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CURRENT ALGEBRA IN TWO PARTICLE BASIS AND
SINGLE PARTICLE INCLUSIVE PROCESSES

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SINGLE PARTICLE INCLUSIVE PROCESSES*

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ABSTRACT

We discuss several problems involved in the current algebra sum rules derived by Suzuki and his collaborators. In particular we discuss the ambiguities related to soft current insertion to the external lines. Alternative prescriptions to get similar sum rules are also discussed. An application of the sum rule to the W-production process is briefly discussed.

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I. INTRODUCTION

Suzuki and his collaborators^{1,2} recently made interesting applications of current algebra³ to single particle inclusive processes, W-production, and also pion production processes. Their results seem to be in agreement with other predictions based on various other models of high energy reactions; the basic assumptions involved are:

- (i) the "low energy theorem" on the soft photon radiation;
- (ii) simple analytic structure of the production amplitude (namely, dominance of the normal singularities at high energies);
- (iii) duality arguments, which is used to motivate the neglect of 3-body scattering amplitudes; and
- (iv) absence of the contact subtraction terms.

The purpose of the present note is to discuss the ambiguity involved in the use of the "low energy theorem" listed above. This ambiguity is related to the need for subtraction terms in the dispersion relation (not the contact subtraction terms) in a subenergy variable of the 3 to 3 scattering amplitude. The presence of such a subtraction term will affect the lower bound¹ for the W-production amplitude in proton-proton scattering. It is thus useful to discuss other possibilities we may get by using different assumptions on the subtraction terms.

It should be noted however that the low energy theorem by Low⁴ is unambiguously defined. The point is that in deriving sum rules, we make stronger assumptions about non-leading terms in the soft photon amplitude than what is needed to get the Low's result.

The current algebra by Gell-Mann³ in the past has proved to be

very useful when applied to single particle states. The development of high energy accelerators however necessitates the analysis of multiparticle processes. The sum rules derived by Suzuki et al. may be regarded as a first step toward the application of current algebra to multiparticle processes. It is hoped that the present note may help one to understand these sum rules and also their limitations better.

In Section II we briefly discuss the derivation of sum rules given by Sanda and Suzuki¹. We point out the ambiguous point related to the "low energy theorem". In Section III we show a detailed calculation of the "low energy theorem." In Section IV we briefly discuss other possibilities we may get and also several problems involved in deriving sum rules. An application of the sum rules is also discussed.

II. SUM RULES

The sum rules we discuss in the following are based on the algebra³

$$\left[\int d^3y e^{-i\vec{q}\cdot\vec{y}} \mathcal{F}_+^0(\vec{y}, t), \int d^3x e^{i\vec{q}\cdot\vec{x}} \mathcal{F}_-^0(\vec{x}, t) \right] = 2 \int d^3x e^{i(\vec{q}-\vec{q}')\cdot\vec{x}} J_3^0(\vec{x}, t) \quad (II.1)$$

where $\mathcal{F}_\pm^0(\vec{x}, t)$ is the time component of vector or axial vector currents and J_3^0 the time component of isospin current. In the limit $\vec{q}' \rightarrow \vec{q}$ the right hand side of Eq. (II.1) reduces to the isospin charge operator. The existence of a class of diagrams, which become singular in the limit $\vec{q}' \rightarrow \vec{q}$ when sandwiched between 2-particle states, makes the calculation of multi-particle sum rules difficult and ambiguous. Specifically we shall want to calculate

$$\langle p_1' p_2' | \left[\int d^3y e^{-i\vec{q}\cdot\vec{y}} \mathcal{F}_+^0, \int d^3x e^{i\vec{q}\cdot\vec{x}} \mathcal{F}_-^0 \right] | p_1 p_2 \rangle = 2 \langle p_1' p_2' | \int d^3x e^{i(\vec{q}-\vec{q}')\cdot\vec{x}} J_3^0 | p_1 p_2 \rangle \quad (II.2)$$

The evaluation of Eq. (II.2) has been done by Sanda and Suzuki¹, and we briefly repeat their arguments in the following. The right hand side of Eq. (II.2) is given by the well known low energy theorem by Low.⁴ For small

$$k \equiv q - q' \equiv p_1' + p_2' - p_1 - p_2, \quad \text{we get}$$

$$\begin{aligned} & \langle p_1' p_2' | \int d^3x e^{-ik \cdot x} J_3^0(\vec{x}, t) d^3x | p_1 p_2 \rangle \\ &= \langle p_1' p_2' | Y_1 p_1 \rangle (Q_{Y_1} + Q_{p_1}) + \end{aligned} \quad (II.3)$$

$$\begin{aligned} & + Q_{p_1} \left\{ \left[\frac{2p_1' \cdot k - k^2}{2(k \cdot p_1) - k^2} - \frac{2p_2' \cdot k + k^2}{2(k \cdot p_2) + k^2} \right] T \right. \\ & \quad \left. - \left[\frac{(k \cdot p_1')}{(k \cdot p_1)} p_2' \cdot k - p_1' \cdot k + \frac{(k \cdot p_2')}{(k \cdot p_2)} p_2' \cdot k - p_2' \cdot k \right] \frac{\partial T}{\partial v} \right\}_{v=0} (2\pi)^3 \delta(k + p_1 + p_2 - p_1' - p_2') \\ & + Q_{p_2} \left\{ Y_1 \leftrightarrow p_2, \quad p_1' \leftrightarrow p_2' \right\}_{v=0} (2\pi)^3 \delta(k + p_1 + p_2 - p_1' - p_2') \end{aligned}$$

where

$$Q_{p_1} = Q_{p_1'} = \text{isocharge of particle 1}$$

$$Q_{p_2} = Q_{p_2'} = \text{isocharge of particle 2}$$

$$T = T(t, v = p_1 \cdot p_2 + p_1' \cdot p_2')$$

= proton-proton scattering amplitude with momentum transfer

t, averaged over spins involved.

The first term in Eq. (II.3) is given by the naive calculation

$$I_3 = \lim_{k \rightarrow 0} \int d^3x e^{-ik \cdot x} J_3^0(\vec{x}, t) d^3x = \text{isospin charge} \quad (II.4)$$

and it survives only in the limit $k = 0$. The formal derivation of Eq. (II.3) is also discussed in the next section.

Sanda and Suzuki have shown that some set of diagrams like Fig. 1, which they call partially connected diagrams, give rise to the contribution which corresponds to the external line insertion of a "soft photon" to the left hand side of Eq. (II.2).

The diagrams like Fig. 1 also give the contribution which corresponds to the first term of Eq. (II.3). When one subtracts these terms from both sides of Eq. (II.2) the right hand side of the equation contains only the nonsingular terms in the limit $\vec{q}' \rightarrow \vec{q}$. Thus Eq. (II.2)

becomes

$$\begin{aligned}
 & [A_{\pi\pi}^{out} - A_{\pi\pi}^{in}]_{c.c.} (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_2) \\
 & = - [A_{\pi\pi}^{out} - A_{\pi\pi}^{in}]_{p.d.} (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_2) \\
 & + \lim_{\epsilon \rightarrow 0} \left\{ \langle p_1' p_2' | \int e^{-ik \cdot x} J_3(\vec{x}, t) d^3x | p_1 p_2 \rangle \right. \\
 & \quad \left. - [A_{\pi\pi}^{out} - A_{\pi\pi}^{in}]_{p.c.} (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_2) \right\} \quad (II.5)
 \end{aligned}$$

where

$$A_{\pi\pi}^{out} = \sum_n \langle p_1' p_2' | \mathcal{T}_\pi^{out}(0) | n \rangle \langle n | \mathcal{T}_\pi^{in}(0) | p_1 p_2 \rangle (2\pi)^3 \delta_3(p_n - p_1 - p_2 - q) \quad (II.6)$$

In Eqs. (II.5) and (II.6), the average over all spins involved is understood.

$[]_{c.c.}$ denotes the contribution from the completely connected diagrams, $[]_{p.d.}$ means the contribution from "partially disconnected" diagrams like Fig. 2. In other words $[]_{p.d.}$ denotes the contribution from all stable partially connected diagrams. Once one knows the last curly bracket in Eq. (II.5) the derivation of the sum rule is straight-forward.¹ Suzuki and Sanda adopt the following form for the last curly bracket in Eq. (II.5)

$$[Q_\beta(p_1' + p_2') + Q_\beta(p_1 + p_2)] \frac{\partial^2}{\partial p^2} (2\pi)^3 \delta_3(p_1 + p_2 - p_1' - p_2') \quad (II.7)$$

As we shall see in the next section, Eq. (II.7) is strongly model dependent. This ambiguity is actually present in all 3 to 3 forward scattering amplitude. For example, one can consider the diagram in Fig. 3. This diagram corresponds to the process

$$p + p + \pi \rightarrow p + p + \pi \quad (II.8)$$

If the pion is scattered in the forward direction, the internal line in Fig. 3 becomes unstable (i.e., $p_n = p_2$). The propagator for the intermediate state n , in the forward scattering limit, is given by

$$(-i) \frac{1}{E_n - E_{p_2} - i\epsilon} = \pi \delta(E_n - E_{p_2}) + (-i) P \frac{1}{E_n - E_{p_2}} \quad (II.9)$$

The first term in Eq. (II.9) represents the completely uncorrelated two successive processes. The pion-proton vertex in Fig. 3 is completely independent of the proton-proton scattering vertex. The principal value part of Eq. (II.9) on the other hand, gives rise to correlations between these two processes. A priori, it is not clear how to handle these correlations in a well defined way, for they will, in general, depend on the detailed coupling scheme of the pion to the proton. The resulting sum rules will reflect definite prescriptions chosen for treating these correlation terms. The lack of a general principle⁵ to do this for multiparticle states, then causes an ambiguity in the sum rules. The principal value part in Eq. (II.9) is, of course, responsible for the low energy theorem⁴ when one considers a soft current attachment.

III. EVALUATION OF SOFT CURRENT ATTACHMENT

In this section we discuss in detail the evaluation of the curly bracket in Eq. (II.5). For this purpose we start with the calculation of Fig. 2 (we now include single particle intermediate states in this diagram).

This diagram corresponds to the expression (or a part of it)

$$\sum_n \langle p_1' p_1' | F_+^{\circ}(\vec{q}) a_p^{+m} | n \rangle \langle n | F_-^{\circ}(\vec{q}) | p_2 \rangle \quad (\text{III.1})$$

where

$$F_{\pm}^{\circ}(\vec{q}) \equiv \int d^4x e^{\mp i \vec{q} \cdot \vec{x}} \mathcal{F}_{\pm}^{\circ}(\vec{x}, 0) \quad (\text{III.2})$$

Note the difference between (III.1) and the expression

$$\sum_n \langle p_1' p_1' | a_p^{+m} F_+^{\circ}(\vec{q}) | n \rangle \langle n | F_-^{\circ}(\vec{q}) | p_2 \rangle \quad (\text{III.3})$$

Eq. (III.3) corresponds to two completely independent interactions, namely, the current interaction and the hadronic vertex are independent two successive processes. On the other hand Eq. (III.1) includes the mutual correlation of these two vertices and it gives rise to the low energy theorem as well as uncorrelated contributions. Eq. (III.1) can be rewritten as

$$\begin{aligned} & \sum_n \langle p_1' p_1' \times p_2' | F_+^{\circ}(\vec{q}) | n \rangle \langle n | F_-^{\circ}(\vec{q}) | p_2 \rangle \\ & + i \sum_n \int d^4x f_p(x) K_x \langle p_1' p_2' | T(F_+^{\circ}(\vec{q}) \mathcal{G}(x)) | n \rangle \langle n | F_-^{\circ}(\vec{q}) | p_2 \rangle \end{aligned} \quad (\text{III.4})$$

For the sake of simplicity we regard the particle 1 in fig. 2 as a neutral scalar field. This does not change the final conclusion. The notation in Eq. (III.4) is standard one⁶; for example $K_x \equiv \square + m_1^2$. We consider the difference between Eq. (III.4) and the corresponding equation obtained by the exchange $F_{\pm}^{\circ} \leftrightarrow F_{\mp}^{\circ}$. Thus we get

$$\begin{aligned} \tilde{G} & \equiv \langle p_1' p_1' \times p_2' | [F_+^{\circ}(\vec{q}), F_-^{\circ}(\vec{q})] | p_2 \rangle \\ & + i \int d^4x f_p(x) K_x \langle p_1' p_2' | \mathcal{G}(x) \theta(x_0) [F_+^{\circ}(\vec{q}), F_-^{\circ}(\vec{q})] | p_2 \rangle \\ & + i \int d^4x f_p(x) K_x \langle p_1' p_2' | F_+^{\circ}(\vec{q}) \mathcal{G}(x) \theta(-x_0) F_-^{\circ}(\vec{q}) | p_2 \rangle \\ & - i \int d^4x f_p(x) K_x \langle p_1' p_2' | F_-^{\circ}(\vec{q}) \mathcal{G}(x) \theta(-x_0) F_+^{\circ}(\vec{q}) | p_2 \rangle \end{aligned} \quad (\text{III.5})$$

The last two terms in Eq. (III.5) contain a "dangerous" diagram like Fig. 4. This diagram belongs to the so-called "class II states"³ and we neglect them in the infinite momentum limit. Alternatively, one may regard Fig. 4 as a part of the Fig. 5.

We shall discuss Fig. 5 separately. Thus we regard the last two terms in Eq. (III.5) as "stable" apart from the ambiguity associated with the contact subtraction terms. We consider only the first two terms in the following. Upon assuming the validity of single particle current algebra sum-rules we get the expression

$$\begin{aligned} \tilde{G} & = 2 \langle p_1' p_1' \times p_2' | J_3^{\circ}(k) | p_2 \rangle \\ & + 2i \int d^4x f_p(x) K_x \langle p_1' p_2' | \mathcal{G}(x) \theta(x_0) J_3^{\circ}(k) | p_2 \rangle \end{aligned} \quad (\text{III.6})$$

where

$$J_3^0(k) \equiv \int d^4x e^{-i\vec{k}\cdot\vec{x}} J_3^0(\vec{x}, 0) \quad (III.7)$$

$$\vec{k} = \vec{q} - \vec{q}'$$

In the limit $\vec{k} \rightarrow 0$, the first term in Eq. (III.6) gives the result

$$\tilde{G}_1 = 2 \langle p_1' | p_1 \rangle \langle p_1' | p_2 \rangle Q_{p_2} \quad (III.8)$$

with Q_{p_2} the isospin charge of the particle 2. The second term in Eq. (III.6) corresponds to the external soft "photon" insertion on the particle p_2 . See Fig. 6.

We evaluate the second term in Eq. (III.6) following the method used by Soloviev,⁷ although we neglect the existence of logarithmically singular terms found by him.

$$\begin{aligned} \tilde{G}_2 &= 2i \sum_n \int d^4x e^{-i p_1 \cdot x} \langle p_1' p_1' | \eta(x) | n \rangle \langle n | J_3^0(x) | p_2 \rangle \theta(x_0) \\ &= 2i \sum_n (2\pi)^3 \delta_3(p_1' + p_1' - p_1 - p_2) (2\pi)^3 \delta_4(p_n - k - p_2) \\ &\quad \times \frac{\langle p_1' p_1' | \eta(0) | n \rangle \langle n | J_3^0(0) | p_2 \rangle}{i(p_1^0 + p_n^0 - p_1'^0 - p_2^0 - i\epsilon)} \end{aligned} \quad (III.9)$$

where

$$\eta(x) \equiv (\square + m_1^2) \varphi(x) \quad (III.10)$$

and we neglect the contact subtraction terms. To evaluate Eq. (III.9)

we assume the overall energy-momentum conservation

$$p_1' + p_2' = p_1 + p_2 + k \quad (III.11)$$

Let us define

$$\begin{aligned} \tilde{G}_2' &\equiv 2 \sum_n (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_2) (2\pi)^3 \delta_4(p_n - k - p_2) \\ &\quad \times \frac{\langle p_1' p_2' | \eta(0) | n \rangle \langle n | J_3^0(0) | p_2 \rangle}{p_n^0 - p_2^0 - k^0 - i\epsilon} \end{aligned} \quad (III.12)$$

where we replaced $J_3^0(0)$ by $J_3^H(0)$ to use the gauge condition effectively.

However note that not all four components of Eq. (III.12) give invariant results. Only the time component gives rise to an invariant expression at the end. This property of Eq. (III.12) turns out to be very important in the following. Note also that only the single particle intermediate state gives nonzero contribution to the time component of Eq. (III.12) in the limit $\vec{k} \rightarrow 0$. The spatial components receive nonzero contribution from multiparticle intermediate states. This can be seen as follows:

$$\begin{aligned} 0 &= \delta_3(\vec{p}_n - \vec{p}_2) \langle n | \frac{1}{i} \partial_\mu J_3^\mu(0) | p_2 \rangle \\ &= \delta_3(\vec{p}_n - \vec{p}_2) \langle n | [\mathcal{P}_\mu, J_3^\mu(0)] | p_2 \rangle \\ &= \delta_3(\vec{p}_n - \vec{p}_2) \langle n | (p_n - p_2)_\mu J_3^\mu(0) | p_2 \rangle \\ &= \delta_3(\vec{p}_n - \vec{p}_2) (p_n^0 - p_2^0) \langle n | J_3^0(0) | p_2 \rangle \end{aligned}$$

Thus

$$\langle n | J_3^0(0) | p_2 \rangle = 0 \quad \text{for } p_n^0 \neq p_2^0$$

and the denominator of Eq. (III.12) is nonsingular for $p_n^0 \neq p_2^0$. Thus we consider the expression (no summation over n)

$$\tilde{G}_2^\mu(n) = 2 (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_n) (2\pi)^3 \delta_3(p_n - p_2 - k) \times \frac{\langle p_1' p_2' | \eta(0) | n \rangle \langle n | J_3^\mu(0) | p_2 \rangle}{p_n^0 - p_2^0 - k^0 - i\epsilon} \quad (\text{III.13})$$

For a single particle state n , we get the expression

$$\langle n | J_3^\mu(0) | p_2 \rangle = Q_2 (p_n + p_2)^\mu f(p_1, p_n) \quad (\text{III.14})$$

Eq. (III.14) is also valid up to the necessary order in our calculation for spinors after spin averaging. The form factor $f(p_n, p_2)$ is regular around $f(m_2^2)$ and the inner product

$$p_2 \cdot p_n = m_2^2 + O(k^2) \quad (\text{III.15})$$

Thus we can neglect the form factor $f(p_2, p_n)$ altogether as $k \rightarrow 0$.

Next we evaluate the amplitude

$$T \equiv \langle p_1' p_2' | \eta(0) | n \rangle \quad (\text{III.16})$$

The different choice of the variables in T gives different results in Eq. (III.13) to the order of $O(k^0)$. Only the most singular term in Eq. (III.13) is uniquely given, regardless of the choice of variables. Notice that whereas momentum is always conserved in Eq. (III.13) across each vertex, energy is not conserved in the intermediate state. Therefore Eq. (III.16) does not give the "physical" amplitude. If one wants to preserve energy momentum in T by defining

$$\tilde{p}_1 \equiv p_1' + p_2' - p_n \quad (\text{III.17})$$

then particle $\tilde{p}_1$ is not on the mass shell, $\tilde{p}_1^2 \neq m_1^2$. We prefer to regard the amplitude T as deviated from the physical amplitude because of the off-energy shell extension of p_n^0 . Then we shall have 3 independent variables in T , with every particle on the mass shell. This is due to energy nonconservation in the intermediate state; this number of independent variables is naturally the same as we encounter in the Feynman diagram approach.⁴ Thus we write T as

$$T = T((p_1 - p_1')^2, p_1 p_n + p_1' p_2', p_n(p_1' + p_2' - p_1)) \quad (\text{III.18})$$

The last variable in T is analogous to the mass variable in the Feynman amplitude. We thus define two new variables

$$\mathcal{V} = p_1' p_2' + p_1 p_n$$

$$M^2 = p_n (p_1' + p_2' - p_1) \quad (\text{III.19})$$

Note that $p_n^2 = m_2^2 \neq M^2$. Using the two δ -functions in Eq. (III.13) we can calculate the small variation of $\mathcal{V}$ and M^2 as follows:

$$\begin{aligned} \mathcal{V} &= p_1' p_2' + p_1 p_n + p_1 (p_n - p_2) \\ &= p_1' p_2' + p_1 p_2 + k p_1 + p_1' (p_n^0 - p_2^0) \end{aligned} \quad (\text{III.20})$$

$$\begin{aligned} M^2 &= p_n (k + p_2) \\ &= m_2^2 + k p_2 + O(k^2) \end{aligned} \quad (\text{III.21})$$

In Eq. (III.20) and Eq. (III.21) we assumed that k is purely space-like, $k^0 = 0$, as it is in our problem. This is not, however, an essential assumption as we shall see later. We can finally write the result as

$$\begin{aligned} &T((p_1 - p_1')^2, p_1 p_n + p_1' p_2', p_n (p_1' + p_2' - p_1)) \\ &= T((p_1 - p_1')^2, p_1 p_2 + p_1' p_2', m_2^2) \\ &+ \frac{\partial T}{\partial \mathcal{V}} [(k p_1) + p_1' (p_n^0 - p_2^0)] + \frac{\partial T}{\partial M^2} (k p_2) \end{aligned} \quad (\text{III.22})$$

Before we insert this expression into Eq. (III.13) we rewrite the denominator of Eq. (III.13) as

$$\frac{1}{p_n^0 - p_2^0 - i\epsilon} = i\pi \delta(p_n^0 - p_2^0) + \mathcal{P} \frac{1}{p_n^0 - p_2^0} \quad (\text{III.23})$$

The δ -functional part in Eq. (III.23) gives the result

$$2i\pi (2\pi)^3 \delta_4(p_1' + p_2' - p_1 - p_2) \langle p_1' p_2' | \eta^{out} | p_2 \rangle Q_{p_2} \quad (\text{III.24})$$

This term survives only for $k = 0$. Eq. (III.24) combined with the final state soft photon insertion added to $\tilde{G}_1$ in Eq. (III.8) (which should not be counted twice because it represents an uncorrelated process, namely no time ordering) gives rise to the result

$$\langle p_1' p_2' | \eta^{out} | p_1 p_2 \rangle \approx 2Q_{p_2} \quad (\text{III.25})$$

The principal value part of Eq. (III.23) combined with Eqs. (III.13), (III.14) and (III.22) gives (after some rearrangement)

$$\begin{aligned} \tilde{G}_2^{\mu} &= 2Q_{p_2} (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_n) \\ &\times \left\{ - \frac{2p_2^{\mu} + k^{\mu}}{2(k p_2) + k^2} T \right. \\ &- \frac{(k p_1)}{2(k p_2)} 2p_2^{\mu} \frac{\partial T}{\partial \mathcal{V}} \\ &+ \frac{p_1^0}{2p_2^0} 2p_2^{\mu} \frac{\partial T}{\partial \mathcal{V}} \\ &- \frac{2[p_2^{\mu} - \eta^{\mu} p_2^0]}{(2p_2^0)^2} T \end{aligned} \quad (\text{III.26})$$

$$- \frac{(k p_2)}{2(k p_2)} 2 p_2^\mu \frac{\partial T}{\partial H^2} \Bigg\}$$

where T now stands for the first term in Eq. (III.22) and we defined

$$\eta^\mu = (0, 0, 0, 1) \quad \text{or} \quad \eta^0 = 1, \quad \vec{\eta} = 0 \quad (\text{III.27})$$

As we have noted before, only the time component of Eq. (III.26) gives the Lorentz invariant result. Finally adding the final state soft photon

insertion we get

$$\begin{aligned} \tilde{G}^\mu &= 2 Q_{p_2} \langle p_1' p_2' | p_1 p_2 \rangle \delta_0^\mu \\ &+ 2 Q_{p_2} (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_2) \\ &\times \left\{ \left[\frac{2 p_2'^\mu - k^\mu}{2(k p_2') - k^2} - \frac{2 p_2'^\mu + k^\mu}{2(k p_2) + k^2} \right] T \right. \\ &- \left[\frac{2(k p_1')}{2(k p_2')} p_2'^\mu + \frac{2(k p_1)}{2(k p_2)} p_2^\mu \right] \frac{\partial T}{\partial p} \\ &- \left[p_2'^\mu \frac{\partial T}{\partial H^2} + p_2^\mu \frac{\partial T}{\partial H^2} \right] \\ &+ \left[\frac{p_1^\mu}{p_2^\mu} p_2'^\mu + \frac{p_1'^\mu}{p_2'^\mu} p_2^\mu \right] \frac{\partial T}{\partial p} \\ &- \left. \left[\frac{2(p_2'^\mu - \eta^\mu p_2^\mu)}{(2 p_2')^2} + \frac{2(p_2'^\mu - \eta^\mu p_2^\mu)}{(2 p_2'')^2} \right] T \right\} \end{aligned} \quad (\text{III.28})$$

where

$$\tilde{M}^2 \equiv m_s^2 - (k p_2') + O(k^2) \quad (\text{III.29})$$

and we added Eq. (III.25) which corresponds to the time component. Next we want to calculate

$$\langle p_1' p_2' | 2 J_3^0(k) | p_1 p_2 \rangle \quad (\text{III.30})$$

The prescription given by Low⁴ tells that one should retain only the first three terms in Eq. (III.28) and add gauge counter terms. Thus we get

$$\begin{aligned} &2 \langle p_1' p_2' | J_3^0(k) | p_1 p_2 \rangle \\ &= 2 Q_{p_2} \langle p_1' p_2' | p_1 p_2 \rangle \\ &+ 2 Q_{p_2} (2\pi)^3 \delta_3(p_1' + p_2' - p_1 - p_2) \\ &\times \left\{ \left[\frac{2 p_2'^\mu - k^\mu}{2(k p_2') - k^2} - \frac{2 p_2'^\mu + k^\mu}{2(k p_2) + k^2} \right] T \right. \\ &- \left. \left[\frac{(k p_1')}{(k p_2')} p_2'^\mu - p_1'^\mu + \frac{(k p_1)}{(k p_2)} p_2^\mu - p_1^\mu \right] \frac{\partial T}{\partial p} \right\} \delta_{\mu,0} \end{aligned} \quad (\text{III.31})$$

Eqns. (III.28) and (III.31) enables us to make the following observations.

Firstly, the self-consistency condition implied by the local current conservation on the density operator j_3^0 requires that the time component of the second group of Eq. (III.31) must vanish in the limit $k = 0$. This is actually the case as one can confirm it by adding an infinitesimally small mass to the "photon". Note that this vanishing of the time component contribution

has nothing to do with the vanishing of infra-red divergence difficulties. Secondly, the time component of the penultimate term in Eq. (III.28) cancels the time component of the gauge counter terms in Eq. (III.31). Since the time component of the last term in Eq. (III.28) vanishes, only the mass derivative terms in Eq. (III.28) survive the cancellation to give contribution to the sum rule. On the other hand, in the calculation by Sanda and Suzuki¹ the gauge counter terms survived and gave the lower bound to their sum rule.

It is much easier to see this cancellation by directly approaching the limit $\vec{k} = 0$ from the time-like photon momentum in Eq. (III.13). Note that the time component of k is always defined by $k^0 = p^0_1 + p^0_2 - p^0_1 - p^0_2$.

If one uses the Feynman diagram approach⁴ (which is the best defined unambiguous way of calculation) one gets both the gauge counter term and mass derivative terms uncanceled. This suggests that the separation of the infrared radiation amplitude into an external radiation part and an internal radiation part is neither unique nor gauge invariant, although the total radiation amplitude is always unambiguously determined by the gauge condition. (Here again we neglect the logarithmically singular terms found by Soloviev⁷).

Next we would like to discuss the results of Sanda and Suzuki⁸ within our framework. They interpreted Fig. 6 in a dispersion-theoretical context. They therefore took the propagator in Eq. (III.13) as a device for analytic continuation in the subenergy variable p^0_n . That is, they put the particle n in $\langle p, p_c | \eta^{(0)} | n \rangle$ and also in $\langle n | \hat{J}_3^{(0)} | p_c \rangle$ on the energy shell. By this choice the amplitude in Eq. (III.16) becomes a physical amplitude, and one can avoid the appearance of the last term

in Eq. (III.20) and also the "off-mass variable" M^2 in Eq. (III.19). Note that every particle is on the mass shell in our approach. Their results then followed.

Apparently their choice of variables corresponds to the assumption that there is no need for subtraction terms in the analytic continuation in the subenergy variable p^0_n . What we evaluated in Eq. (III.28) are precisely these subtraction terms, indicating the ambiguities inherent in such continuations. The energy nonconservation in the intermediate state was responsible for both the absence of the energy derivative terms and also the presence of the "mass derivative" terms. In this sense the prescription taken by Sanda and Suzuki may be regarded as a simplifying assumption on such mass-derivative terms rather than a derivation.

The evaluation of "mass derivative" terms is extremely difficult. Therefore, the sum rule with "mass derivative" terms is practically of no use. In the next section we discuss another simplifying assumption which corresponds to the vanishing of "mass derivative" terms.

IV. DISCUSSIONS

It may be interesting to see what kind of sum rules we get if we follow the simplest prescription possible. We start with the relation

$$\langle p_1 p_2 | [\int e^{i\vec{q}\cdot\vec{x}} \mathcal{T}_\tau^0(\vec{x}, t), \int e^{i\vec{q}\cdot\vec{x}} \mathcal{T}_\tau^0(\vec{x}, t)] | p_1 p_2 \rangle = 2 \langle p_1 p_2 | I_3 | p_1 p_2 \rangle \quad (IV.1)$$

where I_3 is the isospin operator. The right hand side gives

$$2(Q_{p_1} + Q_{p_2}) \langle p_1 p_2 | p_1 p_2 \rangle \quad (IV.2)$$

The connected part in the left hand side (namely, connected diagrams even in the forward limit) gives rise to the result which delta-functionally mismatches the right hand side given by Eq. (IV.2). One can therefore think that external line insertion of the commutator in the left hand side cancels the right hand side. Thus one gets the result

$$[A_{\tau+}^{\circ\circ} - A_{\tau-}^{\circ\circ}]_{c.c.} (2\pi)^3 \delta_3(p_1 + p_2 - p_1 - p_2) = - [A_{\tau+}^{\circ\circ} - A_{\tau-}^{\circ\circ}]_{p.p.} (2\pi)^3 \delta_3(p_1 + p_2 - p_1 - p_2) \quad (IV.3)$$

See Eq. (II.5). The prescription we shall take corresponds to the simple minded replacement

$$(-i) \frac{1}{E_n - E_{p_2} - i\epsilon} \Rightarrow \pi \delta(E_n - E_{p_2}) \quad (IV.4)$$

in the forward limit. See Eq. (II.9). This is also equivalent to neglecting "mass" derivative terms in Eq. (III.28). One formal advantage of this prescription is that it is manifestly gauge invariant. Eq. (IV.3) is homogeneous and does not give rise to any inequality in contrast to the result obtained by Sanda and Suzuki.¹ Eq. (IV.3) is not particularly useful. However, if one sandwiches the algebra by two in-states and follows the same prescription, one obtains the result⁹

$$[B_{\tau+}^{\circ\circ} - B_{\tau-}^{\circ\circ}]_{c.c.} (2\pi)^3 \delta_3(p_1 + p_2 - p_1 - p_2) = - [B_{\tau+}^{\circ\circ} - B_{\tau-}^{\circ\circ}]_{p.p.} (2\pi)^3 \delta_3(p_1 + p_2 - p_1 - p_2) \quad (IV.5)$$

where

$$B_{\tau\pm}^{\circ\circ} \equiv \sum_n \langle p_1 p_2 | \mathcal{T}_\tau^0(0) | n \rangle \langle n | \mathcal{T}_\tau^0(0) | p_1 p_2 \rangle (2\pi)^3 \delta_3(p_n - p_1 - p_2) \quad (IV.6)$$

See also Eqs. (II.5) and (II.6).

Eq. (IV.5) is analogous to the unitarity sum rules (namely, the optical theorem). It is also the statement that current algebras are saturated by single particle states. When one uses multiparticle states one simply obtains homogeneous sum rules (i.e., a sort of superconvergent sum rule). Eq. (IV.5) was discussed by Shei and Suzuki.² They apply it to a pion production process. In the case of W^+ production, Eq. (IV.5) gives rise to the result

$$\int_{M_W}^{V/\sqrt{s}} \frac{1}{|q|} \left[\frac{d\sigma_{\perp}^{W^+}}{dE d\omega d\theta} + \frac{|q|^2}{M_W^2} \frac{d\sigma_{\perp}^{W^+}}{dE d\omega d\theta} \right]_{\cos\theta=0} dE$$

$$\approx \int_{M_W}^{V/\sqrt{s}} \frac{1}{|q|} \left[\frac{d\sigma_{\perp}^{W^-}}{dE d\omega d\theta} + \frac{|q|^2}{M_W^2} \frac{d\sigma_{\perp}^{W^-}}{dE d\omega d\theta} \right]_{\cos\theta=0} dE \quad (IV.7)$$

where every quantity is defined in the C. M. of the colliding proton system.

$\vec{q}$ = momentum of the W-boson

$E = q^0$ = energy of the W-boson

M_W = mass of the W-boson

$s = (p_1 + p_2)^2$

$v_0 = (s + q^2 - 4m_p^2)/2$

σ_L = cross section for the longitudinally polarized boson

$\sigma_{\perp}$ = cross section for the boson polarized perpendicular to both $\vec{p}_1 - \vec{p}_2$ and $\vec{q}$.

See also Ref. (1) for the detailed assumptions necessary to get Eq. (IV.7).

However, note the condition $\cos\theta = 0$; this is the only direction consistent with the infinite momentum boost. Then what Eq. (IV.7) suggests is the near equality

$$\left. \frac{d\sigma_{\perp}^{W^+}}{d\omega d\theta} \right|_{\cos\theta=0} \approx \left. \frac{d\sigma_{\perp}^{W^-}}{d\omega d\theta} \right|_{\cos\theta=0} \quad (IV.8)$$

Next we would like to comment on the anomalous singularities in the variable $v = (p_1 + p_2)q$. This variable v is related to the missing mass m^2 by

$$m^2 = s + q^2 - v \quad (IV.9)$$

The existence of anomalous complex singularities in m^2 around $m^2 \sim s$ is well known.¹⁰ These complex singularities are likely to approach the real axis in the case of our interest; large q^2 (W-production) and small q^2 (pion radiation). It is necessary to assume that these complex singularities cancel when one takes isospin antisymmetric combination.

Finally, we would like to give a plausible argument for the agreement of Eq. (IV.5), with the "finite energy sum rule" in the inclusive process¹¹ suggested by several models of high energy reactions. The "finite energy sum rule" reads

$$\int_{4M_p^2}^{m_c^2} \left[\frac{d\sigma}{dt dm^2} (p + p \rightarrow \pi^+ + X) - \frac{d\sigma}{dt dm^2} (p + \bar{p} \rightarrow \pi^+ + X) \right] \approx s^{2\alpha(t)-2} \quad (IV.10)$$

where $\alpha(t)$ is the Regge trajectory which couples to the proton and π^+ , and m_c^2 is a cut-off mass ($m_c^2 \ll s$). Eq. (IV.10) is supposed to be good for $t \approx 0$. If one continues it in t to large negative t and takes the limit $s \rightarrow \infty$, one gets

$$\int_{4M_p^2}^{m_c^2} \frac{d\sigma}{dt dm^2} (p + p \rightarrow \pi^+ + X) \Big|_{\cos\theta=0} \approx \int_{4M_p^2}^{m_c^2} \frac{d\sigma}{dt dm^2} (p + \bar{p} \rightarrow \pi^+ + X) \Big|_{\cos\theta=0} \quad (IV.11)$$

At $\cos\theta = 0$ (90° in the C.M.), Eq. (IV.11) also suggests

$$\int_{4M_p^2}^{m_c^2} \frac{d\sigma}{dt dm^2} (p + \bar{p} \rightarrow \pi^+ + X) \Big|_{\cos\theta=0} \approx \int_{4M_p^2}^{m_c^2} \frac{d\sigma}{dt dm^2} (\bar{p} + \bar{p} \rightarrow \pi^+ + X) \Big|_{\cos\theta=0} \quad (IV.12)$$

Thus charge conjugation invariance tells us that

$$\int_{4M_n^2}^{M_c^2} \frac{d\sigma}{d^4k d m^2} (p+p \rightarrow \pi^+ + \chi) \Big|_{\epsilon \rightarrow 0} \approx \int_{4M_n^2}^{M_c^2} \frac{d\sigma}{d^4k d m^2} (p+p \rightarrow \pi^- + \chi) \Big|_{\epsilon \rightarrow 0} \quad (\text{IV.13})$$

This is essentially the result discussed by Shei and Suzuki² based on Eq. (IV.5).

We are grateful to Professor Y. Nambu for helpful discussions.

One of us (K. F.) also thanks Drs. A. Sanda and M. Suzuki for a discussion.

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5. We are grateful to Professor Y. Nambu for a discussion on this point.
6. Our metric is the same as that in J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York 1964).
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9. Even in this case it is possible to perform a formal manipulation similar to that in Section III based on the following relation

$$\begin{aligned} \langle p_1' p_2' | J_3^0(k) | p_1 p_2 \rangle &= \langle p_1' p_2' | J_3^0(k) [a_{p_1}^{+in} - a_{p_1}^{+out} + a_{p_1}^{+out}] | p_2 \rangle \\ &= \langle p_1' p_2' | J_3^0(k) | p_1 p_2 \rangle^{out} \\ &\quad + i \int f_{p_1}(k) K_x \langle p_1' p_2' | J_3^0(k) \phi(x)^\dagger | p_2 \rangle \end{aligned}$$
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However it does not seem to be particularly fruitful to pursue this line. We would rather take Eqs. (IV.4) and (IV.5) as a simplifying assumption.

Figure Captions

Fig. 1: A partially connected diagram.

Fig. 2: A partially disconnected diagram.

Fig. 3: Proton-proton-pion scattering.

Fig. 4: A diagram corresponding to a "class II" state.

Fig. 5: A partially connected diagram; current is attached to "particle 1."

Fig. 6: External soft-current insertion.

Fig.1

Fig.2

Fig.3

Fig.4

Fig.5

Fig.6