THE STABILITY OF ANISOTROPIC EQUILIBRIA IN CLOSED LINE TORI

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THE STABILITY OF ANISOTROPIC EQUILIBRIA IN CLOSED LINE TORI

D. B. Nelson, G. O. Spies,* and C. L. Hedrick

1. INTRODUCTION

In a previous memorandum¹ we examined the stability of anisotropic plasmas in axisymmetric geometries. The resultant two dimensional problem is considerably simpler than the general three dimensional problem. In this memorandum we consider truly toroidal, nonaxisymmetric closed line tori (bumpy tori). However, by studying only a subclass of equilibria we are able to achieve much of the simplicity of the axisymmetric problem.

Recall that if we label magnetic lines by a pair of flux coordinates (α, β) , then in general the functional form of the equilibrium pressure tensor is $P(B, \alpha, \beta)$. In this work we shall demand that P depend on only one flux coordinate, i.e., $P(B, \psi)$, and we shall assume that the constant ψ contours form nested tori. The restrictiveness of this assumption and its implications with regard to properties of the equilibrium have been extensively analyzed elsewhere.^{2,3} Such equilibria are only partly two dimensional because the magnetic field is assuredly not constant on a ψ contour. However, by employing an extension of Hamada coordinates to anisotropic equilibria, we arrive at a stability problem which resembles closely the two dimensional axisymmetric case.

To determine the stability of these equilibria we consider the second variation of potential energy, δW , and define stability as

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positivity of δW for all allowable perturbations, instability as negativity for some allowable perturbation. We are unable to find a single condition on the equilibrium which is both necessary and sufficient for stability. By considering special perturbations we determine necessary conditions for stability, while by relaxing constraints on the perturbations (i.e., considering unallowable perturbations), we find sufficient conditions for stability. The resultant expressions are presented in parallel forms which allow estimates of the uncertainty gap between the strongest necessary condition and the weakest sufficient condition.

The underlying model which we assume to describe the plasma is the zero order guiding center plasma, although some of the results are valid for the guiding center fluid (or double adiabatic fluid) as well, and in the scalar pressure limit are comparable to infinite conductivity scalar pressure MHD. None of the analysis requires an assumption of low β , although in this limiting case certain of the expressions are simplified.

2. EQUILIBRIUM

While this note does not treat the equilibrium problem, it is worthwhile to list a few properties of guiding center equilibria. The equilibrium equation can be written as

$$\vec{K} \times \vec{B} = \tilde{\nabla} p_{\parallel} \quad (2.1)$$

where $\vec{K} = \text{curl}(\sigma \vec{B})$, $\sigma = 1 - B^{-1} \partial p_{\parallel} / \partial B$, and $\tilde{\nabla}$ is an operator defined for functions $f(B, \alpha, \beta)$ by $\tilde{\nabla} = \nabla - \nabla B (\partial / \partial B)$ or equivalently by $\tilde{\nabla} = \nabla \alpha \partial / \partial \alpha + \nabla \beta \partial / \partial \beta$. For the restricted class of equilibria considered here, $p_{\parallel} = p_{\parallel}(B, \psi)$, and (2.1) becomes

$$\vec{K} \times \vec{B} = \frac{\partial p_{\parallel}}{\partial \psi} \nabla \psi \quad (2.2)$$

Since $\vec{K} \cdot \nabla \psi = 0$ and $\vec{B} \cdot \nabla \psi = 0$, both vectors lie in constant ψ surfaces. If we assume reflection symmetry about a plane, then the $\vec{B}$ lines close the long way around the torus and the $\vec{K}$ lines close the short way.² Since in equilibrium $p_{\perp} = -B^2 \partial(p_{\parallel}/B) / \partial B$, it follows that if $p_{\parallel}$ depends on B and ψ alone, $p_{\perp}$ also has the functional form $p_{\perp}(B, \psi)$. Further $\tilde{\nabla} p_{\parallel}$ and $\tilde{\nabla} p_{\perp}$ are then parallel, both having the direction of $\nabla \psi$. Since $\tilde{\nabla} \sigma = \tilde{\nabla}(p_{\perp} - p_{\parallel})/B^2$, $\vec{K} \cdot \tilde{\nabla} \sigma = 0$ also.

In the scalar pressure limit, $\sigma = 1$, $\vec{K}$ is identical to current $\vec{J}$, and $p_{\parallel}$ is independent of B . Equation (2.1) is then

$$\vec{J} \times \vec{B} = \nabla p \quad (2.3)$$

and the surfaces of constant ψ agree with surfaces of constant p . In the limit of axisymmetry, B also depends only on ψ so even for anisotropic equilibria, $p_{\perp}$ and $p_{\parallel}$ are constant on ψ surfaces. For the

general case one cannot properly regard ψ surfaces as pressure surfaces because $p_{\parallel}$ and $p_{\perp}$ vary through their B dependence. Nevertheless, because these surfaces are defined by the pressure and not by the structure of the magnetic field, we shall refer to them as pressure surfaces. Indeed it is not necessary to label the surfaces by the amount of magnetic flux ψ contained within them. They can as well be characterized by the volume v contained within each toroidal surface as described in the following section.

3. HAMADA COORDINATES

The extension of Hamada coordinates^{4,5} to anisotropic toroidal equilibria with pressure surfaces is treated in reference 3. For the special case of closed lines with reflection symmetry this coordinate system has a particularly simple structure. The toroidal pressure surfaces are labeled by their volume, v . Within each surface an angular coordinate, θ , runs along the closed $\vec{K}$ lines increasing by unity the short way around the torus. Another angular coordinate, ζ , runs along the closed $\vec{B}$ lines increasing by unity the long way around. The quantities (v, θ, ζ) form a right-handed, nonorthogonal coordinate system.

The $\vec{B}$ and $\vec{K}$ fields are described by their fluxes. Define ψ and I by

$$\psi(v) = \int_{\tau \leq v} \vec{B} \cdot \nabla \zeta \, d\tau \quad (3.1)$$

$$I(v) = \int_{\tau \leq v} \vec{K} \cdot \nabla \theta \, d\tau \quad (3.2)$$

By construction, ψ and I are surface quantities. (They depend only on v , not on θ or ζ .) In fact, they are fluxes of $\vec{B}$ and $\vec{K}$, respectively, for it can be shown that⁶

$$\psi(v) = \int_{S_1} \vec{B} \cdot d\vec{S} \quad (3.3)$$

$$I(v) = \int_{S_2} \vec{K} \cdot d\vec{S} \quad (3.4)$$

The figure indicates the surfaces S_1 and S_2 through which ψ and I are computed:

S_1 is a cut through the torus at constant ζ . S_2 is a ribbon from 0 to v at constant θ . $\vec{B}$ and $\vec{K}$ are then given by³

$$\vec{B} = \dot{\psi} \nabla v \times \nabla \theta \quad (3.5)$$

$$\vec{K} = \dot{I} \nabla \zeta \times \nabla v \quad (3.6)$$

where $\dot{\psi} = d\psi/dv$ and $\dot{I} = dI/dv$.

For scalar pressure equilibria the Jacobian relating (v, θ, ζ) to (x, y, z) is unity, but for anisotropic equilibria this is not true.

Denote the Jacobian by D :

$$D = \nabla v \cdot \nabla \theta \times \nabla \zeta \quad (3.7)$$

To evaluate D it is useful to introduce the reciprocal basis vectors, $\vec{e}_i$:

$$\begin{aligned} \vec{e}_v &= \frac{1}{D} \nabla \theta \times \nabla \zeta \\ \vec{e}_\theta &= \frac{1}{D} \nabla \zeta \times \nabla v \\ \vec{e}_\zeta &= \frac{1}{D} \nabla v \times \nabla \theta \end{aligned} \quad (3.8)$$

Then, for example, $\nabla v = D \vec{e}_\theta \times \vec{e}_v$. Further, the scalar products between the two bases are simple: $\vec{e}_v \cdot \nabla v = 1$, $\vec{e}_v \cdot \nabla \theta = 0$, etc.

With this notation we find

$$\vec{B} = \dot{\psi} D \vec{e}_\zeta \quad \text{or} \quad \vec{e}_\zeta = \vec{B} / \dot{\psi} D \quad (3.9)$$

$$\vec{K} = \dot{I} D \vec{e}_\theta \quad \text{or} \quad \vec{e}_\theta = \vec{K} / \dot{I} D \quad (3.10)$$

Since $\vec{K} \times \vec{B} = \dot{I} \dot{\psi} D^2 \vec{e}_\theta \times \vec{e}_\zeta = \dot{I} \dot{\psi} D \nabla v$, the equilibrium equation (2.2) becomes simply

$$\dot{I} \dot{\psi} D = \left. \frac{\partial p_\parallel}{\partial v} \right|_B \quad (3.11)$$

If we extend the definition of "." to functions $f(B, v)$ so that

$$\dot{f} = \left. \frac{\partial f}{\partial v} \right|_B \quad (3.12)$$

then the equilibrium equation is $\dot{I} \dot{\psi} D = \dot{p}_\parallel$, which specifies the Jacobian D to be

$$D(B, v) = \dot{I} \dot{\psi} / \dot{p}_\parallel \quad (3.13)$$

With scalar pressure $p = p(v)$, the corresponding Jacobian is $D(v)$ only. It is easy to prove that in scalar pressure $D \equiv 1$. Since the volume inside a torus of constant v is just

$$v = \int dx dy dz = \int \frac{dv d\theta d\zeta}{D}$$

we can differentiate to find

$$1 = \dot{v} = \int \frac{d\theta d\zeta}{D} \quad .$$

However, D is independent of θ and ζ and these have been normalized to run from 0 to 1 going the long way (ζ) or the short way (θ) around. Thus,

$$D = \int d\theta d\zeta = 1 \quad .$$

In tensor pressure we have again

$$1 = \int \frac{d\theta d\zeta}{D} \quad (3.14)$$

but cannot go past this relation unless B is independent of θ or ζ because (3.13) shows that D depends on B as well as v .

All of the stability criteria will be expressed in terms of line integrals. The relationship between $d\zeta$ and $d\ell$ taken along a $\vec{B}$ line is readily found using $d\vec{\ell} = \vec{e}_\zeta d\zeta$. Thus

$$(d\ell)^2 - \vec{e}_\zeta \cdot \vec{e}_\zeta (d\zeta)^2 = \frac{B^2}{\dot{\psi} D^2} (d\zeta)^2 \quad (3.15)$$

so

$$d\zeta = \dot{\psi} D \, d\ell / B \quad (3.16)$$

In scalar pressure, since $D \equiv 1$, integrating (3.16) leads to

$$1 = \dot{\psi} \oint \frac{d\ell}{B} \quad (3.17)$$

Therefore, defining $q = \oint d\ell/B$, we have $q = 1/\dot{\psi}$. The average along field lines is denoted by $\langle f \rangle = \oint f d\zeta$ or $\langle f \rangle = q^{-1} \oint f d\ell/B$.

Motivated by these scalar pressure results, we note that in tensor pressure integrating (3.16) leads to

$$1 = \dot{\psi} \int \frac{d\ell}{B} D \quad (3.18)$$

so we define

$$q = \oint \frac{d\ell}{B} D \quad (3.19)$$

to recover $q = 1/\dot{\psi}$. Then the appropriate extension of the average is

$$\langle f \rangle = \int f d\zeta = \frac{1}{q} \oint \frac{d\ell}{B} D f \quad (3.20)$$

4. THE SECOND VARIATION OF ENERGY

In reference 7 (which should be consulted for notation) we wrote δW as

$$\delta W = \int [\sigma Q_{\perp}^2 + \tau Q_{\parallel}^2 - \sigma J_{\parallel} \vec{\xi} \cdot \hat{b} \times \vec{Q}_{\perp} - \left(\frac{2Q_{\parallel}}{B} + S\right) \vec{\xi} \cdot \vec{\nabla} p_{\perp} + g \vec{\xi} \cdot \vec{\nabla} p_{\parallel}] d^3x + W_K \quad (4.1)$$

Local stability requires that σ and τ are everywhere positive and ensures that W_K is positive.⁸ We shall always assume local stability. The difference between the guiding center plasma, the guiding center fluid, and (in the appropriate limit) the scalar pressure plasma is contained in the positive term, W_K , which has different definitions depending upon the model chosen. For much of the present analysis the stabilizing contribution of W_K will be ignored. Its inclusion complicates the results and numerical calculations on axisymmetric equilibria have shown that in the cases tested, the enhanced stability was slight. (This should not be interpreted as a general result.)

In this section we derive two forms for δW suitable for analysis of equilibria with pressure surfaces. The first of these, given by (4.11), is used in Section 7. The second form, (4.22), is used in Sections 5 and 6.

As a starting point we employ a form for δW which is equivalent to (4.1) but slightly easier to manipulate. Since

$$g = Q_{\parallel}/B + \nabla \cdot \vec{\xi} + S \quad (4.2)$$

it follows that

$$g \vec{\xi} \cdot \vec{\nabla} p_{\parallel} = \left(\frac{Q_{\parallel}}{B} + S\right) \vec{\xi} \cdot \vec{\nabla} p_{\parallel} + \vec{\xi} \cdot \vec{\nabla} p_{\parallel} \operatorname{div} \vec{\xi} \quad (4.3)$$

Also from the equilibrium equation (2.1) we derive

$$-\sigma J_{\parallel} \vec{\xi} \cdot \hat{b} \times \vec{Q}_{\perp} = \vec{K} \times \vec{\xi} \cdot \vec{Q} + \frac{Q_{\parallel}}{B} \vec{\xi} \cdot \vec{\nabla} p_{\parallel} \quad (4.4)$$

by expanding $\vec{K}$ in components parallel and perpendicular to $\vec{B}$. Thus

$$\begin{aligned} \delta W = \int & \left[\sigma Q_{\perp}^2 + \tau Q_{\parallel}^2 + \vec{K} \times \vec{\xi} \cdot \vec{Q} - \left(\frac{2Q_{\parallel}}{B} + S \right) (\vec{\xi} \cdot \vec{\nabla} p_{\perp} - \vec{\xi} \cdot \vec{\nabla} p_{\parallel}) \right. \\ & \left. + \vec{\xi} \cdot \vec{\nabla} p_{\parallel} \operatorname{div} \vec{\xi} \right] d^3x + W_K \end{aligned} \quad (4.5)$$

However, since $\vec{\xi} \cdot \vec{\nabla} p_{\perp} - \vec{\xi} \cdot \vec{\nabla} p_{\parallel} = B^2 \vec{\xi} \cdot \vec{\nabla} u$, this version can be written as

$$\begin{aligned} \delta W = \int & \left[\sigma Q_{\perp}^2 + \tau Q_{\parallel}^2 + \vec{K} \times \vec{\xi} \cdot \vec{Q} + \vec{\xi} \cdot \vec{\nabla} p_{\parallel} \operatorname{div} \vec{\xi} - B(2Q_{\parallel} + \vec{\xi} \cdot \nabla B) \vec{\xi} \cdot \vec{\nabla} u \right] d^3x \\ & + W_K \end{aligned} \quad (4.6)$$

This form resembles more closely than (4.1) the "usual" way in which the scalar pressure δW is written.

Now we introduce as basis vectors, $\vec{B}$, $\vec{K}$, and a third vector $\vec{e}$, unspecified except by the constraint $\vec{e} \cdot \nabla v = 1$. The perturbation $\vec{\xi}$ is decomposed into components along $\vec{e}$, $\vec{K}$, and $\vec{B}$ written as

$$\vec{\xi} = X\vec{e} + Y\vec{K} + Z\vec{B} \quad (4.7)$$

For the guiding center plasma, $\vec{\xi} \cdot \vec{B} = 0$, but it can be shown in this case that all of δW (including W_K) is independent of $\vec{\xi} \cdot \vec{B}$. Thus it is correct to write $\vec{\xi}$ as in (4.7) and to ignore the awkward constraint $\vec{\xi} \cdot \vec{B} = 0$. For the guiding center fluid and the scalar pressure plasma, δW is also independent of $\vec{\xi} \cdot \vec{B}$ except in the term W_K .

Next, use is made of the following identity valid only for equilibria with pressure surfaces (it relies on the fact that $\vec{K} \cdot \vec{\nabla}\sigma = 0$)

$$\begin{aligned} & \vec{K} \times \vec{\xi} \cdot \vec{Q} + \vec{\xi} \cdot \vec{\nabla}p_{\parallel} \operatorname{div} \vec{\xi} - B(\vec{\xi} \cdot \vec{\nabla}B)\vec{\xi} \cdot \vec{\nabla}\sigma = \\ & = 2X\vec{F} \cdot \vec{Q} + X^2[\vec{F} \cdot \operatorname{curl} \vec{G} + \vec{e} \cdot \vec{\nabla}p_{\parallel} - B(\vec{e} \cdot \vec{\nabla}B)\vec{e} \cdot \vec{\nabla}\sigma] \\ & \quad + \operatorname{div}[XZ(\vec{e} \cdot \vec{\nabla}p_{\parallel})\vec{B}] \end{aligned} \quad (4.8)$$

The proof of (4.8) is given in the Appendix. Here X and Z are defined in (4.7) and

$$\begin{aligned} \vec{F} &= \vec{K} \times \vec{e} \\ \vec{G} &= \vec{B} \times \vec{e} \end{aligned} \quad (4.9)$$

The use of (4.8) results in a family of new forms for δW (depending upon the choice of $\vec{e}$):

$$\begin{aligned} \delta W &= \int \left\{ \sigma Q_{\perp}^2 + \tau Q_{\parallel}^2 + 2X(\vec{F} \cdot \vec{Q} - BQ_{\parallel}\vec{e} \cdot \vec{\nabla}\sigma) \right. \\ & \quad \left. + X^2[\vec{F} \cdot \operatorname{curl} \vec{G} + \vec{e} \cdot \vec{\nabla}p_{\parallel} \operatorname{div} \vec{e} - B(\vec{e} \cdot \vec{\nabla}B)\vec{e} \cdot \vec{\nabla}\sigma] \right\} d^3x + W_K \end{aligned} \quad (4.10)$$

The divergence term in (4.8) integrates to zero in a torus. Excluding W_K , (4.10) is obviously independent of Z and depends on Y only through Q . This is a considerable simplification over the original form for δW and demonstrates the utility of the $(\vec{e}, \vec{K}, \vec{B})$ basis for equilibria with pressure surfaces. An alternative form for δW is obtained after completing squares on $Q_{\perp}$ and $Q_{\parallel}$:

$$\begin{aligned} \delta W &= \int \left\{ \sigma \left[\vec{Q}_{\perp} + \frac{1}{\sigma} X \vec{F}_{\perp} \right]^2 + \tau \left[Q_{\parallel} + \frac{1}{\tau} X(F_{\parallel} - B\vec{e} \cdot \vec{\nabla}\sigma) \right]^2 + NX^2 \right\} d^3x \\ & \quad + W_K \end{aligned} \quad (4.11)$$

where

$$N = \vec{F} \cdot \text{curl } \vec{G} + \vec{e} \cdot \vec{\nabla}_p \parallel \text{div } \vec{e} - B(\vec{e} \cdot \vec{\nabla} B) \vec{e} \cdot \vec{\nabla} \sigma - \frac{1}{\sigma} F_{\perp}^2 - \frac{1}{\tau} (F_{\parallel} - B \vec{e} \cdot \vec{\nabla} \sigma)^2 \quad (4.12)$$

Note that if N , σ , and τ are everywhere positive, δW is also. Since $\vec{e}$ is unspecified and N depends on $\vec{e}$, one might seek an $\vec{e}$ which makes $N > 0$ for a given equilibrium. If such an $\vec{e}$ can be found, the equilibrium is stable. This observation is the basis for the sufficient condition for stability to be derived in Section 7.

In obtaining these forms for δW we have not used Hamada coordinates; we have merely assumed pressure surfaces and employed a basis $(\vec{e}, \vec{K}, \vec{B})$. To derive a form for δW using Hamada coordinates explicitly, one sets

$$\vec{e} = \vec{e}_v = \frac{1}{D} \nabla \theta \times \nabla \zeta \quad (4.13)$$

and writes $\vec{\xi}$ in terms of $(\vec{e}_v, \vec{e}_\theta, \vec{e}_\zeta)$:

$$\vec{\xi} = X \vec{e}_v + Y \vec{e}_\theta + Z \vec{e}_\zeta \quad (4.14)$$

where $X = \vec{\xi} \cdot \nabla v$, $Y = \vec{\xi} \cdot \nabla \theta$, $Z = \vec{\xi} \cdot \nabla \zeta$. To compute $\vec{Q} = \text{curl}(\vec{\xi} \times \vec{B})$, we note that since $\vec{B} = \psi D \vec{e}_\zeta$

$$\vec{\xi} \times \vec{B} = \dot{\psi} Y \nabla v - \dot{\psi} X \nabla \theta \quad (4.15)$$

Hence,

$$\text{curl}(\vec{\xi} \times \vec{B}) = \nabla(\dot{\psi} Y) \times \nabla v - \nabla(\dot{\psi} X) \times \nabla \theta \quad (4.16)$$

Now regard $\dot{\psi} Y$ and $\dot{\psi} X$ as functions of v , θ , ζ ; for example

$$\nabla(\dot{\psi} Y) = \frac{\partial \dot{\psi} Y}{\partial v} \nabla v + \frac{\partial \dot{\psi} Y}{\partial \theta} \nabla \theta + \frac{\partial \dot{\psi} Y}{\partial \zeta} \nabla \zeta$$

Because $\dot{\psi}$ is independent of θ and ζ , $\vec{Q}$ is given by

$$\vec{Q} = \text{curl}(\vec{\xi} \times \vec{B}) = \dot{\psi} D \frac{\partial X}{\partial \zeta} \vec{e}_v + \dot{\psi} D \frac{\partial Y}{\partial \theta} \vec{e}_\theta - \left(\frac{\partial \dot{\psi} X}{\partial v} + \dot{\psi} \frac{\partial Y}{\partial \theta} \right) D \vec{e}_\zeta \quad (4.17)$$

If we write $\vec{Q}$ as

$$\vec{Q} = Q^v \vec{e}_v + Q^\theta \vec{e}_\theta + Q^\zeta \vec{e}_\zeta \quad (4.18)$$

then the contravariant components of $\vec{Q}$ are

$$\begin{aligned} Q^v &= \dot{\psi} D \frac{\partial X}{\partial \zeta} \\ Q^\theta &= \dot{\psi} D \frac{\partial Y}{\partial \zeta} \\ Q^\zeta &= - \dot{\psi} D \left(\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial \theta} \right) - \ddot{\psi} D X \end{aligned} \quad (4.19)$$

The other quantities required in equation (4.10) for δW are likewise computed in terms of $\vec{e}_v$:

$$\vec{F} = \vec{K} \times \vec{e}_v = D \dot{\psi} \vec{e}_\theta \times \vec{e}_v = - \dot{\psi} \nabla \zeta$$

$$\vec{F} \cdot \vec{Q} = - \dot{\psi} \nabla \zeta \cdot \vec{Q} = - \dot{\psi} Q^\zeta$$

$$\vec{G} = \vec{B} \times \vec{e}_v = D \dot{\psi} \vec{e}_\zeta \times \vec{e}_v = \dot{\psi} \nabla \theta$$

$$\text{curl } \vec{G} = \nabla \dot{\psi} \times \nabla \theta = \ddot{\psi} \nabla v \times \nabla \theta = D \ddot{\psi} \vec{e}_\zeta$$

$$\vec{F} \cdot \text{curl } \vec{G} = - D \dot{\psi} \ddot{\psi} \quad (4.20)$$

$$\vec{e}_v \cdot \tilde{\nabla} p_{\parallel} = \dot{p}_{\parallel} \vec{e}_v \cdot \nabla v = \dot{p}_{\parallel}$$

$$\text{div } \vec{e}_v = \text{div } \frac{1}{D} D \vec{e}_v = D \vec{e}_v \cdot \nabla \left(\frac{1}{D} \right) = - \frac{1}{D} \frac{\partial D}{\partial v} \Big|_{\theta, \zeta}$$

$$\vec{e}_v \cdot \nabla B = \frac{\partial B}{\partial v} \Big|_{\theta, \zeta}$$

$$\vec{e}_v \cdot \tilde{\nabla} \sigma = \dot{\sigma} = - \frac{1}{B} \frac{\partial \dot{p}_{\parallel}}{\partial B} \Big|_v$$

(This last expression follows from the relation $\sigma = 1 - \frac{1}{B} \frac{\partial p_{\parallel}}{\partial B}$.) The coefficient of X^2 in (4.10) is then

$$- D \dot{\psi} - \frac{\dot{p}_{\parallel}}{D} \frac{\partial D}{\partial v} \Big|_{\theta, \zeta} + \frac{\partial B}{\partial v} \Big|_{\theta, \zeta} \frac{\partial \dot{p}_{\parallel}}{\partial B} \Big|_v$$

However, $\dot{p}_{\parallel} = \dot{\psi} D$, so

$$- \frac{\dot{p}_{\parallel}}{D} \frac{\partial D}{\partial v} \Big|_{\theta, \zeta} + \frac{\partial \dot{p}_{\parallel}}{\partial B} \frac{\partial B}{\partial v} \Big|_{\theta, \zeta} = - \dot{\psi} \left[\frac{\partial D}{\partial v} \Big|_{\theta, \zeta} - \frac{\partial D}{\partial B} \Big|_v \frac{\partial B}{\partial v} \Big|_{\theta, \zeta} \right] = - \dot{\psi} \dot{D} \quad (4.21)$$

Thus, expression (4.10) for δW is simply

$$\delta W = \int \left[\sigma Q_{\perp}^2 + \tau Q_{\parallel}^2 - 2X(\dot{\psi} Q_{\perp}^2 + B Q_{\parallel} \dot{\sigma}) - X^2(\dot{\psi} \dot{D} + \dot{\psi} \dot{D}) \right] d^3x + W_K \quad (4.22)$$

In general it is tedious to write out the components of $Q_{\perp}$ and $Q_{\parallel}$ because $\vec{e}_v$ and $\vec{e}_{\theta}$ are not perpendicular to $\vec{B}$. For the perturbations which we consider this will not be necessary.

5. STABILITY TO INTERCHANGES

In this section we ignore the positive term W_K in δW to derive a sufficient condition for stability against perturbations in which X and Y are constant along field lines. Assuming

$$\frac{\partial X}{\partial \zeta} = \frac{\partial Y}{\partial \zeta} = 0 \quad (5.1)$$

then Q^v and Q^θ vanish leaving

$$\vec{Q} = Q^\zeta \vec{e}_\zeta = \frac{Q^\zeta}{\dot{\psi} D} \vec{B} \quad (5.2)$$

Thus, $Q_\perp = 0$ and

$$\frac{Q_\parallel}{B} = \frac{Q^\zeta}{\dot{\psi} D} = - \frac{\partial X}{\partial v} - \frac{\partial Y}{\partial \theta} - \frac{\ddot{\psi}}{\dot{\psi}} X \quad (5.3)$$

These perturbations we have previously denoted general interchanges,¹ because by virtue of $Q_\perp = 0$ they carry magnetic lines into magnetic lines changing the magnitude of $\vec{B}$ but not its direction.

For these perturbations, expression (4.22) for δW becomes

$$\delta W = \int \left[\tau B^2 \left(\frac{Q_\parallel}{B} \right)^2 - 2X \frac{Q_\parallel}{B} (\dot{I}\dot{\psi}D + B^2\dot{\sigma}) - X^2 (\dot{I}\ddot{\psi}D + \dot{I}\dot{\psi}D) \right] d^3x + W_K \quad (5.4)$$

but

$$\dot{I}\dot{\psi}D + B^2\dot{\sigma} = \dot{p}_\parallel - B \frac{\partial \dot{p}_\parallel}{\partial B} = - \dot{p}_\perp \quad (5.5)$$

so

$$\delta W = \int \left[\tau B^2 \left(\frac{Q_\parallel}{B} \right)^2 + 2\dot{p}_\perp \frac{Q_\parallel}{B} X - (\dot{I}\ddot{\psi}D + \dot{I}\dot{\psi}D) X^2 \right] \frac{dv d\theta d\zeta}{D} + W_K \quad (5.6)$$

The integrand of the first term of (5.6) is a quadratic form in $Q_\parallel/B$ and X , both of which are constant along field lines.

Consider first simple interchanges for which $Q_{||} = 0$. Then

$$\delta W = - \int dv d\theta x^2 \oint [I\ddot{\psi}D + \dot{I}\dot{\psi}D] \frac{d\zeta}{D} + W_K \quad (5.7)$$

Sufficient for stability to simple interchanges (dropping W_K) is that on each line (i.e., for each value of v and θ)

$$\oint \frac{d\zeta}{D} [I\ddot{\psi}D + \dot{I}\dot{\psi}D] \leq 0 \quad (5.8)$$

However, $\dot{I}$, $\dot{\psi}$, and $\ddot{\psi}$ are all independent of ζ so (5.8) is

$$\dot{I}\ddot{\psi} + \dot{I}\dot{\psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq 0 \quad (5.9)$$

where the average $\langle \dots \rangle$ was defined in (3.18). Since I and ψ are surface quantities, only $\dot{D}/D$ need be evaluated for different θ values.

Further the only derivatives are normal to the v surfaces, so we have attained the goal of a simple stability criterion for nonaxisymmetric tori.

For the general case, $Q_{||} \neq 0$, positivity of the quadratic form requires that for each v and θ

$$\dot{I}\ddot{\psi} + \dot{I}\dot{\psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq - \frac{\left\langle \frac{\dot{p}_{\perp}}{D} \right\rangle^2}{\left\langle \frac{\tau_B^2}{D} \right\rangle} \quad (5.10)$$

Expression (5.9) appears totally unrelated to the Rosenbluth-Langmuir-Grad expression^{9,10,1} obtained with axisymmetry. In fact, in the limit of axisymmetry they are identical as we now show. Denote by R the RLG form so

$$R = \int \frac{d\ell}{B} \left[\frac{1}{B} \frac{\partial B}{\partial \psi} \frac{\partial p_{\perp}}{\partial \psi} + \frac{\kappa}{rB} \frac{\partial p_{||}}{\partial \psi} \right] \quad (5.11)$$

where $\partial p_{\perp}/\partial \psi$ is the functional derivative holding B constant. Then also

$$R = \int \frac{d\ell}{B} \left[\frac{\partial p_{\parallel}}{\partial \psi} \left(\frac{\kappa}{rB} + \frac{1}{B} \frac{\partial B}{\partial \psi} \right) + \frac{1}{B} \frac{\partial B}{\partial \psi} \left(\frac{\partial p_{\perp}}{\partial \psi} - \frac{\partial p_{\parallel}}{\partial \psi} \right) \right] \quad (5.12)$$

Now $\sigma = 1 + \frac{p_{\perp} - p_{\parallel}}{B^2}$, $\frac{\partial \sigma}{\partial \psi} = \frac{1}{B^2} \frac{\partial}{\partial \psi} (p_{\perp} - p_{\parallel})$, so

$$R = \int \frac{d\ell}{B} \left[\frac{\partial p_{\parallel}}{\partial \psi} \left(\frac{\kappa}{rB} + \frac{1}{B} \frac{\partial B}{\partial \psi} \right) + B \frac{\partial B}{\partial \psi} \frac{\partial \sigma}{\partial \psi} \right] \quad (5.13)$$

Let "." denote $\frac{\partial}{\partial v} \Big|_B$, then $\frac{\partial f}{\partial \psi} = \dot{f}/\dot{\psi}$, so

$$R = \int \frac{d\ell}{B} \left[\frac{\dot{p}_{\parallel}}{\dot{\psi}} \left(\frac{\kappa}{rB} + \frac{1}{B} \frac{\partial B}{\partial \psi} \right) + B \frac{\partial B}{\partial v} \frac{\dot{\sigma}}{\dot{\psi}^2} \right] \quad (5.14)$$

From the equilibrium equation, $\dot{p}_{\parallel}/\dot{\psi} = \dot{I}D$, and $\dot{I}$ is constant along field lines, so

$$R = \dot{I} \int \frac{d\ell}{B} D \left(\frac{\kappa}{rB} + \frac{1}{B} \frac{\partial B}{\partial \psi} \right) + \int \frac{d\ell}{B} \frac{\partial B}{\partial v} \frac{\dot{\sigma}}{\dot{\psi}^2} \quad (5.15)$$

Now $\dot{q} = (1/\dot{\psi}) \frac{\partial}{\partial v} \int \frac{d\ell}{B} D = \int \frac{d\ell}{B} \frac{\partial D}{\partial v} \Big|_{\theta, \zeta} + \int D \frac{\partial}{\partial v} \left(\frac{d\ell}{B} \right)$, but in axial symmetry¹

$$\frac{\partial}{\partial v} \left(\frac{d\ell}{B} \right) = \frac{d\ell}{B} \left(\frac{\kappa}{rB} + \frac{1}{B} \frac{\partial B}{\partial \psi} \right)$$

so

$$\dot{q} = \int \frac{d\ell}{B} \frac{\partial D}{\partial v} \Big|_{\theta, \zeta} + \int \frac{d\ell}{B} D \left(\frac{\kappa}{rB} + \frac{1}{B} \frac{\partial B}{\partial \psi} \right)$$

Inserting this expression into (5.15)

$$R = - \frac{\dot{I}}{\dot{\psi}} \dot{q} + \frac{1}{\dot{\psi}^2} \int \frac{d\ell}{B} \left(\dot{I} \dot{\psi} \frac{\partial D}{\partial v} \Big|_{\theta, \zeta} + B \frac{\partial B}{\partial v} \Big|_{\theta, \zeta} \dot{\sigma} \right) \quad (5.16)$$

Now $\dot{\sigma} = - \frac{1}{B} \frac{\partial \dot{p}_{\parallel}}{\partial B} = - \frac{1}{B} \dot{I} \dot{\psi} \frac{\partial D}{\partial B} \Big|_v$ so the parentheses in (5.16) are

$$\dot{I}\dot{\Psi} \left. \frac{\partial D}{\partial v} \right|_{\theta, \zeta} + B \left. \frac{\partial B}{\partial v} \right|_{\theta, \zeta} \dot{\sigma} = \dot{I}\dot{\Psi} \left(\left. \frac{\partial D}{\partial v} \right|_{\theta, \zeta} - \left. \frac{\partial D}{\partial B} \right|_v \left. \frac{\partial B}{\partial v} \right|_{\theta, \zeta} \right) = \dot{I}\dot{\Psi}\dot{D}$$

and therefore

$$R = - \frac{\dot{I}}{\dot{\Psi}} \dot{q} + \frac{1}{\dot{\Psi}^2} \int \frac{d\ell}{B} \dot{I}\dot{\Psi}\dot{D}.$$

Finally, $\dot{q} = (1/\dot{\Psi})' = -\ddot{\Psi}/\dot{\Psi}^2$ so

$$R = \frac{1}{\dot{\Psi}^2} \left(\ddot{I}\dot{\Psi} + \dot{I}\ddot{\Psi} \left\langle \frac{\dot{D}}{D} \right\rangle \right) \quad (5.17)$$

Hence, the criteria are identical because $R < 0$ implies (5.9) is satisfied.

6. STABILITY TO A CLASS OF LOCALIZED PERTURBATIONS

For axisymmetric equilibria a stability criterion stronger than that derived from interchanges was obtained by considering $\vec{\xi} \sim \cos m \theta$ and studying the limit $m \rightarrow \infty$.¹¹ The appropriate generalization to the present case is to consider perturbations which are highly localized in θ , i.e., which vanish unless θ is almost equal to some θ_0 .

As in the preceding section we employ Hamada coordinates and write $\vec{\xi}$ in terms of components

$$\vec{\xi} = X\vec{e}_v + Y\vec{e}_\theta + Z\vec{e}_\zeta \quad (4.14)$$

The degree of localization is measured by a parameter ϵ : a stretched coordinate $\bar{\theta} = (\theta - \theta_0)/\epsilon$ is introduced, and $\vec{\xi}$ is assumed to vanish outside the interval $|\bar{\theta}| < 1/2$. Since δW is independent of Z , we restrict attention to test functions X and Y and write them as series in ϵ :

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \sum_{v \geq 0} \epsilon^v \begin{bmatrix} X_v(v, \bar{\theta}, \zeta) \\ Y_v(v, \bar{\theta}, \zeta) \end{bmatrix} \quad (6.1)$$

Equilibrium quantities are also expanded in powers of ϵ . Then δW is written as an expansion in ϵ which is minimized order by order up to the lowest order which is not automatically positive. The condition for positivity of this order is then necessary for stability because δW itself can be made negative by choosing a sufficiently small ϵ if this condition is violated. This procedure was previously applied to scalar pressure equilibria with shear⁵ and with closed lines,¹² but with localization in v instead of in θ .

We shall make one additional restriction on ξ which greatly simplifies the analysis but which weakens the stability criterion, namely that X is constant along field lines. Relaxing this constraint adds a destabilizing term in $\vec{J} \cdot \vec{B}$. For the bumpy torus this is expected to be small, but for equilibria with significant parallel current the difference could be important.

To lowest order, δW given by (4.22) is

$$\delta W_{-1} = \int \frac{dv d\bar{\theta} d\zeta}{D} \sigma B^2 \left(\frac{\partial Y_0}{\partial \bar{\theta}} \right)^2 + W_K \quad (6.2)$$

Again we ignore W_K in this section to find $Y_0 = 0$ minimizes δW_{-1} to zero. The stability criterion is obtained in the next order. To zero order in ϵ , $\vec{Q}$ is given by

$$\begin{aligned} Q_0^v &= 0 \\ Q_0^u &= 0 \\ Q_0^\zeta &= -\dot{\psi} D \left(\frac{\partial X_0}{\partial v} + \frac{\partial Y_1}{\partial \bar{\theta}} \right) - \ddot{\psi} D X_0 \end{aligned} \quad (6.3)$$

Thus $Q_{\perp 0} = 0$ and $Q_{\parallel 0}/B = Q_0^\zeta/\dot{\psi} D$. Hence to first order in ϵ , δW is given by

$$\delta W_1 = \int \left[B^2 \left(\frac{Q_{\parallel}}{B} \right)^2 + \epsilon \dot{P}_{\perp} \frac{Q_{\parallel}}{B} X - (\dot{I} \ddot{\psi} D + \dot{I} \dot{\psi} \dot{D}) X^2 \right] \frac{dv d\bar{\theta} d\zeta}{D} + W_K \quad (6.4)$$

We have dropped subscripts on $Q_{\parallel}$ and X and have used relation (5.5) to simplify the integrand. All equilibrium quantities in this order are taken at $\theta = \theta_0$. Expression (6.4) appears identical to (5.6) derived for interchanges, but here $Q_{\parallel}/B$ is not required to be constant along field lines. Hence, we expect the resulting stability criterion to be stronger than (5.10).

For a sufficient condition ensuring stability against the perturbations defined in (6.1), we ignore W_K . The procedure is then the same as in axisymmetry. We complete squares on $Q_{||}/B$ and show that the resulting term can be minimized to zero. Write δW_1 as a quadratic form in $Q_{||}/B$ and X :

$$\delta W_1 = \int \left[\alpha \left(\frac{Q_{||}}{B} \right)^2 + 2\beta \frac{Q_{||}}{B} X - \gamma X^2 \right] d^3x$$

where $\alpha = \tau_B^2$, $\beta = \dot{p}_\perp$, $\gamma = \ddot{I}\ddot{\psi}D + \dot{I}\dot{\psi}\dot{D}$. Complete squares:

$$\delta W_1 = \int \left[\alpha \left(\frac{Q_{||}}{B} - \frac{\beta}{\alpha} X \right)^2 - \left(\frac{\beta^2}{\alpha} + \gamma \right) X^2 \right] d^3x \quad (6.5)$$

The first term can be minimized to zero by proper choice of Y :

$$\frac{Q_{||}}{B} - \frac{\beta}{\alpha} X = - \frac{\partial X}{\partial v} - \frac{\partial Y}{\partial \bar{\theta}} - \left(\frac{\ddot{\psi}}{\dot{\psi}} - \frac{\dot{p}_\perp}{\tau_B^2} \right) X$$

So if we choose

$$\frac{\partial Y}{\partial \bar{\theta}} = - \frac{\partial X}{\partial v} - \left(\frac{\ddot{\psi}}{\dot{\psi}} - \frac{\dot{p}_\perp}{\tau_B^2} \right) X$$

then the term vanishes. This minimizing Y can be found if we impose the additional constraint on X : $\int X d\bar{\theta} = 0$. With this choice of Y , (6.5) becomes

$$\delta W_1 = - \int dv d\bar{\theta} X^2 \int \frac{d\zeta}{D} \left(\frac{\dot{p}_\perp^2}{\tau_B^2} + \ddot{I}\ddot{\psi}D + \dot{I}\dot{\psi}\dot{D} \right) \quad (6.6)$$

Sufficient for stability to these perturbations is thus

$$\ddot{I}\ddot{\psi} + \dot{I}\dot{\psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq - \left\langle \frac{\dot{p}_\perp^2}{\tau_B^2 D} \right\rangle \quad (6.7)$$

7. SUFFICIENT CONDITION FOR STABILITY

In the preceding sections we have derived criteria for stability against certain types of perturbations. In this section we establish a sufficient condition for stability against all perturbations, i.e., a true lower bound on the minimum of δW attainable with any admissible perturbation. If this condition is satisfied, the equilibrium is guaranteed to be exponentially stable within either the single fluid or the two fluid guiding center theory.

There are five important steps in establishing this sufficient condition:

1. We drop the positive term w_K and write δW in terms of the basis $(\vec{e}, \vec{K}, \vec{B})$ with unspecified $\vec{e}$.
2. We note that δW is a quadratic form in X and $\vec{Q}$.
3. We derive a sufficient condition for positivity of this form by employing linear algebra and considering X and $\vec{Q}$ to be independent.
4. We note that the sufficient condition depends on the choice of basis vector $\vec{e}$ with the arbitrariness of $\vec{e}$ equivalent to one free scalar function.
5. The optimization of $\vec{e}$ to obtain the best condition (easiest to satisfy) is equivalent to finding the lowest eigenvalue of a Hill's equation. This equation can be solved directly or else available methods can be used to approximate the eigenvalue from below.

The condition derived is sufficient but not necessary because a positive term is dropped from δW and because constraints connecting $\vec{\xi}$ and its

derivatives in $\vec{Q}$ are ignored.

Recall that after dropping W_K , δW can be written as in (4.11)

$$\delta W = \int \left\{ \sigma \left[\vec{Q}_\perp + \frac{X}{\sigma} \vec{F}_\perp \right]^2 + \tau \left[Q_\parallel + \frac{X}{\tau} (F_\parallel - B \vec{e} \cdot \vec{\nabla} \sigma) \right]^2 + N X^2 \right\} d^3 x \quad (4.11)$$

where

$$\vec{F} = \vec{K} \times \vec{e}$$

$$\vec{G} = \vec{B} \times \vec{e}$$

$$N = \vec{F} \cdot \text{curl } \vec{G} + \vec{e} \cdot \vec{\nabla} p_\parallel \text{div } \vec{e} - B(\vec{e} \cdot \nabla B)(\vec{e} \cdot \vec{\nabla} \sigma) - \frac{1}{\sigma} F_\perp^2 - \frac{1}{\tau} (F_\parallel - B \vec{e} \cdot \vec{\nabla} \sigma)^2$$

δW is a quadratic form in $\vec{Q}$ and X . It is obviously positive if σ , τ , and N are everywhere positive. The positivity of σ and τ is already required by local stability, thus $N > 0$ is the only new condition.

The exact structure of N depends upon the particular choice of basis vector $\vec{e}$. It is reasonable to search for a best criterion (easiest to satisfy in a given equilibrium) by a suitably optimum choice of $\vec{e}$. N has two properties which aid the search for this optimum. As a function of $\vec{e}$ we denote N by $N(\vec{e})$, then if c is any scalar function

$$1) \quad N(c\vec{e}) = c^2 N(\vec{e})$$

$$2) \quad N(\vec{e} + c\vec{K}) = N(\vec{e})$$

The proof of 1) is trivial. To prove 2) define $\vec{F}^1 = \vec{K} \times (\vec{e} + c\vec{K})$ and $\vec{G}^1 = \vec{B} \times (\vec{e} + c\vec{K})$. Then the proof rests on the following identity

$$\begin{aligned} \vec{F}^1 \cdot \text{curl } \vec{G}^1 &= \vec{F} \cdot \text{curl } \vec{G} - c\vec{F} \cdot \text{curl } (\vec{\nabla} p_\parallel) - \vec{F} \cdot \nabla c \times \vec{\nabla} p_\parallel \\ &= \vec{F} \cdot \text{curl } \vec{G} - cB\vec{F} \cdot (\vec{\nabla} \sigma \times \nabla B) - (\vec{K} \cdot \nabla c)(\vec{e} \cdot \vec{\nabla} p_\parallel) \\ &= \vec{F} \cdot \text{curl } \vec{G} - cB(\vec{e} \cdot \vec{\nabla} \sigma)(\vec{K} \cdot \nabla B) - (\vec{K} \cdot \nabla c)(\vec{e} \cdot \vec{\nabla} p_\parallel) \end{aligned}$$

Property (1) shows that the positivity of N is unchanged by multiplication of $\vec{e}$ with a scalar function, while property (2) implies that N is independent of the component of $\vec{e}$ along $\vec{K}$. Thus without loss of generality we can write $\vec{e}$ as

$$\vec{e} = \frac{\nabla v}{|\nabla v|^2} + \frac{\Lambda}{p_{\parallel}} \vec{B} \quad (7.1)$$

The arbitrariness of $\vec{e}$ is now reflected in the choice of the function $\Lambda(\vec{x})$. We shall see that the optimal Λ satisfies a differential equation along field lines.

Of course for every choice of Λ , $N(\Lambda) \geq 0$ gives a sufficient condition for stability. As an aside, we remark that in scalar pressure theory the choice of $\Lambda = 0$ corresponds to the sufficient condition implicit in equation (4.5) of Ref. 13, while the choice of Λ to make $\vec{e} = \vec{e}_v$ leads to the sufficient condition of Solovén.¹⁴ The generalization of these conditions to tensor pressure is immediate, but not very useful because they are too hard to satisfy.

If we insert the form given for $\vec{e}$ by (7.1) into N we find

$$\begin{aligned} \vec{F} &= \vec{K} \times \vec{e} = \frac{\vec{K} \times \nabla v}{\nabla v^2} + \frac{\Lambda}{p_{\parallel}} \vec{\nabla} p_{\parallel} = \frac{\vec{K} \times \nabla v}{\nabla v^2} + \Lambda \nabla v \\ F_{\parallel} &= \frac{\vec{F} \cdot \vec{B}}{B} = \frac{\vec{K} \times \nabla v \cdot \vec{B}}{B \nabla v^2} = - \frac{\vec{\nabla} p_{\parallel} \cdot \nabla v}{B \nabla v^2} = - \frac{p_{\parallel}}{B} \\ \vec{F}_{\perp} &= \frac{(\vec{B} \times \vec{F}) \times \vec{B}}{B^2} = \frac{[\vec{B} \times (\vec{K} \times \nabla v)] \times \vec{B}}{B^2 \nabla v^2} + \Lambda \nabla v = - \frac{\vec{K} \cdot \vec{B}}{B^2 \nabla v^2} \nabla v \times \vec{B} + \Lambda \nabla v \\ F_{\perp}^2 &= \frac{K_{\parallel}^2}{\nabla v^2} + \Lambda^2 \nabla v^2 \end{aligned}$$

$$\vec{e} \cdot \tilde{\nabla} p_{\parallel} = \dot{p}_{\parallel} \vec{e} \cdot \nabla v = \dot{p}_{\parallel}$$

$$\text{div } \vec{e} = \text{div} \frac{\nabla v}{\nabla v^2} + \vec{B} \cdot \nabla \frac{\Lambda}{\dot{p}_{\parallel}}$$

$$\vec{e} \cdot \nabla B = \frac{\nabla v \cdot \nabla B}{\nabla v^2} + \frac{\Lambda}{\dot{p}_{\parallel}} \vec{B} \cdot \nabla B$$

$$\vec{e} \cdot \tilde{\nabla} \sigma = \dot{\sigma} \vec{e} \cdot \nabla v = \dot{\sigma} = \left(1 - \frac{1}{B} \frac{\partial p_{\parallel}}{\partial B}\right) \dot{\sigma} = - \frac{1}{B} \frac{\partial \dot{p}_{\parallel}}{\partial B}$$

$$F_{\parallel} - B e \cdot \tilde{\nabla} \sigma = - \frac{\dot{p}_{\parallel}}{B} + \frac{\partial \dot{p}_{\parallel}}{\partial B} = \frac{\dot{p}_{\perp}}{B}$$

$$\vec{F} \cdot \text{curl } \vec{G} = \frac{\vec{K} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{B} \times \nabla v}{\nabla v^2} + \Lambda \nabla v \cdot \text{curl} \frac{\vec{B} \times \nabla v}{\nabla v^2}$$

Now

$$\begin{aligned} \nabla v \cdot \text{curl} \frac{\vec{B} \times \nabla v}{\nabla v^2} &= \text{div} \left[\frac{(\vec{B} \times \nabla v) \times \nabla v}{\nabla v^2} \right] + \frac{\vec{B} \times \nabla v}{\nabla v^2} \cdot \text{curl } \nabla v \\ &= \text{div } \vec{B} \\ &= 0 \end{aligned}$$

and

$$\frac{\vec{K} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{B} \times \nabla v}{\nabla v^2} = \text{div} \left[\frac{\vec{B} \times \nabla v}{\nabla v^2} \times \frac{\vec{K} \times \nabla v}{\nabla v^2} \right] + \frac{\vec{B} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{K} \times \nabla v}{\nabla v^2}$$

but

$$\begin{aligned} \text{div} \left[\frac{\vec{B} \times \nabla v}{\nabla v^2} \times \frac{\vec{K} \times \nabla v}{\nabla v^2} \right] &= - \text{div} \left[\frac{\vec{K} \times \vec{B} \cdot \nabla v}{\nabla v^2} \nabla v \right] = - \text{div} \left[\frac{\dot{p}_{\parallel}}{\nabla v^2} \nabla v \right] \\ &= - \dot{p}_{\parallel} \text{div} \frac{\nabla v}{\nabla v^2} - \ddot{p}_{\parallel} - \frac{\partial \dot{p}_{\parallel}}{\partial B} \frac{\nabla v \cdot \nabla B}{\nabla v^2} \end{aligned}$$

Hence

$$\vec{F} \cdot \text{curl } \vec{G} = - \dot{p}_{\parallel} \text{div} \frac{\nabla v}{\nabla v^2} - \ddot{p}_{\parallel} - \frac{\partial \dot{p}_{\parallel}}{\partial B} \frac{\nabla v \cdot \nabla B}{\nabla v^2} + \frac{\vec{B} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{K} \times \nabla v}{\nabla v^2}$$

Collecting these results, simplifying and canceling, we find

$$N = \vec{B} \cdot \nabla \Lambda - \frac{\nabla v^2}{\sigma} \Lambda^2 + \frac{\vec{B} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{K} \times \nabla v}{\nabla v^2} - \ddot{p}_{\parallel} - \frac{K_{\parallel}^2}{\sigma \nabla v^2} - \frac{\dot{p}_{\perp}^2}{\tau_B^2} \quad (7.2)$$

The stability condition $N(\Lambda) \geq 0$ is thus a differential inequality for Λ along field lines. We ask: does there exist a function Λ such that $N(\Lambda) \geq 0$? To have a well defined coordinate system, Λ must be finite, periodic, and continuously differentiable.

To answer this question, it is useful to put (7.2) into dimensionless form. We divide by the (positive) coefficient of Λ^2 to obtain

$$\frac{\sigma B}{\nabla v^2} \frac{d\Lambda}{d\ell} - \Lambda^2 = A \geq 0 \quad (7.3)$$

with $d/d\ell \equiv \hat{b} \cdot \nabla$ and

$$A = - \frac{\sigma}{\nabla v^2} \left[\frac{\vec{B} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{K} \times \nabla v}{\nabla v^2} - \ddot{p}_{\parallel} - \frac{K_{\parallel}^2}{\sigma \nabla v^2} - \frac{\dot{p}_{\perp}^2}{\tau_B^2} \right] \quad (7.4)$$

Next we introduce a dimensionless coordinate x by

$$dx = \frac{1}{X} \frac{\nabla v^2}{\sigma B} d\ell; \quad X = \oint \frac{d\ell}{B} \frac{\nabla v^2}{\sigma} \quad (7.5)$$

Then $\frac{\sigma B}{\nabla v^2} \frac{d\Lambda}{d\ell} = X \frac{d\Lambda}{dx}$, and if we replace Λ , $N(\Lambda)$, and A by the dimensionless quantities

$$\lambda = X\Lambda; \quad n_{\lambda} = X^2 N(\Lambda); \quad a = X^2 A \quad (7.6)$$

(7.3) is equivalent on each magnetic line to the ordinary differential inequality

$$n_{\lambda} = \lambda' - \lambda^2 - a \geq 0 \quad (7.7)$$

with $\lambda' = d\lambda/dx$. The marginal condition, $n_\lambda = 0$, is Ricotti's differential equation. It can be transformed to a second order linear equation with periodic coefficients by the change of variable $\lambda = -y'/y$. Hence we are led to consider solving (7.7) by studying the related Hill's equation

$$y'' + \tilde{a}y + \alpha y = 0, \quad y(x+1) = y(x) \quad (7.8)$$

where, with a defined by (7.6),

$$\tilde{a} = a - \bar{a}, \quad \bar{a} = \int_0^1 a dx$$

This problem is of Sturm Liouville type with periodic boundary conditions and therefore has a finite lowest eigenvalue α_0 whose eigenfunction y_0 has no zeros. Choosing $\lambda = -y'_0/y_0$ as a possible solution to (7.7), we find $n_\lambda = \alpha_0 - \bar{a}$. Hence, $\bar{a} \leq \alpha_0$ is sufficient for stability. It can be shown that this method gives the optimal choice for λ . In terms of the original variables

$$\begin{aligned} \bar{a} &= X \int \frac{\nabla v^2}{\sigma B} A d\ell \\ &= -q^2 \left\langle \frac{\nabla v^2}{\sigma D} \right\rangle \left\langle \frac{1}{D} \left[\frac{\vec{B} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{K} \times \nabla v}{\nabla v^2} - \ddot{p}_{\parallel} - \frac{K_{\parallel}^2}{\sigma \nabla v^2} - \frac{\dot{p}_{\perp}^2}{\tau B^2} \right] \right\rangle \end{aligned}$$

with $\langle \dots \rangle$ defined as in (3.18). It can be shown that³

$$\left\langle \frac{1}{D} \left[\frac{\vec{B} \times \nabla v}{\nabla v^2} \cdot \text{curl} \frac{\vec{K} \times \nabla v}{\nabla v^2} - \ddot{p}_{\parallel} \right] \right\rangle = -\ddot{I}\ddot{\Psi} - \left\langle \frac{\dot{p}_{\parallel} \dot{D}}{D^2} \right\rangle \quad (7.9)$$

and $\langle \dot{p}_{\parallel} \dot{D}/D^2 \rangle = \dot{I}\dot{\Psi} \langle \dot{D}/D \rangle$ so

$$\bar{a} = q^2 \left\langle \frac{\nabla v^2}{\sigma D} \right\rangle \left[\dot{I}\ddot{\Psi} + \dot{I}\dot{\Psi} \left\langle \frac{\dot{D}}{D} \right\rangle + \left\langle \frac{K_{\parallel}^2}{\sigma D \nabla v^2} + \frac{\dot{p}_{\perp}^2}{\tau B^2 D} \right\rangle \right]$$

Thus the stability criterion $\bar{a} \leq \alpha_0$ is equivalent to

$$\ddot{\Psi} + \dot{\Psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq \frac{\alpha_0}{q^2 \left\langle \frac{\nabla v^2}{\sigma D} \right\rangle} - \left\langle \frac{K_{\parallel}^2}{\sigma D \nabla v^2} + \frac{\dot{p}_{\perp}^2}{\tau D B^2} \right\rangle \quad (7.10)$$

The eigenvalue α_0 can be found by direct solution of the Hill's equation

(7.8) or else available methods can be used to find a lower bound for

α_0 in terms of A.¹⁵

8. SUMMARY

We have examined the stability of anisotropic closed line tori with pressure surfaces and have found simple conditions for stability to various perturbations. Because we have ignored the positive kinetic term W_K , each of these is only sufficient for stability to the given perturbation. W_K can also be included in all but the final condition at no cost in analytic difficulty (but some in complexity).

We repeat here the various conditions to emphasize their parallelism.

1. Stability to simple interchanges

$$\ddot{I}\ddot{\Psi} + \dot{I}\dot{\Psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq 0 \quad (5.9)$$

2. Stability to general interchanges

$$\ddot{I}\ddot{\Psi} + \dot{I}\dot{\Psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq - \left\langle \frac{\dot{p}_1^2}{D} \right\rangle / \left\langle \frac{\tau_B^2}{D} \right\rangle \quad (5.10)$$

3. Stability to a class of localized perturbations

$$\ddot{I}\ddot{\Psi} + \dot{I}\dot{\Psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq - \left\langle \frac{\dot{p}_1^2}{\tau_B^2 D} \right\rangle \quad (6.7)$$

4. Stability to all perturbations

$$\ddot{I}\ddot{\Psi} + \dot{I}\dot{\Psi} \left\langle \frac{\dot{D}}{D} \right\rangle \leq \frac{\alpha_0}{q^2 \left\langle \frac{\nabla v^2}{\sigma D} \right\rangle} - \left\langle \frac{\dot{p}_1^2}{\tau_B^2 D} \right\rangle - \left\langle \frac{K_{\parallel}^2}{\sigma D \nabla v^2} \right\rangle \quad (7.10)$$

APPENDIX

We give here the proof of the identity quoted in Section 4:

$$\begin{aligned}
 & \vec{K} \times \vec{\xi} \cdot \vec{Q} + \vec{\xi} \cdot \tilde{\nabla} p_{\parallel} \operatorname{div} \vec{\xi} - B(\vec{\xi} \cdot \nabla B) \vec{\xi} \cdot \tilde{\nabla} \sigma = \\
 & = 2X\vec{F} \cdot \vec{Q} + X^2[\vec{F} \cdot \operatorname{curl} \vec{G} + \vec{e} \cdot \tilde{\nabla} p_{\parallel} - B(\vec{e} \cdot \nabla B)\vec{e} \cdot \tilde{\nabla} \sigma] \quad (4.8) \\
 & \quad + \operatorname{div} [XZ(\vec{e} \cdot \tilde{\nabla} p_{\parallel})\vec{B}]
 \end{aligned}$$

where X , Y , and Z are defined by

$$\vec{\xi} = X\vec{e} + Y\vec{K} + Z\vec{B} \quad (4.9)$$

and $\vec{F} = \vec{K} \times \vec{e}$, $\vec{G} = \vec{B} \times \vec{e}$. The basis vector $\vec{e}$ is constrained by $\vec{e} \cdot \nabla v = 1$. The identity (4.8) is only true for equilibria in which $\vec{K} \cdot \tilde{\nabla} \sigma = 0$, and we shall require this property in the following proof.

The proof of (4.8) is straightforward, but tedious. We proceed term by term to calculate the left hand side. First to compute $\vec{Q} = \operatorname{curl}(\vec{\xi} \times \vec{B})$, we note that

$$\vec{\xi} \times \vec{B} = -X\vec{G} + Y\tilde{\nabla} p_{\parallel}$$

so

$$\vec{Q} = -\nabla X \times \vec{G} - X \operatorname{curl} \vec{G} + \nabla Y \times \tilde{\nabla} p_{\parallel} + Y \operatorname{curl} \tilde{\nabla} p_{\parallel} \quad (A.1)$$

Now $\vec{K} \times \vec{\xi} = X\vec{F} + Z\tilde{\nabla} p_{\parallel}$ so

$$\vec{K} \times \vec{\xi} \cdot \vec{Q} = X\vec{F} \cdot \vec{Q} + Z\tilde{\nabla} p_{\parallel} \cdot \vec{Q}$$

The second term of this is evaluated using

$$\begin{aligned}
 \tilde{\nabla} p_{\parallel} \cdot \operatorname{curl}(\vec{\xi} \times \vec{B}) &= \operatorname{div}[(\vec{\xi} \times \vec{B}) \times \tilde{\nabla} p_{\parallel}] + \vec{\xi} \times \vec{B} \cdot \operatorname{curl} \tilde{\nabla} p_{\parallel} \\
 &= \operatorname{div}[X(\vec{e} \cdot \tilde{\nabla} p_{\parallel})\vec{B}] + \vec{\xi} \times \vec{B} \cdot \operatorname{curl} \tilde{\nabla} p_{\parallel}
 \end{aligned}$$

There is a further simplification gained by noting that since $\partial p_{\parallel} / \partial B = B(1 - \sigma)$,

$$\tilde{\nabla} p_{\parallel} = \nabla p_{\parallel} - B(1 - \sigma) \nabla B$$

$$\text{curl } \tilde{\nabla} p_{\parallel} = - \text{curl} [B(1 - \sigma) \nabla B] = B \nabla \sigma \times \nabla B = B \tilde{\nabla} \sigma \times \nabla B$$

the last equality stemming from $(\nabla \sigma - \tilde{\nabla} \sigma) \times \nabla B = 0$. Hence

$$\vec{\xi} \times \vec{B} \cdot \text{curl } \tilde{\nabla} p_{\parallel} = XB(\vec{e} \cdot \tilde{\nabla} \sigma) \vec{B} \cdot \nabla B$$

and the first term of (4.8) is

$$\vec{K} \times \vec{\xi} \cdot \vec{Q} = X \vec{F} \cdot \vec{Q} + XZB(\vec{e} \cdot \tilde{\nabla} \sigma) \vec{B} \cdot \nabla B + Z \text{div} [X(\vec{e} \cdot \tilde{\nabla} p_{\parallel}) \vec{B}] \quad (\text{A.2})$$

The second term of (4.8) is evaluated using

$$\vec{\xi} \cdot \tilde{\nabla} p_{\parallel} = X \vec{e} \cdot \tilde{\nabla} p_{\parallel}$$

$$\text{div } \vec{\xi} = \vec{e} \cdot \nabla X + X \text{div } \vec{e} + \vec{K} \cdot \nabla Y + \vec{B} \cdot \nabla Z$$

so

$$\begin{aligned} \vec{\xi} \cdot \tilde{\nabla} p_{\parallel} \text{div } \vec{\xi} &= X(\vec{e} \cdot \tilde{\nabla} p_{\parallel})(\vec{e} \cdot \nabla X + \vec{K} \cdot \nabla Y) + \\ &+ X^2 \vec{e} \cdot \tilde{\nabla} p_{\parallel} \text{div } \vec{e} + X \vec{e} \cdot \tilde{\nabla} p_{\parallel} \vec{B} \cdot \nabla Z \end{aligned}$$

but direct computation of $\vec{F} \cdot \vec{Q}$ shows that

$$(\vec{e} \cdot \tilde{\nabla} p_{\parallel})(\vec{e} \cdot \nabla X + \vec{K} \cdot \nabla Y) = \vec{F} \cdot \vec{Q} - Y \vec{F} \cdot \text{curl } \tilde{\nabla} p_{\parallel} + X \vec{F} \cdot \text{curl } \vec{G}$$

so the second term of (4.8) is

$$\begin{aligned} \vec{\xi} \cdot \tilde{\nabla} p_{\parallel} \text{div } \vec{\xi} &= X \vec{F} \cdot \vec{Q} - XY \vec{F} \cdot \text{curl } \tilde{\nabla} p_{\parallel} + X^2 (\vec{F} \cdot \text{curl } \vec{G} + \vec{e} \cdot \tilde{\nabla} p_{\parallel} \text{div } \vec{e}) \\ &+ X \vec{e} \cdot \tilde{\nabla} p_{\parallel} \vec{B} \cdot \nabla Z \end{aligned} \quad (\text{A.3})$$

The final term in (4.8) is easily seen to be

$$\begin{aligned}
- B(\vec{\xi} \cdot \nabla B)(\vec{\xi} \cdot \tilde{\nabla} \sigma) &= - XZB(\vec{B} \cdot \nabla B)\vec{e} \cdot \tilde{\nabla} \sigma - X^2B(\vec{e} \cdot \nabla B)\vec{e} \cdot \tilde{\nabla} \sigma \\
&- XYB(\vec{K} \cdot \nabla B)\vec{e} \cdot \tilde{\nabla} \sigma
\end{aligned} \tag{A.4}$$

(A.2), (A.3), and (A.4) contain all the ingredients of the identity (4.8), but there is some cancellation left to perform. Consider the terms in Y. These are

$$- XY\vec{F} \cdot \text{curl } \tilde{\nabla} p_{\parallel} - XYB(\vec{K} \cdot \nabla B)\vec{e} \cdot \tilde{\nabla} \sigma$$

However, $\vec{F} \cdot \text{curl } \tilde{\nabla} p_{\parallel} = B\vec{F} \cdot \tilde{\nabla} \sigma \times \nabla B = - B(\vec{e} \cdot \tilde{\nabla} \sigma)\vec{K} \cdot \nabla B$ so the terms in Y cancel. The terms in Z are

$$\begin{aligned}
&XZB(\vec{e} \cdot \tilde{\nabla} \sigma)\vec{B} \cdot \nabla B + Z \text{div}[X(\vec{e} \cdot \tilde{\nabla} p_{\parallel})\vec{B}] + X(\vec{e} \cdot \tilde{\nabla} p_{\parallel})\vec{B} \cdot \nabla Z \\
&- XZB(\vec{e} \cdot \tilde{\nabla} \sigma)\vec{B} \cdot \nabla B
\end{aligned}$$

After trivial cancelling and grouping we are left only with

$$\text{div}[XZ(\vec{e} \cdot \tilde{\nabla} p_{\parallel})\vec{B}]$$

The remaining terms in $\vec{Q}$ and X yield the right hand side of (4.8), so we are finished.

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