C00-2280-10

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SERIES IN APPLIED MATHEMATICS

Report No. 74-5 MASTER June 1974

ISOPERIMETRIC INEQUALITIES FOR A
CRITICAL REACTOR

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ISOPERIMETRIC INEQUALITIES FOR A CRITICAL REACTOR

by

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and

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^{*}Research supported by the National Science Foundation under Grant NSF GP-33031X1.

^{**}Research supported by AEC Grant AT(11-1)-2280 Mod. 1.

Isoperimetric Inequalities for a Critical Reactor

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Lawrence E. Payne^{*}, Cornell University

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We shall obtain various bounds relating to the fundamental eigenfunction $u(x)$ of the Helmholtz equation $\Delta u + \lambda u = 0$ in a domain D with $(\partial u / \partial \nu) + hu = 0$ on its boundary ∂D . Here h is a given positive constant and ν is the outward normal. It is known that the fundamental eigenvalue λ is positive, simple, increasing with h , and that the corresponding eigenfunction does not vanish on $D + \partial D$. We are therefore able to choose $u(x)$ positive in keeping with the applications we have in mind. One of our principal goals is to obtain isoperimetric inequalities for the average-to-peak eigenfunction ratio $E = u_{\text{ave}} / u_{\text{max}}$. The interest in this ratio stems from its interpretation in nuclear engineering where $u(x)$ is the neutron density in a homogeneous, monoenergetic, critical reactor occupying the domain D . A relatively high value of E (smaller than 1, of course!) is desirable as it insures adequate power without exceeding the maximum temperature allowed by metallurgical limitations. Along with these estimates for E we derive bounds for $\text{grad } u$ and relations among various norms of u (E can itself be regarded as proportional to the ratio of the L_1 norm to the L_∞ norm). Our results do, of course, also apply to the many other areas where the same boundary value problem arises.

Previously, in [4], we considered instead the Dirichlet boundary condition

^{*} Research supported by the National Science Foundation under Grant NSF GP-33031X1.

^{**} Research supported by AEC Grant AT(11-1)-2280 Mod. 1. The author wishes to thank the members of the Mathematical Institute, Oxford University, for their hospitality.

$u = 0$ which is somewhat less realistic in the nuclear context. Moreover, results for the Dirichlet problem can be recovered from the present ones by letting h tend to infinity. Although we shall follow the same broad outline as in [4], we now encounter additional technical difficulties at nearly every step. First, we obtain bounds for $\text{grad } u$ by using Hopf's second maximum principle for elliptic inequalities [1]. This has to be accomplished in an indirect way since the Helmholtz equation with positive λ is the prototype of equations whose solutions do not satisfy the maximum principle. Next, we introduce the level surfaces for u , which, in contrast to the Dirichlet problem, are not necessarily closed. With the level value of u as a parameter, we derive differential inequalities that yield comparisons among various norms of u . Finally, by passing to the independent variable v , the volume enclosed within a level surface, we are led to a differential inequality which gives the desired isoperimetric bound for E .

Formulation of the problem.

Let D be a domain in R_n with boundary ∂D which, at first, we take to be of class $C^{2+\epsilon}$. Since D is not necessarily convex nor even contractible to a point, there may be parts of the boundary where the mean curvature K is negative; let $K_0 \geq 0$ be such that

$$(1) \quad K_0 \geq \max_{x \text{ on } \partial D} (-K) (n-1).$$

Clearly K_0 may be taken equal to 0 either for a convex domain, or more generally, for a domain whose mean curvature is everywhere nonnegative. Let $u(x)$ be the positive eigenfunction, with maximum value equal to 1, corresponding to the fundamental eigenvalue λ , of the boundary value problem

$$(2) \quad \begin{aligned} \Delta u + \lambda u &= 0, & x \in D, \\ \frac{\partial u}{\partial \nu} + hu &= 0, & x \in \partial D, \end{aligned}$$

where ν is the outward normal on ∂D , h is a given positive constant, and Δ is the n -dimensional Laplacian.

The average-to-peak ratio E is independent of the scalar multiple present in the fundamental eigenfunction of (2) so that, in terms of the normalized eigenfunction u ,

$$(3) \quad E = \frac{\int_D u \, dx}{V},$$

where V is the volume of D .

Bounds for the gradient.

The estimate we shall obtain turns out to be the same as for the Dirichlet problem in [4]. The following argument makes this plausible. The boundary condition in (2) implies that the linear extrapolation of u vanishes at a distance $1/h$ normal to ∂D . This suggests that, for h large compared to a typical dimension of D , we could extend u as a solution of the same Helmholtz equation to a larger domain D' on whose boundary (approximately a distance $1/h$ from ∂D) u vanishes. Estimates obtained for the Dirichlet problem would then apply equally well to (2). The argument loses some of its thrust when h is not large since such an extension may then be impossible, for instance in the case of a spherical annulus whose inner radius is small compared to the outer one. In any event, the proof below is based on different arguments.

Theorem 1. Let $u(x)$ be the fundamental eigenfunction of (2) with maximum value 1.

a) If $h \geq K_0$ and $n = 2$, then

$$(4) \quad |\text{grad } u|^2 \leq \lambda [(1-u^2) + (G^2-1)(1-u)]$$

$$(5) \quad = \lambda [G^2 - (G^2-1)u - u^2],$$

where the dimensionless parameter G is defined by

$$(6) \quad G = \gamma + (1+\gamma^2)^{\frac{1}{2}}, \quad \gamma = K_0 \lambda^{-\frac{1}{2}}; \quad G = 1 \text{ for convex } D.$$

b) Let $n \geq 2$ and let τ be the maximum of u on ∂D ; assume that the domain $u > \tau$ is convex. Then (5) holds for $u \geq \tau$ with $G = 1$.

Proof. a) Set

$$(7) \quad J = |\text{grad } u|^2 + \lambda u^2 + \alpha u,$$

with $\alpha \geq 0$ yet to be specified. Straightforward calculations and use of the Schwarz inequality lead to

$$(8) \quad \Delta J + \frac{\sum_{k=1}^n a_k (\partial J / \partial x_k)}{|\text{grad } u|^2} \geq 0,$$

where $a_1, \dots, a_n$ are bounded on $D + \partial D$. After excising small balls around the points where $\text{grad } u$ vanishes, we can apply Hopf's second maximum principle [1]. Allowing the radii of the spheres to go to zero, we are left with three possibilities: 1) the maximum of J occurs only at points P on ∂D and $(\partial J / \partial \nu) > 0$ at any such point; 2) J is identically constant on D ; 3) the maximum of J occurs at a point where $\text{grad } u = 0$. Of course, if 2) occurs so does 3). We now show that for sufficiently large α , we have $\partial J / \partial \nu \leq 0$ so that case 3) must occur.

We introduce, for $n = 2$, a normal-tangential coordinate system in a neighborhood of ∂D : ν is the normal coordinate and s is a tangential coordinate which reduces to the arclength on ∂D when $\nu = 0$. In terms of these coordinates,

$$(9) \quad J = \left(\frac{\partial u}{\partial \nu} \right)^2 + \left(\frac{1}{k} \frac{\partial u}{\partial s} \right)^2 + \lambda u^2 + 2\alpha u,$$

where the metric coefficient k is identically 1 on ∂D . Using the boundary condition and the relation $K = \partial k / \partial \nu$, we find

$$\frac{1}{2} \left(\frac{\partial J}{\partial v} \right)_{\partial D} = \frac{\partial u}{\partial v} \left(\frac{\partial^2 u}{\partial v^2} + \lambda u + \alpha \right) - \left(\frac{\partial u}{\partial s} \right)^2 \left(h + \frac{\partial k}{\partial v} \right).$$

If $h \geq K_0$ (it is therefore enough to have $h \geq 0$ if D is convex), we can drop the last term to obtain the inequality

$$(10) \quad \frac{1}{2} \left(\frac{\partial J}{\partial v} \right)_{\partial D} \leq \frac{\partial u}{\partial v} \left(\frac{\partial^2 u}{\partial v^2} + \lambda u + \alpha \right).$$

In view of the smoothness assumptions on ∂D , the differential equation (2) may be applied at the boundary where it takes the form

$$(11) \quad \frac{\partial^2 u}{\partial v^2} + (n-1) K \frac{\partial u}{\partial v} + \Delta_s u + \lambda u = 0,$$

where Δ_s is the surface Laplacian which is just $\partial^2/\partial s^2$ when $n = 2$. In this case (10) simplifies to

$$(12) \quad \frac{1}{2} \left(\frac{\partial J}{\partial v} \right)_{\partial D} \leq -q \left(Kq + \alpha - \frac{\partial^2 u}{\partial s^2} \right),$$

where $q = -(\partial u/\partial v)$ is strictly positive on ∂D . If we could show that $\partial^2 u/\partial s^2$ is nonpositive we could drop that term from (12). From (9) we find

$$(13) \quad \frac{1}{2} \left(\frac{\partial J}{\partial s} \right)_{\partial D} = \frac{\partial u}{\partial s} \left(h^2 u + \lambda u + \alpha + \frac{\partial^2 u}{\partial s^2} \right)$$

and

$$(14) \quad \frac{1}{2} \left(\frac{\partial^2 J}{\partial s^2} \right)_{\partial D} = \frac{\partial u}{\partial s} \left(h^2 \frac{\partial u}{\partial s} + \lambda \frac{\partial u}{\partial s} + \frac{\partial^3 u}{\partial s^3} \right) + \frac{\partial^2 u}{\partial s^2} \left(h^2 u + \lambda u + \alpha + \frac{\partial^2 u}{\partial s^2} \right).$$

If J has a maximum at a boundary point P , then $\frac{\partial J}{\partial s} = 0$ and $\frac{\partial^2 J}{\partial s^2} \leq 0$ at that point. If $\frac{\partial^2 u}{\partial s^2} > 0$ at P , (13) would imply $\frac{\partial u}{\partial s} = 0$ at P which yields a contradiction when inserted in (14). Thus $\frac{\partial^2 u}{\partial s^2} \leq 0$ at P and (12) becomes

$$(15) \quad \frac{1}{2} \left(\frac{\partial J}{\partial v} \right)_P \leq -q(Kq + \alpha) .$$

Let q_0 be the (unknown) maximum value of q on ∂D . The choice $\alpha = K_0 q_0$ then gives $(\partial J / \partial v) \leq 0$ at P so that the maximum of J must occur at an interior point where $\text{grad } u = 0$. Hence

$$(16) \quad |\text{grad } u|^2 + \lambda u^2 + 2\alpha u \leq \max(\lambda u^2 + 2\alpha u) = \lambda + 2\alpha$$

or

$$(17) \quad |\text{grad } u|^2 \leq \lambda(1-u^2) + 2\alpha(1-u) .$$

It follows that

$$(18) \quad q_0^2 \leq \lambda + 2K_0 q_0$$

which implies that

$$(19) \quad q_0 \leq \sqrt{\lambda} G \text{ and } \alpha = K_0 q_0 \leq \frac{\lambda}{2} (G^2 - 1) .$$

Substituting this upper bound for α in (17), we obtain the desired estimate.

The proof does not carry over easily to higher dimensions because the tangential terms in (9) and (11) are considerably more complicated.

b) We now deal with $n \geq 2$. Let τ be the maximum of u on ∂D . The boundary condition implies $\tau < 1$. Let $D(\tau)$ be the domain where u exceeds τ and let the boundary of this domain be denoted by $\partial D(\tau)$. On $\partial D(\tau)$ the tangential component of $\text{grad } u$ and the surface Laplacian both vanish; this makes the calculation of $\partial J / \partial v$ on $\partial D(\tau)$ very simple. The equivalent of (12) is

$$\left(\frac{1}{2} \frac{\partial J}{\partial v} \right)_{\partial D(\tau)} \leq -(n-1) \left(\frac{\partial u}{\partial v} \right)_{\partial D(\tau)}^2 K_\tau + \alpha \frac{\partial u}{\partial v} ,$$

where K_τ is the mean curvature of $\partial D(\tau)$ and $\frac{\partial u}{\partial v}$ is evaluated on $\partial D(\tau)$. If $\partial D(\tau)$ is convex, the choice $\alpha = 0$ shows that $\partial J / \partial v$ is nonpositive so that the Hopf principle gives the desired inequality $|\text{grad } u|^2 \leq \lambda(1-u^2)$ for $u \geq \tau$.

Remark. By standard arguments our results can be extended to domains with Lipschitz boundaries, as long as K is uniformly bounded from below for the

smooth approximating surfaces.

The inequality (5) has some interesting immediate consequences for two-dimensional domains. First, we obtain an upper bound for the values of u on the boundary. On ∂D ,

$$h^2 u^2 = \left(\frac{\partial u}{\partial \nu} \right)^2 \leq |\text{grad } u|^2 \leq \lambda [G^2 - (G^2 - 1)u - u^2],$$

or

$$(h^2 + \lambda)u^2 + \lambda(G^2 - 1)u \leq \lambda G^2.$$

Dropping the nonnegative term $\lambda(G^2 - 1)u$, we find

$$(20) \quad u \leq G(1 + \mu^2)^{-\frac{1}{2}} \quad \text{on } \partial D,$$

where

$$(21) \quad \mu^2 = h^2 / \lambda.$$

If $G > 1$, retention of the dropped term would improve the bound at the cost of making it more complicated.

Integrating (2) over D and using (20), we are led to the inequality

$$(22) \quad E \leq \mu(1 + \mu^2)^{-\frac{1}{2}} (LV^{-1} \lambda^{-\frac{1}{2}}) G,$$

where L is the length of ∂D and V is the area of D . Next, we multiply (2) by u , integrate over D and use (20) to obtain

$$\frac{1}{V} \int_D u^2 dx \leq \frac{G^2}{2} - \frac{(G^2 - 1)}{2V} \int_D u dx + \frac{1}{2} \frac{\mu}{1 + \mu^2} (LV^{-1} \lambda^{-\frac{1}{2}}) G^2,$$

which, in the convex case, reduces to

$$(23) \quad \frac{1}{V} \int_D u^2 dx \leq \frac{1}{2} + \frac{1}{2} \frac{\mu}{1 + \mu^2} (LV^{-1} \lambda^{-\frac{1}{2}}).$$

As $h \rightarrow \infty$, we tend to the Dirichlet problem. Since λ remains finite, μ approaches infinity and (23) becomes

$$\frac{1}{V} \int_D u^2 dx \leq \frac{1}{2},$$

a result already obtained in [4].

Remark. The one-dimensional problem for (2) can be solved explicitly. Let D be the interval $(-a/2) < x < (a/2)$. Then $u(x) = \cos \lambda^{\frac{1}{2}} x$, with $\lambda = 4z^2/a^2$ where z is the lowest positive root of $z \tan z = (ha)/2$. Elementary calculations show that (5), (20), (22) and (23) are exact in this case.

Norm Estimates.

In the next two sections we take $n \geq 2$ but confine ourselves to the convex case ($K_0 = 0$, $G = 1$). Let us introduce the level surfaces of u . In general u varies on ∂D but its maximum value τ on ∂D is smaller than 1. Only the level surfaces with u -values greater than or equal to τ will be closed. In any case let $D(t)$ be the domain where $u > t$; its boundary, which may include part of ∂D , will be denoted by $\partial D(t)$ and its volume by $v(t)$. It is clear that $v(t)$ is a decreasing function of t with $v(t_{\min}) = V$ and $v(1) = 0$. We shall assume henceforth that the domain $D(\tau)$ where $u > \tau$ is convex. By repeating the derivation preceding (20) we can show that

$$(24) \quad \tau \leq (1 + \mu^2)^{-\frac{1}{2}}, \quad \mu^2 = h^2/\lambda.$$

The first two lemmas below are proved by elementary calculus.

Lemma 1. For any continuous function $h(z)$, define

$$H(t) = \int_{D(t)} h(u) dx.$$

Then

$$H'(t) = h(t)v'(t), \quad H(t) = - \int_t^1 h(z)v'(z) dz.$$

Lemma 2. For $t \geq \tau$

$$v'(t) = - \int_{\partial D(t)} |\text{grad } u|^{-1} dS.$$

Lemma 3. Let

$$(25) \quad f(t) = \int_{D(t)} u(x) dx ,$$

then

$$(26) \quad f(t) \leq (1-t^2)^{\frac{1}{2}} (1-\tau^2)^{-\frac{1}{2}} f(\tau), \quad t \geq \tau .$$

Proof. Integrating the differential equation (2) over $D(t)$, we find

$$\lambda f(t) = \int_{\partial D(t)} |\text{grad } u| dS ,$$

and, from Lemmas 1 and 2,

$$f'(t) = tv'(t) = -t \int_{\partial D(t)} |\text{grad } u|^{-1} dS ,$$

so that

$$-f'/f = \lambda t \frac{\int_{\partial D(t)} |\text{grad } u|^{-1} dS}{\int_{\partial D(t)} |\text{grad } u| dS} \geq \frac{\lambda t}{\max_{\partial D(t)} |\text{grad } u|^2}$$

For $t \geq \tau$, $\partial D(t)$ is the level surface $u = t$ so that (5) can be used with $u = t$ on the right side to give

$$-f'/f \geq t/(1-t^2) ; (\log f)' \leq [\log(1-t^2)^{\frac{1}{2}}]' .$$

An integration from τ to t then yields the inequality (26).

Theorem 2. Let $g(t)$ be a nonnegative, increasing function. Then

$$(27) \quad \int_D u g(u) dx \leq (1-\tau^2)^{-\frac{1}{2}} I(\tau) \int_D u dx \leq \frac{(\mu^2+1)^{\frac{1}{2}}}{\mu} I\left(\frac{1}{\sqrt{1+\mu^2}}\right) \int_D u dx ,$$

where

$$(28) \quad I(z) = \int_z^1 t g(t) (1-t^2)^{-\frac{1}{2}} dt, \quad \mu = h\lambda^{-\frac{1}{2}} .$$

Proof. Since g is increasing, we have

$$(29) \quad \int_D ug(u)dx \leq g(\tau) \int_{D-D(\tau)} udx + \int_{D(\tau)} ug(u)dx.$$

To estimate the last term, multiply (26) by $g'(t)$ and integrate from τ to 1. Performing the integral on the left by parts and using Lemma 1, we find

$$\int_{D(\tau)} ug(u)dx \leq g(\tau)f(\tau) + (1-\tau^2)^{-\frac{1}{2}}f(\tau) \int_{\tau}^1 (1-t^2)^{\frac{1}{2}}g'(t)dt,$$

which, when substituted in (29), gives

$$\int_D ug(u)dx \leq g(\tau) \int_D udx + (1-\tau^2)^{-\frac{1}{2}}f(\tau) \int_{\tau}^1 (1-t^2)^{\frac{1}{2}}g'(t)dt.$$

In the last term we integrate by parts and replace $f(\tau)$ by its upper bound $\int_D udx$ to obtain the first inequality in (27). The second inequality is a consequence of (24) and of the observation that $(1-z^2)^{-\frac{1}{2}}I(z)$ is an increasing function of z .

Corollary. For p a nonnegative integer,

$$(30) \quad \int_D u^{p+2}dx \leq \left[\frac{(\mu^2+1)^{\frac{1}{2}}}{\mu} \int_{(1+\mu^2)^{-\frac{1}{2}}}^1 t^{p+2}(1-t^2)^{-\frac{1}{2}}dt \right] \int_D udx.$$

Remarks. 1. One can verify that the inequalities (27) and (30) turn into equalities for the one-dimensional problem.

2. As $h \rightarrow \infty$, we have results for the Dirichlet problem. Since λ remains finite, $\mu \rightarrow \infty$ so that (30) becomes

$$\int_D u^{p+2}dx \leq \left[\int_0^1 t^{p+2}(1-t^2)^{-\frac{1}{2}}dt \right] \int_D udx = \left[\int_0^{\pi/2} \cos^{p+2}zdz \right] \int_D udx,$$

which, for the case $p = 0$, gives

$$\int_D u^2dx \leq \frac{\pi}{4} \int_D udx.$$

3. Since $[\int_D u^{p+2} dx]^{1/(p+2)}$ tends as $p \rightarrow \infty$ to

the sup norm of u (which is equal to 1), it is tempting to try to use (30) as $p \rightarrow \infty$ for a lower bound to $\int_D u dx$ and hence to E .

Unfortunately, a simple asymptotic analysis shows that we get only the trivial lower bound 0, except in the one-dimensional case where we recover the exact value.

As an example of how the gradient estimates may be used together with other standard inequalities to obtain Payne-Rayner type inequalities (see [2], [3]) in the problem considered in this paper, we look at the special case of $n = 2$ and D convex.

Let $f(t)$ be defined as in (25). Then for $t > \tau$

$$\frac{-\lambda f f'(t)}{t} = \int_{\partial D(t)} |\text{grad } u| dS \int_{\partial D(t)} \frac{dS}{|\text{grad } u|} \geq S^2(t).$$

Integrating from $t = \tau$ to $t = 1$ we find, using the classical isoperimetric inequality,

$$\frac{\lambda}{2} \left(\int_{D(\tau)} u dx \right)^2 \geq \int_{\tau}^1 t S^2(t) dt \geq 4\pi \int_{\tau}^1 t v(t) dt = 2\pi \int_{D(\tau)} u^2 dx - 2\pi \tau^2 v(\tau).$$

The last result has been obtained by integration by parts.

Now

$$\begin{aligned} 2\pi \int_D u^2 dx &= 2\pi \int_{D(\tau)} u^2 dx + 2\pi \int_{D-D(\tau)} u^2 dx \\ &\leq \frac{\lambda}{2} \left(\int_{D(\tau)} u dx \right)^2 + 2\pi \tau^2 v(\tau) + 2\pi \tau^2 [V - v(\tau)] \leq \frac{\lambda}{2} \left(\int_D u dx \right)^2 + 2\pi \tau^2 V. \end{aligned}$$

Thus from (24)

$$2\pi \int_D u^2 dx \leq \frac{\lambda}{2} \left(\int_D u dx \right)^2 + \frac{2\pi V}{1+\mu}.$$

This reduces to the inequality of Payne and Rayner [2] in the limit as $\mu \rightarrow \infty$.

Isoperimetric inequalities for the average-to-peak ratio.

Again let $v(t)$ be the volume of the domain $D(t)$ where $u > t$, and let τ be the maximum of u on ∂D . We shall denote $v(\tau)$ by $\bar{v}$. Since $v(t)$ is decreasing we can define its inverse $t(v)$; clearly $t(v)$ is a decreasing function on $0 \leq v \leq V$. Let $F(v) = f(t(v))$, where $f(t)$ is defined from (25).

Lemma 4.

$$(31) \quad -\left[\left(\frac{F'}{F} \right)^2 \right]' \leq -\left(\frac{1}{F^2} \right)', \quad v \leq \bar{v}.$$

Proof. Using Lemmas 1 and 2, we find

$$(32) \quad F'(v) = f'(t(v))t'(v) = t(v),$$

$$(33) \quad F''(v) = t'(v) = 1/v' = -\left[\int_{\partial D(t(v))} |\text{grad } u|^{-1} dS \right]^{-1}, \quad v \leq \bar{v}.$$

Integrating the differential equation (2) over $D(t)$, we obtain

$$(34) \quad \lambda F(v) = \int_{\partial D(t(v))} |\text{grad } u| dS.$$

Combining (33) and (34), and using (5) and (32), we find

$$-FF'' \leq [1-t^2(v)] = 1-(F')^2; \quad (F')^2 - FF'' \leq 1,$$

which, when multiplied by F'/F^3 gives (31).

Theorem 3.

$$(35) \quad E \leq \frac{(1-\tau^2)^{\frac{1}{2}}}{\arcsin(1-\tau^2)^{\frac{1}{2}}} \leq \frac{\mu(1+\mu^2)^{-\frac{1}{2}}}{\arcsin[\mu(1+\mu^2)^{-\frac{1}{2}}]}.$$

Proof. Integrate (31) from v to $\bar{v}$; since $F'(\bar{v}) = \tau$, we have

$$F'(v) \leq \left[1 - \frac{F^2(v)}{F^2(\bar{v})} (1-\tau^2) \right]^{\frac{1}{2}}.$$

Integration from $v = 0$ to $\bar{v}$ yields

$$\arcsin(1-\tau^2)^{\frac{1}{2}} \leq \frac{\bar{v}}{F(\bar{v})} (1-\tau^2)^{\frac{1}{2}}.$$

From the definition (3) of E , it follows that

$$\begin{aligned} VE &= \int_{D-D(\tau)} u dx + \int_{D(\tau)} u dx \leq \tau(V-\bar{v}) + F(\bar{v}) \\ &\leq \tau(V-\bar{v}) + \bar{v} \frac{(1-\tau^2)^{\frac{1}{2}}}{\arcsin(1-\tau^2)^{\frac{1}{2}}} \\ &\leq \tau V + \bar{v} \left[\frac{(1-\tau^2)^{\frac{1}{2}}}{\arcsin(1-\tau^2)^{\frac{1}{2}}} - \tau \right]. \end{aligned}$$

Since the last bracketed term is positive, we may substitute V for $\bar{v}$ to obtain the first inequality in (35). The middle term in (35) is an increasing function of τ so that we may substitute the upper bound (24) for τ , leading to the second inequality in (35).

Remarks. 1. Again the inequalities in (35) become equalities for the one-dimensional case.

2. As $h \rightarrow \infty$, we obtain $E \leq 2/\pi$ for the Dirichlet problem. This was one of the principal results of [4].

3. In [6] we studied the Dirichlet problem for the nonlinear equation $\Delta u + \omega(u) = 0$. The gradient estimate obtained there was based on using instead of J in (7), the expression

$$|\text{grad } u|^2 + 2W(u) + \alpha u,$$

where

$$W(u) = \int_0^u \omega(\eta) d\eta .$$

This led to

$$|\text{grad } u|^2 \leq 2 [W(u_M) - W(u)] , \quad D \text{ convex} ,$$

where u_M = maximum value of u .

The same result can be obtained for the nonlinear equation with the boundary condition $(\partial u / \partial \nu) + hu = 0$ by using the methods of the present article.

4. The inhomogeneous reactor is of great interest in applications since it may permit a flatter neutron profile. The neutron density in this case is the fundamental eigenfunction of $\Delta u + \lambda \rho(x)u = 0$, where $\rho(x) > 0$ and appropriate boundary conditions are imposed. In a recent paper [5], Protter and Weinberger obtained gradient bounds for similar equations. Their results are however best suited to the case $\lambda \leq 0$ when u itself obeys a maximum principle; as expected, boundary conditions and boundary curvature play no role in their treatment. It might, however, be possible to combine their approach and ours to derive gradient bounds for the critical inhomogeneous reactor.

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