \vec{v}_1(\tau) = \vec{v}_1 \\ \vec{x}_2(\tau) &= \vec{x}_2 + \vec{v}_2(\tau - t), \quad \vec{v}_2(\tau) = \vec{v}_2 \end{aligned} \quad (\text{II-13})$$

and use the Fourier transform of the interaction potential

$$\frac{\partial \phi_{12}}{\partial \vec{x}_1} = \int \frac{i\vec{K}}{K^2} e^{i\vec{K} \cdot \vec{x}_{12}} \frac{e^2}{\pi L} d^2\vec{K} \quad (\text{II-14})$$

In solving Eq. (II-12), we essentially calculate the energy exchange between two particles along the free flight orbits and end up with the pair correlation function:

$$p(1,2) = \frac{e^2}{\pi L m} \int \frac{i\vec{k}}{k^2} e^{i\vec{k} \cdot \vec{x}_{12}} \frac{1 - e^{-i\vec{k} \cdot \vec{v}_{12} t}}{i\vec{k} \cdot \vec{v}_{12}} d^2\vec{k} \cdot \frac{\partial}{\partial \vec{v}_1} f_1(1) f_1(2) + (1 \leftrightarrow 2) \quad (\text{II-15})$$

Noting that

$$\lim_{t \rightarrow \infty} \frac{1 - e^{-i\vec{k} \cdot \vec{v}_{12} t}}{\vec{k} \cdot (\vec{v}_{12})} = \pi i \delta^+(\vec{k} \cdot \vec{v}_{12}) ,$$

the evolution of the one body distribution function is then straight forward.

$$\frac{\partial f_1}{\partial t} = \frac{\partial}{\partial \vec{v}_1} \cdot \int d\vec{v}_2 \vec{\Phi}(\vec{v}_1 \cdot \vec{v}_2) \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) f_1(1) f_1(2) \quad (\text{II-16})$$

Here, the dyadic $\vec{\Phi}$ is defined as

$$\vec{\Phi}(\vec{v}_1 \cdot \vec{v}_2) = - \frac{4\pi n_0 e^4}{L^2 m^2} \int d^2\vec{k} \frac{\vec{k}\vec{k}}{k^4} \delta(\vec{k} \cdot \vec{v}_{12}) \quad (\text{II-17})$$

One can show that the K-integral diverges for small K and therefore we need a cutoff, $K_D \left(\equiv \frac{1}{\lambda_D} \right)$ in the lower limit corresponding to the shielding of the Coulomb force. Then, carrying out the integration

$$\frac{d\mathbf{v}_1}{dt} = - \frac{8 \pi^2 n_o e^4}{m_L^2} \frac{1}{K_D} \frac{\vec{v}_{12} \vec{v}_{12}}{v_{12}^3} \quad (II-18)$$

The above derivation gives the time scale of any dynamical variables which are based upon one body distribution proportional to $n_o \lambda_D^2$.

B. Strongly-Magnetized Plasma

In general, the particles move in the phase space according to Eqs. (II-1). The electric field is that due to all the other particles. Notice that if the one body distribution function is spatially uniform, the first term on the RHS of Eq. (II-11) vanishes. We then split the distribution function into two parts.

$$f_1(\vec{x}_1, \vec{v}_1, t) = f_1^{(0)}(\vec{v}_1^2, t) + f_1^{(1)}(\vec{x}_1, \vec{v}_1, t) \quad (II-19)$$

The material from here through Eq. (II-29) is intended as an heuristic argument to justify a cutoff for the temporary divergent kinetic equation of Vahala and Montgomery². We shall argue that the appropriate cutoff time is that necessary for the gyrocenters of two thermal particles to diffuse through a gyroradius. We assume very

small macroscopic spatial inhomogeneity so that the second term of Eq. (II-19) is much smaller than the first term, and we will be mainly interested in the Fokker-Planck equation of the first term. From Eqs. (II-9) and (II-10), we obtain the following three equations:

$$\frac{\partial}{\partial t} f_1^{(0)}(v_1^2, t) = \frac{n_0}{m} \int dx_2 \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} p(1,2) \quad (\text{II-20})$$

$$\left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_1 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_1} \right) f_1^{(1)} = \frac{n_0}{m} \int dx_2 \frac{\partial \varphi_{12}}{\partial \vec{x}_1} f_1^{(1)}(2) \cdot \frac{\partial f_1^{(0)}}{\partial \vec{v}_1} \quad (\text{II-21})$$

and

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{x}_2} + \vec{v}_1 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_1} + \vec{v}_2 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_2} \right. \\ & \quad \left. - \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) \right] p(1,2) \\ & = \frac{n_0}{m} \int dx_3 \frac{\partial \varphi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} (f_1^{(0)}(1)p(2,3) + f_1^{(1)}(3)p(1,2)) \\ & \quad + \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_1^{(0)}(1)f_1^{(0)}(2) + (1 \leftrightarrow 2) \quad (\text{II-22}) \end{aligned}$$

The second equation [Eq. (II-21)] is recognized as the linear Vlasov equation which takes care of the evolution of the inhomogeneity of the one body distribution function. Equation (II-22) can be simplified to

$$\begin{aligned}
 & \left[\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{x}_2} + \vec{v}_1 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_1} + \vec{v}_2 \times \vec{\Omega} \cdot \frac{\partial}{\partial \vec{v}_2} \right. \\
 & \quad \left. + \frac{e\vec{E}(\vec{x}_1, t)}{m} \cdot \frac{\partial}{\partial \vec{v}_1} + \frac{e\vec{E}(\vec{x}_2, t)}{m} \cdot \frac{\partial}{\partial \vec{v}_2} \right] p(1,2) \\
 & = \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_1^{(0)}(1) f_1^{(0)}(2) + \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{v}_2} f_1^{(0)}(1) f_1^{(0)}(2)
 \end{aligned}
 \tag{II-23}$$

where

$$\vec{E}(\vec{x}_1, t) = -\frac{1}{e} \frac{\partial \phi_{12}}{\partial \vec{x}_1} - \frac{n_0}{e} \int d\vec{x}_3 \frac{\partial \phi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_1^{(1)}(3)$$

is the electric field due to all the other particles in the system.

In Eq. (II-23), we neglect the energy exchange of particle 1 (or 2)

with the background while we keep the characteristics of Eq. (II-23)

in a complete form, which are the Eqs. (II-1) now labeled and 1 and 2.

Vahala and Montgomery² showed that the unperturbed orbit will cause secularity. So we must go one step further to avoid that secular behavior.

We expand the electric field w.r.t. the guiding center.

$$\frac{d\vec{x}_r}{dt} \equiv \vec{v}_r, \quad \frac{d\vec{x}_c}{dt} \equiv \vec{v}_c \equiv \frac{e\vec{\Omega} \times \vec{E}(\vec{x}_c, t)}{m\Omega^2}, \quad \vec{v} \equiv \vec{v}_r + \vec{v}_c. \quad (\text{II-24})$$

Substitute (II-24) into (II-1), we obtain

$$\frac{d\vec{v}_r}{dt} \approx \vec{v}_r \times \vec{\Omega} + \frac{e}{m} \vec{x}_r \cdot \vec{\nabla} \vec{E}(\vec{x}_c, t) - \frac{1}{\Omega} \frac{e}{m} \frac{d\vec{E}(\vec{x}_c, t)}{dt}. \quad (\text{II-25})$$

In case the last two terms are negligible, then

$$\frac{d\vec{v}_r}{dt} \approx \vec{v}_r \times \vec{\Omega} \quad (\text{II-26})$$

and

$$\vec{v} \approx \vec{v}_r + \frac{e}{m} \frac{\vec{\Omega} \times \vec{E}(\vec{x}_c, t)}{\Omega^2}. \quad (\text{II-27})$$

This is essentially assuming that the characteristic frequencies are all less than the gyrofrequency. The picture is now clear. The $\vec{v}_r$ represents the cyclotron motion which is the spirit of the unperturbed finite gyroradius model. In this model, particles keep on stationary orbits and will not separate. This is the cause of secularity. If particle 1 is giving energy to particle 2, since their orbits are fixed, the energy exchanged is proportional to time. The drift of the guiding center provides a natural cutoff to the energy exchange process.

From Eq. (II-23)

$$p(1,2) = \int dt \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_1^{(0)}(1) f_1^{(0)}(2) + (1 \leftrightarrow 2) \quad (\text{II-28})$$

The time integration is now along the circular orbit plus a drift of the guiding center.

The recipe we now want to adopt is to freeze the one body distribution function, calculate the energy exchange along the unperturbed orbit, and cut the time limit at which two particles are drifted away by a "reasonable" distance.

Equation (II-28) becomes

$$p(1,2) = \frac{2e^2}{Im} \int_0^{t^*} dt \frac{\vec{x}_{12}(t) \cdot \vec{v}_1(t)}{x_{12}^2(t)} f_1'^{(0)}(1) f_1^{(0)}(2) \\ (\text{along } \vec{x}_r) + (1 \leftrightarrow 2) \quad (\text{II-29})$$

(t^* is the cutoff time for the Vahala-Montgomery³ kinetic equation). To avoid the difficulty of dealing with the explicit equations of motion of many body systems, we then make use of ensemble average to evaluate $\vec{x}_c^2$ as function of time. The unperturbed trajectory can be written as

$$\vec{v}_1 = v_{10} \cos \Omega t \hat{e}_x + v_{10} \sin \Omega t \hat{e}_y$$

$$\vec{v}_2 = v_{20} \cos (\Omega t + \alpha) \hat{e}_x + v_{20} \sin (\Omega t + \alpha) \hat{e}_y$$

$$\vec{x}_1 = \frac{v_{10}}{\Omega} \sin \Omega t \hat{e}_x - \frac{v_{10}}{\Omega} \cos \Omega t \hat{e}_y$$

$$\vec{x}_2 = \left[A + \frac{v_{20}}{\Omega} \sin(\Omega t + \alpha) \right] \hat{e}_x - \frac{v_{20}}{\Omega} \cos(\Omega t + \alpha) \hat{e}_y \quad . \quad (\text{II-30})$$

Placing Eq. (II-30) into Eq. (II-29) and carrying out the integration, we obtain²

$$p(1,2) = \begin{cases} 0 & x_{12}^2 + \frac{2 \left[\vec{x}_{12} \cdot (\hat{b} \times \vec{v}_{12}) \right]}{\Omega} > 0 \\ -\frac{t^*}{T_0} \frac{4\pi e^2}{\text{Im}} \frac{\vec{v}_1 \cdot (\vec{v}_{12} \times \hat{b})}{v_{12}^2} f_1^{(0)}(1) f_1^{(0)}(2) + (1 \leftrightarrow 2) & x_{12}^2 + \frac{2 \left[\vec{x}_{12} \cdot (\hat{b} \times \vec{v}_{12}) \right]}{\Omega} < 0 \end{cases}$$

$$\text{where } T_0 = \frac{2\pi}{\Omega} \quad .$$

(II-31)

Following Taylor and McNamara's³ derivation of diffusion coefficient D of guiding center plasma, the following is obtained.

$$\langle \vec{x}_c^2 \rangle = 2Dt$$

$$D = \frac{CK_B T}{eB} \left(\frac{\epsilon}{\pi} \right)^{\frac{1}{2}} \left[\ln \left(\frac{\ell}{2\pi\lambda_D} \right) \right]^{\frac{1}{2}}$$

$$\epsilon^{-1} = n_o \lambda_D^2 \quad . \quad (II-32)$$

Here, ℓ is the size of the system, the bracket means ensemble average, the diffusion coefficient has $\frac{1}{B}$ dependence. Equation (II-31) infers that no pair correlation exists when two particle orbits are not overlapping, so that the natural cutoff t^* can be found. We assume

$$t^* \equiv nT_o \approx \frac{r_c^2}{2D} \quad . \quad \text{After a simple calculation, we then end up with}$$

$$t^* = \frac{(\pi/\epsilon)^{\frac{1}{2}}}{\Omega} \left[\ell n \left(\frac{\ell}{2\pi\lambda_D} \right) \right]^{-\frac{1}{2}} \text{ and}$$

$$p(1,2) = -\frac{1}{2} \left[\pi \epsilon \ell n \left(\frac{L}{2\pi\lambda_D} \right) \right]^{-\frac{1}{2}} \frac{4\pi e^2}{Lm} \frac{\vec{v}_1 \cdot \vec{v}_{12} \times \hat{b}}{v_{12}^2} \cdot f_1^{(0)'}(1) f_1^{(0)}(2) + (1 \leftrightarrow 2) \quad (\text{II-33})$$

The Fokker-Planck equation in a strongly magnetized plasma is then

$$\frac{\partial f_1^{(0)}}{\partial t} = -\frac{\partial}{\partial \vec{v}_1} \cdot \int d\vec{v}_2 \vec{\varphi}_s \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) f_1^{(0)}(1) f_1^{(0)}(2) \quad (\text{II-34})$$

where the dyadic $\vec{\varphi}_s$ is given by

$$\vec{\varphi}_s = -n \frac{8\pi^2 n_o e^4}{m^2 L^2 \Omega} \frac{v_{12}^2 \vec{1} - \vec{v}_{12} \vec{v}_{12}}{v_{12}^2} \quad (\text{II-35})$$

with

$$n = \frac{1}{2} \left[\pi \epsilon \ell n \left(\frac{\ell}{2\pi\lambda_D} \right) \right]^{-\frac{1}{2}}$$

A dimensional analysis shows that the time scale is proportional to B and $\sqrt{n \lambda_D^2}$, and also dependent upon the size of the system.

III. RELAXATION TIMES AND H-THEOREM

Consider a "test particle" problem. We assume the test particles have the same mass and charge as the background plasma, but the density is much smaller than the background. The test particle distribution function is f_t ; that of the background is f_b . The evolution of the velocity moments is governed by the Fokker-Planck equation which, in general, can be put in the form

$$\frac{\partial f_t}{\partial t} = - \frac{\partial}{\partial \vec{v}} \cdot (\vec{F}_t(\vec{v}) f_t) + \frac{1}{2} \frac{\partial^2}{\partial \vec{v} \partial \vec{v}} : (\vec{T}_t f_t) \quad (\text{III-1})$$

where

$$\vec{F}_t(\vec{v}) = - 2 \int \vec{\Phi}_{tb} \cdot \frac{\partial f_b}{\partial \vec{v}'} d\vec{v}' \quad (\text{III-2})$$

$$\vec{T}_t(\vec{v}) = - 2 \int \vec{\Phi}_{tb} f_b d\vec{v}' \quad (\text{III-3})$$

Here, $\vec{\Phi}_{tb}$ is the dyadic tensor from the multispecies kinetic equation and takes different forms for unmagnetized and magnetized plasma.

The moments of f_t are defined as

$$\begin{aligned}\vec{u} &\equiv \int \vec{v} f_t d\vec{v} \\ \vec{u}\vec{u} &\equiv \int \vec{v}\vec{v} f_t d\vec{v}\end{aligned}\quad (\text{III-4})$$

etc.

Also, we define the slowing down time as

$$t_s \equiv \left| \frac{\vec{u}}{\partial \vec{u} / \partial t} \right|_{t=0} \quad (\text{III-5})$$

and the energy relaxation time as

$$t_e \equiv \left| \frac{u^2}{\partial u^2 / \partial t} \right|_{t=0} \quad (\text{III-6})$$

The Boltzmann H-function is defined by

$$H = \int d\vec{v} f(\vec{v}, t) \log f(\vec{v}, t) \quad (\text{III-7})$$

The H-theorem⁹ states that if at a given instant t , the state of the particles satisfies the assumption of molecular chaos, then at the instant $t + \epsilon$ ($\epsilon \rightarrow 0$), $\frac{dH}{dt} \leq 0$. $\frac{dH}{dt} = 0$, if and only if $f(\vec{v}, t)$ is the Maxwell-Boltzmann distribution.

It is not difficult to show the kinetic equations we derived satisfy an H-theorem and conservation laws. We here present the slowing down time t_s for the unmagnetized case and the energy relaxation time t_e for the strongly magnetized plasma. The detailed calculation is shown in Appendix 1.

For the interest of comparison with computer simulation, we will write below the functional dependence of t_s and t_e on the strength of magnetic field and the plasma density. We include 2 and 3 dimensional unmagnetized plasma⁵ results for comparison.

$B = 0$ <div style="display: flex; justify-content: space-around; margin-top: 10px;"> 2D 3D </div> $t_s \sim n_o \lambda_D^2, n_o \lambda_D^3 \ln(n_o \lambda_D^3)$	$\text{Strong } B$ $t_e \sim \sqrt{n_o \lambda_D^2}, t_e \sim B$
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IV. COMPUTER SIMULATION

A. Particle-in-cell Model

The numerical computations are performed using the PIC model. A particle simulation promotes, in time, the particles along the characteristic equations

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{e}{m} \vec{E}(\vec{x}, t) + \frac{e\vec{v} \times \vec{B}}{mc} \\ \frac{d\vec{x}}{dt} &= \vec{v}\end{aligned}\tag{IV-1}$$

The distribution function is simply the density of the particles in phase space.

The first step in determining the electric field is to calculate the charge density on the grids. The particle positions are used to calculate the "grid charge density", ρ , by the method of area weighting. A simple illustration results when a particle lies in a cell with corners labeled by A, B, C, D (clockwise). The sides of the cell are normalized to unity with corner A as the origin. Suppose a particle is located at (x, y) , then

$$\rho_A = (1 - x)(1 - y)$$

$$\rho_B = (1 - y)x$$

$$\rho_C = y(1 - x)$$

$$\rho_D = xy \quad . \quad (IV-2)$$

After this kind of calculation, we solve the electric field by Gauss' law using fast Fourier transform (Appendix 2), which restricts us to 32 or 64 grid points in each dimension. The Gauss law

$$\vec{\nabla} \cdot \vec{E}(x,y) = -4\pi\rho(x,y) \quad (IV-3)$$

is solved in $\vec{k}$ -space, and transformed back to $\vec{x}$ space. By the inverse procedure of area weighting, the electric field at the particle is

$$\vec{E}(x,y) = (1 - x)(1 - y)\vec{E}_A + (1 - y)x\vec{E}_B + y(1 - x)\vec{E}_C + xy\vec{E}_D .$$

(IV-4)

After the electric field at all particle positions is determined, the new velocity $\vec{v}_{+\frac{1}{2}}$ is calculated from the old $\vec{v}_{-\frac{1}{2}}$ according to the following scheme:⁶

$$\vec{v}_{+\frac{1}{2}} = \vec{v}_{-\frac{1}{2}} + \frac{\Delta t}{1 + \left(\frac{\Delta t}{2}\right)^2 B^2} \left[\vec{E} + \vec{v}_{-\frac{1}{2}} \times \vec{B} + \frac{\Delta t}{2} (\vec{E} \times \vec{B} - \vec{v}_{-\frac{1}{2}} B^2) \right] \quad .$$

(IV-5)

The new position $\vec{x}_{+1}$ is determined from the old $\vec{x}_0$ according to

$$\vec{x}_{+1} = \vec{x}_0 + \vec{v}_{+\frac{1}{2}} \Delta t \quad . \quad (IV-6)$$

B. Initial Setup and Output

We are looking for the H-value as a function of time by varying the density or magnetic field strength to check with the theory. The initial conditions are set up in the following manner. The particle position is randomly selected. The velocity distribution is set up uniformly distributed from 0 to 2 in units of V_{th} ($\sqrt{K_B T / 2m}$) as described in Appendix 2. The H-value is calculated by summing the histogram to evaluate the integral of Eq. (III-7). The uniform distribution set up initially is as follows:

$$f(\vec{v}^2) = \begin{cases} 0 & v > 2 \\ \frac{1}{4\pi} & v < 2 \end{cases} \quad (IV-7)$$

where v is in dimensionless units. For this uniform distribution, $H = -2.531$. For a Maxwellian distribution, the H value is equal to $H = -2.837$, the mean energy of the system has been normalized to 1. The number of the particles is usually 4000. We load the particles in the 2-D velocity space in a spiral fashion.

The distribution function is printed out with a velocity interval $\Delta V = 0.023$. The initial H -value in the experiment is -0.2534 , which departs from theoretical value by only 0.1%. The relaxation time is determined (somewhat arbitrarily) to be the time at which $H = -2.70$.

V. COMPUTER SIMULATION RESULTS

The simulation results are consistent with the theories in strong and zero magnetic field cases.

Figure 1 shows that by changing ΔV , the velocity interval for evaluation of the H-value, the result is not affected as far as we can tell. Also shown are the results of change ΔT , the time step. The relaxation process may be assumed not to differ provided the accumulated error is less than 1% of the total kinetic energy. If energy is not accurately conserved, the H-value can change significantly, and that obscures the determination of the relaxation time. The value α is defined as the Debye length/the cell size (λ_D/m_s). (For details see Appendix 2.) We see a significant difference for different α values, but the convergence when $\alpha > 1.5$ is obvious.

Figure 2 shows the relaxation process for different values of α and different random loadings. The number IX is the index number which generates a sequence of random numbers. These initial random loadings of particle positions give different relaxation times. The reason is that the different loadings give different electric field spectrum.

Figure 3 shows what happens when the magnetic field is applied. The independence of α disappears. We were expecting the

convergence could be achieved for large enough α . Unfortunately, our program was unable to show this. In general, the larger the value α , the faster the relaxation, but the longer the computer time in one computational step because we have more grids to compute the electric field. Larger α values mean that the closer collisions are taken into account more accurately. Although we expect that a larger α gives a closer physical simulation, it also turns out that larger α makes energy conservation more difficult. We therefore need to cut down the time step.

Figure 4 shows the velocity distribution function $f(\vec{v}, t)$. It was intended to simulate the uniform distribution function as described by Eq. (IV-7) at the initial time. For large times, the distribution begins to build up a thermal tail and also a bump at low velocity. The thermal tail is particularly important because higher energy particles contribute more to the H-value. The main body of the distribution function always has wiggles and does not change significantly to the end of the computation.

Figure 5 shows our results consistent with those of Montgomery and Nielson¹ for $\alpha = 1$. $\alpha = 2$ gives a smaller slope, $t_{\text{rel}} \approx 1.95 n_o \lambda_D^2$.

Figure 6 shows that with a stronger magnetic field, t_{rel} has a dependence $\sqrt{n_o \lambda_D^2}$. One curve is for $\alpha = 2.0$, $H_F = -2.70$, $\Omega = 4 \omega_p$, $t_{\text{rel}} = 3.75 \sqrt{n_o \lambda_D^2}$. Another curve is for $\alpha = 2.0$, $H_F = -2.65$, $\Omega = 2 \omega_p$, $t_{\text{rel}} \approx 1.82 \sqrt{n_o \lambda_D^2}$. Two points do not lie

on the solid line. The reason is that these two points refer to different sizes of the system. If the dependence $\left[\ln \left(\frac{\ell}{2\pi\lambda_D} \right) \right]^{\frac{1}{2}}$ is allowed for, the result is then below the solid curve.

Figure 7 shows the relaxation time vs Ω/ω_p when $\Omega \geq 3 \omega_p$, the curve is linear as predicted by theory (see Appendix I). For weaker magnetic fields, an enhancement is apparent. Around $\Omega = 2 \omega_p$, the relaxation is most rapid. We will discuss these two points later.

Figure 8 shows a particle trajectory in a weak magnetic field. Any particle may come back to exchange energy with the original particle. For an unmagnetized plasma, particles only meet once and for all, while in a magnetized plasma, particles always come back unless they have drifted outside the force range of the original particle.

Figure 9 shows another two trajectories for $\Omega = 2.0 \omega_p$. The particle orbits are highly perturbed. In order to describe the orbits properly, we need to consider not only the drifting of the guiding center but also the action due to the other cyclotron modes. We will discuss this more later.

Figure 10 shows the electric field spectral density for $\omega = 0$ and $\omega = \Omega$, i.e., the dc mode and 1st cyclotron mode at $K \cong 0.4 K_D$. The higher modes are negligible. We will discuss Fig. 10 based upon the uncorrelated finite gyroradius model. For very weak B region, the zeroth and first cyclotron modes are not distinguishable.

VI. THEORETICAL INTERPRETATION

In order to neglect the last two terms of Eq. (II-25), we restrict ourselves to strongly magnetized plasma. We have shown the energy exchange time, t_e , is proportional to B and $\sqrt{n_o \lambda_D^2}$. These functional dependences are also seen in the evolution of the H -value by computer simulation. When the magnetic field is weaker, we do not expect a simple kinetic equation to apply because this is a transition regime where relaxation times change from $\sqrt{n_o \lambda_D^2}$ dependence to $n_o \lambda_D^2$ of unmagnetized plasma. Our discussion of the weak magnetic field region will be based on the uncorrelated finite gyro-radius model in an effort to determine the functional dependence of relaxation times on B . Equation (II-26) amounts to assuming a constant electric field. One can imagine a pseudo-charge-center when the gyrating part is averaged out. This PCC corresponds to guiding center and is almost static in time scale Ω^{-1} .

Suppose the electron rods are uncorrelated and moving under the influence of the magnetic field only. The equation of motion of the particle i is

$$\frac{d\vec{v}_i}{dt} = \vec{v}_i \times \vec{\Omega}, \quad \frac{d\vec{x}_i}{dt} = \vec{v}_i \quad . \quad (\text{VI-1})$$

The electric field at position $\vec{x}$ produced by particle i at position $\vec{x}_i$ is

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = - \sum_i \frac{4\pi e}{L} \delta [\vec{x} - \vec{x}_i(t)] \quad . \quad (\text{VI-2})$$

We assume the problem is electrostatic so that

$$\vec{\nabla} \times \vec{E}(\vec{x}, t) = 0 \quad . \quad (\text{VI-3})$$

Taking the Fourier transform of Eq. (VI-2), we have

$$i\vec{K} \cdot \vec{E}(\vec{K}, t) = - \sum_i 4\pi e e^{-i\vec{K} \cdot \vec{x}_i(t)} / (2\pi)^2 L \quad . \quad (\text{VI-4})$$

The solutions of Eq. (VI-1) are as the following:

$$\begin{aligned} \vec{v}_i(t) &= \vec{v}_{oi} \cos \Omega t + \frac{\vec{v}_{oi} \times \vec{\Omega}}{\Omega} \sin \Omega t \\ \vec{x}_i(t) &= \vec{x}_{oi} + \frac{\vec{v}_{oi}}{\Omega} \sin \Omega t - \frac{\vec{v}_{oi} \times \vec{\Omega}}{\Omega^2} (\cos \Omega t - 1) \quad . \quad (\text{VI-5}) \end{aligned}$$

Thus from Eq. (VI-4),

$$\vec{E}(\vec{K}, t) = \sum_i \frac{4\pi e i \vec{K}}{(2\pi)^2 K^2 L} e^{-i\vec{K} \cdot \vec{x}_i(t)} \quad . \quad (\text{VI-6})$$

Putting (VI-5) into (VI-6),

$$\vec{E}(\vec{K}, t) = \sum_i \frac{4\pi e i \vec{K}}{(2\pi)^2 K_L^2} e^{-i\vec{K} \cdot \vec{x}_{oi}} e^{-i\vec{K} \cdot \left[\frac{\vec{v}_{oi}}{\Omega} \sin \Omega t + \frac{\vec{v}_{oi} \times \vec{\Omega}}{\Omega^2} (\cos \Omega t - 1) \right]}$$

Let

$$\vec{K} \cdot \left(\frac{\vec{v}_{oi}}{\Omega} \sin \Omega t + \frac{\vec{v}_{oi} \times \vec{\Omega}}{\Omega^2} (\cos \Omega t - 1) \right) = \frac{K v_{oi} \sin(\Omega t + \varphi_{oi})}{\Omega} - \frac{K v_{oi}}{\Omega} \sin \varphi_{oi}$$

Also, make use of the Bessel function identity, i.e.,

$$e^{\frac{-i K v_{oi} \sin(\Omega t + \varphi_{oi})}{\Omega}} = \sum_m J_m \left(\frac{K v_{oi}}{\Omega} \right) e^{-i m (\Omega t + \varphi_{oi})}$$

We thus arrive at

$$\vec{E}(\vec{K}, t) = \sum_i \frac{4\pi e i \vec{K}}{(2\pi)^2 K_L^2} e^{-i\vec{K} \cdot \vec{x}_{oi}} \sum_m J_m \left(\frac{K v_{oi}}{\Omega} \right) e^{-i m (\Omega t + \varphi_{oi})} e^{+i \frac{K v_{oi}}{\Omega} \sin \varphi_{oi}}$$

(VI-7)

The two-time ensemble average of the electric field fluctuation is then

$$\begin{aligned}
 \langle \vec{E}(\vec{K}, t+\tau) \vec{E}(\vec{K}', t) \rangle &= \left\langle \frac{-16\pi^2 e^2}{(2\pi)^4 K^4 L^2} \vec{K} \vec{K}' \sum_{i,j} e^{-i\vec{K} \cdot \vec{x}_{oi} - i\vec{K}' \cdot \vec{x}_{oj}} \right. \\
 &\quad \times \sum_{m,n} J_m \left(\frac{K v_{oi}}{\Omega} \right) J_n \left(\frac{K' v_{oj}}{\Omega} \right) e^{in(\Omega t + \varphi_{oj})} \\
 &\quad \left. \times e^{im(\Omega t + \Omega \tau + \varphi_{oi})} e^{i \frac{K v_{oi}}{\Omega} \sin \varphi_{oi} - i \frac{K' v_{oj}}{\Omega} \sin \varphi_{oj}} \right\rangle
 \end{aligned}
 \tag{VI-8}$$

The following relation is to be used:

$$\left\langle \vec{K} \vec{K}' e^{-i\vec{K} \cdot \vec{x}_{oi} - i\vec{K}' \cdot \vec{x}_{oj}} \right\rangle = \frac{(2\pi)^2}{\ell^2} \vec{K} \vec{K}' \delta(\vec{K} + \vec{K}') \delta_{ij} \tag{VI-9}$$

The integration over φ_{oi} gives

$$\left\langle e^{in\varphi_{oj} + im\varphi_{oi}} \delta_{ij} \right\rangle = 2\pi \delta_{m, -n} \tag{VI-9a}$$

Equation (VI-8) then becomes

$$\begin{aligned} \langle \vec{E}(\vec{K}, t+\tau) \vec{E}(\vec{K}', t) \rangle &= \frac{16\pi^2 e^2}{(2\pi)^4 K^4} \vec{K} \vec{K}' \delta(\vec{K} + \vec{K}') \frac{(2\pi)^2}{L^2} \\ &\times 2\pi \sum_{\mathbf{m}} \left\langle \sum_{\mathbf{i}} J_{\mathbf{m}}^2 \left(\frac{K_{\mathbf{v} \mathbf{o} \mathbf{i}}}{\Omega} \right) \right\rangle e^{i\mathbf{m}\Omega\tau} \quad (\text{VI-10}) \end{aligned}$$

The spectral density is defined by

$$\begin{aligned} \vec{S}_{\mathbf{E}}(\vec{K}, \omega) &= \frac{1}{(2\pi)^3} \int d\vec{r} dt e^{-i(\vec{K} \cdot \vec{r} - \omega\tau)} \langle \vec{E}(\vec{x}, t) \vec{E}(\vec{x} + \vec{r}, t + \tau) \rangle \\ &\quad (\text{VI-11}) \end{aligned}$$

is then equal to the following:

$$\begin{aligned} \vec{S}_{\mathbf{E}}(\vec{K}, \omega) &= \frac{16\pi^2 e^2}{(2\pi)^4 K^4} \vec{K} \vec{K}' \frac{(2\pi)^2}{L^2} \sum_{\mathbf{n}} 2\pi \delta(\omega - n\Omega) \left\langle \sum_{\mathbf{i}} J_{\mathbf{n}}^2 \left(\frac{K_{\mathbf{v} \mathbf{o} \mathbf{i}}}{\Omega} \right) \right\rangle \\ &= 2\pi \frac{8\pi^2 e^2 \vec{K} \vec{K}' n_0}{K^4 L^2} \sum_{\mathbf{n}} \delta(\omega - n\Omega) \int_0^\infty d^2\vec{v} J_{\mathbf{n}}^2 \left(\frac{K_{\mathbf{v}}}{\Omega} \right) f^{(0)}(v^2) \quad (\text{VI-12}) \end{aligned}$$

The above equation differs from that derived from Eq. (II-23), the linearized Vlasov equation, only by the dielectric function. The dressed test particle model is well-known (13) but has not been justified for two dimensional magnetized plasma. Because there is no Landau damping in 2-D magnetized plasma, the poles of the dielectric function lie on the real axis. We will not know how to deal with the infinite number of real poles unless we assume some collisional effects to displace the poles from the real axis. Our conjecture is the dielectric function just smears out the peaks at the Bernstein modes. Because of the analytical simplicity, we treat the problem in terms of the uncorrelated test particle model rather than solving the linearized Vlasov equation.

Suppose

$$f^{(0)}(v^2) = \frac{m}{2\pi K_B T} e^{-mv^2/2K_B T}$$

and make use of the following identity

$$\frac{m}{KT} \int_0^\infty v dv J_n^2\left(\frac{Kv}{\Omega}\right) e^{-mv^2/2K_B T} = e^{-\lambda} I_n(\lambda)$$

where

$$\lambda = \frac{K_B T K^2}{m\Omega^2} = \frac{K^2 v_{th}^2}{\Omega^2}$$

Equation (VI-12) then becomes

$$\vec{S}_E(\vec{K}, \omega) = \frac{16\pi^2 e^2 \vec{K} \vec{K}_0}{K^4 L^2} \sum_n \delta(\omega - n\Omega) e^{-\lambda} I_n(\lambda) \quad (VI-13)$$

First, take $\Omega \rightarrow \infty$. So that $\lambda \rightarrow 0$

$$I_n(\lambda) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

So that

$$\vec{S}_E(\vec{K}, \omega) = \frac{16\pi^2 \vec{K} \vec{K}_0 e^2}{K^4 L^2} \delta(\omega) \quad (VI-14)$$

This result is related to Eq. (B-3) in Taylor and McNamara's³ calculation of diffusion coefficient based upon the guiding-center plasma model for random distribution. We do not see the damping behaviour of the guiding center spectrum in (VI-14) because the diffusion coefficient is zero in this uncorrelated model. We see that the zeroth mode (the dc component) is associated with the PCC, the guiding center, in the strong B limit. As λ becomes larger the other modes become more important. The neglect of the last two terms of Eq. (II-27) is then not allowed. Equation (II-28) is essentially derived from taking zeroth mode as the whole

electric field. As we go to a weaker B field, we take the first mode into consideration so that Eq. (II-1) can be approximated by

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\Omega} + \vec{E}_0(\vec{x}) + \vec{E}_1(\vec{x})e^{i\Omega t} \quad (\text{VI-15})$$

where $\vec{E}_0$, $\vec{E}_1$ corresponding to the electric field amplitudes of zeroth and first cyclotron modes, respectively. If we neglect the spatial dependence of $\vec{E}_1(\vec{x})$, the resonance phenomenon is obvious, since the driving force has the same frequency as the gyrofrequency. In the simulations, $\vec{E}_1(\vec{x})$ is concentrated in the longest wavelength. From Bessel function tables,¹⁰ $e^{-\lambda} I_1(\lambda)$ has a maximum at $\lambda \cong 0.935$. This corresponds to $\Omega \cong 2.5 \omega_p$, assuming $K \cong 0.4 K_D$. This partly explains that around $\Omega = 2 \omega_p$, the relaxation is fastest. We do not expect a perfect agreement because we have neglected the interactions between particles and the contributions from higher modes as well as spatial dependence. Also different is that the simulation is PIC model. Our distribution function is not Maxwellian.

Since zeroth mode will not do the job of relaxation, at least true for strong B case, we expand $e^{-\lambda} I_1(\lambda)$ w.r.t. λ , as λ is small. Since

$$e^{-\lambda} I_1(\lambda) \rightarrow \frac{\lambda}{2} \text{ as } \lambda \ll 1 \quad (\text{VI-16})$$

This gives $\vec{E}_1(\vec{x})$ in (VI-15), a dependence of $\sqrt{\lambda} \cong \frac{\omega_p}{\Omega}$. Thus we see the relaxation time is proportional to Ω , as we have proved more rigorously by the Fokker-Planck equation. A side comment here is that the dc component has a dependence

$$e^{-\lambda} I_0(\lambda) \cong (1 - \lambda)$$

consistent to lowest order in ω_p^2/Ω^2 with Okuda and Dawson.¹¹ Furthermore, as $\lambda > 1$, in a weak B regime, we first sum

$$\sum_{K=1}^{\infty} I_K(z) = \frac{e^z - I_0(z)}{2} \quad (\text{VI-17})$$

because the total electric field might give the relaxation. The following asymptotic form is useful:

$$e^{-\lambda} I_n(\lambda) \rightarrow \frac{1}{\sqrt{2\pi\lambda}}, \quad \text{all } n \text{ as } \lambda \gg 1 \quad (\text{VI-18})$$

So that from Eq. (VI-13), the spectral density becomes

$$\vec{S}_E(\vec{K}, \omega) \cong \frac{16\pi^2 e^2 \vec{K} \vec{K} n_0}{K_L^2} \sum_n \delta(\omega - n\Omega) \left[\frac{1}{2} + \frac{1}{2} \frac{1}{\sqrt{2\pi\lambda}} \right] \quad (\text{VI-19})$$

And the relaxation time has a dependence

$$\tau \sim 1 - \frac{\Omega}{\omega_p} \frac{1}{\sqrt{2\pi}} \quad . \quad (\text{VI-20})$$

The conclusion is restrictive because the justification of the uncorrelated finite gyroradius model is dubious for the weaker B region. Qualitatively, this dependence indicates the enhancement of pair correlation. A dashed line is drawn in Fig. 7 for comparison between the theory and the experiment. In Fig. 10 we have the electric field spectral density for the dc mode and the 1st cyclotron mode at $K \cong 0.4 K_D$ (in our computer simulation most energy is in the K mode). The curves show a ratio of $K^2 v_{th}^2 / \Omega^2$ consistent with the theoretical value 0 (λ) in the stronger B region. The 1st mode has a maximum at $\Omega = 2 \omega_p$ corresponding to the resonance. For $\Omega \gg \omega_p$ we see almost all the electrical field energy is in the dc mode which just corresponds to the guiding center model. We already see in Fig. 9 the particle trajectory is highly noncircular, that is due to the fact that more modes contribute to the particle motion.

VII. CONCLUSION

The relaxation time of the strongly magnetized two-dimensional plasma is shown to be proportional to B and $\sqrt{n_o \lambda_D^2}$. In the transition region, the cyclotron modes cause the resonance because the driving force has the same frequency as that of the gyration. According to the computer simulation, the minimum of the relaxation time vs the magnetic field strength occurs at $\Omega \approx 2 \omega_p$. The prediction of the uncorrelated finite gyroradius model for them is at $\Omega \approx 2.5 \omega_p$. In the weaker B case, both theory and computer simulation show an enhancement of the relaxation as the magnetic field strength is increased.

Figure 1 H-value as a function of time varying ΔT (time interval),
 ΔV (velocity interval), and α . The magnetic field is zero.

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Figure 1

Figure 2 H-value as a function of time ($B = 0$) varying IX and α .

Figure 2

Figure 3 H-value as a function of time ($\Omega = 0.5 \omega_p$), showing lack of convergence with changes in α .

Figure 3

Figure 4 Distribution function vs time at $H = -2.531$ (initial),
 $H = -2.60$, and $H = -2.716$.

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Figure 4

Figure 5 Relaxation time vs $n_o \lambda_D^2$ ($B = 0$)

$\alpha = 1$	$H_f = -2.70$	$t_{rel} \cong 3.04 n_o \lambda_D^2$
$\alpha = 2$	$H_f = -2.70$	$t_{rel} \cong 1.95 n_o \lambda_D^2$

Figure 5

Figure 6 Relaxation time vs $n_o \lambda_D^2$ (strong B)

$$\begin{array}{llll} \Omega = 4\omega_p & H_f = -2.70 & \alpha = 2.0 & t_{\text{rel}} \approx 3.75 n_o \lambda_D^2 \\ \Omega = 2\omega_p & H_f = -2.65 & \alpha = 2.0 & t_{\text{rel}} \approx 1.82 n_o \lambda_D^2 \end{array}$$

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Figure 6

Figure 7 Relaxation time vs B

Strong B $t_{\text{rel}} \approx 1.64 \Omega / \omega_p^2$

Resonance at $\Omega \approx 2 \omega_p$

Weak B $t_{\text{rel}} \approx 1 - \frac{1}{\sqrt{2\pi}} \frac{\Omega}{\omega_p}$

✓

Figure 7

Figure 8 Particle trajectories, $n_o \lambda_D^2 = 15.2$, $t_f = 15 \omega_p^{-1}$, $\alpha = 2$. For $\Omega = 0.75, 0.25, 0.125 \omega_p$. The box represents the boundary of the system. We use the periodic boundary conditions.

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PARTICLE TRAJECTORY (WEAK B)

Figure 8

Figure 9 Particle trajectories, $n_o \lambda_D^2 = 60.8$, $\alpha = 2$ for
 $\Omega = 2.0$.

Figure 9

Figure 10 Components of the electric field spectrum for $\omega = 0$ and $\omega = \Omega$. Notice the initial distribution was uniform instead of Maxwellian. The results are quantitatively in agreement with the theory. $K = 0.4 K_D$. (This mode contains most of the electric field energy.)

Figure 10

APPENDIX I

RELAXATION TIMES

A. Slowing Down Time for Unmagnetized Plasma

The test particle distribution function can be chosen initially as

$$f_t(\vec{v}, 0) = \delta(\vec{v} - \vec{u}_0) \quad (A-1)$$

Put this into Eq. (III-4) and make use of Eq. (III-1) to obtain

$$\left. \frac{\partial \vec{u}}{\partial t} \right|_{t=0} = -2 \int \vec{\phi}_{tb} \cdot \frac{\partial f_b}{\partial \vec{v}'} d\vec{v}' \bigg|_{\vec{v}_t = \vec{u}_0} \quad (A-2)$$

where

$$\vec{\phi}_{tb} = - \frac{8\pi^2 n_0 e^4}{m^2 L^2} \frac{1}{K_D} \frac{(\vec{u}_0 - \vec{v}')(\vec{u}_0 - \vec{v}')}{|\vec{v}' - \vec{u}_0|^3}$$

Assuming $f_b(\vec{v})$ vanishes on large $\vec{v}'$, after an integration by parts, (A-2) becomes

$$\left. \frac{\partial \vec{u}}{\partial t} \right|_{t=0} = \frac{16\pi^2 n_o e^4}{m^2 L^2 K_D} \frac{\partial}{\partial \vec{v}_t} \int \frac{f_b(\vec{v}')}{|\vec{v}_t - \vec{v}'|} d\vec{v}' \bigg|_{\vec{v}_t = \vec{u}_o} \quad (\text{A-3})$$

In order to obtain explicit form we assume

$$f_b(\vec{v}) = \frac{m}{2\pi KT} e^{-\frac{m\vec{v}^2}{2KT}} \quad (\text{A-4})$$

Noting that

$$\begin{aligned} \int_{-\infty}^{+\infty} d^2\vec{v} \frac{e^{-a\vec{v}^2}}{|\vec{u} - \vec{v}|} &= \int_0^\infty \int d^2\vec{v} \frac{e^{-a(\vec{v} + \vec{u})^2}}{|\vec{v}|} \\ &= \int_0^\infty dv \int d\theta e^{-a(v^2 + 2uv \cos \theta + u^2)} \\ &= \frac{\pi^{3/2}}{\sqrt{a}} \exp\left(-\frac{au^2}{2}\right) I_0\left(\frac{au^2}{2}\right) \end{aligned}$$

so that

$$\frac{\partial \vec{u}}{\partial t} = \frac{16\pi^2 n_o e^4}{m^2 L^2 K_D V_{th}^3} \vec{v}_t \exp\left(\frac{-v_t^2}{2V_{th}^2}\right) \left[I_1\left(\frac{v_t^2}{2V_{th}^2}\right) - I_0\left(\frac{v_t^2}{2V_{th}^2}\right) \right] \quad (\text{A-5})$$

where $v_{th}^2 = 2KT/m$, and

$$t_s = \left| \frac{\vec{u}}{\frac{\partial u}{\partial t}} \right| \cong n_o \left(\frac{LK_B T}{n_o e^2} \right) \left(\frac{mL}{n_o e^2} \right) \sim n_o \lambda_D^2 \omega_p^{-1}.$$

B. Energy Relaxation Time for Strongly Magnetized Plasma

The Fokker-Planck equation in this case is

$$\frac{\partial f_t(\vec{v}_1)}{\partial t} = - \frac{\partial}{\partial \vec{v}_1} \cdot \int d\vec{v}_2 \vec{\Phi}_S \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) f_t \left(\frac{1}{2} v_1^2, t \right) f_1 \left(\frac{1}{2} v_2^2, t \right) \quad (A-6)$$

where

$$\vec{\Phi}_S = - A \frac{(\vec{v}_{12} \times \hat{b})(\vec{v}_{12} \times \hat{b})}{v_{12}^2}, \quad A \equiv \frac{2\pi}{\Omega} \frac{4\pi n_o e^4}{m_L^2} n$$

with

$$n = \frac{1}{2} \left[\pi \epsilon \ln \left(\frac{L}{2\pi \lambda_D} \right) \right]^{-\frac{1}{2}}.$$

Notice that in our derivation, we assume $f_1^{(0)}$ depends on v_1^2 only, so the test particle distribution function is assumed also to depend only on v_1^2 . Let us assume

$$f_t(\vec{v}_1^2, 0) = \frac{2\delta(\vec{v}_1^2 - \vec{u}_0^2)}{2\pi} \quad \text{and} \quad f_1(\vec{v}_2^2) = \frac{m}{2\pi KT} e^{-m\vec{v}_2^2/2KT} \quad (\text{A-7})$$

so that

$$\int f_t(\vec{v}_1^2, 0) d^2\vec{v}_1 = 1 \quad (\text{A-8})$$

We write (A-6) in the following form:

$$\frac{\partial f_t}{\partial t} = - \frac{\partial}{\partial \vec{v}_1} \cdot (\vec{F} f_t) + \frac{1}{2} \frac{\partial^2}{\partial \vec{v}_1 \partial \vec{v}_1} : (\vec{T} f_t)$$

where

$$\vec{F} = - 2 \int \vec{\Phi}_S \cdot \frac{\partial f_1(\vec{v}_2^2)}{\partial \vec{v}_2} d\vec{v}_2$$

$$\vec{T} = - 2 \int \vec{\Phi}_S f_1(\vec{v}_2^2) d\vec{v}_2$$

We then calculate the second moments,

$$\begin{aligned}
\frac{m_t}{2} \frac{\partial \vec{u}^2}{\partial t} &= \frac{\partial}{\partial t} \int \frac{m_t}{2} v_1^2 f_t(\vec{v}_1) d\vec{v}_1 \\
&= \frac{m_t}{2} \int \left(- \frac{\partial}{\partial \vec{v}_1} \cdot \vec{F} f_t \right) v_1^2 d\vec{v}_1 + \frac{m_t}{4} \int \left(\frac{\partial^2}{\partial \vec{v}_1 \partial \vec{v}_1} : \vec{T} f_t \right) v_1^2 d\vec{v}_1 \\
&= m_t \int \vec{F} \cdot \vec{v}_1 f_t d\vec{v}_1 + \frac{m_t}{2} \int (\vec{T} : \vec{1}) f_t d\vec{v}_1 \quad (A-9)
\end{aligned}$$

Noting that

$$\vec{T} = 2A \int \frac{(\vec{v}_{12} \times \hat{b})(\vec{v}_{12} \times \hat{b})}{v_{12}^2} f_1(\vec{v}_2) d\vec{v}_2$$

we then have

$$\vec{T} \cdot \vec{1} = 2A \int f_1(\vec{v}_2) d\vec{v}_2 = 2A$$

The second term of (A-9) is equal to $m_t A$. The first term in (A-9) is more complicated but straightforward. Assuming $\vec{v}_1 = (v_1, \theta_1)$ $\vec{v}_2 = (v_2, \theta_2)$ in polar coordinates, the first term becomes

$$\begin{aligned}
m_t \int \vec{F} \cdot \vec{v}_1 f_t d\vec{v}_1 &= \frac{m_t^2 A}{2\pi KT} \int \frac{u_o^2 v_2^2 [\sin(\theta_1 - \theta_2)]^2}{u_o^2 + v_2^2 - 2u_o v_2 \cos(\theta_1 - \theta_2)} \\
&\times d\theta_1 d\theta_2 f_1(\vec{v}_2) v_2 dv_2 \quad . \quad (A-10)
\end{aligned}$$

Using $d\theta_1 d\theta_2 = d(\theta_1 - \theta_2) d \frac{\theta_1 + \theta_2}{2} = d\theta_\gamma d\theta$, the above becomes

$$\frac{m_t^2 A}{KT} \int \frac{v_2^2 \sin^2 \theta_\gamma d\theta_\gamma}{1 + \frac{v_2^2}{u_o^2} - \frac{2v_2}{u_o} \cos \theta_\gamma} f_1(v_2) \frac{dv_2^2}{2} \quad .$$

The integration over θ_γ gives

$$\int \frac{\sin^2 \theta_\gamma d\theta_\gamma}{1 + \frac{v_2^2}{u_o^2} - \frac{2v_2}{u_o} \cos \theta_\gamma} = \begin{cases} \pi & \frac{v_2^2}{u_o^2} < 1 \\ \frac{\pi u_o^2}{v_2^2} & \frac{v_2^2}{u_o^2} > 1 \end{cases}$$

So that (A-10) splits into two terms

$$\frac{m_t^2 A}{KT} \left[\int_0^{u_o} \pi v_2^2 f_1(\vec{v}_2) v_2 dv_2 + \int_{u_o}^{\infty} \frac{\pi u_o^2}{v_2^2} v_2^2 f_1(\vec{v}_2) v_2 dv_2 \right] \quad .$$

The second term in the square bracket equals

$$\frac{u_o^2}{2} e^{-m_t u_o^2 / 2KT}$$

The first term in the square bracket equals

$$\frac{KT}{m_t} \left[1 - e^{-m_t u_o^2 / 2KT} - \frac{m_t u_o^2}{2KT} e^{-m_t u_o^2 / 2KT} \right]$$

Adding these together, we find

$$m_t \int \vec{F} \cdot \vec{v}_1 f_t d\vec{v}_1 = m_t A \left(1 - e^{-m_t u_o^2 / 2KT} \right) \quad (A-11)$$

So that

$$\frac{m_t}{2} \frac{\partial u_o^2}{\partial t} = \left(2 - e^{-m_t u_o^2 / 2KT} \right) m_t A$$

and

$$\left| \frac{u_o^2}{\partial u_o^2 / \partial t} \right|_{t=0} \cong \frac{u_o^2}{4A} = \frac{m^2 \Omega L^2}{32 \pi^2 n_o e^4 \cdot n}$$

This gives

$$t_e \sim \Omega \text{ and } t_e \sim \frac{m_L^2}{n_o e^4} n \sim \sqrt{n_o \lambda_D^2} .$$

We calculated the theoretical value of the slope in Fig. 7, it is about twice the value of the experiment. The slope in the figure could be larger if the final H value is cut at a larger value.

There is quite an amount of difference between the theory and the computer model; we do not expect a perfect agreement.

APPENDIX II

DETAILS OF COMPUTER PROGRAM

A. Difference Scheme for $\vec{v}$

The following scheme is used to solve Eq. (IV-1). The subscripts of v denote the time steps. We first calculate

$$\vec{v}_0 = \vec{v}_{-\frac{1}{2}} + \frac{1}{2} \vec{E}_0 \Delta t \quad . \quad (A-12)$$

The $\vec{v}_0$ is taken as the average of $\vec{v}_{+\frac{1}{2}}$ and $\vec{v}_{-\frac{1}{2}}$, so

$$\vec{v}_0 = \frac{1}{2} (\vec{v}_{+\frac{1}{2}} + \vec{v}_{-\frac{1}{2}}) \quad . \quad (A-13)$$

The $\vec{v}_{+\frac{1}{2}}$ is calculated by taking the magnetic field into account.

$$\vec{v}_{+\frac{1}{2}} = \vec{v}_0 + \frac{1}{2} \vec{E}_0 \Delta t + \vec{v}_0 \times \vec{B} \Delta t \quad (A-14)$$

where (A-12) was used once. Take (A-14) $\times \vec{B}$

$$\vec{v}_{+\frac{1}{2}} \times \vec{B} = \vec{v}_{-\frac{1}{2}} \times \vec{B} + \vec{E}_0 \times \vec{B} \Delta t - \frac{1}{2} (\vec{v}_{+\frac{1}{2}} + \vec{v}_{-\frac{1}{2}}) \times \vec{B} \Delta t \quad . \quad (A-15)$$

From the above three equations, we solve $\vec{v}_{+\frac{1}{2}}$ in terms of $\vec{v}_{-\frac{1}{2}}$. The final equation is

$$\vec{v}_{+\frac{1}{2}} = \vec{v}_{-\frac{1}{2}} + \frac{1}{1 + \frac{(\Delta t)^2}{2} B^2}$$

$$\left[\vec{v}_{-\frac{1}{2}} \times \vec{B} \Delta t + \vec{E}_0 \Delta t + \frac{\vec{E}_0 \times \vec{B}}{2} (\Delta t)^2 - \frac{\Delta t}{2} \vec{v}_{-\frac{1}{2}} B^2 \right] \quad (A-16)$$

B. Dimensionless Variables

We define the units in the computer program assuming

$$\vec{v} = v_0 \underline{v}, \quad \vec{x} = x_0 \underline{x}, \quad t = t_0 \tilde{t}, \quad \vec{E} = E_0 \underline{E}, \quad \vec{B} = B_0 \underline{b}, \quad (A-17)$$

where all underlined quantities are vectors, and $\tilde{t}$ is a scalar.

From Eq. (IV-1)

$$\frac{d\underline{v}}{d\tilde{t}} = \frac{eE_0 t_0}{mV_0} \underline{E} + \frac{et_0 B_0}{mc} \underline{v} \times \underline{b} \quad (A-18)$$

$$\frac{d\underline{x}}{d\tilde{t}} = \frac{v_0 t_0}{x_0} \underline{v}$$

and from Eq. (IV-3)

$$\frac{d}{d\tilde{t}} \cdot \underline{E} = \frac{-4\pi n_0 e x_0}{E_0} \left(1 - \frac{n}{n_0} \right) \quad (A-19)$$

Let

$$\frac{eE_0 t_0}{mv_0} = 1, \quad \frac{v_0 t_0}{x_0} = 1, \quad \frac{eB_0}{mc} t_0 = \Omega t_0.$$

We then choose

$$x_0 = \frac{\lambda_D}{\alpha}, \quad v_0 = \beta V_{th}, \quad t_0 = \frac{x_0}{v_0} = \frac{1}{\omega_p \alpha \beta}. \quad (A-20)$$

Then

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{E} = -\omega_p^2 \frac{x_0 t_0}{v_0} \left(1 - \frac{n}{n_0}\right) = -\frac{\omega_p^2}{\alpha^2 \beta^2} \left(1 - \frac{n}{n_0}\right) \quad (A-21)$$

$$\frac{d\underline{v}}{d\tau} = \underline{E} + \underline{v} \times \underline{b}, \quad \frac{d\underline{x}}{d\tau} = \underline{v}. \quad (A-22)$$

In practice, we choose $\beta = 1$, and change the size of the system by changing α . We fix x_0 as the cell size and normalize to unity.

LIST OF REFERENCES

- ¹D. Montgomery and C. W. Nielson, *Phys. Fluids* 13, 1405 (1970).
- ²G. Vahala and D. Montgomery, *J. Plasma Phys.* 6, 425 (1971).
- ³I. R. Bernstein, *Phys. Rev.* 109, 10 (1958). Our calculations are based on the uncorrelated finite gyroradius model and do not involve the dielectric function. These cyclotron modes remind us of the Bernstein modes.
- ⁴J. B. Taylor and B. McNamara, *Phys. Fluids* 14, 1492 (1971).
- ⁵D. Montgomery, Theory of the Unmagnetized Plasma (Gordon and Breach, New York, 1971).
- ⁶R. L. Morse, in Methods of Computational Physics, ed. B. Alder (Academic Press, New York, 1970), Vol. 9, p. 213.
- ⁷D. Montgomery, in Kinetic Theory: Lectures in Theoretical Physics, ed. W. Brittin, A. Barut, and M. Guenin (Gordon and Breach, New York, 1967), Vol. 9c.
- ⁸D. Montgomery and D. A. Tidman, Plasma Kinetic Theory (McGraw-Hill, New York, 1964).
- ⁹K. Huang, Statistical Mechanics (John Wiley and Sons, Inc., New York, 1963).
- ¹⁰Handbook of Mathematical Functions (National Bureau of Standards, Ninth Printing, 1970), p. 416.
- ¹¹H. M. Okuda and J. M. Dawson, *Phys. Fluids* 16, 408 (1973).
- ¹²D. Swift, Univ. of Iowa research report 72-13.
- ¹³N. Rostoker, *Nucl. Fusion* 101, 1 (1961).