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FUNDAMENTAL INVESTIGATION OF LIQUID-METAL LUBRICATED JOURNAL BEARINGS

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U. S. ATOMIC ENERGY COMMISSION
WASHINGTON 25, D. C.

January 12, 1967

SOUTHWEST RESEARCH INSTITUTE
SAN ANTONIO HOUSTON

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R. A. Burton
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LIST OF SYMBOLS

a	Arbitrary constant
b	Local half-thickness of film
C_f	Friction coefficient
D	Bearing diameter
H	Dimensionless film thickness
h	Local film thickness
L	Bearing length
m	Velocity profile exponent
n	Velocity profile exponent
P	Dimensionless pressure
$\bar{P}$	Mean pressure in bearing
p	Pressure
R	Local Reynolds number, bu_b/ν
r	Bearing radius
U	u/u_{bc}
U^+	Dimensionless velocity, $u/\sqrt{\tau/\rho}$
u	Velocity in direction of turning
u'	Turbulent velocity component in direction of turning
V	Velocity vector modulus, $\sqrt{u^2 + w^2}$
W	w/u_{bc}
w	Velocity in axial direction

LIST OF SYMBOLS (Cont'd)

X	Dimensionless coordinate in direction of sliding x/h
x	Coordinate in direction of turning
Y^+	Dimensionless wall distance, $y\sqrt{\tau/\rho}/\nu$
y	Coordinate normal to boundary
Z	Dimensionless axial coordinate
z	Axial coordinate
δ^*	Displacement thickness
μ	Absolute viscosity
ν	Kinematic viscosity
ρ	Fluid density
τ	Shear stress
T	Shear stress at wall
$()_b$	Refers to midchannel
$()_{bc}$	Refers to Couette flow at midchannel
$()_c$	Refers to Couette flow
$()_o$	Refers to stationary wall
$()_s$	Refers to surface velocity of journal

INTRODUCTION

The purpose of this report is to examine experimental data and analytical approaches applicable to thin turbulent films, and to determine how these might best be used in the confident prediction of bearing behavior. Such prediction has been complicated by the absence of an axiomatic theory of turbulence in sufficiently explicit form to be used in the calculation of flow behavior. Any approach to the problem must be based upon empirical information and must draw upon dimensional analysis, along with a few conceptual guides, to generalize from the experimental situations to the subject flows. Earlier developments of such analyses were concerned with the turbulent boundary layer and with displacement flow in pipes. When recent interest was turned to the turbulent lubricant film, the earlier techniques were applied and new ones were originated to deal with the problem (which may be idealized as the interaction of a Couette flow with both codirectional and transverse pressure flows).

Some analysts have drawn upon special approximations in addition to those inherent in the empirical/analytical approach. Others have originated new techniques of data generalization and have applied these to bearing calculations without stopping to show how these differ from better known approaches. Withal, divergences are noted among the predictions. In some cases, special correction factors have been introduced to make the predictions compatible with experimental results on lubricants, but with no assurance that these factors have any generality. Furthermore, the available directly-applicable experimental data have consisted largely of pressures and torques in bearing configurations, and are so far removed from the details of the flow as to be of little use in the critical testing of theoretical predictions.

In view of the need for reliable prediction techniques for numerous recent developments (especially in connection with liquid metal lubrication), the work reported here was undertaken with the following objectives: (1) to provide a critical comparison of the analytical techniques available, (2) to determine the sensitivity of the overall equations of flow to the different analytical approaches, and (3) to provide whatever experimental information is needed to guide the development of analyses.

This has led to the writing of the momentum and continuity equations in simplified form suitable for near-parallel flow between plates. When these are integrated across the film they acquire a form analogous to the momentum-integral equations used in boundary-layer analysis. The quantities which must be predicted by the analytical/empirical techniques appear in these equations as coefficients, given the symbol ζ_i . In the

case of codirectional shear and displacement flows, calculation of the ζ_i coefficient by different techniques will lead to closely comparable answers, provided care is exerted to avoid errors in the near vicinity of the walls.

On the other hand, for shear flow and displacement flow acting across one another, as with end-leakage in a bearing, the problem is more difficult, largely because experimental data have not been available to check the analytical approaches. In this report, one approach to the analysis is carried through to provide detailed predictions, and these are compared with new experimental data. These data also provide information on the details of the flow.

As an entirely separate consideration, experimental data have given evidence of pressure jumps at boundary discontinuities, especially in stepped pads, tilted pads and spiral grooves. Data on these are summarized in this report and are applied to the analysis of tilted pads. The computed results indicate possible conditions of unstable operation.

MOMENTUM AND CONTINUITY RELATIONSHIPS IN A THIN FILM

To set into perspective the discussion which follows, it is desirable to write the Navier Stokes equations with Reynolds-stress terms, and to simplify these to apply to thin films. In this form, they may be integrated to produce useful equations for bearing analysis into which empirical information can be inserted by means of dimensionless coefficients. In Cartesian coordinates, the dynamic equations for an incompressible fluid may be expressed by

$$\frac{\partial \bar{u}_a}{\partial t} + \sum_{i=1,2,3} u_i \frac{\partial \bar{u}_a}{\partial x_i} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_a} + \nu \sum_{i=1,2,3} \frac{\partial^2 \bar{u}_a}{\partial x_i^2} - \frac{\partial^2 \bar{u}_a' \bar{u}_i'}{\partial x_i} \quad (1)$$

and the continuity equation is:

$$\sum \frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (2)$$

In the above equations, the quantities with superscript bar and prime indicate time average and fluctuating components, respectively. Let $x_1 = x$, $x_2 = y$, $x_3 = z$, and let the bearing film thickness be measured along the y -axis, which would be normal to a boundary surface. For a very thin film, simplifications based on order of magnitude considerations may be made as follows⁽²⁾:

$$\frac{\partial^2 \bar{u}}{\partial x^2}, \frac{\partial^2 \bar{u}}{\partial z^2}, \frac{\partial^2 \bar{w}}{\partial x^2}, \frac{\partial^2 \bar{w}}{\partial z^2} \ll \frac{\partial^2 \bar{u}}{\partial y^2}, \frac{\partial^2 \bar{w}}{\partial y^2} \quad (3)$$

$$\frac{\partial^2 \bar{u}'^2}{\partial x^2}, \frac{\partial^2 \bar{u}' w'}{\partial z^2}, \frac{\partial^2 \bar{u}' w'}{\partial x^2}, \frac{\partial^2 \bar{w}'^2}{\partial z^2}, \frac{\partial \bar{u}' w'}{\partial x}, \frac{\partial \bar{v}' w'}{\partial z} \ll \frac{\partial \bar{u}' v'}{\partial y}, \frac{\partial \bar{u}' w'}{\partial y}, \frac{\partial \bar{v}'^2}{\partial y} \quad (4)$$

Assume that the smaller quantities may be allowed to vanish from the equations, and that $v' = 0$ at $y = h$ and $y = 0$. If the mean flow is at steady

state, $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, $\frac{\partial w}{\partial t}$ vanish. In addition, it is consistent with the thin film simplifications to assume that

$$\frac{\partial \bar{p}}{\partial y} = 0$$

Integrating with respect to y , and drawing upon the above simplifying assumptions, the momentum and continuity equations become

$$\rho \left[\int_0^h \frac{\partial u^2}{\partial x} dy + \int_0^h \frac{\partial(uw)}{\partial z} dy \right] = - \frac{\partial p}{\partial x} h + \nu \left[\left(\frac{\partial u}{\partial y} \right)_h - \left(\frac{\partial u}{\partial y} \right)_0 \right] \quad (5)$$

$$\rho \left[\int_0^h \frac{\partial w^2}{\partial z} dy + \int_0^h \frac{\partial(uw)}{\partial x} dy \right] = - \frac{\partial p}{\partial x} h + \nu \left[\left(\frac{\partial w}{\partial y} \right)_h - \left(\frac{\partial w}{\partial y} \right)_0 \right] \quad (6)$$

$$\int_0^h \frac{\partial u}{\partial x} dy + \int_0^h \frac{\partial w}{\partial z} dy = 0 \quad (7)$$

At this point, it is convenient to introduce dimensionless coefficients into the integral terms, so defined as to contain those items which must be computed from outside experimental or theoretical information. The dimensionless groups may be defined as follows:

$$\begin{aligned} \zeta_1 &\equiv \int_0^h \frac{u^2 dy}{u_b h} & \zeta_2 &\equiv \frac{\int_0^h w u dy}{w_b u_b h} \\ \zeta_3 &\equiv \int_0^h \frac{w^2 dy}{w_b^2 h} & \zeta_7 &\equiv \frac{\int_0^h u dy}{u_b h} \end{aligned} \quad (8)$$

$$\zeta_8 \equiv \frac{\int_0^h w \, dy}{w_b h}$$

$$\zeta_4 \equiv \frac{\zeta_8}{\zeta_7}$$

$$C_{xc} \zeta_5 \equiv \frac{\nu \left[\left(\frac{\partial u}{\partial y} \right)_h - \left(\frac{\partial u}{\partial y} \right)_0 \right]}{u_{bc} (u_b - u_{bc})}$$

$$C_{xc} \zeta_6 \equiv \frac{\nu \left[\left(\frac{\partial w}{\partial y} \right)_h - \left(\frac{\partial w}{\partial y} \right)_0 \right]}{u_{bc} w}$$

(8) (Cont'd)

Here ζ_1 , ζ_2 , and ζ_3 enter into inertial terms; ζ_7 , ζ_8 , and ζ_4 are concerned with the transport of matter. All six are determined by the velocity profile shape. The parameters ζ_5 and ζ_6 are concerned with wall shear stress and have acquired their stated form after much numerical experimentation. As will be shown, they provide well behaved and useful coefficients.

In terms of the above coefficients, Equations (5), (6), and (7) become

$$\frac{\partial}{\partial x} (u_b^2 h \zeta_1) + \frac{\partial}{\partial z} (u_b w_b h \zeta_2) = - \frac{1}{\rho} \frac{\partial p}{\partial x} h + 2 C_{xc} \zeta_5 u_{bc} (u_b - u_{bc}) \quad (9)$$

$$\frac{\partial}{\partial x} (w_b^2 h \zeta_3) + \frac{\partial}{\partial z} (w_b u_b h \zeta_2) = - \frac{1}{\rho} \frac{\partial p}{\partial z} h + C_{xc} \zeta_6 u_{bc} (w) \quad (10)$$

$$\frac{\partial}{\partial x} (u_b h \zeta_7) + \frac{\partial}{\partial z} (w_b h \zeta_8) = 0 \quad (11)$$

These equations may serve as the basis for computation of turbulent film behavior.

Again, we reiterate that ζ_5 and ζ_6 are determined by the wall shear stress relationship, while the remainder is concerned with velocity profile shape. The shape coefficients will be discussed in the paragraphs which follow immediately, and the shear stress coefficients will be discussed after these. By any system of calculation, they are near in magnitude to unity.

NUMERICAL ESTIMATES OF VELOCITY PROFILE EFFECTS

Useful insights into the variation of the ζ_i quantities can be had through the application of approximate velocity profile data to their estimation. To obtain such numerical estimates of the general order of magnitude of ζ_1 , ζ_2 , ζ_3 , ζ_4 , ζ_7 , and ζ_8 , it is possible, for example, to assume that the velocity profile in the half-channel (i. e., between a wall and midstream) can be represented by a power-law profile such that

$$\frac{u}{u_b} = \left(\frac{y}{b}\right)^n \quad ; \quad \frac{w}{w_b} = \left(\frac{y}{b}\right)^m \quad (12)$$

Here, y , w , and u are measured, in each case, from the nether wall in the half-channel. These functions have been established experimentally as good approximations of fully turbulent flows, be they displacement flow, Couette flow, or combinations of these. (1) To provide full-channel profile expressions, referenced to the wall where $y = 0$, Equation (12) leads to

$$u = \left[(u_{bc} + \Delta u_b) \left(\frac{y}{b}\right)^n \right]_{y < b} + \left[2U_{bc} - (u_{bc} - \Delta u_b) \left(h - \frac{y}{b}\right)^n \right]_{y > b}$$

$$w = \left[w_b \left(\frac{y}{b}\right)^m \right]_{y < b} + \left[\left(\frac{h - y}{b}\right)^m \right]_{y > b} \quad (13)$$

Here, y is measured across the channel from the stationary wall, and $w = 0$, $y = 0$ and $y = h$.

If one wishes to express the s-shaped profile for the Couette flow, for the entire channel with reference to a single wall, the appropriate equation would become

$$\frac{u}{u_{bc}} = \left[\left(\frac{y}{b}\right)^n \right]_{y < b} + \left[2 - \left(h - \frac{y}{b}\right)^n \right]_{y > b}$$

Typical numerical values for ζ_1 , ζ_2 , ζ_3 , and ζ_4 have been computed for such profiles and are reproduced in Tables 1 through 4. Here, the exponents n and m are related to the half-channel displacement thickness, δ^*/b , by:

$$\frac{\delta_x^*}{b} = \frac{n}{1+n} \quad , \quad \frac{\delta_z^*}{b} = \frac{m}{1+m} \quad (14)$$

Hence, the range of n and m chosen is representative of a range of δ^*/b from 0.1667 to 0.100 which is the range encountered in experimental measurements for fully turbulent flow⁽¹⁾ except in the immediate vicinity of transition as illustrated in Figure 1 and Figure 2. A convenient non-dimensionalization is achieved by letting $\Delta u_b/u_{bc} = U$ in which case the midstream velocity relative to one wall becomes $1 + U$, and, relative to the opposite wall, it becomes $1 - U$.

Note that, for a given value of U , in any of the tables, the departure of ζ_1 from its mean value, for the range of δ^*/b , is at worst about 5 percent. Thus, small errors in δ^* , which may result from inaccuracies in computation of the velocity profile, need not seriously prejudice the calculated solution to the equations.

One seemingly important feature of the tables is that when $U = -1$, ζ_1 and ζ_2 become infinite. When it is remembered, however, that these variables ultimately enter the equations as $\zeta_1(1+U)^2$ and $\zeta_2(1+U)$, it is seen that, on the basis of Equations (13) and (14), the product remains a small but finite value. Inspection will show that the dependence of this magnitude on n or m is as well behaved as that of the other quantities in Tables 1 through 4. We may conclude at this point that errors in the calculated velocity profile produce much smaller errors in the final ζ_i terms which ultimately enter the film equations. We may also conclude that, for modest variations of U , the ζ_i terms do not vary greatly. In many problems, inertial effects contribute only a small portion of the pressure distribution (except at discontinuities, where they may be dealt with separately as a boundary condition). If inertial effects are small, little error will result in treating ζ_1 , ζ_2 , and ζ_3 as constants, so as to provide a simple estimate of the inertial effects. The parameter ζ_4 is more influential in the overall solution and may be treated as a constant only if its range of variation is not expected to be great. If better accuracy is desired, then assumed values of n and m may be used in Equation (18) to provide an expression ζ_7 and ζ_8 which takes account of U . If further improved accuracy is desired and if one is convinced that he can obtain a more accurate representation of the velocity profile by some calculation procedure, then a return to Equation (11) will provide a more general form for computational purposes. As will be shown in the section which follows, it is debatable whether presently available procedures will give better estimates than will Equation (18), using realistic values of the exponents.

FIGURE 1. DISPLACEMENT-THICKNESS VARIATION NEAR TRANSITION

FIGURE 2. EFFECT OF ECCENTRICITY ON DISPLACEMENT THICKNESS IN LARGE-SCALE TURBULENCE APPARATUS

TABLE 1. DEPENDENCE OF ζ_1 , ON n AND U FOR
FULL-CHANNEL PROFILES

δ^*x/b	ζ_1				
	$U = -1$	$U = -1/2$	$U = 0$	$U = 1/2$	$U = 1$
0.1666	∞	1.57	1.048	0.914	0.857
0.125	∞	1.39	1.028	0.932	0.889
0.1	∞	1.29	1.020	0.943	0.909
Mean		1.42	1.030	0.929	0.884

TABLE 2. DEPENDENCE OF ζ_2 , ON n , m AND U
FOR FULL-CHANNEL PROFILES

δ^*x/b	δ^*z/b	ζ_2				
		$U = -1$	$U = -1/2$	$U = 0$	$U = 1/2$	$U = 1$
0.1666	0.1666	∞	0.952	0.833	0.793	0.773
0.125	0.100	∞	0.982	0.857	0.815	0.794
0.100	0.100	∞	0.972	0.875	0.842	0.826
0.125	0.111	∞	0.989	0.889	0.855	0.838
Mean			0.973	0.863	0.826	0.807

TABLE 3. DEPENDENCE OF ζ_3 ON n FOR FULL-CHANNEL PROFILES

δ^*x/b	ζ_3
0.1666	0.715
0.125	0.778
0.1000	0.818
Mean	0.769

TABLE 4. DEPENDENCE OF ζ_4 ON n , m AND U

δ^*x/b	δ^*z/b	ζ_4				
		$U = -1$	$U = -1/2$	$U = 0$	$U = 1/2$	$U = 1$
0.166	0.166	2.00	0.714	0.833	0.882	0.909
0.166	0.125	1.69	0.734	0.857	0.9076	0.935
0.125	0.125	2.06	0.778	0.875	0.913	0.933
0.125	0.111	2.00	6.790	0.888	0.927	0.948
Mean		1.98	0.754	0.863	0.907	0.931

ANALYTICAL METHODS FOR GENERATION FOR VELOCITY PROFILES

A brief reference to prior analytical studies will serve to show the varieties of approach that have been tried. This will be followed by comparison between several promising techniques, showing the degree to which they differ from one another in their predictions.

In a pioneering study (both experimental and analytical), Smith and Fuller⁽²⁾ have developed an approximate analysis of turbulent flows, assuming that the tractions produced by displacement flows are independent of those produced by shear flows. Tao has also exploited this assumption for flows without⁽³⁾ and with⁽⁴⁾ end leakage. This decoupling of the component flows is not supportable as a workable approximation, in view of later work. It totally neglects, for example, the most important effects of a strong Couette flow upon the resistance to superposed weak axial and tangential displacement flows. Such combinations are encountered in the case of small shaft eccentricities in bearings and improper accounting for the tractions would lead to major errors in such tasks as the calculation of dynamic stability of the shaft-bearing system.

Chou and Saibel⁽⁵⁾ were first to apply the mixing length approach to the bearing without end leakage and, in so doing, have taken account of the coupling effects between tangential shear and displacement flows acting together. On the other hand, for convenience in the calculation, they have assumed a mixing length distribution which departs seriously from experimental distributions, especially in the regions near the walls where the velocity gradient is steep and errors in mixing length lead to sizable effects on the overall profile.

In a series of papers, and more recently in a book⁽⁶⁾, Constantinescu has also exploited the mixing length approach. He assumed in his earlier work that axial flow was independent of tangential flow. Since then, a more proper basis for dealing with axial flow has been pointed out^(7,8), and Constantinescu has discussed this in his book.

Sternlicht and Arwas⁽⁹⁾ have also applied the mixing length approach, assuming that $\ell = ky$ out from each wall, and simultaneously accounting for the viscous shear stress throughout the flow, or

$$\tau_{xy} = \left(\rho \ell^2 \left[\frac{\partial u}{\partial y} \right] + \mu \right) \frac{\partial u}{\partial y} \quad (15)$$

where

$$\ell = ky$$

It has been shown⁽⁸⁾ that, by their ignoring the lapse in mixing length near the wall, errors in computed midstream velocity as great as 40 percent were encountered, when the generally accepted value of $k = 0.4$ was used. Consequently, to make the analytical results agree with experiments, it was necessary to allow the proportionality constant k to be a strong function of Reynolds number. Furthermore, the correction produced was the same order of magnitude as the prediction, a fact which has marred confidence in application of the mixing length approach.

More recently, Wheeler⁽¹⁰⁾ has reexamined this approach and found that $k = 0.2$, rather than the commonly used 0.4, would be more suitable for his bearing film calculations. Had the author used a more nearly realistic mixing length distribution, however, taking account of the lapse near the wall, the change of k would not have been required.

Figure 3 illustrates experimental measurements of eddy viscosity near a wall⁽¹¹⁾, along with the relationship called for by $\ell = ky$ (or, in nondimensional terms, $L^+ = kY^+$). Also shown is an approximate fit derived from a modified power-law expression for the dimensionless universal (or wall-law) velocity profile. It is clearly shown here that, for $Y^+ < 20$, the expression $L^+ = 0.4Y^+$ greatly overestimates the eddy viscosity.

This illustrates also why good results may be obtained from the primitive technique of letting $\epsilon/\nu = 0$ for $Y^+ < 11$, and $(\epsilon/\nu)/(1 + \epsilon/\nu) = 1$ for $Y^+ > 11$, thus providing a step-shaped curve to approximate the continuous variation.

Reynolds⁽¹²⁾ has introduced additional refinements of the mixing length approach by taking account of modification of the mixing length distribution near midstream. He has divided the flow, quite properly, into a sublayer, a wall-law region, and a core flow. The relative importance of the core correction will be discussed further in paragraphs which follow.

In essence, the mixing length distribution serves as a means for computation of the eddy viscosity of the flow where $\rho(\epsilon + \nu) \frac{\partial u}{\partial y} = \tau_{xy}$ and $\epsilon = \ell^2 \left(\frac{\partial u}{\partial y} \right)$. It is possible to use other means to generate the profile of ϵ/ν in the film. One of these would be to assume that the distribution of

FIGURE 3. EDDY-VISCOSITY VARIATION

ϵ/ν in the half-channel does not change from the distribution which would prevail for Couette flow at the same wall shear stress. As will be shown, this is not a bad approximation. It is far better, indeed, than some of the above approximations of the mixing length approach. An improvement on this has been introduced by Pan and Ng⁽¹³⁾, where it is assumed that ϵ/ν is a function of the dimensionless group Y^+ or $y \frac{\sqrt{\tau \rho}}{\mu}$ where τ is the local shear stress acting on a plane parallel to the wall, ρ is the fluid density, μ is the fluid viscosity and y is the distance from the nether wall. This functionality is determined from experimental wall-law or Couette-flow measurements. This approximation takes account automatically of the coupling effects between Couette flow and displacement flow in either the axial or tangential direction. Furthermore, for codirectional displacement flows and Couette flows, the eddy viscosity distribution given by this technique is close to that given by the mixing length technique, when the proper variation in the sublayer as well as in the turbulent stream are taken into account.

In many ways, it appears that analyses of the subject flows have been divided into a multitude of approaches with inherent approximations or suspected errors in each. It is significant, however, that the mixing length approach and the eddy viscosity approach give closely comparable results when put to test in sample problems. Such comparisons will be discussed in the paragraphs which follow.

NUMERICAL COMPARISON OF PROFILE- GENERATION TECHNIQUES

To provide a critical comparison of profile-computation techniques, numerical analyses were undertaken, wherein data from the wall-law profile were used in several schemes to generate eddy-viscosity distributions and a velocity profile for pure displacement flow. In the wall-law flow, as in Couette flow, the shear stress component parallel to the wall remains constant across the channel. For the displacement flow, shear stress varies linearly outward from zero at midstream, where $y = b$. A valid profile generation scheme should permit the use of data from one case to predict the other. Comparisons are made on the basis of identical wall shear stress τ_0 . The results for eddy-viscosity variation are summarized in Figure 2 where the computed viscosity is expressed in ratio for ϵ_{WL} , which would prevail for a wall-law profile having the same wall shear stress. The computation schemes used are outlined below:

- (1) Wall-Law Profile. This has been approximated by the equation

$$Y^+ = U^+ + \left(\frac{U^+}{8.74} \right)^7 \quad (16)$$

which has been shown to agree with experimental data⁽¹⁴⁾. This agreement is also illustrated in Figure 3. The eddy-viscosity distribution which goes with this is $\epsilon/\nu = 0.801 (U^+/8.74)^6$ and is shown in Figure 3 to agree well with the experimental results of Schubauer⁽¹¹⁾. The mixing length distribution for the stated profile is

$$(L^+)^2 = 0.801(U^+/8.74)^6 [1 + 0.801(U^+/8.74)^6] \quad (17)$$

- (2) Displacement Flow, Experimental. Experimental mixing length data of Nikuradse⁽¹⁵⁾ have been used to calculate the eddy viscosity in the turbulent portion of the flow. This has been set in ratio to the wall-law eddy-viscosity distribution, in Figure 2, and shows that ϵ is diminished in almost linear fashion, going to zero as shear stress goes to zero.
- (3) $L^+ = f(Y_0^+)$, where $L^+ = \ell \frac{\sqrt{T\rho}}{\mu}$ and $Y^+ = y \frac{\sqrt{T\rho}}{\mu}$, the quantity T representing wall shear stress. In essence, it is assumed that, for a given wall shear stress, the mixing length is a

fixed function of wall distance y , independent of variation of τ away from the wall. This variation is obtained for Equations (16) and (18).

- (4) $\epsilon/\nu = f(Y^+)$, where $Y^+ = y \frac{\sqrt{\tau}\rho}{\mu}$. Here, τ is the local shear stress and is a function of y . As shown, this does not differ greatly from Item (3) in its prediction. This variation is obtained from Equations (16) and (17).
- (5) $\epsilon/\nu = f(U^+)$, where $U^+ = u\sqrt{\rho/\tau_0}$. Here, τ_0 is wall shear stress; U is local velocity and it is a function of y . In Figure 2, it is seen that this gives prediction of ϵ/ν , closest to the measured values. This variation is given by Equation (17).
- (6) $\epsilon/\nu = f(Y_0^+)$, where $Y_0^+ = y \frac{\sqrt{T}\rho}{\mu}$, and T is the wall shear stress. This assumes that the eddy-viscosity distribution is the same as ϵ_{WL} , which is the reference condition in Figure 4.

To compare the velocity profile predictions of each, refer to Table 5 where computed midstream velocity is compared with the experimental value, for the same wall shear stress, and $R = 10,000$.

TABLE 5. COMPARISON OF APPROXIMATIONS OF
DISPLACEMENT FLOW PROFILES [WHERE
 $\tau/T = (1 - y/b)]$, FOR $R \approx 10,000$

	$U_b/U_b, \text{ exp}$	δ^*/b	n
Wall-law profile	0.985	0.125	0.142
Experimental	1.000	0.128	0.146
$L^+ = f(Y_0^+)$	0.893	0.084	0.092
$\epsilon/\nu = f(Y^+)$	0.889	0.081	0.088
$\epsilon/\nu = f(U^+)$	0.881	0.0784	0.085
$\epsilon/\nu = f(Y_0^+)$	0.862	0.068	0.073
$L^+ = f(U^+)$	0.930	0.102	0.113

Here it is seen that midstream velocity differs at its worst by only 13 percent. On the other hand, it is particularly significant that the experimental eddy-viscosity depression in the core flow is both great and almost precisely the correct amount to cause the profile of velocity to be nearly

FIGURE 4. EDDY-VISCOSITY PREDICTION
BY SEVERAL SCHEMES

invariant under the change of stress distribution. This was earlier pointed out in a note⁽⁸⁾ by direct comparison of experimental data for the velocity profiles but is more dramatically presented here. It suggests further that some of the operations used to calculate profiles may be wasted effort since the final result could have been predicted by simply assuming the profile to be that of the wall-law shape, irrespective of stress distribution. The error in displacement thicknesses for the customarily assumed $L^+ = f(Y_0^+)$ is about 30 percent. On the other hand, the ζ_i coefficients are not so strongly affected by the choice of method, as will be discussed in connection with crossed flows, where both x and z velocity profiles may be taken into account.

A second observation is less comforting, however. Since wall shear stress varies as $T = C_f \rho u_b^2 / 2$, and since C_f is not a strong variable by any of the indicated methods of calculation, computation of wall shear stress from a given midstream velocity produces greater errors than does the inverse problem dealt with above. For example, if it is assumed that $\epsilon / \nu = f(Y_0^+)$, Table 4 predicts that centerline velocity will be 0.862 of the experimental value for a given wall shear stress. Yet, for a given midstream velocity, the calculated shear stress would be 1.345 times the experimental value.

AN ANALYTICAL APPROACH TO THE INTERACTIONS BETWEEN TURBULENT SHEAR FLOW AND DISPLACEMENT FLOW BETWEEN PARALLEL WALLS

In essence, this section applies the relationship $L^+ = f(U^+)$ to the problem of a displacement flow acting across a Couette flow. In reviewing the history of this problem, we note that Tao⁽²⁾ and Constantinescu⁽¹⁵⁾ have solved the turbulent film bearing problem by neglecting the coupling effects between orthogonal shear and pressure flows. Pan and Ng⁽¹³⁾ and more recently Ng and Elrod⁽¹⁴⁾ have applied the ad hoc assumption that $\epsilon/\nu = fY^+$ to such flows, where $Y^+ = y\sqrt{(\tau/\rho)}/\nu$ and τ is local shear stress parallel to the wall. Prandtl⁽¹⁵⁾, however, has derived a mixing length equation, taking account of the coupling effect in such flows where the shear stress varies in direction with distance from a wall. Burton^(7, 8) has applied this equation to the short bearing problem for the special case of a weak pressure (or displacement) flow at right angles to a strong Couette (or shear) flow.

This is extended here to the interactions between a strong fully turbulent displacement flow acting across the shear flow. Solutions are obtained for the set of differential equations recommended by Prandtl, by means of a digital computer. To provide the necessary expression for mixing length variation required to complete these equations, an empirical expression is used which gives a continuous statement for mixing length variation in the turbulent core and through the sublayer up to the boundary wall. The purpose of these solutions is to provide a frame of reference to aid in interpreting experimental measurement on such flows.

Basic Equations

In the presence of coupling effects, the mixing length equations⁽¹⁶⁾ for shear stress variation in Cartesian coordinates are:

$$\tau_{yx} = (\rho \ell^2 \left| J \right| + \mu) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (18)$$

$$\tau_{yz} = (\rho \ell^2 \left| J \right| + \mu) \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \quad (19)$$

where

$$|J| = \left| \left\{ 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right]^2 + \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right]^2 + \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]^2 \right\}^{1/2} \right| \quad (20)$$

For the case of parallel flow, it is assumed that $\partial u / \partial x = \partial w / \partial z = \partial u / \partial z = \partial w / \partial x = v = 0$. Using this assumption, Equations (18) and (19) reduce to

$$\tau_{yx} = \left\{ \mu + \rho \ell^2 u_y \left[1 + \left(\frac{w_y}{u_y} \right)^2 \right]^{1/2} \right\} u_y \quad (21)$$

$$\tau_{yz} = \left\{ \mu + \rho \ell^2 u_y \left[1 + \left(\frac{w_y}{u_y} \right)^2 \right]^{1/2} \right\} w_y \quad (22)$$

where

$$u_y = \frac{du}{dy} \quad (23)$$

$$w_y = \frac{dw}{dy} \quad (24)$$

Here, x - and z -coordinates are parallel to the boundary surfaces, and the y -coordinate is perpendicular to those. The velocity of u and w represents the flow in x - and z -directions, respectively. The viscosity, fluid density and mixing length are denoted by μ , ρ and ℓ , respectively. The components in x - and z -directions of shear stress variation across the fluid film are denoted by τ_{yx} and τ_{yz} , respectively.

A one-dimensioned empirical dimensionless mixing length equation has been proposed⁽¹⁶⁾ based on a good direct fit of the whole range of empirical velocity profile. For the very small Y^+ [where $Y^+ = \frac{y}{\nu} \left(\frac{\tau_{yx}}{\rho} \right)^{1/2}$], this equation follows a cubic relation for which Emmons⁽⁶⁾ has provided some measurements in terms of eddy dimension. As for the higher values of Y^+ , this becomes nearly linear and corresponds to the ordinary assumption in mixing length theory. Furthermore, as shown in Figure 3, it closely describes experimental measurements near the wall. Hence,

this equation will be employed and also extended to the two-dimensional case as follows:

$$(L^+)^2 = 0.801 \left(\frac{V^+}{8.74} \right)^6 \left[1 + 0.801 \left(\frac{V^+}{8.74} \right)^6 \right] \quad (25)$$

where

$$V^+ = [(U^+)^2 + (W^+)^2]^{1/2} \quad (26)$$

This is valid for wall-law flow where $W = 0$ and $\tau = T$. In a previous section, it was shown to be valid for displacement flow with only a small error. Thus, the assumption that it will apply to the combined flow has significant precedent. The above-mentioned dimensionless quantities are defined as follows:

$$(L^+)^2 = T\ell^2/\rho\nu^2 \quad (27)$$

$$U^+ = u\sqrt{\rho/T} \quad (28)$$

$$W^+ = w\sqrt{\rho/T} \quad (29)$$

$$T = \sqrt{(T_{yx})^2 + (T_{yz})^2} \quad (30)$$

The momentum and continuity equations for a thin incompressible fluid under a steady-state condition reduce to

$$\rho \left[u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right] = - \frac{\partial p}{\partial x} + \frac{\partial T_{yx}}{\partial y} \quad (31)$$

$$\rho \left[u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right] = - \frac{\partial p}{\partial z} + \frac{\partial T_{yz}}{\partial y} \quad (32)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} = 0 \quad (33)$$

Introduction of the parallel flow assumptions do satisfy Equation (33) and can reduce Equations (31) and (32) to

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} \quad (34)$$

$$\frac{\partial p}{\partial z} = \frac{\partial \tau_{yz}}{\partial y} \quad (35)$$

Given local shear stresses (τ_{yx} and τ_{yz}) and mixing length in Equation (25), Equations (21), (22), (34), and (35) constitute the basic equations to solve for u , w , and p .

Solutions for u , w , and p

Dividing Equation (22) by (21), w_y is

$$w_y = Gu_y \quad (36)$$

where

$$G = \tau_{yz}/\tau_{yx} \quad (37)$$

Substituting Equation (36) into (21), the governing differential equation of u_y becomes

$$\left(\frac{\partial u}{\partial y} \right)^2 + A \left(\frac{\partial u}{\partial y} \right) - \frac{B}{4} = 0 \quad (38)$$

where

$$A = \mu / \left\{ \rho \ell^2 \left| (1 + G^2)^{1/2} \right| \right\} \quad (39)$$

$$B = 4\tau_{yx} / \left\{ \rho \ell^2 \left| (1 + G^2)^{1/2} \right| \right\} \quad (40)$$

Solving Equation (38) for u_y as

$$u_y = \frac{1}{2} [-A + (A^2 + B)^{1/2}] \quad (41)$$

Because of positive value of u_y across fluid film, the value of τ_{yx} is, hence positive too. Thus, B is a definitely positive quantity. Owing to $(A^2 + B)^{1/2} > A$ and positive value of u_y , a "plus" sign in front of $(A^2 + B)^{1/2}$ is chosen in Equation (41) as

$$u_y = \frac{1}{2} [-A + (A^2 + B)^{1/2}] \quad (42)$$

Integrating u_y in Equation (42) with respect to y and using conditions of $u = 0$ at $y = 0$, u is obtained in terms of integral form

$$u = \frac{\mu}{2\rho} \int_0^y \frac{-1 + \left[1 + \frac{4\rho\ell^2\tau_{yx}}{\mu^2} \left| (1 + G^2)^{1/2} \right| \right]^{1/2}}{\ell^2 \left| (1 + G^2)^{1/2} \right|} dy \quad (43)$$

Substituting Equation (42) into (36) and integrating it with respect to y with $w = 0$ at $y = 0$, w in integral form is

$$w = \frac{\mu}{2\rho} \int_0^y G \frac{-1 + \left[1 + \frac{4\rho\ell^2\tau_{yx}}{\mu^2} \left| (1 + G^2)^{1/2} \right| \right]^{1/2}}{\ell^2 \left| (1 + G^2)^{1/2} \right|} dy \quad (44)$$

Using the quantities in Equations (27) to (30) and defining $q = \sqrt{u^2 + w^2}$, $R = \frac{bq}{\nu}$ and $C_f = \frac{2}{\rho} \frac{T}{q^2}$, Equations (43) and (44) in dimensionless form are, respectively,

$$U^+ = \beta \int_0^{y/b} K d\left(\frac{y}{b}\right) \quad \left(0 \leq \frac{y}{b} \leq 1\right) \quad (45)$$

$$W^+ = \beta \int_0^{y/b} G K d\left(\frac{y}{b}\right) \quad \left(0 \leq \frac{y}{b} \leq 1\right) \quad (46)$$

where

$$\beta = \frac{R\sqrt{C_f}}{2\sqrt{2}} \quad (47)$$

$$K = \frac{-1 + \left[1 + \frac{4\tau_{yx}}{T} (L^+) \left| (1 + G^2)^{1/2} \right| \right]^{1/2}}{(L^+)^2 \left| (1 + G^2)^{1/2} \right|} \quad (48)$$

and L^+ and G are identical to the same forms as in Equations (25) and (37), respectively.

Equations (45) and (46) are the general solutions in integral form for U^+ and W^+ . Of interest here is the case of a Couette flow with orthogonal superposed pressure flows in varying strengths.

Solution for Orthogonal Couette Flow and Pressure Flow

If only Couette flow acts in x-direction, τ_{yx} may be assumed to be constant across the fluid film. Hence

$$\tau_{yx} = T_{yx} \quad \left(0 \leq \frac{y}{b} \leq 2 \right) \quad (49)$$

Because of symmetry of w about $y = b$, τ_{yz} is an odd linear function about $y = b$. Hence

$$\tau_{yz} = T_{yz} \left(1 - \frac{y}{b} \right) \quad \left(0 \leq \frac{y}{b} \leq 2 \right) \quad (50)$$

where T_{yx} and T_{yz} are the wall shear stresses.

By means of Equations (49) and (50), τ_{yx}/T and G are, respectively,

$$\frac{\tau_{yx}}{T} = \frac{1}{[1 + (T_{yz}/T_{yx})^2]^{1/2}} \quad (51)$$

$$G = \frac{T_{yz}}{T_{yx}} \left(1 - \frac{y}{b} \right) \quad (52)$$

Equations (45) and (46) with (51) and (52) are the solutions of U^+ and W^+ for Case 1. The values of U^+ and W^+ near the wall are

$$U^+ = \frac{2\beta}{[1 + (T_{yz}/T_{yx})^2]^{1/2}} \cdot \frac{y}{b} \quad (53)$$

$$W^+ = \frac{2\beta(T_{yz}/T_{yx})}{[1 + (T_{yz}/T_{yx})^2]^{1/2}} \cdot \frac{y}{b} \quad (54)$$

Using Equations (49) and (50), Equations (34) and (35) reduce to

$$p = - \frac{2T_{yz}}{h} z \quad (55)$$

With condition of $p = p_{\max}$ at $z = 0$, Δp is

$$\Delta p = 2 \frac{T_{yz}}{h} z \quad (56)$$

The pressure drop in the z -direction in Equation (56) is a linear function in z . At corresponding values of T_{yz} and z , Δp increases with decreasing thickness of fluid film.

Numerical Evaluation of u and w

The following parameters were chosen for a numerical study: $R = 3 \times 10^3$, $C_f = 0.0057$ and $T_{yz}/T_{yx} = 0.2, 1.0$ and 1.8 , the dimensionless velocity profiles were computed. These are plotted in Figures 5 and 6. In the range of $0 \leq (y/b) \leq 0.2$, the stronger the pressure flow in z -direction, the higher the quantities of u/u_{bc} and w/w_b . The opposite is true for u/u_{bc} in the range of $0.2 \leq (y/b) \leq 1$. The ratio of w/w_b is independent with T_{yz}/T_{yx} in the range of $0.2 \leq (y/b) \leq 1$.

Dimensionless Displacement Thickness

The dimensionless displacement thickness in the half-fluid film in x - and z -directions are, respectively,

$$\frac{\delta_x^*}{b} = 1 - \int_0^1 \left(\frac{u}{u_{bc}} \right) d\left(\frac{y}{b} \right) \quad (57)$$

$$\frac{\delta_z^*}{b} = 1 - \int_0^1 \left(\frac{w}{w_b} \right) d\left(\frac{y}{b} \right) \quad (58)$$

Using the same parameters in the last section and performing numerical integration of the second term on the right side in Equations (57) and (58), the values of δ_x^*/b and δ_z^*/b are listed in Table 6.

TABLE 6. CALCULATION OF DIMENSIONLESS
DISPLACEMENT THICKNESSES

T_{yz}/T_{yx}	δ_x^*/b	δ_z^*/b
0.2	0.1991	0.1300
1.0	0.1991	0.1211
1.8	0.2082	0.1211

For the orthogonal weak pressure flows in the range of $0.2 \leq (T_{yz}/T_{yx}) \leq 1$, δ_x^*/b remains unchanged, and δ_z^*/b decreases with increasing intensity of orthogonal pressure flow. In the range of orthogonal strong pressure flows such that $1 \leq (T_{yz}/T_{yx}) \leq 1.8$, δ_x^*/b remains unchanged, and δ_z^*/b decreases with increasing intensity of pressure flow.

Evaluation of ζ_1 , ζ_2 , and ζ_3

Transport integrals of the flow may be designated ζ_1 , ζ_2 , and ζ_3 and are defined as follows:

$$\zeta_1 = \frac{\int_0^h u^2 dy}{h(u_{bc})^2} \quad (59)$$

$$\zeta_2 = \frac{\int_0^h u w dy}{h u_{bc} w_b} \quad (60)$$

and

$$\zeta_3 = \frac{\int_0^h w^2 dy}{h w_b^2} \quad (61)$$

Because of the symmetries inherent in the Couette and pressure flow relative to $y/b = 1$, Equations (59) to (61) are reduced to

$$\zeta_1 = 2 + \int_0^1 \left(\frac{u}{u_{bc}} \right)^2 d\left(\frac{y}{b} \right) - 2 \int_0^1 \left(\frac{u}{u_{bc}} \right) d\left(\frac{y}{b} \right) \quad (62)$$

$$\zeta_2 = \int_0^1 \left(\frac{w}{w_b} \right) d\left(\frac{y}{b} \right) \quad (63)$$

and

$$\zeta_3 = \int_0^1 \left(\frac{w}{w_b} \right)^2 d\left(\frac{y}{b} \right) \quad (64)$$

The quantities ζ_1 to ζ_3 , which were numerically evaluated directly from Equations (62) to (64), are listed in Table 8.

Instead of using Equations (62) to (64) and the actual velocity profiles in Figures 5 and 6, a simple integrated form for ζ_1 to ζ_3 can be obtained by approximating the actual velocity profiles by means of power-law velocity distributions. The turbulent power law for substitutions into Equations (60) and (63) are

u-profile:

$$\frac{u}{u_{bc}} = \left(\frac{y_o}{b} \right)^n \quad \left(0 \leq \frac{y_o}{b} \leq 1 \right) \quad (65)$$

$$\frac{u}{u_{bc}} = 2 - \left(\frac{y_h}{b} \right)^n \quad \left(0 \leq \frac{y_h}{b} \leq 1 \right) \quad (66)$$

w-profile:

$$\frac{w}{w_b} = \left(\frac{y_o}{b} \right)^m \quad \left(0 \leq \frac{y_o}{b} \leq 1 \right) \quad (67)$$

$$\frac{w}{w_b} = \left(\frac{y_h}{b} \right)^m \quad \left(0 \leq \frac{y_h}{b} \leq 1 \right) \quad (68)$$

Denote y_o and y_h as the reference coordinate, measured across the film with respect to the lower and upper boundary surfaces, respectively. The

FIGURE 5. NORMALIZED THEORETICAL TANGENTIAL VELOCITY PROFILE COMPARED WITH EXPERIMENTAL ONE, MADE TO CORRESPOND AS TO WALL DISTANCE, AND GIVEN CORRESPONDING VELOCITIES AT TWO SELECTED POINTS FOR PURPOSES OF COMPARISON

FIGURE 6. NORMALIZED THEORETICAL AXIAL VELOCITY PROFILES COMPARED WITH EXPERIMENTAL PROFILES, MADE TO CORRESPOND AT $Y = 0.15$

values of n and m refer to the power-law velocity profiles in the x - and z -directions, respectively. Substituting Equations (65) to (68) into (59) and (61) and integrating, the quantities of ζ_1 to ζ_3 are

$$\zeta_1 = \frac{1}{1 + 2n} + \frac{2n}{1 + n} \quad (69)$$

$$\zeta_2 = \frac{1}{1 + m} \quad (70)$$

and

$$\zeta_3 = \frac{1}{1 + 2m} \quad (71)$$

Substituting Equations (65) and (66) into (57), the relation between n and δ_x^*/b is

$$n = \frac{\delta_x^*/b}{1 - (\delta_x^*/b)} \quad (72)$$

Similarly, from Equations (58), (67) and (68), m is

$$m = \frac{\delta_z^*/b}{1 - (\delta_z^*/b)} \quad (73)$$

Substituting the values of δ_x^*/b and δ_z^*/b in Table 6 into Equations (72) and (84), the values of n and m are calculated and listed in Table 7.

TABLE 7. POWER-LAW VELOCITY PROFILE EXPONENTS

T_{yz}/T_{yx}	n	m
0.2	0.2486	0.1494
1.0	0.2486	0.1378
1.8	0.2630	0.1378

Using the values of n and m in Table 7 and Equations (69) to (71), the values of ζ_1 to ζ_3 are calculated and listed in Table 8.

The calculated values of ζ_1 to ζ_3 by means of simple power-law forms are shown to agree well with those by means of numerical evaluation from Equations (62) to (64).

TABLE 8. VALUES OF ζ_1 , ζ_2 AND ζ_3

T_{yz}/T_{yx}	ζ_1		ζ_2		ζ_3	
	<u>D.E.*</u>	<u>S.F.**</u>	<u>D.E.</u>	<u>S.F.</u>	<u>D.E.</u>	<u>S.F.</u>
0.2	1.0798	1.0661	0.8736	0.8700	0.8028	0.7699
1.0	1.0786	1.0661	0.8765	0.8774	0.8046	0.7839
1.8	1.0773	1.0718	0.8801	0.8774	0.8069	0.7839

*D.E. denotes the direct numerical evaluation of ζ_1 to ζ_3 from the actual velocity profiles in Equations (62) to (64).

**S.F. denotes the calculated values of ζ_1 to ζ_3 , using the best fit power-law exponent.

Experimental Comparisons

Experimental velocity profiles were generated in an annulus between a stationary outer cylinder and a turning inner cylinder, of 6-ft diameter. The clearance was 0.5 inch. This apparatus has been described in detail elsewhere⁽¹⁾. The axial length of the cylinder was 20 in., and leakage flow was blocked at one end but free at the other. A dam extended 15 in. out from the blockage in the axial direction and caused the flow to acquire an axial displacement-flow component. The working fluid was air, and velocities were measured by hot-wire anemometry. By exploration, a point was found where tangential flow was almost purely Couette flow, and, at the same time, there was a strong axial displacement flow. The Reynolds number R was 2500, and thus comparable to that used in the reported calculations. To compare the normalized velocity profiles, so obtained were calculated ones, two bases were tried, and the results are illustrated in Figure 5.

In one case, both were matched at midstream, and the general shapes appear to conform. In the other case, they were matched near the wall, and, on this basis, the experimental curve leads to a large midstream velocity. The latter procedure can be justified if it is assumed that the turbulent profile approaches the wall-law profile near the wall. Thus, matching the two near the wall would occur for corresponding shear stress. If this postulate is valid, then the computation procedure has underestimated the midstream velocity by about 5 percent. Attempts to compute shear stress from the velocity would lead to errors of about 10 percent since shear stress depends approximately on V^2 .

Whether or not this discrepancy is judged large depends upon what the user requires of his calculations. The important thing is that a logical mathematical procedure has been used to generate axial and tangential profiles of a skewed shear stress distribution, and these have been shown to give results which compare with experimental measurements. For many applications, one might apply the analytical approach used here and scale up the velocities so obtained in accordance with the discrepancies which have been illustrated.

Yet one more interesting comparison can be derived from the data, this being the ratio $(w_b/u_{bc})/(T_z/T_x)$ which gives a measure of the departure of the midstream velocity vector from the wall shear-stress vector. This is illustrated in Table 9.

TABLE 9. RELATIONSHIP BETWEEN WALL SHEAR STRESS DIRECTION AT MIDSTREAM VELOCITY VECTOR

T_z/T_x	$(w_b/u_{bc})/(T_z/T_x)$
0.2	0.81
1.0	0.80
1.8	0.796
0.69 (experimental)	0.777

Again, a satisfying relationship between theory and experiment has been demonstrated.

AN EXPERIMENTAL STUDY OF THE INTERACTION BETWEEN DISPLACEMENT FLOWS AND A COUETTE FLOW

In the preceding sections, reference has been made to experimental measurements of crossed Couette and displacement flows. In this section, these measurements will be analyzed and discussed in detail.

In a lubricant film, the relative movement between the boundary surfaces produces what is termed a "Couette flow" or a "shear flow." Because of changes of film thickness, an additional type of flow known as "displacement flow" or "pressure flow" is superimposed upon this Couette flow. It may act across the Couette flow in which case it would correspond to end leakage, or it may act along the Couette flow and give rise to pressure gradients in the direction of turning. Any approach to the analytical treatment of a turbulent lubricant film provides, in essence, a means for predicting the effect of the Couette flow upon the pressure gradients produced by the displacement flow. Up until now, experimental measurements have been largely restricted to Couette flow alone and displacement flow alone and a few isolated data on the combined, codirectional Couette and pressure flow. Except for a few sketchy items, there is, however, very little directly applicable information on the interaction of a Couette flow and a pressure flow moving across it. Consequently, any attempt to analyze bearings with dominant end leakage effects has been couched in a sizeable degree of uncertainty. To provide the necessary information to check various methods for treating such flows, several experiments were undertaken, of which the kind reported here has been the most successful in producing reliable information.

Experimental Technique

The large-scale turbulent film apparatus was used to generate the desired combinations of flows. The journal was set concentric in the bearing. End leakage was blocked by a plastic curtain at one end, and a dam was extended across the clearance from this blockage, spanning three-fourths of the width of the bearing, and leaving only a very small clearance for the journal to pass beneath. This dam diverted the flow as shown in Figure 1, and produced combinations of end-leakage flow, Couette flow, and tangential-pressure flow. By probing various regions with a hot-wire anemometer, a position was found where only a very weak pressure flow acted tangentially, and a strong pressure flow acted in the axial direction. This position was approximately 2 in. axially outward from the dam-tip and 1 in. tangentially upstream.

Two techniques were possible for applying hot-wire anemometer to the measurement of the skewed velocity profiles. One was to select a fixed distance out from the wall for a given reading and to orient the wire so as to be normal to the local velocity vector. This was found to be very difficult since the precise angle of flow could not be determined accurately. A second technique was to orient the probe first in the axial direction and then in the tangential direction at each position out from a wall, and thereby to read the two velocity components separately. Unfortunately, there is some coupling effect between the readings. To determine the correction factor, pure Couette flow was generated, and the wire was placed at mid-channel where the velocity was known to be half the sliding speed of the journal. The wire was then oriented at different known angles to the direction of flow. Readings for each of these orientations were plotted against the sine of the angle between the wire and the actual velocity direction, which would represent the velocity component normal to the wire. In this plot (Fig. 7), the velocity calculated from the hot-wire circuit was equal to the true normal velocity component, only when the hot wire was exactly normal to the total velocity vector. Where the wire was parallel to the total velocity vector, there was still a reading of approximately 15 percent of the total velocity magnitude. An equation which expresses the relationship is as follows:

$$\frac{V_{n, \text{ calc}}}{V} - 0.15 = 0.085 \frac{V_n}{V}; \text{ where } V = \sqrt{U^2 + W^2} \quad (74)$$

(Note that we need not concern ourselves with a third velocity component since the flow is essentially parallel to the walls.) Applying Equation (74) to the flows where the velocity acts in some unknown direction, it is possible to take the measured axial velocity and measured tangential velocity and to compute the true axial, U , and tangential, W , from these using the following equations:

$$\begin{aligned} 0.85U &= U_{\text{calc}} - 0.15\sqrt{U^2 + W^2} \\ 0.85W &= W_{\text{calc}} - 0.15\sqrt{U^2 + W^2} \end{aligned} \quad (75)$$

Typical data are shown in Table 10 where U represents the velocity in the tangential direction expressed in a ratio to the midchannel Couette velocity ($U_s/2$), and W represents the axial velocity similarly nondimensionalized. y/h is the wall distance measured from the stationary wall. A, B, C, and D represent independent runs on different days. Looking at the profile of U , it is seen that the velocity is almost unity midway between the walls, in which case, the profile would be nearly symmetrical, and one could assume that there was only very weak displacement flow in the direction

FIGURE 7. DIRECTIONAL SENSITIVITY OF HOT-WIRE PROBE

TABLE 10. PROFILE DATA FOR ORTHOGONAL
COUETTE AND PRESSURE FLOWS

<u>y/h</u>	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>Avg</u>	<u>Symm</u>
Tangential velocity, U						
0.0000	0	0	0	0	0	0
0.0625	0.4061	0.4796	0.4910	0.4287	0.451	0.539
0.1875	0.6772	0.6517	0.6530	0.5648	0.637	0.771
0.3125	0.7641	0.7449	0.8104	0.7054	0.757	0.885
0.4375	0.8255	0.8712	0.9426	0.7954	0.859	0.960
0.5625	0.9503	0.9562	0.9840	0.8636	0.938	1.040
0.6875	0.9812	1.0088	0.9840	0.9707	0.986	1.114
0.8125	1.0462	1.1020	1.1101	1.1209	1.095	1.228
0.9375	1.3227	1.3888	1.3903	1.3854	1.372	1.460
1.000	2.000	2.000	2.000	2.000	2.000	2.000
Axial velocity, W						
0	0	0	0	0	0	0
0.0625	0.3580	0.4594	0.4256	0.5265	0.442	0.421
0.1875	0.4462	0.6667	0.5571	0.6856	0.589	0.624
0.3125	0.5106	0.6794	0.6075	0.7589	0.639	0.668
0.4375	0.5150	0.6862	0.6075	0.7930	0.651	0.665
0.5625	0.5431	0.7183	0.6135	0.8425	0.679	0.665
0.6875	0.5406	0.7331	0.7045	0.8104	0.697	0.668
0.8125	0.4772	0.7223	0.6177	0.8236	0.660	0.624
0.9375	0.2885	0.3854	0.3734	0.5581	0.401	0.421
1.000	0	0	0	0	0	0

of the turning. Similarly, the W profile is almost symmetrical, though slightly distorted by the presence of the weak displacement flow in the axial direction. Averages of the runs are indicated at each wall distance position. To draw on as many points as possible, velocities at corresponding wall distances from both the moving and stationary walls were adjusted so as to be relative to the referenced wall in each case, and these were averaged. Thus, the corresponding velocities on the two sides of the channel were averaged with one another to generate symmetrical profiles, eliminating, to some extent, the small effect of the tangential displacement flow, and the small effect of asymmetry in the axial flow. This is not a large correction, and one would find that, if one plotted the flow in the half-channel normalized to have a value of unity at $y/h = 0.5$, the averaged and symmetrical profiles correspond very closely. The latter however has been used in subsequent calculations, and it is felt to be more accurate, drawing as it does on more data points. Figures 8 and 9 illustrate the adjusted profiles of U and W.

Analysis of Data

Several fundamental questions can be answered through cross comparison and analysis of the profile data. These include (1) verification of the eddy-viscosity concept for skewed flows, (2) determination of the relationship between midstream eddy viscosity in the skewed flow and that in wall-law flow of the same wall shear stress, and (3) determination of the friction coefficient and the direction of the friction force in the skewed flow. These will be discussed separately below.

The concept of the eddy viscosity requires that, at any wall distance in a turbulent flow, the direction of the velocity gradient is the same as the direction of the local shear stress on a surface parallel to the wall. It is of interest to test this concept in a flow where the shear stress direction varies significantly on moving outward from the wall to midstream. In the experimental situation, this direction rotates approximately 45° over this distance. At a given wall distance, irrespective of how eddy viscosity is determined physically, it should be possible to write, at a given value of y

$$\begin{aligned}\tau_{xy} &= y \frac{\partial u}{\partial y} (\epsilon + \nu) \rho \\ \tau_{zy} &= y \frac{\partial w}{\partial z} (\epsilon + \nu) \rho\end{aligned}\tag{76}$$

or, combining these and assuming that ϵ has the same magnitude in both the z- and x-directions, at any given value of y:

FIGURE 8 PROFILE OF TANGENTIAL VELOCITY COMPONENT FOR CROSSED SHEAR AND DISPLACEMENT FLOWS

FIGURE 9 PROFILE OF AXIAL COMPONENT OF CROSSED SHEAR AND DISPLACEMENT FLOWS

$$W = \int_0^y \frac{\tau_{zy}}{\tau_{xy}} \frac{\partial u}{\partial y} dy \quad (77)$$

For pure shear flow, where there is no pressure gradient along the flow, where streamlines are parallel, and where viscosity (eddy or otherwise) varies symmetrically with respect to the channel centerline, the shear stress will be constant across the channel, and it is possible to write

$$\tau_{xy} = T_{xy} \quad (78)$$

Then, similarly for displacement flow under corresponding conditions (but where there is no relative motion of the walls in the direction of the flow), it is possible to write

$$\tau_{zy} = T_{zy} (1 - 2y/h) \quad (79)$$

Using this information, the above integral can be written as

$$\frac{T_x}{T_z} W = \int (1 - 2y/h) \frac{\partial U}{\partial y} dy \quad (80)$$

Thus, it is possible to take data for the variation of U and use it to predict the variation of W . This conversion is fairly accurate since it involves no differentiation of the data but, rather, integration. The results of such a calculation are illustrated in Table 11 where wall distance and the adjusted measured velocity are shown as the first two columns. The calculated value of the integral of Equation (80) is shown in column 3. Averaging these quantities and assuming ($W_{calc, mean} = W_{symm, mean}$), it is possible to get a representative value of $T_x/T_z = 1.17$ as indicated. Inserting this into the values of column 3, the column of W_{calc} was produced. In Figure 9, these calculated points have been shown relative to the measured points, and it is seen that, within what is apparently experimental scatter, both serve to fit the same curve closely. This indicates that, to a good degree of accuracy, it is valid to assume that the eddy viscosity is not directional at a given wall distance y .

Determination of the Friction Coefficient

To determine the friction coefficient, it is possible to take any pair of velocity and wall distance points, to insert these into an equation for the universal velocity profile, and to determine what wall friction coefficient, C_f , would be called for if the measured velocity profile did indeed correspond through to the wall-law profile. By carrying out this

TABLE 11. COMPARISON OF MEASURED W_{symm}
WITH THAT CALCULATED FROM V_{symm}

<u>y/h</u>	<u>W_{symm}</u>	<u>$\frac{T_x}{T_z} W_{\text{calc}}$</u>	<u>W_{calc}^*</u>
0	0	0	0
0.0625	0.421	0.524	0.448
0.1875	0.624	0.713	0.610
0.3125	0.668	0.774	0.662
0.4375	<u>0.665</u>	<u>0.791</u>	0.676
Mean	0.5945	0.6955	

*On basis of $\frac{T_x}{T_z} = \frac{0.6955}{0.5945} = 1.17$

calculation at several wall distances, it is possible, then, to make a plot of calculated C_f , which can be extrapolated to the wall, where actual physical profiles tend to approach the idealized wall-law profile. The wall-law equation used was as follows:

$$U^+ = U^+ + (U^+/8.74)^7 \quad (81)$$

This has been demonstrated to serve as an accurate representation of wall-law experimental data within the range in which it will be applied here. In terms of C_f this becomes:

$$R = \left(\frac{2u^2}{u_b^2 C_f} \right) + \left(\frac{2u^2}{u_b^2 C_f} \right)^4 / (8.74)^7 \quad (82)$$

where $R = uy/\nu$, and $T_o = C_f \rho u_b^2 / 2$.

If a velocity profile follows the wall-law shape precisely, then the same magnitude of C_f would be calculated from any combination of (u, y) data points. For the present experimental data, using the above equations, the results obtained are shown in Figure 10. This figure displays values of C_f calculated from simple Couette flow, and from the full velocity vector, $V = \sqrt{W^2 + U^2}$, from the adjusted profiles under consideration here. Note that the Couette flow acting alone has a somewhat higher friction coefficient (wall intercept at $y/h = 0$) than if it followed the wall-law profile precisely. Indeed, the point at the extreme right would be the friction coefficient for wall-law Couette flow with the same midstream velocity. The added friction is believed to be attributable to the vortex flow which supplements turbulent mixing in transferring momentum in the flow. This is a consequence of the film thickness used here, and has been discussed elsewhere.

For the skewed flow, where $T = \sqrt{T_x^2 + T_z^2}$

$$T_x = \left(\frac{T_x}{T} \right) C_f \rho \frac{v_b^2}{2} \quad (83)$$

For Couette flow alone

$$T_{xc} = C_{fc} \rho \frac{u_{bc}^2}{2} \quad (84)$$

The ratio would be:

FIGURE 10. VARIATION OF CALCULATED C_f WITH WALL DISTANCE FOR THREE PROFILE TYPES

$$\frac{T_x}{T_{xc}} = \left(\frac{T_x}{T} \right) \frac{C_f}{C_{fc}} \frac{v_b^2}{u_{bc}^2} \quad (85)$$

Substituting for these quantities,

$$\frac{T_x}{T_{xc}} = \frac{1.17}{\sqrt{(1.17)^2 + 1}} \frac{0.0055}{0.008} [1 + (0.665)^2] = 0.7508 \quad (86)$$

This is somewhat surprising in that one would anticipate that the presence of the axial flow would have increased eddy viscosity. Yet, the predictions here show that little effect on turning resistance would have occurred, and this effect appears to be a reduction. Careful reference to the shape of the velocity profile shows that midstream slope is considerably reduced from that of Couette flow; thus the velocity in the tangential direction near the wall is diminished, and this accounts for the lower wall shear stress, even with an increase in eddy viscosity near the wall. To further interpret this, note that if the tangential velocity profile for the combined flow were multiplied by scaling factor so as to correspond to the Couette flow profile at the wall (and, consequently, have the same wall shear stress), the midstream velocity would be almost half again that for the Couette flow. Thus, for the same tangential wall shear stress, a considerably higher turning speed would be called for in the combined flow than the Couette flow alone. The reason for this difference is that the eddy viscosity in midstream is far less than the eddy viscosity called for in the wall law.

Figure 11 shows an additional comparison between axial and the tangential profile of the combined flow. In this case, the tangential velocity profile of the combined flow corresponds to an equation of approximately

$$U = Y^{1/4} \quad (87)$$

whereas, for the experimental Couette flow,⁽¹⁾ this would be

$$U = Y^{1/8} \quad (88)$$

Again, this indicates that, for corresponding relative velocities, the gradient is much the steeper, and the eddy viscosity is much the smaller near midstream in the combined flow.

Midstream Eddy Viscosity

To obtain some measure of the midstream eddy viscosity, note that, if the small effect of the viscosity ν is neglected, the shear stress in the tangential direction is given by

$$\rho \epsilon \frac{\partial u}{\partial y} = \tau \quad (89)$$

For Couette flow, this would be

$$\rho \epsilon_c \left(\frac{\partial u}{\partial y} \right) = T_c \quad (90)$$

Setting these in ratio to one another, we find

$$\frac{\epsilon}{\epsilon_c} = \frac{\tau}{T_c} \left[\frac{\partial u}{\partial y} \right] / \left(\frac{\partial u}{\partial y} \right)_c \quad (91)$$

Substituting data from the measured profiles for the two cases, it is found that the eddy viscosity near midstream would be 0.4288 times the eddy viscosity in midstream expected for a Couette flow of the same sliding velocity. Thus, it is seen that the midstream eddy viscosity is diminished even more than the wall shear stress is reduced by the pressure of the axial flow. A check to determine how the mixing length might differ from that of wall-law flow draws upon the following equation:

$$\ell^2 \frac{\partial u}{\partial y} = \epsilon \quad (92)$$

or set in ratio as above

$$\frac{\epsilon}{\epsilon_c} \frac{\left(\frac{\partial u}{\partial y} \right)_c}{\left(\frac{\partial u}{\partial y} \right)} = \frac{\ell^2}{\ell_c^2} = (0.4288) \left(\frac{4}{7} \right) = 0.2418 \quad (93)$$

This leads to the requirement that, in the combined flow, the midstream mixing length be reduced to 0.48 of the mixing length in the Couette flow.

Scheme for Generating Profiles

The above observations are such that existing schemes for generating profiles cannot be made to account for them without introduction of

an arbitrary correction or law to account for the midstream depression of eddy viscosity. On the other hand, for purely tangential flows where displacement flows may act codirectionally with Couette flows, profile shape rather than mixing length distribution or eddy viscosity distribution has been shown experimentally to be preserved with good accuracy. This appears to be true of the combined flows, when dealt with on a suitable basis, as illustrated in Figure 11 for example.

The distribution of calculated C_f , when plotted against wall distance, gives an indication of conformance of profile shape to the wall-law profile, as has been discussed above.

The simplest approach to produce this approximate result would be to assume $\epsilon/\epsilon_{WL} = |\tau|/T$, or that eddy viscosity would be the wall-law eddy viscosity for the given wall shear stress T and wall distance, this being diminished by the ratio of local absolute shear stress (parallel to the wall) in ratio to wall shear stress. By definition, $\tau = \left| \sqrt{\tau_{xy}^2 + \tau_{zy}^2} \right|$ and $T = \left| \sqrt{T_{xy}^2 + T_{zy}^2} \right|$

If

$$\epsilon \frac{\partial u}{\partial y} = \tau_{xy} \quad (94)$$

and

$$\epsilon_{WL} \left(\frac{\partial u}{\partial y} \right)_{WL} = T \quad (95)$$

Then

$$\frac{(\partial u / \partial y)}{(\partial u / \partial y)_{WL}} = \frac{\epsilon_{WL}}{\epsilon} \frac{\tau_{xy}}{T} \quad (96)$$

where

$$\epsilon_{WL} = f(Y_O^+), \text{ and } (\partial u / \partial y)_{WL} = \frac{\rho v}{T} f(Y_O^+)$$

If

$$\frac{\epsilon}{\epsilon_{WL}} = \frac{\left| \sqrt{\tau_{xy}^2 + \tau_{zy}^2} \right|}{T} \quad (97)$$

FIGURE 11. COMPARISON OF AXIAL
AND TANGENTIAL PROFILES ON
LOG-LOG COORDINATES

then

$$\frac{\partial u / \partial y}{(\partial u / \partial y)_{WL}} = \sqrt{1 + \frac{\tau_{zy}^2}{\tau_{xy}^2}} \quad (98)$$

It would be of interest to explore the advantages and disadvantages in application of this approach to bearing film calculations.

DISCONTINUITY EFFECTS IN TURBULENT BEARING FILMS

Over and above the effects described in the above analyses, one must account for inertial effects which contribute significantly to pressure rises. These are particularly apparent at the leading edge of pads and at steps or grooves. The section which follows will serve to outline some of the experimental observations on the influences of such discontinuities.

Apparatus

The apparatus was based upon the previously described 6-ft diameter wheel, 20 in. long. About this surface (which could be turned at speeds above 100 rpm), various stationary bearing configurations could be supported, including tilted pads, stepped pads, spiral grooves and the full bearing. End leakage could be restricted by plastic curtains, thus making possible the simulation of long or short bearings, depending upon the presence or absence of the curtains. Reynolds numbers in excess of $hu_s/2\nu = R = 2500$ were possible where h is film thickness, u_s is journal surface speed, and ν is kinematic viscosity. This is well within the turbulent regime.

Performance of Tilted Pads

Tilted pads were 3 ft in circumferential length, 20 in. in axial dimension and were pivoted about a film thickness of 0.5 in. at a point 0.75 ft forward of the trailing edge. In these experiments, end leakage was (1) free, (2) partially restricted, and (3) totally restricted. Pressures were measured at axial midspan. Figure 12 shows two conditions for a fixed angle of tilt (1.0-in. film thickness at leading edge, 0.26 in. at trailing edge). Note that a significant "ram effect" or "impact pressure" appears at the leading edge. In this figure and others, the dimensionless pressure P is defined as $P = p/\rho u_{bc}^2$, where p is pressure, ρ is density and u_{bc} is half the journal surface speed u_s .

Figure 13 shows pressure variations for two tilt angles and end leakage fully restricted. Again, the leading edge pressure plays a dominant role.

Leading Edge Pressures

When entrance region velocity measurements were made for different leading edge pressures in tilted pads and stepped pads, it was found that the correlation presented in Figure 14 prevailed. Here, the

FIGURE 12. PRESSURE VARIATION FOR TILTED PADS FOR UNRESTRICTED AND PARTIALLY RESTRICTED END LEAKAGE

FIGURE 13. PRESSURE VARIATION FOR TWO TILT ANGLES, FOR A PAD WITHOUT END LEAKAGE

FIGURE 14. VARIATION OF PAD-ENTRY PRESSURE WITH INLET VELOCITY

dimensionless pressure is related to the velocity in such a way that results can be correlated in terms of a dimensionless "impact" pressure according to the equation shown. If the entering flow had been carried at wheel speed (u_s , where $u_s = 2u_{bc}$) then brought to a halt in a frictionless process, as on a Pitot impact tube, the following relationship would prevail:

$$\frac{p}{\rho u_{bc}^2} = \frac{u^2}{2u_{bc}^2} \quad (99)$$

or

$$\frac{p}{\rho u_{bc}^2} + \frac{u^2}{2u_{bc}^2} = 2 \quad (100)$$

Thus, the apparent dimensionless stagnation pressure of the entering flow as described in Figure 14 is much less than if the fluid were carried at wheel speed, shown in Equation (100). This implies that a boundary layer exists on the rotor surface with velocity dropping off with distance from the surface, and with a mean velocity $< u_{bc}$. Even if a thin film of liquid were transported on the journal, for the fully flooded entry condition, a globule of liquid would be expected to accumulate at the entry region. The slowing of the liquid (movement into an adverse pressure gradient) would be in a process of irreversible mixing, and stagnation pressure would be less than the amount based on the assumption that the flow enters with the velocity u_s . It is believed that the reported data are representative of the fully flooded entry condition and are meaningful beyond the particular experiment. It is also felt that much more is still to be done with the entry-pressure problem.

Stepped Pads

Pressure distributions for stepped pads are shown in Figures 15 and 16. The "unshielded" case represents unrestricted end leakage, and the "shielded" one represents zero end leakage. Of special interest is the large pressure drop at the step where film thickness changes from 0.5 in. to 0.25 inch. For the pad with leakage, the pressure drops to a sizeable negative value at this point. This dimensionless pressure would correspond to -14.7 psig when $(14.7)(144) = \rho u_{bc}^2 (0.1)$ or $\rho u_{bc}^2 = 21,200$. For a fluid with the density of water, this would occur at approximately $u_s = 200$ ft/sec. This is not an excessive speed for projected high-performance applications. Thus, it would not be surprising to see cavitation under a pad in the turbulent operating region.

FIGURE 15. PRESSURE VARIATION UNDER A SHORT STEPPED PAD

FIGURE 16. PRESSURE VARIATION UNDER A STEPPED PAD WITHOUT END LEAKAGE

Conclusion

The above examples serve to illustrate the rather sizeable inertial effects in turbulent lubricant film. These are: (1) leading edge "ram effect" which moves center of pressure forward and otherwise significantly affects pressure distribution under pads, and (2) convergence effect at a step which causes a strong pressure drop and may lead to cavitation.

SPIRAL GROOVE BEARINGS AND SEALS

The spiral groove configuration was investigated as part of the general program of simulation of different bearing types in the large-scale turbulence apparatus. As will be shown, the measurements demonstrate the overwhelming influence of inertial effects. They also give indications of regions where the pressure is significantly below ambient. When these negative pressures are sufficiently great in liquid-lubricated bearings, the film will cavitate and disrupt, and the bearings will cease to perform as intended. Whereas the numerical magnitudes of the results here are to an extent specific to the apparatus, they nevertheless point up design limitations additional to those previously applied to such configurations. The results further suggest that any analyses of turbulent screw-thread seals and spiral-groove bearings, which do not take account of the inertial effects at the discontinuities, may be quite far removed from reality. In addition to dramatizing the effects of the steps, the present work also gives some insights into how to deal with the changes which occur across these discontinuities.

Description of Apparatus

The general configuration of the bearing in the large-scale turbulence apparatus has been described previously. Into this configuration, spiral raised areas, made of 1/4-in. thick rubber sheet were cemented to the stationary surfaces to provide the lands of the spiral configuration. This is illustrated in Figure 17. Looking down on the developed surface of the bearing, one sees the rubber sections making an angle of 45° with the direction of turning. One end of the gas film is blocked by a seal made of a plastic sheet which rubs against the edge of the journal. The other end is open. Locations for pressure taps (or velocity probes) are illustrated in the figure. In the "groove" regions, the film thickness was 0.54 in.; in the regions over the "lands", it was approximately 0.27 in., these dimensions prevailing when the bearing was held concentric with the journal. Upon displacement of the bearing relative to the journal, the relative dimensions could be shifted locally.

Pressures were measured by the tracer technique also described previously⁽¹⁾. Both to generalize these and to convert them to a meaningful scale, these pressures had been nondimensionalized ($P = p/\rho u_{bc}^2$) in Figures 18, 19, and 20 which show pressure distribution on such a dimensionless basis for three eccentricities. In Figure 18 (for concentric operation), one finds in the thin film regions that pressures are predominately negative and are of the order of approximately 0.1. In Figure 19,

FIGURE 17. LAYOUT OF EXPERIMENTAL GROOVE AND LAND ARRANGEMENTS, SHOWING LOCATION OF PRESSURE TAPS

FIGURE 18. DIMENSIONLESS PRESSURE, P , FOR CONCENTRIC OPERATION

FIGURE 19. DIMENSIONLESS PRESSURE, P , FOR DISPLACEMENT TOWARD JOURNAL

FIGURE 20. DIMENSIONLESS PRESSURE, P , FOR DISPLACEMENT FROM JOURNAL

the bearing has been displaced toward the journal, keeping the step height constant but thinning the film throughout by approximately 1/8 inch. Again, pressures over the land are predominately negative, going to even lower values than for concentric operation. In Figure 20, the bearing has been displaced away from the journal by approximately 1/8 in. from concentric operation, and the magnitudes of the negative pressures are reduced. The pressure distributions in all three drawings were measured at a Reynolds number $R = 2300$, where R is based upon a nominal 0.5-in. film thickness and a velocity equal to half the turning speed of the wheel. In previous work, it has been shown that, for other geometrically similar bearings at the same Reynolds number, the same dimensionless pressure should prevail. Consequently, the values observed would be indicative of the pressures to be encountered in practice but would be altered somewhat by spiral angle and other factors.

In Figure 21, the maximum negative pressure has been taken from each of the three distributions and plotted against film thickness ratio. To illustrate the significance of this, assume that the film thickness ratio is approximately one-half and P is approximately 0.09. If fluid had a density equivalent to water (2 slugs/ft³) then the velocity, u_s , at which the actual pressure would reach -15 psi would be 220 ft/sec. This is not a particularly high velocity in many machinery applications. Furthermore, if the configuration were used as a bearing and radial displacement took place under load, the negative pressure would be increased at any given speed, and the permissible velocity would be decreased. One can visualize such a bearing operating without difficulty under a light load and then under impact conditions, displacing and causing disruption of the film.

Load-Carrying Capacity

The pressure distributions in Figures 3, 4 and 5 can be used to provide an estimate of load-carrying capacity when each pressure is assumed to attribute to a parallelogram drawn about the point with boundaries equally spaced between pressure taps. In this way, an area-weighted average has been obtained for the pressure, and this is plotted in Figures 22 and 23.

Figure 22 shows the dimensionless pressure as a function of ratio of minimum to maximum film thickness and indicates that there is a condition of maximum load support. Should the ratio $H_{\min}/H_{\max}$ be reduced below approximately 0.5, load support decreases. Decrease of load support as the surfaces approach one another would represent the condition of negative-spring or instability. This effect is more clearly demonstrated in Figure 23 where the minimum film thickness is expressed

FIGURE 21. EFFECT OF FILM THICKNESS RATIO ON MAXIMUM NEGATIVE PRESSURE

FIGURE 22. EFFECT OF FILM THICKNESS RATIO ON MEAN PRESSURE, $\bar{P}$

FIGURE 23. EFFECT OF MINIMUM CLEARANCE ON MEAN PRESSURE, $\bar{P}$

in ratio to journal radius, and again it is seen that, if this thickness becomes too small, the bearing becomes unstable. This is because the negative pressures in the thin film regions begin to predominate under such a circumstance. If the spiral grooves were made long enough in the axial direction, there is reason to believe that pressure gradient effects would overcome this and shift the peak to a lower magnitude. Nevertheless, the observations reported here must serve as a warning regarding the stability of the spiral groove bearing under turbulent operation conditions.

Velocity Measurements

A hot-wire anemometer was inserted into each of the pressure holes and was used to measure the velocity at midchannel. It was also used to obtain a rough estimate of the direction of the velocity but could not resolve this better than within 10° . In this latter capacity, it indicated that the flow was essentially tangential at all of the ports except possibly the one at the lower right-hand corner, within a few degrees of uncertainty of measurement. Typical velocities are shown in Figure 24, expressed in ratio to u_{bc} , as V . When $V = 1$, pure Couette flow would prevail in the space. From the figure, it is seen that $V < 1$ in the thick-film portions thereby indicating a displacement flow in the direction of turning. This is as would have been anticipated from the theory of the stepped pad where a similar augmentation of flow occurs in the thin film region and diminution occurs in the thick film region. In Figure 25, the quantity $V^2/2$ is tabulated, showing the order of magnitude of the changes of this factor.

It is of interest to compute a stagnation pressure which, in our dimensionless terminology, would be $P_o = P + \zeta_1 V^2/2$. This would represent the condition of the flow at any position in the channel if ζ were

defined as $\zeta_1 = \left(\int_0^h u^2 dy \right) / (u_b^2 h)$. Earlier in this report, this has been dis-

cussed, and tables have been prepared based upon an assumed shape for the turbulent velocity profile and the magnitude of the dimensionless V . For example, if $V = 0.5$, $\zeta_1 = 1.33$, and, if $V = 1$, $\zeta_1 = 1.022$. Using such values, the data in Figures 26 and 27 were computed. There is some scatter in the values reported, but this does not obscure the important relationship. For example, at the central step, from thick film to thin film, the mean change of P_o is 0.055 or approximately 10 percent. On the other hand, in going from the upper edge of the thin film region to one corresponding to the lowest edge of the thick-film region, it is 0.149, or nearly three times the other value.

FIGURE 24. DIMENSIONLESS VELOCITY, V/u_{bc}

FIGURE 25. DIMENSIONLESS $V^2/2$

FIGURE 26. DIMENSIONLESS MOMENTUM FLUX INTENSITY, $\zeta V^2/2$

FIGURE 27. DIMENSIONLESS TOTAL PRESSURE, $P + \zeta V^2/2$

Effect of Step Shape

The steps in the large-scale turbulence apparatus were almost true geometric discontinuities, while the steps in actual bearings must undoubtedly be rounded, consequently, a separate set of measurements was called for to determine the effects of step-abruptness. These were carried out for steps, normal to the flow, and the results are illustrated in Figure 28 where the change across a vertical step and a 45° beveled step is compared. In each case, the flat boundary moves in the direction shown in the figure. For converging flow over the step, there was no measured change of pressure drop. For diverging flow, the pressure recovery was greater for the 45° step, thereby indicating some diffuser effect. On the other hand, the pressure change was not greatly different. In this figure, the pressure magnitudes are shown to four significant figures, although they were not accurate to this degree.

Continuity Considerations

In discussing the measured velocities, it was noted that the flow was nearly tangential at each measuring point. Recognizing that the volume flow in any section of channel is approximately VH , it is seen that the velocity changes across the step cannot be reconciled with conservation-of-volume, if the restriction of no axial movement is held rigid. For example, before the convergence $VH \approx 0.75$; and just after it, $VH \approx 0.5$. Further reflection provides a reconciliation of this observation with other factors. It requires that the condition of absolute-tangential flow be relaxed in the neighborhood of the step. The pressure change across the step occurs within a very short distance in the tangential direction, in the neighborhood of the step. If this drop occurred all along the step, it would necessitate a large tangential pressure gradient. Because the step is inclined to the axis of the cylinder, this would also necessitate a large axial pressure gradient and tend to give rise to an axial flow. On the other hand, if the velocity component tangential to the step is preserved, and velocity changes occur only normal to the step, there will be no pressure gradients along the step.

Figure 29 shows a case where a unit velocity to the left is reduced on entering a region of increased film thickness. The component of this velocity normal to the step is indicated as 0.707. If this is foreshortened by one-half in passing through, then the normal component becomes 0.35, and the tangential component remains 0.707. The overall velocity becomes 0.79. These velocities correspond roughly to those measured in the experimental arrangement.

FIGURE 28. EFFECT OF STEP ABRUPTNESS ON PRESSURE CHANGE

FIGURE 29. EFFECT OF STEP UPON VELOCITIES, ON MOVING FROM THIN FILM TO THICK FILM

In Figure 30, the opposite situation is shown where fluid moves from left to right going from thick film to thin film with an increase in velocity normal to the step. Here, the velocity normal to the step is 0.5 and increases to unity. The tangential component is 0.5 in either case, and the magnitude of the overall velocity vector is increased to 1.1. The angular change is not great and would be less noticeable for the actual case which appears to be between these two examples. After all, one would expect that outward fluid movement in one zone must be balanced by inward movement in the next.

In the region following this step, there is a possibility of some bending of the velocity vectors to become more nearly tangential, and the overall pressure changes must be governed by the overall velocity changes. In essence, what this means is that there is a shift of the velocity pattern downward in one case and upward in the other, undoubtedly with standing vortexes compensating for this shift in the regions near the end where the barrier is placed. At the open end, compensation is provided simply by movement of the fluid into or out of the clearance space.

FIGURE 30. EFFECT OF STEP UPON VELOCITIES,
ON MOVING FROM THICK FILM TO THIN FILM

ANALYSES OF LAMINAR AND TURBULENT FLOWS UNDER TILTED PADS

Comparisons between Laminar and Turbulent Films

In earlier sections, turbulent film equations have been derived in the following form:

$$\frac{\partial P}{\partial X} = - \frac{2\zeta_5}{H} C_{xc} U \quad (101)$$

$$\frac{\partial P}{\partial Z} = - \frac{\zeta_6 C_{xc}}{H} W \quad (102)$$

$$\frac{\partial H(1 + U)}{\partial X} + \frac{\zeta_8}{\zeta_7} \frac{\partial WH}{\partial Z} = 0 \quad (103)$$

where C_{xc} , ζ_5 , ζ_6 and ζ_8/ζ_7 can be treated as constants to a good first approximation.

For laminar flow with quadratic velocity profiles, under the same assumptions that go into the Reynolds equation, a similar set may be derived

$$\frac{\partial P}{\partial X} = - 8 \frac{U}{H^2} \quad (104)$$

$$\frac{\partial P}{\partial Z} = - 8 \frac{W}{H^2} \quad (105)$$

$$\frac{\partial H\left(\frac{3}{4} + U\right)}{\partial X} + \frac{\partial WH}{\partial Z} = 0 \quad (106)$$

When these are combined, they do indeed become Reynolds equation, but, when left separate as they are, they may be used in a computation procedure similar to that which has been developed for Equations (101), (102), and (103).

The similarities between the two sets of equations make possible easy comparison between the two types of flow. For example, the continuity equations, Equations (103) and (106), are seen to be very nearly the same when it is recognized that ζ_7 is near unity. To make possible

additional comparison, let $P' = P/8$ and $P'' = P/2\zeta_5 C_{xc}$ in which the remaining equations become

$$\text{Turbulent} \quad \left\{ \begin{array}{l} \frac{\partial P''}{\partial X} = -\frac{U}{H} \\ \frac{\partial P''}{\partial Z} = -\frac{W}{2H} \frac{\zeta_6}{\zeta_5} \end{array} \right. \quad (107)$$

$$(108)$$

$$\text{Laminar} \quad \left\{ \begin{array}{l} \frac{\partial P'}{\partial X} = -\frac{U}{H^2} \\ \frac{\partial P'}{\partial Z} = -\frac{W}{H^2} \end{array} \right. \quad (109)$$

$$(110)$$

Here, it is seen that the equations differ in their sensitivity to H and in the presence of $\zeta_6/2\zeta_5$ in Equation (108). For very small amounts of tilt, $H \rightarrow 1$; if also $\zeta_6/\zeta_5 \rightarrow 1$, then the turbulent film bearing shows greater leakage in the Z direction than does the laminar film, on the selected dimensionless basis. Of course, the real implications of these differences can best be demonstrated by complete calculations of pressure for the given bearing configurations.

Iterative Procedure

Linearity of Equation (103) is achieved by similarly dropping account of U in this equation in the first iteration, and then supplying the approximate value of this variable for the second iteration. Letting $2\zeta_5 C_{xc}$ be defined as A , $\zeta_6 C_{xc}$ be defined as B , and letting $(1 + U) \frac{dH/dX}{H}$ be defined as F , Equations (101), (102), and (103) may be written in reverse order to become

$$-F = \frac{\partial U}{\partial X} + \zeta_7 \frac{\partial W}{\partial Z} \quad (111)$$

$$\frac{\partial P}{\partial X} = -\frac{AU}{H} \quad (112)$$

$$\frac{\partial P}{\partial Z} = -\frac{BW}{H} \quad (113)$$

Solutions may be obtained that satisfy Equation (111) and that may or may not satisfy Equations (112) and (113); however, if a systematic group of solutions can be obtained, it is permissible to sum these to

achieve a combined solution which satisfies all three equations. The problem is to devise a system of generation and summing such solutions which is economical of computer time. The method chosen here is to begin with a primitive solution of Equation (111) and allow the discrepancy between the pressures computed from Equations (112) and (113) from this solution to serve as a guide in generation of successive solutions. For example, the following represents a valid solution for Equation (111):

$$\frac{\partial U}{\partial X} = - \frac{F}{2} \quad (114)$$

with the complementary statement

$$\zeta_7 \frac{\partial W}{\partial Z} = - \frac{F}{2} \quad (115)$$

It follows from (114) that

$$U = U_0 - \frac{1}{2} \int_0^X F dX \quad (116)$$

where U_0 is an initial value at $X = 0$. It also follows from (120) that

$$P_x = P_0 - \int_0^X \frac{A}{H} U dX \quad (117)$$

where P_0 is an initial value, and P_x is P as computed by the X integration. Combining Equations (116) and (117)

$$P_x = P_0 - \int_0^X \frac{A}{H} \left[U_0 - \frac{1}{2} \int_0^X F dX \right] dX \quad (118)$$

but U_0 is determined by the requirement that the trailing edge pressure be zero; hence,

$$U_0 = \left\{ P_0 + \int_0^L \left[\frac{A}{2H} \int_0^X F dX \right] dX \right\} / \left\{ \int_0^L \frac{A}{H} dX \right\} \quad (119)$$

From Equation (115)

$$\frac{\partial W}{\partial Z} = - \frac{F}{2\zeta_7} \quad (120)$$

or

$$W = - \frac{1}{2} \int_0^Z \frac{F}{\zeta_7} dZ \quad (121)$$

because $W = 0$ at $Z = 0$.

Since $P = 0$ at $Z_{\max}$ and P_z is computed by integration in the Z direction

$$P_z = - \frac{1}{2} \int_{Z_{\max}}^Z \frac{B}{H} \int_0^Z \frac{F}{\zeta_7} dZ dZ \quad (122)$$

it would only be fortuitous that P_z equal P_x throughout the entire field. A solution can be obtained, however, which, when added to the prior solution, tends to produce values of P_z closer to those of P_x , than in the primitive solution. This is obtained by letting, at each point,

$$\frac{\partial U}{\partial X} = a(P_z - P_x) \quad (123)$$

and noting that, to satisfy Equation (111), there must be a complementary solution

$$\zeta_7 \frac{\partial W}{\partial Z} = - a(P_z - P_x) \quad (124)$$

In Equations (123) and (124), $a \ll 1$, say 0.001, and is called the relaxation coefficient.

For the next approximate solution, the boundary conditions become

$$\begin{aligned} W &= 0 \text{ at } Z = 0 \\ P &= 0 \text{ at } Z = Z_{\max} \\ P &= 0 \text{ at } X = 0 \\ P &= 0 \text{ at } X = L \end{aligned}$$

A solution which satisfies these boundary conditions may be added to the previous solution, and thus the sum will satisfy all of the previous boundary conditions as stated. For the incremental solution, this is equivalent to substituting $-a(P_z - P_x)$ for $F/2$, in Equations (114), (116), (118) and $a(P_z - P_x)$ for $F/2$ in Equations (120), (121), (122), and letting $P_0 = 0$. By this means, the new P_z and P_x magnitudes may be produced for all (X, Z) points and may be added to the previously obtained matrices of P_x and P_z . The new matrices of P_x and P_z may be used in Equations (123) and (124) for additional repeats by the process of successive approximation.

Pressures under a Finite Tilted Pad

The laminar film Equations (104), (105), and (106) are close enough mathematically to the turbulent ones to serve as a test case for the iterative procedure which has been developed for turbulent films. This is of interest since exact solutions exist for finite tilted pads with laminar flow⁽¹⁷⁾, and these may be compared with corresponding computed solutions. To this end, calculations were carried out for $h_1/\ell_x = 0.001$, $h_1/h_2 = 2$, $L/W = 1$. A grid size of fifteen spaces in the X direction and nine spaces in the Z direction was selected, and 100 iterations were carried out, though this was considerably more than was necessary. The computer time per case on a CDC 3600 digital computer was 4 sec, printing out a complete solution for dimensionless pressures and velocities at 135 grid points. The relatively coarse grid was justified for the integration techniques used since these tend to smooth irregularities.

Table 12 illustrates the convergence of the mean pressure $\bar{P}$ for successive groups of iterations. (Primes will be dropped from the P quantities in the remaining discussion.) Figure 31 shows the pressure distribution for length/width ratio of unity as computed, this being compared with the pressure distributions obtained by Hays⁽¹⁷⁾ in an exact solution. In Figure 32, the axial distribution near the pressure peak is plotted out and is also compared with that calculated by Hays. In both plots, the calculated pressures are slightly high. However, it can be shown that, with coarser grids, pressures would indeed tend to be high, converging toward the exact solution as grid number was increased. Though increased number of grid points could have been used here to obtain a more precise fit, this was not done since it was felt that, with the practical grid size used, the agreement was quite good enough for the type of comparisons to be made in subsequent calculations and that the economies resulting from the small grid size would be justified.

TABLE 12. CONVERGENCE OF COMPUTATION

<u>Iteration</u>	<u>$\bar{P}$</u>	<u>$\bar{X}$</u>
5	306.8	0.577
10	275.4	0.586
15	269.8	0.587
20	269.8	0.586

FIGURE 31. VARIATION OF PRESSURE IN DIRECTION OF SLIDING (AT MIDSPAN) SHOWING EXACT SOLUTIONS (CURVES) AND COMPUTED DATA (POINTS) FOR $R = 1$. WHERE $R = L/W$

FIGURE 32. VARIATION OF PRESSURE NORMAL TO DIRECTION OF SLIDING, AT $X/B = 0.7$, SHOWING EXACT SOLUTIONS (CURVES) AND COMPUTED DATA (POINTS) FOR $R = 1$, WHERE $R = L/W$

Computed Performance of a Turbulent Film Bearing of Comparable Dimensions to the Above

To illustrate the difference between turbulent-film and laminar-film behavior, even when the leading-edge pressure jump is ignored, a calculation was carried out for the same bearing geometry, but using Equations (107), (108), and (103). Here, the two quantities ζ were allowed to be unity, the friction coefficient was taken as 0.0075, and the dimensionless pressure as used in Equations (107) and (108) serves to permit the plotting of the turbulent film results on the laminar-flow dimensionless plots. Figures 33 and 34 show that the turbulent-film bearing behaves rather like a much shorter laminar-film bearing on this basis. Though the length to width ratio is unity, the turbulent-film bearing behaved more nearly like a laminar one of width 0.6 times the length. Otherwise, the general distribution of pressure is not significantly different from that for laminar flow. This would have been anticipated from our earlier reported work⁽¹⁾, where it was shown that the relation resistance to axial and tangential pressure flows was unity for laminar operation but was in the ratio 1/2.59 for turbulent flow.

Computed Results for Tilted Pads with Leading-Edge Stagnation

To see how computed results would be affected by the assumed leading-edge pressure relationship of Figure 14, this was used to set leading-edge pressure distribution of bearings. This has made possible comparisons of effects of film thickness variation for the same length/width ratio, and length/width variations for the same film thickness, all accompanied by tilt angle variation. Figure 35 shows the sensitivity of the several parameters of the bearing performance to leading-edge pressure. Calculations were made in this case for assumed dimensionless leading-edge pressure of 0, 0.1, 0.2 and 0.5. The circled points on the diagram indicate the solution which corresponds to the combination of centerline leading-edge pressure and velocity given by the curve in Figure 14.

Figures 36, 37 and 38 show computed results for a set of pads for which the dimensionless length L is 750, and for which L/W is assigned the values of 1/2, 1 and 3/2. The length is expressed in ratio to the leading-edge film thickness, and this tilt angle is measured in radians. Figure 36 shows that, at small tilt angles, all three bearings may have negative leading-edge pressure. This is in accord with the experimental findings.

FIGURE 33 VARIATION OF PRESSURE IN DIRECTION OF SLIDING (AT MIDSPAN) COMPARING LAMINAR SOLUTIONS WITH COMPUTED RESULTS FOR A TURBULENT FILM, WHERE $R=1$ AND LEADING-EDGE PRESSURE IS ZERO

FIGURE 34. VARIATION OF PRESSURE NORMAL TO DIRECTION OF SLIDING, AT $X/B = 0.7$, COMPARING LAMINAR SOLUTIONS WITH COMPUTED RESULTS FOR A TURBULENT FILM, WHERE $R=1$ AND LEADING-EDGE PRESSURE IS ZERO

FIGURE 35. SENSITIVITY OF MEAN PRESSURE, $\bar{P}$, AND CENTER-OF-PRESSURE POSITION $\bar{X}$ TO VARIATION OF LEADING-EDGE PRESSURE P_1

FIGURE 36. VARIATION OF LEADING-EDGE PRESSURE WITH TILT ANGLES FOR TILTED PADS, AS INFLUENCED BY LENGTH/WIDTH RATIO

FIGURE 37. VARIATION OF MEAN PRESSURE WITH TILT ANGLES FOR TILTED PADS, AS INFLUENCED BY LENGTH/WIDTH RATIO

FIGURE 38. VARIATION OF CENTER-OF-PRESSURE LOCATION FOR TILTED PADS, AS INFLUENCED BY LENGTH/WIDTH RATIO AND TILT ANGLE

Figure 37 shows the corresponding variation of the mean pressure which is a measure of load support. In the range plotted, this is positive, but it goes negative near zero tilt.

Figure 38 shows the movement of the center-of-pressure under variation of tilt angle. Each curve shows negative slope in the region where P_1 is negative. If such pads were mounted on pivots and if the tilt angle were allowed to become small, the toe of the pad would tend to drop down and contact the shaft. The possible consequences of this condition are sufficiently serious as to suggest this as a prime cause of failure of tilted pads operating in the turbulent regime.

To explore further the effects of design parameters on bearing performance, the data reported in Figures 39, 40 and 41 were computed. In this series, the length/width ratio is fixed, but two values of film thickness are used so as to make the dimensionless length take on the magnitudes of 750 and 500. Since the thicker film can tolerate a greater tilt without contact, it has been desirable to set the tilt angle in ratio to the maximum allowable value, and maximum in order to facilitate comparisons.

Figure 39 shows that leading-edge pressure is not greatly influenced by film thickness for a corresponding ratio of leading edge/trailing edge clearance. [Note that $a/a_{\max} = (h_1 - h_2)/h_1$.]

Figure 40 shows that the load support curves do not as closely follow one another, there being a definite penalty for the thicker film ($L = 500$).

In Figure 41, the variation for center-of-pressure location is plotted. Though the two curves diverge toward the right, they lie upon one another at the left. Both show the negative slope for a sizeable part of the expected operating range.

A corresponding set of curves to the above is shown in Figures 42, 43 and 44 for a relatively wide bearing. Though the magnitudes plotted differ for those of the prior set, the general behavior of the curve is the same.

FIGURE 39. EFFECT OF L AND TILT ANGLE ON LEADING EDGE PRESSURE

FIGURE 40. EFFECT OF L AND TILT ANGLE OF MEAN PRESSURE

FIGURE 41. EFFECT OF L AND TILT ANGLE ON CENTER-OF-PRESSURE LOCATION

FIGURE 42. LEADING-EDGE
PRESSURE VARIATION FOR
A LONG BEARING

FIGURE 43. MEAN PRESSURE
VARIATION FOR A LONG
BEARING

FIGURE 44. VARIATION OF CENTER-
OF-PRESSURE LOCATION FOR A
LONG BEARING

CONCLUSION

In the preceding sections, we have shown several analytical techniques to deal adequately with codirectional shear and displacement flows. By reference to new experimental data and by numerical calculations which exploit and extend the applicability of one of the more promising techniques, we have shown that the behavior of crossed Couette and pressure flows can be predicted with good accuracy. In so doing, we have added significantly to the confidence in analyses of bearings with end leakage.

Drawing further upon experimental results, we have demonstrated large pressure changes near discontinuities in the bearing surfaces such as found in stepped pads and in spiral-groove bearings. Generalizing the pressure rise data for the entry region of pads, we have been able to include this effect as a boundary condition in computations of the performance of tilted pads. In view of the foregoing critical discussions, we feel safe in saying that the calculated performance is realistic. The fact that pad-tipping is predicted by these calculations, under some circumstances, should serve as a warning to designers.

The negative pressure observed in stepped pads and spiral-groove configurations indicates conditions under which films would be expected to cavitate. Since this may also lead to malfunction of the bearing or seal, such observations are expected to lead, ultimately, to statements of design limitations. They also serve to provide explanations of previously observed film-breakup in such configurations as the screw-thread seal, a topic which has been the subject of considerable speculation and mystery.

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