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Effect of Exchange and the Pauli Principle^{*†}
on Nucleon-Nucleus Scattering I.

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I. Introduction

This paper is concerned with the elastic and inelastic scattering of nucleons by nuclei and particularly with the effects which arise because of the identity of the incident and emergent nucleon with the nucleons making up the target nucleus. It has been a common practice to describe this scattering process in terms of the interaction of an A particle system, considered as a unit, the target nucleus, with a nucleon. The direct attempt to solve the $(A + 1)$ particle problem has rarely been employed for the calculation of scattering states. It is perforce restricted to light nuclei. On the other hand, the completeness of the target nucleus plus nucleon description is limited according to the extent to which a detailed description of the target nucleus exists. If only the ground state of the target nucleus is included in the description, one is restricted to an optical model. However, for many nuclei information is available with respect to the wave function of the low lying levels of the target nucleus generally expressed in terms of some model; but as the energy of excitation becomes greater our information becomes much more fragmentary and unreliable. Under these circumstances it becomes possible to employ a coupled channel calculation including those reliably known states of the target nucleus in the calculation which are important for the process under consideration. In most treatments, the exclusion principle and the exchange scattering are not considered. Rather it is hoped that the effective energy dependent potentials employed will in some way simulate their effects. If collective coordinates rather than a "microscopic" description of the target nuclear states is used, a discussion of the identical particle effects is automatically precluded.

In this paper we shall discuss a method for the explicit consideration of identical particle effects in the case of the N coupled channel problem. We shall discuss in a qualitative way the problems which anti-symmetrization and exchange scattering raise and how the N coupled channel problem should be formulated so as to eliminate these difficulties. In a subsequent paper one of us (W.F.) will develop this formulation further and apply it to neutron-nucleus scattering.

To introduce the subject in its least complex form, let us first consider the "pure" elastic scattering in which the only open channel is the elastic one. Moreover, let us first examine the impact of the exclusion principle on the properties of the scattering amplitude. It will be a relatively simple manner to generalize to the case of N coupled channels including inelastic open channels as well as the open elastic channel.

Perhaps the most straightforward way of dealing with the anti-symmetrization problem is that outlined by Mott and Massey (1). In this procedure one first obtains a solution ϕ of the Schroedinger equation disregarding the identity of the incident particle and the target nucleons. In this first problem in the asymptotic region in which the nucleon and the target no longer interact, one finds a direct scattering amplitude in which the incident and emergent particles are the same and an exchange scattering amplitude in which the incident particle and target nucleon exchange so that the emergent particle is a target nucleon. Once ϕ is obtained then because of the symmetry of the nuclear Hamiltonian any permutation of particle coordinates in ϕ yields a wave function which is also a solution. The particular one that is of physical interest is

the totally anti-symmetric one. Anti-symmetrizing ϕ produces a solution of the Schroedinger equation which satisfies the Pauli exclusion principle.

Let us describe this procedure more formally. Let $\phi_t(r_1 \dots r_A)$ describe the anti-symmetrized eigenfunctions of the target nucleus with Hamiltonian H_t :

$$H_t \phi_t = \epsilon_t \phi_t \quad (1)$$

Let the solution, ϕ , of the full problem

$$H \phi = \epsilon \phi \quad (2)$$

not satisfying the Pauli principle be in the asymptotic region expanded in terms of ϕ_t as follows:

$$\phi(r_0, r_1 \dots r_A) \rightarrow \sum_t u_t(r_0) \phi_t(r_1 \dots r_A) \quad (3)$$

This sum includes an integration over the continuum eigenstates of H_t as well as the bound states. As indicated in Eq. (3), we are concerned with expansion (3) in the asymptotic region, in which one of the coordinates, r_i approaches infinity. If the limit r_0 to infinity is examined ϕ is found to consist of an incident wave plus an outgoing spherical wave whose amplitude is then the direct scattering amplitude. If on the other hand, if one of the target nucleon coordinates, say r_1 , is allowed to go to infinity, the first terms

in expansion (3) go to zero exponentially. However, the continuum part of the series will in general be singular in such a way as to yield an outgoing wave in variable r_1 . If only elastic scattering is energetically possible this outgoing wave must in the limit of large r_1 have the form

$$\phi \rightarrow v_1(r_1) \phi_0(r_0, r_2, \dots, r_A) \quad r_1 \rightarrow \infty$$

The incident particle and particle r_1 of the target nucleus have exchanged roles and for this reason the corresponding amplitude is referred to as the exchange scattering amplitude. A similar term exists for each coordinate r_i . It is clear that for the traditional method in this relatively naive form, truncation of the series (3) at N terms is not permissible as the exchange scattering would be completely omitted.

The resolution of this difficulty is suggested by the form of the asymptotic amplitude which is obtained upon anti-symmetrization. Suppose asymptotically the direct contribution to the scattering is given by $u_0(r_0) \phi_0(r_1, r_2, \dots, r_A)$ and the exchange amplitudes by $v_i(r_i) \phi_0(r_1, r_2, \dots, r_{i-1}, r_0, r_{i+1}, \dots, r_A)$. Then the anti-symmetrized asymptotic wave function is asymptotically proportional to

$$\psi \rightarrow A \left[u_0(r_0) \phi_0(r_1, \dots, r_A) + \sum_i v_i(r_i) \phi_0(r_1, r_2, \dots, r_{i-1}, r_0, r_{i+1}, \dots, r_A) \right] \quad (4)$$

where A is the anti-symmetrization operator. This can be written

$$\psi \rightarrow A \left[u_0(r_0) - \sum_i v_i(r_i) \right] \phi_0(r_1, \dots, r_A) \quad (5)$$

If we examine the term in the anti-symmetrized Ψ which contains $\phi_0(r_1 \dots r_A)$ by letting r_0 approach infinity we find that the consequent amplitude contains both the direct and exchange scattering amplitudes. Moreover, those terms not included in the asymptotic Ψ of Eq. (5) will be free of singularities. They go to zero exponentially as r_i increases to infinity.

Equation (5) suggests that the appropriate form to be used for the anti-symmetrized solution of Eq. (2) is one with the asymptotic form

$$\Psi \rightarrow A \chi_0(r_0) \phi_0(r_1 \dots r_A) \quad (6)$$

By requiring Ψ to have this asymptotic form, we have not yet guaranteed that χ_0 will contain completely both the direct and exchange scattering amplitudes as in Eq. (5). We must exclude the possibility the terms not explicitly shown in Eq. (6) could contribute in the asymptotic region. For this purpose it is sufficient to require that the difference $[\Psi - A(\chi_0 \phi_0)]$ not asymptotically contain any dependence upon ϕ_0 where the argument of ϕ_0 is any A coordinates taken from the $(A + 1)$ coordinates $(r_0, r_1, \dots, r_A)$. When this supplementary condition is met the amplitude $\chi_0(r_0)$ must contain all of the elastic scattering amplitude. Instead of the $(A + 1)$ amplitudes $u_0, v_1 \dots v_A$ only one amplitude χ_0 is required in order to specify the elastic scattering. The problem of calculating the exchange amplitudes is now replaced by the problem of enforcing the supplementary condition.

It should be noted that the function χ_0 is not unique. It is always possible to add to χ_0 a function χ'_0 which satisfies the condition

$$A \chi'_0(r_0) \phi_0(r_1 \dots r_A) \equiv 0 \quad (7)$$

For example if ϕ_0 were a Slater determinant of single particle wave functions, then a χ'_0 equal to any of these single particle wave functions would satisfy Eq. (7). It is important to know the functions χ'_0 at least implicitly, for a number of reasons. For the present (other reasons will be given below) note that the most efficient description of χ_0 is obtained if it is taken to be orthogonal to all the χ'_0 . For the simple case of the Slater determinant for ϕ_0 this amounts to the trivial remark that the $(A + 1)$ st particle should be put into an excited single particle state not included among the A single particle wave functions making up ϕ_0 . For a more elaborate wave function involving dynamical correlations for ϕ_0 , χ'_0 satisfying Eq. (7) will exist but will not be as readily ascertained.

To summarize, in the single channel case we look for a wave function Ψ of the form Eq. (6) which satisfies the supplementary condition stated immediately below Eq. (6). In addition, we need the solutions χ'_0 of Eq. (7).

These problems were solved in Ref. (2) and (3). A projection operator P was found which when acting in an anti-symmetrized wave function Ψ yields:

$$P\Psi = \mathcal{A} \left[\chi_0 \phi_0 (\underline{r}_1 \dots \underline{r}_A) \right] \quad (8)$$

as well as

$$\langle \phi_0 (\underline{r}_1 \dots \underline{r}_A) (1 - P) \Psi \rangle = 0 \quad \text{for all } \underline{r}_0. \quad (9)$$

We see that by virtue of Eq. (9) that the supplementary condition is satisfied. The Schroedinger equation satisfied by $P\Psi$ involving an "effective Hamiltonian," H_{eff} can be easily obtained (2). Its solution gives a complete solution of the

original problem Eq. (2) satisfying the Pauli exclusion principle. We shall not review the details of this method as it can be easily extracted from the N coupled channel discussion to be given below by putting $N = 1$.

One more point before we leave the one channel case. It is generally more convenient as far as calculations are concerned to deal with the amplitude U_0 rather than χ_0 where U_0 is defined by

$$U_0(r_0) \equiv \langle \phi_0(r_1, r_2, \dots, r_A), \psi(r_0, r_1, \dots, r_A) \rangle \quad (10)$$

From Eq. (9) it follows that

$$U_0(r_0) = \langle \phi_0(r_1, r_2, \dots, r_A), P\psi(r_0, r_1, \dots, r_A) \rangle$$

or

$$U_0(r_0) = \langle \phi_0(r_1, r_2, \dots, r_A), A[\chi_0(r_0) \phi_0(r_1, r_2, \dots, r_A)] \rangle \quad (11)$$

For large values of r_0 , U_0 and χ_0 approach each other so that the scattering amplitude can be obtained from the asymptotic behavior of U_0 or χ_0 . Because χ_0 is not unique in virtue of (7), the relation between U_0 and χ_0 given in Eq. (11) is singular. We can, however, make χ_0 unique by insisting that it be orthogonal to all χ'_0 . Then it becomes possible to invert Eq. (11) and obtain the more interesting χ_0 in terms of the more convenient U_0 .

In the N channel approximation, the expansion of the exact solution in terms of the states of the target nucleus defined by Eq. (1):

$$\Psi = A \sum_{t=0}^{\infty} \chi_t(\underline{r}_0) \phi_t(\underline{r}_1 \dots \underline{r}_A) \quad (12)$$

is truncated at N terms. These N terms should include the elastic channel, the inelastic channels, and important bound states such as doorway states. Let P_N be a projection operator which exactly projects out these N states:

$$P_N \Psi = A \sum_{t=0}^{N-1} \chi_t(\underline{r}_0) \phi_t(\underline{r}_1 \dots \underline{r}_A) \quad (13)$$

The P of Eq. (8) is just P_1 . Indeed we may consider the determination of P_N to be given below as the generalization of the procedure used in the one channel case (2). Just as in the one channel case, supplementary conditions on $P_N \Psi$ are required or are implied by Eq. (13). In particular, we require that the difference $\Psi - P_N \Psi$ not contain any dependence on ϕ_t , $0 \leq t \leq N-1$, where the argument of each ϕ_t is allowed to be any selection of the A coordinates from the $(A+1)$ coordinates, $(\underline{r}_0, \underline{r}_1 \dots \underline{r}_A)$. As in the $(N=1)$ case, this condition guarantees that the amplitudes of physical interest, the elastic scattering as well as the inelastic scattering amplitudes can be obtained from the behavior of $P_N \Psi$ for large r_0 . Moreover, it is now certain that the remainder of Ψ , $(1 - P_N) \Psi$ is well behaved, decays exponentially with large values of any of the coordinates r_i and so does not contribute to the scattering amplitude. Once this separation has been accomplished a valid truncation becomes possible, i.e., one can now approximate Ψ by $P_N \Psi$. Or one can to some approximation take the effect of the states $(1 - P_N) \Psi$ into account by considering the exact equation satisfied by $P_N \Psi$ and in that equation approximating the

terms arising from the coupling of $P_N \Psi$ to $(1 - P_N) \Psi$. This leads to the same form for the equation satisfied by $P_N \Psi$ as that which obtains when truncation is applied except that effective interactions (complex if we are dealing with energy averaged amplitudes) are used.

As in the one channel case it should be realized the set $\{\chi_t\}$ may not be unique. In other words, χ'_t may exist which satisfy the analogue of Eq. (7):

$$\sum_{t=0}^{N-1} \chi'_t(r_0) \phi_t(r_1 \dots r_A) = 0 \quad (14)$$

As in the one channel case, knowledge of these χ'_t is important for two reasons. One is to obtain the most succinct description of χ_t by requiring for example that $P_N \Psi$ be orthogonal to all wave functions satisfying Eq. (14). Secondly, it is more convenient to employ instead of the set χ_t the set U_t defined by

$$\begin{aligned} U_t(r_0) &\equiv \langle \phi_t(r_1 \dots r_A), \Psi \rangle = \langle \phi_t(r_1 \dots r_A), P_N \Psi \rangle \\ &\equiv \sum_{\tau=0}^{A-1} \langle \phi_t(r_1 \dots r_A), \chi_\tau(r_0) \phi_\tau(r_1 \dots r_A) \rangle \end{aligned} \quad (15)$$

In order to invert Eq. (15) to obtain χ_τ from U_t it is necessary to determine all the sets χ'_t . Then by insisting that

$$\left\langle \sum \chi_\tau \phi_\tau, \sum \chi'_t \phi_t \right\rangle = 0 \quad (16)$$

for all sets χ'_t , Eq. (14) can be inverted to obtain a unique χ_t in terms of U_t .

The determination of P_N , satisfying the supplementary condition, as well as the determination of χ_t' satisfying Eq. (14) will be the subject of the next section. As in the one channel case, the determination of P_N automatically yields the solution of the second problem, the evaluation of the χ_t' . The effective Hamiltonian as well as the Schroedinger equation for $P_N\Psi$ including the changes which are a consequence of the identity of the particles and of the Pauli principle will be explicitly derived. This is an exact Schroedinger equation. The terms which arise because of the coupling between $P_N\Psi$ and $(1 - P_N)\Psi$ are completely visible and approximations can be readily inserted.

This problem has been considered by many authors. A recent review has been made by Mittelman (4). Some researches like this one have been concerned with an explicit treatment of the Pauli principle for all channels. We mention here the paper of Coester and Kummel (5) and Lipperheide (6). These authors obtain an infinite set of coupled channels, but they do not solve the problem of truncation which, as we have pointed out here and as Villars (7) emphasized earlier, require special attention. Bloch and Gillet (8) use a new channel formalism employing a restricted class of closed channel wave functions: those which may be expressed as Slater determinants of bound single particle levels. They do not use as in this paper closed channels which are associated with excited target states. Another approach is that of Hahn (9) who treats each permutation in the anti-symmetrized wave function as an independent channel. Levin (10) Lippmann, Mittelman and Watson (11) use the Lippmann-Schwinger integral equation where the source term is the anti-symmetrized incident wave. A somewhat different method is used by MacDonald (21). Finally, we mention the perturbation theory approach of Bell and Squire (12) and Vagrado (13) who derive an optical potential which includes the effect of the Pauli exclusion principle.

In the Wigner $\mathcal{R}$ matrix formalism (14) the problem of anti-symmetrization is resolved by postulating that the right hand of Eq. (6) holds exactly for r_i larger than or equal to a channel radius a . Joining this $r \geq a$ form to the inside ($r_i < a$) wave function guarantees the satisfaction of the supplementary condition. It has been pointed out by A. K. Kerman (15) that this procedure is approximate even if, as the $\mathcal{R}$ matrix theory requires, it is assumed that the nucleon-nucleus interaction vanishes outside $r_i = a$. Form (6) does not hold precisely for any finite r_i . The other components of the exact wave function Ψ are not zero outside a but rather decrease exponentially. Upon anti-symmetrization these will contribute in the elastic channel and affect the joining of the inside and outside wave functions. This in addition beclouds the question as to whether or not exchange scattering has been correctly accounted for. These effects become smaller as a is increased but then the $\mathcal{R}$ matrix parameters become correspondingly less physical. For a of the order of the nuclear radius which it is claimed is the "best" value of a the overlap is appreciable and therefore the anti-symmetrization problem is serious. The $\mathcal{R}$ matrix theory has been reformulated in Ref. (2) in 1962 where a projection operator P appropriate for $\mathcal{R}$ matrix theory is exhibited and the Bloch (16) interaction term derived. If the analysis of the present paper is applied only to $P\Psi$ valid anti-symmetrization and truncation can be obtained.

II. General Theory

Let us first consider the problem of satisfying formally what we have referred to as the supplementary condition. This condition, which is required to hold only asymptotically, is certainly satisfied if we require it to be

valid everywhere. Let Ψ be any anti-symmetrized wave function of the $A+1$ variables $(r_0, \dots, r_A)$. It need not be the solution of Schroedinger Eq. (2). We require that

$$\left\langle \phi_t(r_1, \dots, r_A) \left[\Psi - A \sum_{t=0}^{N-1} \chi_t(r_0) \phi_t(r_1, \dots, r_A) \right] \right\rangle = 0 \quad (17)$$

for $t = 0, 1, \dots, N-1$

Eq. (17) must hold for all r_0 and therefore it follows that

$$\left\langle A \sum f_t(r_0) \phi_t(r_1, \dots, r_A), \left[\Psi - A \sum \chi_t(r_0) \phi_t(r_1, \dots, r_A) \right] \right\rangle = 0 \quad (18)$$

for arbitrary f_t

Eq. (17) is the statement in analytic form of the requirement that the part of Ψ not included in $A \sum \chi_t \phi_t$, namely $\Psi - A \sum \chi_t \phi_t$, be orthogonal to ϕ_t where the argument of ϕ_t can be any selection of A coordinates from the $A+1$ coordinates $(r_0, r_1, \dots, r_A)$. Eq. (17) can be solved for χ_t , in terms of Ψ . The functions χ_t depend linearly on Ψ so that a projection operator P_N exists which when acting on an anti-symmetrized wave function projects out the dependence on ϕ_t , $t = 0 \dots N-1$:

$$P_N \Psi = A \sum_{t=0}^{N-1} \chi_t(r_0) \phi_t(r_1, \dots, r_A) \quad (19)$$

Equation (17) may be written

$$U_t(\underline{r}) = \chi_t(\underline{r}) - \sum_{t'=0}^{N-1} \langle K_{tt'}(\underline{r}', \underline{r}), \chi_{t'}(\underline{r}') \rangle \quad (20)$$

where U_t has been defined in Eq. (15) and

$$K_{tt'}(\underline{r}', \underline{r}) \equiv A \langle \phi_t(\underline{r}', \underline{r}_2 \dots \underline{r}_A), \phi_{t'}(\underline{r}, \underline{r}_2 \dots \underline{r}_A) \rangle \quad (21)$$

It is convenient to define columnar matrices of N elements,

$$\chi = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{N-1} \end{pmatrix} \quad U = \begin{pmatrix} U_0 \\ \vdots \\ U_{N-1} \end{pmatrix}$$

and a square $N \times N$ matrix operator K whose matrix element is $\int K_{tt'}(\underline{r}', \underline{r})$
Eq. (20) becomes

$$U = (1 - K)\chi$$

where

$$1 \equiv \int \delta(\underline{r} - \underline{r}') \delta_{tt'} \quad (22)$$

The problems of satisfying the supplementary condition or equivalently of determining P_N , and of inverting $(1 - K)$ to obtain χ in terms of U are seen to be identical. That is, once $(1 - K)$ is inverted, it is a simple matter to extract P_N . Moreover, the problem of determining the inverse $(1 - K)^{-1}$ requires the determination of its singularities. We shall now show that these

singularities occur because there are the "supererogatory" χ' satisfying Eq. (14). In solving Eq. (22) for χ it will be necessary to determine χ' .

The inversion of $(1 - K)$ may be expressed in terms of the eigenfunctions $\omega^{(\alpha)}$ of the K with eigenvalue λ_{α} namely

$$\lambda_{\alpha} K \omega^{(\alpha)} = \omega^{(\alpha)} \quad (23)$$

K is hermitian so that λ_{α} is real. The corresponding eigenfunctions $\omega^{(\alpha)}$ form an orthogonal (but not complete) set which we shall also normalize. K may be expressed in terms of $\omega^{(\alpha)}$:

$$K = \sum_{\alpha} \frac{\omega^{(\alpha)} \omega^{(\alpha)\dagger}}{\lambda_{\alpha}} \quad (24)$$

where $\omega^{(\alpha)}$ is a single column matrix while $\omega^{(\alpha)\dagger}$ is a single row matrix with elements $\omega_t^{(\alpha)*}$. Hence,

$$K_{tt'}(\underline{r}', \underline{r}) = \sum_{\alpha} \frac{\omega_t^{(\alpha)}(\underline{r}') \omega_{t'}^{(\alpha)*}(\underline{r})}{\lambda_{\alpha}}$$

The eigenfunctions with $\lambda_{\alpha} = 1$ are of special importance. They will be denoted by $\chi^{(\alpha)}$.

Turning now to Eq. (22), we must investigate the meaning of

$$\frac{1}{1 - K} U$$

before we can solve (22) for χ . We note that $(1/1 - K)$ is singular in the

sense that if the overlap $\langle \chi^{(\alpha)'}, U \rangle$ were not zero, $(1 - K)^{-1}U$ would be infinite. As we shall show (see Eq. (30)) $\langle \chi^{(\alpha)'}, U \rangle$ is in fact zero so that $(1 - K)^{-1}U$ is finite. However, then the amplitude $\chi^{(\alpha)'}$ in the final result for $(1 - K)^{-1}U$ is arbitrary although it is not infinite. Suppose that we define a new K operator, K' which omits the terms $\chi_{\alpha} = 1$:

$$K' \equiv \sum_{\alpha, \lambda_{\alpha} \neq 1} \frac{\omega^{(\alpha)} \omega^{(\alpha)\dagger}}{\lambda_{\alpha}} \quad (25)$$

Then $[1 - K']^{-1}$ is singular free and is given by:

$$\left(\frac{1}{1 - K'} \right)_{tt'} = \delta_{tt'} \delta(\underline{r} - \underline{r}') + \sum_{\alpha, \lambda_{\alpha} \neq 1} \frac{\omega_t^{(\alpha)}(\underline{r}') \omega_{t'}^{(\alpha)*}(\underline{r})}{\lambda_{\alpha} - 1} \quad (26)$$

The solution of Eq. (22) is then

$$\chi = \frac{1}{1 - K'} U + \sum_{\alpha} C_{\alpha} \chi^{(\alpha)'} \quad (27)$$

where C_{α} are arbitrary.

This degree of arbitrariness is a consequence of the existence of the supererogatory solutions. As we shall now show the $\chi^{(\alpha)'}$ are just the χ' alluded to in Eq. (14), i.e.

$$A \sum \left(\chi^{(\alpha)'}(r_{m0}) \right)_t \phi_t(r_{m1} \dots r_A) = 0 \quad (28)$$

To prove this, we note that $\chi^{(\alpha)'}$ is a solution of Eq. (23), $\lambda_\alpha = 1$

$$K \chi^{(\alpha)'} = \chi^{(\alpha)'} \quad (29)$$

or, retracing our steps back to Eq. (17):

$$\left\langle \phi_t(r_{m1} \dots r_A), A \sum \chi_t^{(\alpha)'}(r_{m0}) \phi_t(r_{m1} \dots r_A) \right\rangle = 0$$

for all r_{m0} and for $0 \leq t \leq N - 1$. It immediately follows that

$$\left\langle A \sum \chi_t^{(\alpha)'}(r_{m0}) \phi_t(r_{m1} \dots r_A), A \sum \chi_t^{(\alpha)'}(r_{m0}) \phi_t(r_{m1} \dots r_A) \right\rangle = 0$$

Eq. (28) follows from this result.

In a similar way we can show that

$$\left\langle \chi^{(\alpha)'}(r_{m0}), U(r_{m0}) \right\rangle = 0 \quad (30)$$

Inserting Eq. (22) for U , Eq. (30) becomes

$$\left\langle \chi^{(\alpha)'}(r_{m0}), (1 - K) \chi \right\rangle = \left\langle (1 - K) \chi^{(\alpha)'}, \chi \right\rangle = 0$$

because of Eq. (29).

To summarize, the function χ' satisfying Eq. (28) are given by the eigenvectors of K with eigenvalue 1. The vector χ which satisfies Eq. (17) and is therefore a solution of Eq. (22) and which is orthogonal to $\chi^{(\alpha) '}$ is given by Eq. (27) with $C_\alpha = 0$

$$\chi = \frac{1}{1 - K'} U ; \quad \langle \chi^{(\alpha) '}, \chi \rangle = 0 \quad (31)$$

Finally we note that

$$P_N \Psi = A \sum_t \chi_t \phi_t = A \left[\sum_{t, t'} \left(\frac{1}{1 - K'} \right)_{tt'} U_{t'} \right] \phi_t \quad (32)$$

where we have used Eq. (31). Note that Eq. (32) holds for arbitrary values of C_α in Eq. (27) as a consequence of Eq. (28). Substituting for U_t , from its defining Eq. (15) we obtain

$$P_N = A \left[\sum_{tt'} \phi_t \right] \left(\frac{1}{1 - K'} \right)_{tt'} \left[\phi_{t'} \right] \quad (33)$$

$\left(\frac{1}{1 - K'} \right)_{tt'}$ is given explicitly in Eq. (26). P_N projects out of an anti-symmetric Ψ all the dependence on ϕ_t , for all t between 0 and $N - 1$. We can obtain the operator derived in Ref. (2) for the one channel case by limiting the sum in Eq. (33) to $t = t' = 0$. Note that P_N , $\omega^{(\alpha)}$, $\chi^{(\alpha)}$ are determined

by the properties of the target nucleus and particularly by the properties of the N states of the target nucleus.

To complete the discussion we need to present the coupled equations satisfied by χ_t or U_t . Toward this end premultiply the Schroedinger equation

$$H \Psi = E \Psi$$

by ϕ_t and integrate. We obtain

$$\langle \phi_t, H \Psi \rangle = E U_t \quad (34)$$

Let

$$H = H_t + T + V \quad (35)$$

where H_t is the target Hamiltonian, T is the kinetic energy operator for the incident nucleon and V is the potential between the nucleon and those of the target nucleus. Eq. (34) becomes

$$[E - E_t] U_t = [T U_t + \langle \phi_t, V \Psi \rangle] \quad (36)$$

If we now approximate Ψ by $P_N \Psi$, Eq. (36) may be rewritten in several forms:

$$\text{when } \Psi \rightarrow P_N \Psi$$

$$[E - E_t] U_t = T U_t + \left\langle \phi_t, V \sum_{t', t''} \phi_{t'} \left(\frac{1}{1 - K'} \right)_{t', t''} U_{t''} \right\rangle \quad (37a)$$

or

$$[E - E_t] [1 - K'] \chi_t = T [1 - K'] \chi_t + \left\langle \phi_t, V \sum_{t'} \chi_{t'} \phi_{t'} \right\rangle \quad (37b)$$

or

$$E \Omega_t = \left\langle \phi_t \left(\frac{1}{\sqrt{1 - K'}} \right), H \sum_{t'} \phi_{t'} \frac{1}{\sqrt{1 - K'}} \Omega_{t'} \right\rangle \quad (37c)$$

where

$$\Omega_t \equiv \left(\frac{1}{\sqrt{1 - K'}} \right)_{tt'} U_{t'} = \left(\sqrt{1 - K'} \right)_{tt'} \chi_{t'} \quad (38)$$

We note that since the eigenvalues $(1/\lambda_\alpha)$ of K' are smaller than unity (for proof see Ref. (17)) the operator $(1 - K')^{1/2}$ is hermitian. As a consequence, the effective Hamiltonian in Eq. (37c) is hermitian, so that the solution of the coupled Schroedinger equations for Ω_t will not involve any false absorption or production. This must also be the case for the equation for U_t in which the effective Hamiltonian is not directly hermitian. The equation for U_t is, however, probably the most convenient of the three forms given in Eq. (37) since it concentrates the effects of the identity of the

particles in the effective coupling potential which is energy independent but non-local.

Eq. (37) is approximate only because we have replaced Ψ by $P_N \Psi$, i.e. we have truncated the coupled equations describing the scattering. The exact equation for $P_N \Psi$ is (see 2)

$$\left[E - P_N H P_N - P_N H Q_N \frac{1}{E - Q_N H Q_N} Q_N H P_N \right] (P_N \Psi) = 0 \quad (39)$$

where

$$Q_N \equiv 1 - P_N$$

and

$$Q_N \Psi = \frac{1}{E - Q_N H Q_N} Q_N H P_N (P_N \Psi) \quad (40)$$

Eq. (37) is obtained by omitting the third term in the square bracket in Eq. (39) which as Eq. (40) demonstrates involves the coupling of $P_N \Psi$ with the rest of Ψ given by $Q_N \Psi$. By our particular choice of P_N , we have guaranteed that $Q_N \Psi$ will not make any contributions asymptotically ($r_t \rightarrow 0$). By choosing the set $\{\phi_t\}$ properly so as to include important states such as the doorway states (18) which couple importantly with the entrance channel or with the exit channels, we reduce the contribution $Q_N \Psi$ can make within the interaction region. However, important effects may still remain. Much of the compound nucleus contribution, except for light nuclei, remains in $Q_N \Psi$. Possible improvements include replacing $Q_N \Psi$ by some appropriate average

(18), (19) and (20). This introduces complex terms in the interaction, so that the coupled equations are those of a generalization of the single channel optical model. For further details see Ref. (18) and (20).

III. Summary

Equation (37) provides a formulation of the N coupled channel approximation to elastic and inelastic scattering which takes correctly into account effects which are a consequence of the identity of the incident and emergent nucleon with those forming the target nucleus. To include these effects within the coupled channel equations it is necessary to invert operator $(1 - K')$ which can be accomplished by determining the eigenvectors of K and their corresponding eigenvalues. The solution of this problem requires the solving of a set of N coupled equations, the operator K (see Eq. (21)) depending only upon the properties of the target nuclear states involved in the description of the N channels. The solution of these equations as well as the properties of the K matrix for the case where the target nucleus states are expressed in terms of a general number of particle and hole levels in a self consistent Hartree-Fock potential are given in Ref. (17). In this reference, cases in which the states consist of single Slater determinants or linear combinations of such determinants where the number of mixing configurations is N are discussed.

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