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Perturbation Theory of Many-Boson Systems*

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ABSTRACT

A noncanonical transformation of the Boson creation and annihilation operators is performed in order to obtain a Hamiltonian which can be treated by the standard methods of field-theoretic perturbation theory. The standard results¹ (with a slight modification) are rederived by this technique.

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I. INTRODUCTION

Belyaev¹ has shown that the many-boson system can be treated by the methods of field-theoretic perturbation theory. His proof of this fact, however, is outside the realm of field-theoretic perturbation theory. The purpose of this paper is an alternative derivation of Belyaev's result, using only standard perturbative techniques. This new derivation is somewhat more exact in its treatment of the zero-momentum state, and a slight correction to Belyaev's result is found. Further light is thrown on the nature of the approximation of large numbers.

II. FORMULATION OF THE PROBLEM

We consider a system consisting of a large number of identical bosons interacting via two-body forces. The units are chosen so that $\hbar = 2m = 1$, where m is the mass of a single boson. Then the Hamiltonian for the system is

$$\underline{E} = \sum_k k^2 a_k^\dagger a_k + (2V)^{-1} \sum v(k_1 - k_3) \delta_{k_1 + k_2, k_3 + k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}. \quad (1)$$

All k 's are vectors; we use bold face type for operators, except the a_k , $a_k^\dagger$, where the a_k , $a_k^\dagger$ are the annihilation and creation operators, respectively, for a boson in the single-particle state with wave number k and wave function $V^{-1/2} e^{ik \cdot r}$. V is the volume of the system and only those k 's necessary for completeness of the set of single-particle

functions are included, i. e., the k 's satisfying the usual periodic boundary conditions. $v(k)$ is the Fourier transform of the two-body interaction $u(r)$:

$$v(k) = \int u(r) e^{ik \cdot r} d^3r. \quad (2)$$

In order that $\underline{E}$ be Hermitian, we must have

$$v(-k) = v^*(k). \quad (3)$$

Since the a_k, a_k^+ are Bose operators, they obey the commutation relations

$$[a_k, a_{k'}] = 0, \quad [a_k, a_{k'}^+] = \delta_{k, k'} \quad (4)$$

From these, it follows that $n_k = a_k^+ a_k$ is the operator for the number of particles in the single-particle state k and has the eigenvalues $0, 1, 2, \dots$. The total number operator is

$$\underline{N} = \sum_k n_k. \quad (5)$$

$\underline{N}$ and $\underline{E}$ commute, so that we can now formulate the problem as follows: find the simultaneous eigenvectors, $|E_i, N\rangle$ of $\underline{E}$ and $\underline{N}$ and, in particular, the set of eigenvalues E_i which occurs for a fixed value of N :

$$\begin{aligned} \underline{E} |E_i, N\rangle &= E_i |E_i, N\rangle \\ \underline{N} |E_i, N\rangle &= N |E_i, N\rangle \end{aligned} \quad (6)$$

Of course, we shall actually consider just the lowest eigenvalues of the energy for a fixed number of particles N ; the lowest energy eigenvalue, the ground-state energy, and its eigenvector are of particular

interest.

Consider first just the kinetic energy part of the Hamiltonian:

$$\underline{T} = \sum k^2 a_k^\dagger a_k = \sum k^2 \underline{n}_k. \quad (7)$$

We know the eigenvectors of $\underline{T}$: they are just the simultaneous eigenvectors $|n_k\rangle$ of all the $\underline{n}_k$ (the $\underline{n}_k$ commute with each other)

$$\begin{aligned} \underline{n}_j |n_k\rangle &= n_j |n_k\rangle, \\ \underline{T} |n_k\rangle &= \sum_j n_j j^2 |n_k\rangle. \end{aligned} \quad (8)$$

Thus, the ground-state eigenvector for the system consisting of N noninteracting bosons ($v=0$) is just the vector

$$|n_0 = N, n_k = 0\rangle, \quad k \neq 0 \quad (9)$$

with eigenvalue $T = E = 0$.

The idea of perturbation theory is then to expand a state $|E_i, N\rangle$ of the interacting system in terms of the unperturbed states (eigenstates of the interactionless Hamiltonian):

$$|E_i, N\rangle = \sum_{\sum n_k = N} |n_k\rangle \langle n_k | E_i, N\rangle, \quad (10)$$

and find the coefficients $\langle n_k | E_i, N\rangle$ and eigenvalues E_i as power series in the interaction coupling constant.

Field-theoretic techniques have greatly simplified the solution of this problem. However, the application of such techniques is not feasible unless the unperturbed ground state $|0\rangle_u$ is a "vacuum"; that is, a state such that

$$a_k |0\rangle_u = 0 \quad (11)$$

for all k of a complete set. In our case, the state $|0\rangle_u$ satisfies

$$a_k |0, N\rangle_u = \delta_{k,0} N^{1/2} |0, N-1\rangle_u \neq 0. \quad (12)$$

It is not possible, as with fermions, to just relabel annihilation and creation operators and thus avoid the difficulty, since

$$a_0^+ |0, N\rangle_u = (N+1)^{1/2} |0, N+1\rangle_u \neq 0. \quad (13)$$

In the past, the 0-momentum single-particle state has therefore been treated only approximately² (except by Belyaev¹). The standard approximation has been the replacement of a_0 and a_0^+ by a c-number $n_0^{1/2}$ which is later determined so as to minimize the ground state energy of the system. This eliminates the 0-momentum state and makes $|0\rangle_u$ a vacuum for the operators which remain, but it has the disadvantage that terms of the type $n_0^{1/2} (2V)^{-1} a_1^+ a_2 a_3$ arise in $\underline{E}$. These terms do not commute with $\underline{N}$, so that it is no longer possible to find energy eigenvalues for a system with a fixed number of particles.

We therefore introduce a new transformation to eliminate the 0-momentum state. The new transformation has the advantages that a) it involves no approximation, as does the replacement of a_0 and a_0^+ by $n_0^{1/2}$, and b) it therefore introduces no number non-conserving terms into the Hamiltonian.

III. TRANSFORMATION TO EXCITATION OPERATORS

From the commutation relations (4), we have

$$a_0 n_0 = (n_0 + 1) a_0. \quad (14)$$

and therefore, by the spectral theorem,

$$a_0 f(n_0) = f(n_0 + 1) a_0. \quad (15)$$

Consider, then, the operators

$$\begin{aligned} b_k &= n_0^{-1/2} a_0^+ a_k = a_0^+ (n_0 + 1)^{-1/2} a_k \\ b_k^+ &= a_k^+ a_0 n_0^{-1/2} = a_k^+ (n_0 + 1)^{-1/2} a_0. \end{aligned} \quad (16a)$$

Clearly,

$$b_0 = b_0^+ = n_0^{-1/2}. \quad (16b)$$

Also, since $a_0 a_0^+ = n_0 + 1$, we have

$$\begin{aligned} b_k^+ b_{k'} &= a_k^+ a_{k'} \\ b_k b_k^+ &= a_k a_k^+, \quad k \neq 0, \quad k' \neq 0. \end{aligned} \quad (17)$$

Therefore, for $k \neq 0, k' \neq 0$, we have

$$[b_k, b_{k'}] = 0, \quad [b_k, b_{k'}^+] = \delta_{k, k'} \quad (18)$$

and the b_k, b_k^+ are annihilation and creation operators for quasiparticle excitations of the system which obey Bose statistics and which are just like particles in the sense that their annihilation and creation operators satisfy the same commutation relations that Bose particle annihilation and creation operators satisfy.

Equation (17) apparently does not hold if both sides are applied to a state with $n_0 = 0$. However, we will simply specify that products of operators are to be simplified before being applied to state vectors.

This avoids the difficulty and, moreover, just corresponds to the techniques used in field-theoretic perturbation theory, i. e., one deals for the most

part directly with the operators alone. Another way of seeing that Eq. (17) is correct is to note that $\underline{E}$, Eq. (25), in terms of b 's with the commutation relations (18) acts on any state in exactly the same manner as $\underline{E}$, Eq. (1), in terms of the a 's; both forms of $\underline{E}$, therefore, have the same spectrum.

The a 's do not commute with $\underline{N}$, since they change the number of particles; however, the b 's change the number of quasiparticles but leave the total number of particles unchanged and therefore commute with $\underline{N}$ (as can easily be checked). Thus, there are no longer any particle number non-conserving operators present. If we start with a state with N particles, then all the intermediate states which arise after a certain number of b 's have been applied have the same number, N , of particles. That is, we can completely eliminate the 0-momentum state by first replacing the a 's by b 's, and then giving $\underline{n}_0$ in terms of the b_k , $k \neq 0$:

$$\underline{n}_0 = N - \Sigma' \underline{n}_k \quad (19)$$

where N is a c-number; we have introduced the notation

$$\Sigma' = \Sigma_{k \neq 0} \quad (20)$$

which will be used from now on. The subsidiary condition has now been taken care of and we need only translate $\underline{E}$ into b -language.

Obviously, from (17),

$$\underline{T} = \Sigma' k^2 b_k^\dagger b_k = \Sigma' k^2 \underline{n}_k \quad (21)$$

For the interaction term we first write, using (4) and dropping

the k 's for the moment in an obvious notation:

$$\begin{aligned}
 \underline{E}_{\text{int}} &= (4V)^{-1} \Sigma (v(1-3) + v(2-4)) \delta_{1+2, 3+4} a_1^+ a_2^+ a_3 a_4 \\
 &= (4V)^{-1} \Sigma (v(1-3) + v(3-1)) \delta_{1+2, 3+4} a_1^+ a_2^+ a_3 a_4 \\
 &= (2V)^{-1} \Sigma w(1-3) \delta_{1+2, 3+4} a_1^+ a_2^+ a_3 a_4 \quad (22)
 \end{aligned}$$

where

$$w(k) = 1/2(v(k) + v(-k)), \quad w(k) = w(-k) = w^*(k). \quad (23)$$

Now, we use (17) and (18) to write

$$\begin{aligned}
 \underline{E}_{\text{int}} &= (2V)^{-1} \Sigma w(1-3) \delta_{1+2, 3+4} (a_1^+ a_3 a_2^+ a_4 - \delta_{2,3} a_1^+ a_4) \\
 &= (2V)^{-1} \Sigma w(1-3) \delta_{1+2, 3+4} (b_1^+ b_3 b_2^+ b_4 - \delta_{2,3} b_1^+ b_4). \quad (24)
 \end{aligned}$$

Going over from Σ to Σ' requires taking out all the zero momentum subscripts; the result is, with (16b):

$$\begin{aligned}
 \underline{E}_{\text{int}} &= \underline{E}_1 + \underline{E}_2' + \underline{E}_2'' + \underline{E}_3 + \underline{E}_4 \\
 \underline{E}_1 &= (w(0)/2V)N(N-1) \\
 \underline{E}_2' &= (N/V) \Sigma' w(k) \underline{n}_k - V^{-1} \Sigma' \underline{n}_k \Sigma' w(k) \underline{n}_k \quad (25) \\
 \underline{E}_2'' &= \Sigma' (w(k)/2V) (b_k^+ b_{-k}^+ (\underline{n}_0 (\underline{n}_0 - 1))^{1/2} + (\underline{n}_0 (\underline{n}_0 - 1))^{1/2} b_k b_{-k}) \\
 \underline{E}_3 &= \Sigma' (w(k)/V) (\underline{n}_0^{1/2} b_{k+k'}^+ b_k b_{k'} + b_k^+ b_{k'}^+ b_{k+k'} \underline{n}_0^{1/2}) \\
 \underline{E}_4 &= (2V)^{-1} \Sigma'_{k_1 \neq k_3} w(1-3) \delta_{1+2, 3+4} b_1^+ b_2^+ b_3 b_4
 \end{aligned}$$

It is interesting to see just how the subsidiary condition is taken

care of in (25). $\underline{E}'_2$ and $\underline{E}_4$ change neither the number of quasiparticles nor the number of particles. $\underline{E}''_2$ and $\underline{E}_3$, however, do change the number of quasiparticles; the square-root factors just keep the number of quasiparticles from exceeding the number of particles. They thus insure that the states with the total number of quasiparticles less than or equal to N form an invariant subspace under $\underline{E}$. If we start with a state in this subspace, as we shall, none of the terms in (25) will take us out of the subspace. In a moment we shall see that one of the approximations we make amounts to introducing terms which destroy the invariance of the subspace, but only to relative order N^{-1} or V^{-1} .

Before doing perturbation theory on (25), we put it in normal order. $\underline{E}_4$ is already normal-ordered. Using the fact that the normal pairings are

$$b_{\underline{k}} b_{\underline{k}'} = b_{\underline{k}}^+ b_{\underline{k}'}^+ = b_{\underline{k}}^+ b_{\underline{k}'} = 0, \quad b_{\underline{k}} b_{\underline{k}'}^+ = \delta_{\underline{k}, \underline{k}'}, \quad (26)$$

we obtain for $\underline{E}'_2$

$$\underline{E}'_2 = ((N-1)/V) \sum' w(\underline{k}) \underline{n}_{\underline{k}} - V^{-1} : \sum' \underline{n}_{\underline{k}}, \sum' w(\underline{k}) \underline{n}_{\underline{k}} : \quad (27)$$

In order to put $\underline{E}''_2$ and $\underline{E}_3$ in normal order, we must expand the square roots:

$$\underline{n}_0^{1/2} = (N - \sum' \underline{n}_{\underline{k}})^{1/2} = N^{1/2} \sum_r \binom{1/2}{r} (-)^r (\sum' \underline{n}_{\underline{k}}/N)^r. \quad (28)$$

Then

$$(\sum' \underline{n}_{\underline{k}}/N)^r b_{\underline{k}+\underline{k}'}^+ b_{\underline{k}} b_{\underline{k}'} = : (\sum' \underline{n}_{\underline{k}}/N)^r b_{\underline{k}+\underline{k}'}^+ b_{\underline{k}} b_{\underline{k}'} : + \quad (29)$$

$$(r(r+1)/2N) : (\sum' \underline{n}_{\underline{k}}/N)^{r-1} b_{\underline{k}+\underline{k}'}^+ b_{\underline{k}} b_{\underline{k}'} : + \dots$$

so that

$$\begin{aligned}
 n_0^{1/2} b_{k+k'}^+ b_k b_{k'} &= N^{1/2} \sum_0^\infty \binom{\frac{1}{2}}{r} (-)^r : (\sum' n_k / N)^r b_{k+k'}^+ b_k b_{k'} : + \\
 &\quad N^{-1/2} \sum_0^\infty \binom{\frac{1}{2}}{r} (-)^r (r(r+1)/2) : (\sum' n_k / N)^{r-1} b_{k+k'}^+ b_k b_{k'} : + \dots \\
 &= N^{1/2} \sum_0^\infty \left[\binom{\frac{1}{2}}{r} (-)^r + \binom{\frac{1}{2}}{r+1} (-)^{r+1} (r+1)(r+2)/2N \right] \times \\
 &\quad \times : (\sum' n_k / N)^r b_{k+k'}^+ b_k b_{k'} : \quad (30)
 \end{aligned}$$

We now specify that we are going to consider only systems with N and V so large that we can neglect terms of relative order N^{-1} or V^{-1} wherever they appear. Then only the first term in (30) need be retained and we see that

$$E_3 = : \sum' (w(k)/V) (n_0^{1/2} b_{k+k'}^+ b_k b_{k'} + b_k^+ b_{k'}^+ b_{k+k'} n_0^{1/2}) : \quad (31)$$

and, by a similar procedure,

$$E_2'' = : \sum' (w(k)/2V) (b_k^+ b_{-k}^+ n_0 + n_0 b_k b_{-k}) : \quad (32)$$

It is just here that we have destroyed the invariance of the subspace with N or fewer than N quasiparticles. As we have just seen, however, we only introduce errors of relative order N^{-1} or V^{-1} ; the correction only affects that part of a state which has $n_0 \leq 1$. Moreover, there is no particle nonconservation, but the number of quasiparticles is allowed to be greater than the number of particles.

IV. PERTURBATION THEORY

We now treat the Hamiltonian by using field-theoretic perturbation theory. For the time being we will drop E_1 , since it is just a constant.

We let

$$\underline{E}_u = \underline{T} + (N/V) \sum' w(k) \underline{n}_k = \sum' (k^2 + (N/V)w(k)) \underline{n}_k = \sum' \epsilon(k) \underline{n}_k \quad (33)$$

be our "unperturbed" Hamiltonian, where we have used the fact that $N-1$ is the same as N , owing to the size of the system. Again, the eigenvectors of $\underline{E}_u$ are just the $|n_k\rangle'$, where the prime means that only the n_k for $k \neq 0$ are given. The eigenvalues of $\underline{E}_u$ are then

$$\underline{E}_u |n_j\rangle' = \sum_k \epsilon(k) n_k |n_j\rangle'. \quad (34)$$

If $\epsilon(k) > 0$ for $k \neq 0$, then the unperturbed ground state is the state $|0\rangle'$:

$$b_k |0\rangle' = 0 \quad k \neq 0. \quad (35)$$

We treat only the case in which $\epsilon(k) > 0$ for all k ; this means that the two-body force is repulsive, since we have

$$w(0) = \int u(r) d^3r > 0. \quad (36)$$

We now go over directly to the diagrammatic representation of the S operator without giving the intermediate analysis, since the latter is of the standard type. The results of this straight-forward analysis are the following correspondences between the matrix elements of the S operator and Feynman diagrams.

To a directed quasiparticle line with wave number k and frequency λ corresponds an unperturbed quasiparticle Green's function $G_0^u(k, \lambda)$:

$$G_0^u(k, \lambda) = (i/2\pi) (\lambda - \epsilon(k) + i0)^{-1}. \quad (37)$$

The second term in (27) gives the vertex shown in Fig. 1 with corresponding factor

$$(2\pi i w(k)/V) \delta(\lambda''' + \lambda' - \lambda'' - \lambda). \quad (38)$$

The N terms in (32) give Figs. 2a and 2b, each corresponding to the same factor

$$(-2\pi i N w(k)/V) \delta(\lambda + \lambda'), \quad (39)$$

where the factor $1/2$ goes out because the term always contributes twice.

The $\sum_k$ terms in (32) give Figs. 2c and 2d with the common factor

$$(2\pi i w(k)/V) \delta(\lambda + \lambda' + \lambda'' - \lambda'''). \quad (40)$$

In $\underline{E}_3$ (31) the square roots must be expanded in order to obtain the correspondence with Feynman diagrams; the result is Figs. 3a and 3b, each with the corresponding series

$$(-2\pi i w(k_1)/V) \sum_{m=0}^{\infty} [(d/dx)^m (N-x)^{1/2}]_{x=0} \delta(\lambda_1 + \lambda_2 + \sum \lambda^{(i)} - \lambda_3 - \sum \mu^{(i)}). \quad (41)$$

Finally, $\underline{E}_4$ (25) gives Fig. 4 and

$$(-2\pi i w(k_1 - k_3)/V) \delta_{k_1 + k_2, k_3 + k_4} \delta(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4) \quad (42)$$

In addition to these vertex and quasiparticle line factors, there are just a few combinatorial ones: first, a factor $(r!)^{-1}$ for each set of r identical pieces in a diagram, and second, an extra factor of $1/2$ for certain isolated strongly connected graphs (definition later). This latter factor is to compensate for the fact that in certain cases one too many factors of $1/2$ have been taken out. As usual, intermediate k 's are to be summed over their possible values except zero, and intermediate λ 's are to be integrated from $-\infty$ to ∞ .

2a

2b

2c

2d

3a(i)

(ii)

+ --- +

(m+i)

+ ---

3b(i)

(ii)

(m+i)

+ ---

We shall now show that the dotted lines shown in certain of the vertices can be eliminated by a simple procedure. We shall then be left with the vertices without dotted lines, Figs. 2a, 2b, 3a(i), 3b(i) and 4. These latter are just what one has in a theory which starts by replacing a_0^+ and a_0 by $N^{1/2}$ everywhere in $\underline{E}$ (1). However, we shall see that the elimination of the dotted line leaves its mark. First, it changes the single quasiparticle energies, that is to say, it leads to self-energy terms. Second, the factor that remains in the vertices without dotted lines is not $N^{1/2}$, but rather a "renormalized" value which turns out to be $\langle n_0 \rangle^{1/2}$.

V. REMOVAL OF DOTTED LINES; THE RENORMALIZED PROBLEM

Some terminology will now be introduced. A graph is part of a diagram; a connected graph is one with no disconnected parts; a strongly connected graph is a connected graph which cannot be separated into two or more disconnected parts by cutting dotted lines only; and an isolated strongly connected graph is a strongly connected graph with no external quasiparticle lines.

Consider a dotted line within a strongly connected graph, for example, Fig. 5a, which uses the vertex of Fig. 1. By assumption the graph obtained by removing the dotted line is also a strongly connected graph, for example, 5b corresponds to removing the dotted line from 5a. Now the ratio of the factors associated with these two graphs is of order V^{-1} , since there is the same number of k -sums in both and therefore

=

the extra V^{-1} from (38) is not compensated by a V arising from a possible extra k -sum. We may therefore neglect any diagram that has a dotted line entirely within one of its strongly connected graphs.

It is easy to see that to each isolated strongly connected graph without internal dotted lines corresponds a factor of order V . This follows directly from an analysis of the number of factors N and V and the number of independent k -sums associated with such a graph and has been noted previously.^{1,2}

Consider then a strongly connected graph which is connected to the rest of the diagram by a single dotted line, for example, 5c. If both parts which are connected by the single dotted line have external quasiparticle lines, then the argument given just above applies and the diagram is negligible. However, if one or both of the parts is isolated, then the V^{-1} from the dotted line is compensated by a factor V due to the isolated strongly connected graph. Clearly, a diagram in which two strongly connected parts, at least one of which is isolated, are connected by s dotted lines is of relative order V^{-s+1} and therefore negligible for $s > 1$.

We thus have the following general result: the dotted lines need only be considered when they connect strongly connected graphs. To be more precise, a connected graph with dotted lines is nonnegligible if and only if it contains isolated strongly connected graphs and at most one strongly connected graph with external quasiparticle lines and if each of the dotted lines connects two parts of the graph which

would be disconnected if the dotted lines were not present.

We can now go back and examine the effects of the various possible nonnegligible dotted lines. Consider first the one in Fig. 1. According to the rule above, it must appear as shown in Fig. 6a or else as in Fig. 6b (or perhaps both simultaneously, but then only in a "vacuum" energy term where it is just exactly taken into account by considering only Figs. 6a and 6b; investigation shows that the factors of 2 and 1/2 all occur in exactly the proper places). Fig. 6a, summed over all possibilities for the box, gives a self-energy which can be represented graphically by Fig. 7. We will give self-energies μ in such a way that the factor corresponding to Fig. 7 is $-2\pi i\mu$. This means that self-energies μ add directly to the unperturbed self-energy $\epsilon(k)$. The self-energy corresponding to Fig. 6a is

$$\mu_1(k) = -(w(k)/V)(N - \bar{n}_0) \quad (43)$$

where $\bar{n}_0$ is defined by

$$N - \bar{n}_0 = \sum_k \int_{-\infty}^{\infty} d\lambda [G(k, \lambda) - G_0^u(k, \lambda)] = \sum_k \int_{\text{up}} d\lambda G(k, \lambda) \quad (44)$$

and $G(k, \lambda)$ is the total (perturbed) single quasiparticle Green's function.

The notation is appropriate in that it follows from the standard techniques³ that $\bar{n}_0$ is in fact the expectation value of $\underline{n}_0$ in the perturbed ground state.

By $\int_{\text{up}}$ we mean that the integral must be taken around a contour enclosing the upper half λ plane. The second part of (44) follows from the fact that G_0^u has poles only in the lower half plane.⁴

Figure 6b gives the self-energy

$$\begin{aligned}\mu_2 &= -V^{-1} \sum_k \int_{-\infty}^{\infty} d\lambda w(k) [G(k, \lambda) - G_0^u(k, \lambda)] \\ &= -V^{-1} \sum_k \int_{\text{up}} d\lambda w(k) G(k, \lambda).\end{aligned}\quad (45)$$

Figure 2c gives a self-energy, Fig. 8a, of

$$\mu_3 = -(2V)^{-1} \sum_k \int d\lambda w(k) G_{+-}(k, \lambda) \quad (46)$$

where the factor $1/2$ is due to the fact that G_{+-} alone would give every graph in μ_3 (except those with a factor of $1/2$ due to symmetry) twice. Here $G_{+-}(k, \lambda)$ is the Green's function indicated by Fig. 9, i.e., it is the sum of all possible connected Feynman graphs of the form shown in Fig. 9. Figure 2c also gives the vertex correction shown in Fig. 8b, with the factor

$$(2 \pi i w(k)/V)(N - \bar{n}_0) \delta(\lambda + \lambda') . \quad (47)$$

Clearly, since the Green's function indicated in Fig. 10 is equal, term by term, to that in Fig. 9, it follows that the self-energy arising from Fig. 2d is also μ_3 . The vertex correction is of the form Fig. 8c, with the same factor (47).

The dotted lines in Fig. 3 again give both self-energy terms and vertex corrections. The vertex correction due to the dotted lines in Fig. 3a comes from the graphs in which all the lines arising from the factors $\bar{n}_k$ are closed as shown in Fig. 11a. The self-energy is due to graphs of the type shown in Fig. 11b. For the vertex correction we find

11a

11b

11c

12

13

14

15

16

$$\begin{aligned}
& (-2\pi i w(k_1)/V) \sum_1^{\infty} \left[\frac{1}{\ell!} \left(\frac{d}{dx} \right)^{\ell} (N-x)^{\frac{1}{2}} \right]_{x=0} (N-\bar{n}_0)^{\ell} \\
& = -2\pi i \bar{n}_0^{\frac{1}{2}} w(k_1)/V + 2\pi i N^{\frac{1}{2}} w(k_1)/V .
\end{aligned} \tag{48}$$

Fig. 11b gives the self-energy

$$\begin{aligned}
\mu_4 &= V^{-1} \left\{ \sum_1^{\infty} \left[\frac{1}{(\ell-1)!} \left(\frac{d}{dx} \right)^{\ell} (N-x)^{\frac{1}{2}} \right]_{x=0} (N-\bar{n}_0)^{\ell-1} \right\} \cdot \sum_{k_1 k_2} \int_{up} d\lambda_1 \int d\lambda_2 w(k_1) G_a(k_1 k_2, \lambda_1 \lambda_2) \\
&= -(2\bar{n}_0^{\frac{1}{2}} V)^{-1} \sum_{k_1 k_2} \int_{up} d\lambda_1 \int d\lambda_2 w(k_1) G_a(k_1 k_2, \lambda_1 \lambda_2) ,
\end{aligned} \tag{49}$$

where G_a is the Green's function shown in Fig. 12. Figure 3b gives a self-energy μ_4 again and a vertex correction indicated in Fig. 11c with a factor given by (48). Thus, the effect of all the self-energy terms is to change the denominator of the single quasiparticle Green's function from

$$\lambda - k^2 - Nw(k)/V$$

to

$$\lambda - k^2 - Nw(k)/V - \mu_1(k) - \mu_2 - 2\mu_3 - 2\mu_4$$

which is, with (43),

$$\lambda - k^2 - \bar{n}_0 w(k)/V - \mu \tag{50}$$

where

$$\mu = \mu_2 + 2\mu_3 + 2\mu_4 \tag{51}$$

is independent of k .

The vertices that result when similar vertices are combined are shown in Figs. 13, 14 and 4. The factor associated with Fig. 13

is the sum of (39) and (47) and is

$$(-2\pi i \bar{n}_0 w(k)/V) \delta(\lambda + \lambda'). \quad (52)$$

The factor associated with each of the vertices in Fig. 14 is the sum of (48) and the term with $m = 0$ in (41):

$$(-2\pi i \bar{n}_0^{\frac{1}{2}} w(k_1)/V) \delta(\lambda_1 + \lambda_2 - \lambda_3). \quad (53)$$

The factor associated with Fig. 4 is still (42).

In order to simplify the future treatment, we define our new single quasiparticle Green's function $G_0(k, \lambda)$ to be

$$G_0(k, \lambda) = (i/2\pi)(\lambda - k^2 - \mu + i0)^{-1} \quad (54)$$

and incorporate the term $\bar{n}_0 w(k)/V$ in (50) as the vertex shown in Fig. 15 with factor

$$(-2\pi i \bar{n}_0 w(k)/V) \delta(\lambda - \lambda'). \quad (55)$$

We are then able to restate the diagram rules in a very simple form. There are directed quasiparticle lines, Fig. 16a, with corresponding factors G_0 (54) with appropriate wave number and frequency. There are wavy interaction lines, Fig. 16b, with associated factors $D_0(k, \lambda)$:

$$D_0(k, \lambda) = -2\pi i w(k)/V \quad (56)$$

and two types of vertices as shown in Figs. 16c and 16d. The "incomplete" vertices in Fig. 16c have a factor $\bar{n}_0^{\frac{1}{2}}$; the "complete" vertex in Fig. 16d has a factor of unity associated with it. In addition there are

just the few combinatorial factors mentioned directly after (42).

Wave number and frequency are conserved at each vertex.

The previous paragraph defines what we shall call the renormalized problem. What we have done up to this point is to prove that the renormalized problem is equivalent to the problem specified by the original Hamiltonian (1) with the subsidiary condition that the total number of particles be N , the proof being valid if N and the volume of the system are large enough. In the renormalized problem μ is given by (51), (45), (46) and (49), while $\bar{n}_0$ is given by (44). The equations for μ and $\bar{n}_0$ are therefore implicit equations; this is the price that is paid for the diagrammatic simplicity of the renormalized problem.

The interpretation of the renormalized problem as involving "interactons" (the wavy lines) which propagate with infinite velocity, corresponding to D_0 being independent of frequency, is obvious. In this picture $\bar{n}_0^{1/2}$ is just the coupling constant for transformation of an interacton into a quasiparticle or vice versa. Corrections for finite velocity of propagation of the interaction would then introduce a frequency dependence of D_0 .

VI. PROPER PARTS AND GREEN'S FUNCTIONS

Because of the simplicity of the rules for Feynman diagrams in the renormalized problem, it is advantageous to introduce the proper parts shown in Fig. 17, where each of the proper parts represents all

possible connected graphs which cannot be disconnected by cutting a single line of either type. Since k^2 and $w(k)$ are even in k , Π , P , Σ and T are even in k . Also, it is clear that Π and T are even in λ .

We introduce the Green's functions shown in Fig. 18. All are even in k and G_{+-} and D are also even in λ . As usual, the Green's functions include the factors for the end lines while the proper parts do not.

Figure 19 shows diagrammatically the equations for finding some of the G 's in terms of the proper parts; the equations themselves are

$$\begin{aligned} G &= G_0 + G\Sigma G_0 + G_{+0}PG + G_{+-}TG_0, \\ G_{+0} &= GPD_0 + G_{+0}\Pi D_0 + G_{+-}P^{(-)}D_0, \text{ and} \\ G_{+-} &= GTG_0^{(-)} + G_{+0}P^{(-)}G_0^{(-)} + G_{+-}\Sigma^{(-)}G_0^{(-)}, \end{aligned} \quad (57)$$

where a $(-)$ superscript indicates that the arguments of the corresponding function are $-k$ and $-\lambda$. All other arguments are k, λ . The solution of (57) is

$$\begin{aligned} G &= \Delta^{-1} \begin{vmatrix} D_0^{-1} - \Pi & -P^{(-)} \\ -P^{(-)} & G_0^{(-)-1} - \Sigma^{(-)} \end{vmatrix} \\ G_{+0} &= \Delta^{-1} \begin{vmatrix} -P^{(-)} & G_0^{(-)-1} - \Sigma^{(-)} \\ -P & -T \end{vmatrix} \\ G_{+-} &= \Delta^{-1} \begin{vmatrix} -P & D_0^{-1} - \Pi \\ -T & -P^{(-)} \end{vmatrix} \end{aligned} \quad (58)$$

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$$\begin{aligned}
 \leftarrow \bigcirc \leftarrow &= \leftarrow + \leftarrow \bigcirc \leftarrow \square \leftarrow \\
 &+ \leftarrow \bigcirc \text{---} \square \leftarrow \\
 &+ \leftarrow \bigcirc \rightarrow \square \leftarrow
 \end{aligned}$$

$$\leftarrow \bigcirc \text{---} = \leftarrow \bigcirc \leftarrow \square \text{---} + \leftarrow \bigcirc \text{---} \square \text{---} + \leftarrow \bigcirc \rightarrow \square \text{---}$$

$$\leftarrow \bigcirc \rightarrow = \leftarrow \bigcirc \leftarrow \square \rightarrow + \leftarrow \bigcirc \text{---} \square \rightarrow + \leftarrow \bigcirc \rightarrow \square \rightarrow$$

where

$$\Delta = \begin{vmatrix} G_0^{-1} - \Sigma & -P & -T \\ -P & D_0^{-1} - \pi & -P^{(-)} \\ -T & -P^{(-)} & G_0^{(-)-1} - \Sigma^{(-)} \end{vmatrix} \quad (59)$$

Δ is even in k and λ . By a similar procedure we find

$$D = \Delta^{-1} \begin{vmatrix} G_0^{-1} - \Sigma & -T \\ -T & G_0^{(-)-1} - \Sigma^{(-)} \end{vmatrix} \quad (60)$$

As usual, the poles of these Green's functions are related to excitation energies of the system. In the present case, we have Green's functions for quasiparticles which involve no change in the number of particles. The zeros of Δ are therefore excitation energies of the system.

VII. GROUND STATE ENERGY, μ , AND $\bar{n}_0$

In the renormalized problem, every isolated graph is of order VT , where V is as before and the T comes from the fact that such a graph has an extra frequency delta function which is to be interpreted as

$$\delta(0) = \lim_{T \rightarrow \infty} (2\pi)^{-1} \int_{-T/2}^{T/2} e^{i0t} dt = \lim_{T \rightarrow \infty} \frac{T}{2\pi} \quad (61)$$

Let $LT/2\pi$ be the sum of all the different diagrams each consisting of a single isolated graph. Then the factorial factors for diagrams with identical isolated graphs ensure that the sum of all diagrams with no

external lines is $\exp(LT/2\pi)$. This is related to the ground state energy E_0 by

$$e^{-iE_0 T} = e^{LT/2\pi} \quad (62)$$

or

$$E_0 = iL/2\pi .$$

Actually, of course, we must still add E_1 so that

$$E_0 = E_1 + iL/2\pi . \quad (63)$$

Consider now $(\partial E_0 / \partial \mu)_{\bar{n}_0}$, where we mean that μ occurs in each term only in G_0 (54) and $\bar{n}_0$ only at the incomplete vertices, Fig. 16c. Let L_i be a graph of L . Then $(i/2\pi)(\partial L_i / \partial \mu)_{\bar{n}_0}$ is L_i with each quasiparticle Green's function doubled up. From Fig. 6a it is clear that this is just what is needed for $N - \bar{n}_0$:

$$N - \bar{n}_0 = \frac{i}{2\pi} \left(\frac{\partial L}{\partial \mu} \right)_{\bar{n}_0} = \left(\frac{\partial E_0}{\partial \mu} \right)_{\bar{n}_0} . \quad (64)$$

The factor of $1/2$ in certain E_0 graphs just takes care of the fact that some terms in $\bar{n}_0$ would seem to occur more than once in (64).

We now consider μ . Diagrammatically, we can represent μ_2 , $2\mu_3$ and $2\mu_4$ as in Fig. 20a, b and c; each figure must be multiplied by $(2\pi i \bar{n}_0^{-1/2})^{-1}$ and integrated over the k and λ which appear. Clearly the sum is just Fig. 20d, and therefore

$$\mu = (2\pi i \bar{n}_0^{-1/2})^{-1} \sum_{\text{up}} \int d\lambda G_{+0}(k, \lambda) . \quad (65)$$

It is also evident that if G_{+0}^i is a term of G_{+0} then $\bar{n}_0^{-1/2} \sum_{\text{up}} \int G_{+0}^i d\lambda$ is a term in E_0 . However, each term L_i in E_0 corresponds to just

as many terms in $\bar{n}_0^{-1/2} \sum' \int_{\text{up}} G_{+0}^i d\lambda$ as there are incomplete vertices of a particular type in L_i . Since the numbers of incomplete vertices of the two types are equal, the number of either is just the exponent of $\bar{n}_0$:

$$\bar{n}_0^{-1/2} \sum' \int_{\text{up}} G_{+0} d\lambda = \bar{n}_0 \frac{\partial L}{\partial \bar{n}_0} \quad (66)$$

and therefore

$$\mu = \frac{1}{2\pi i} \frac{\partial L}{\partial \bar{n}_0} = - \frac{\partial E_0}{\partial \bar{n}_0} \quad (67)$$

Equation (67) is almost Belyaev's rule for determining μ .

However, Belyaev has $\mu = -\partial E_0 / \partial N$, while the above derivation shows that $\bar{n}_0$ is correct.

VIII. NONEXISTENCE OF ENERGY GAP

We can also describe the limiting values of the proper parts in terms of derivatives of E_0 . For example, $\Sigma(0,0)$ with its external lines removed is just like $L/\bar{n}_0$, except that each term in E_0 gives a² contributions to $\Sigma(0,0)$, where a is the number of incomplete vertices of a given type.

Therefore

$$\Sigma(0,0) = \frac{\partial}{\partial \bar{n}_0} \bar{n}_0 \frac{\partial L}{\partial \bar{n}_0} = \frac{\partial L}{\partial \bar{n}_0} + \bar{n}_0 \frac{\partial^2 L}{\partial \bar{n}_0^2} \quad (68)$$

$P(0,0)$ without its external lines is $L/\bar{n}_0^{1/2}$ with a doubled quasiparticle Green's function (except for $\bar{n}_0^{1/2}$ which appears in lowest order):

$$P(0,0) = \frac{i}{2\pi} \frac{\partial}{\partial \mu} \bar{n}_0^{-1/2} \frac{\partial L}{\partial \bar{n}_0} + \bar{n}_0^{-1/2} \quad (69)$$

Similarly

$$T(0,0) = \bar{n}_0 \frac{\partial^2 L}{\partial \bar{n}_0^2} \quad (70)$$

$$\pi(0,0) = \left(\frac{i}{2\pi} \frac{\partial}{\partial \mu}\right)^2 L \quad (71)$$

Therefore we have

$$\Delta(0,0) = \begin{vmatrix} 2\pi i\mu - \frac{\partial L}{\partial \bar{n}_0} - \bar{n}_0 \frac{\partial^2 L}{\partial \bar{n}_0^2} & (-\bar{n}_0^{-\frac{1}{2}} - \frac{i\bar{n}_0^{\frac{1}{2}}}{2\pi} \frac{\partial^2 L}{\partial \mu \partial \bar{n}_0}) & -\bar{n}_0 \frac{\partial^2 L}{\partial \bar{n}_0^2} \\ -\bar{n}_0^{-\frac{1}{2}} - \frac{i\bar{n}_0^{\frac{1}{2}}}{2\pi} \frac{\partial^2 L}{\partial \mu \partial \bar{n}_0} & \frac{iV}{2\pi w(0)} - \left(\frac{i}{2\pi}\right)^2 \frac{\partial^2 L}{\partial \mu^2} & -\frac{i\bar{n}_0^{\frac{1}{2}}}{2\pi} \frac{\partial^2 L}{\partial \mu \partial \bar{n}_0} - \bar{n}_0^{-\frac{1}{2}} \\ -\bar{n}_0 \frac{\partial^2 L}{\partial \bar{n}_0^2} & -\bar{n}_0^{-\frac{1}{2}} - \frac{i\bar{n}_0^{\frac{1}{2}}}{2\pi} \frac{\partial^2 L}{\partial \mu \partial \bar{n}_0} & 2\pi i\mu - \frac{\partial L}{\partial \bar{n}_0} - \bar{n}_0 \frac{\partial^2 L}{\partial \bar{n}_0^2} \end{vmatrix} = 0 \quad (72)$$

since the first and third rows are equal when (67) is used.

Thus, $\lambda = 0$ is a zero of $\Delta(0, \lambda)$ and therefore the lowest excitation of the system has zero energy for $k \rightarrow 0$. This is the result that was first proved by Hugenholtz and Pines.⁵

IX. LOWEST ORDER

The incomplete vertices are more important than the complete ones by a factor $\bar{n}_0^{-1/2}$. We therefore define the lowest order to be that which considers the incomplete vertices only to all orders. This is very simply done in terms of the proper parts we have introduced.

With the label 1 designating this lowest order, we have

$$\pi_1 = \Sigma_1 = T_1 = 0, \quad P_1(k, \lambda) = \bar{n}_0^{-\frac{1}{2}} \quad (73)$$

Substitution into (59) then gives

$$\Delta_1 = (2\pi iV/w(k)) [\lambda^2 - (k^2 + \mu)^2 - (2\bar{n}_0 w(k)/V)(k^2 + \mu)] . \quad (74)$$

From the preceding proof of the nonexistence of an energy gap, in particular from (68)-(71), it follows that we must take no contributions to E_0 in this order

$$E_{0(1)} = E_1 \quad (75)$$

and therefore, from (64) and (67),

$$\mu_1 = (N - \bar{n}_0)_1 = 0 \quad (76)$$

and

$$\Delta_1 = (2\pi iV/w(k)) [\lambda^2 - k^2(k^2 + (2Nw(k)/V))] \quad (77)$$

with a spectrum

$$\lambda_1 = k(k^2 + 2\rho w(k))^{1/2}, \quad (78)$$

which is the one that has often been given for this system.

Note that it is helpful, as in (75), to use the procedures of the previous section as a guide for determining what selection of graphs will be consistent with the nonexistence of the gap. Of course, it is possible to find a gap by terminating the perturbation series for E_0 and for the proper parts at points which are not consistent with the proof of the previous section. The proof itself depends, clearly, on convergence of the perturbation series.

X. ACKNOWLEDGMENTS

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FOOTNOTES

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⁴Another way of obtaining the result that the λ integral may be taken
around a contour enclosing the upper half λ -plane is to note that the
box in Fig. 6a represents $\lim_{\Delta t \rightarrow 0-} \sum_k G(k, \Delta t)$

where $G(k, t)$ is the Fourier transform of $G(k, \lambda)$. The time interval
 Δt is $t' - t_0$ where t_0 is the time of creation and t' is the time of
annihilation of the quasiparticle. We have

$$\lim_{\Delta t \rightarrow 0-} \sum_k G(k, \Delta t) = \lim_{\Delta t \rightarrow 0-} \sum_k \int_{-\infty}^{\infty} d\lambda G(k, \lambda) e^{-i\lambda \Delta t} = \sum_k \int_{\text{up}} d\lambda G(k, \lambda).$$

Similar analyses are valid for all λ integrations arising from Green's functions whose incoming and outgoing lines have the same time variable.

⁵N. M. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).