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BENDING-MAGNET AND QUADRUPOLE ABERRATIONS
FOR PARAXIAL RAYS

by

M. L. Good*

Physics Division

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*Department of Physics, University of Wisconsin

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TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	iii
1. IDEAL BENDING MAGNET	2
2. REALISTIC BENDING MAGNET	3
3. VERTICAL MOTION IN AN IDEAL QUADRUPOLE	5
4. HORIZONTAL MOTION	8
5. CORRECTION FOR THE EDGE FOCUSSING OF QUADRUPOLES	11

ABSTRACT

Aberrations of bending magnets and quadrupoles for paraxial rays are discussed, with particular emphasis on isolating those effects likely to be of practical importance in high-energy separated beams.

Bending-Magnet and Quadrupole Aberrations for Paraxial Rays

M. L. Good

Consider a magnet with a plane of symmetry (xz plane). Thus either a quadrupole or a bending magnet may be discussed.

The nominal beam direction at entry into the magnet is in the xz plane, and at least approximately in the plus z direction.

We will write the Lorentz force for a field which is describable by its value on the central trajectory and a first derivative. We examine the various small terms that are usually neglected. Thus we are considering a magnet which is ideal in the sense that its field does not contain higher derivatives, but we do consider end effects. The Lorentz force in the y direction is

$$F_y = \frac{e}{c} (v_z H_x - v_x H_z).$$

If we take

$$\ddot{y} = d^2 y / ds^2,$$

where s is the length along the actual path, then we have

$$\ddot{y} = \frac{e}{mvc} \left(\frac{v_z}{v} H_x - \frac{v_x}{v} H_z \right)$$

$$\ddot{y} = \frac{1}{H\rho} (\cos \beta \cos \alpha H_x - \cos \beta \sin \alpha H_z),$$

where the angles are as defined in Fig. 1.

Fig. 1. Definitions of angles.

If we now replace H_x by $y \left| \frac{\partial H_x}{\partial y} \right|_{y=0}$ and H_z by $y \left| \frac{\partial H_z}{\partial y} \right|_{y=0}$ and utilize the condition that $\vec{\nabla} \times \vec{H} = 0$, we have

$$\ddot{y} = \frac{1}{H\rho} \left[\cos \beta \cos \alpha \left| \frac{\partial H}{\partial x} \right|_{y=0} - \cos \beta \sin \alpha \left| \frac{\partial H}{\partial z} \right|_{y=0} \right] y. \quad (1)$$

This is our basic equation and, within the limits stated, it is exact.

The derivatives involved in Eq. (1) may, of course, depend on x and z .

We now apply it to some special cases.

1. IDEAL BENDING MAGNET

The edges of the pole tips are taken to be lines of constant z . The magnet is taken ideal in the sense that the main field H_y is independent of x . Thus, the first term in Eq. (1) drops out. [We shall show in Sec. 2 that this assumption is frequently unjustified, and that the first term in Eq. (1) often represents a severe aberration in bending magnets.]

The second term will be seen to be the edge-focussing term. If $\partial H / \partial z$ is large only in small fringe regions at entry and exit, α is constant to good approximation through the fringe field and so is y . If also $\cos \beta \approx 1$, we have

$$\begin{aligned} \Delta \dot{y} &= \int \ddot{y} \, ds \\ &= - \frac{1}{H\rho} \left[(\sin \alpha_1) y_1 \int \left| \frac{\partial H}{\partial z} \right|_{y=0} \frac{dz}{\cos \alpha_1} + (\sin \alpha_2) y_2 \int \left| \frac{\partial H}{\partial z} \right|_{y=0} \frac{dz}{\cos \alpha_2} \right], \end{aligned}$$

where ds has been approximated by $dz / \cos \alpha_1$ and $dz / \cos \alpha_2$ in the integrals in the second line. Then

$$\Delta \dot{y} \approx - \frac{1}{H\rho} [y_1 H_1 \tan \alpha_1 - y_2 H_2 \tan \alpha_2].$$

This describes a thin lens at each end of the magnet, of strength

$$\frac{1}{f} = \frac{\tan \alpha}{\rho}.$$

This is the usual edge-focussing formula. All the approximations made are likely to be quite good in practice, but the assumption that H_y is independent of x is frequently not justified. This point will now be examined.

2. REALISTIC BENDING MAGNET

In a real bending magnet, $\int B_y dz$ will vary slightly with x . This means that in general $\partial B_y / \partial x$ is not zero. We have then

$$\begin{aligned} \Delta \dot{y} &= (\Delta \dot{y})_{\text{edge}} + \frac{1}{H\rho} \int \cos \beta \cos \alpha \left| \frac{\partial H}{\partial x} y \right|_{y=0} y ds \\ &= (\Delta \dot{y})_{\text{edge}} + \frac{1}{H\rho} \int \left| \frac{\partial H}{\partial x} y \right|_{y=0} y dz \end{aligned}$$

If the second term is small, a preliminary value of $y(z)$ can be calculated with this term neglected; and the second term can then be evaluated by substituting this $y(z)$ in the integral. If the second term is large, Eq. (1) could be integrated directly by computer. Our purpose here is to show that the second term can be surprisingly large. For this purpose we will approximate it crudely by writing

$$\int \left| \frac{\partial H}{\partial x} y \right|_{y=0} y dz \approx \bar{y} \frac{\partial}{\partial x} \int H_y dz,$$

where the integral is supposed to be taken along the path of the particle. Then

$$\Delta \dot{y} \approx (\Delta \dot{y})_{\text{edge}} + \frac{1}{H\rho} \left(\frac{\partial}{\partial x} \int H_y dz \right) \bar{y}.$$

If we now treat the entire magnet as a thin lens (a fairly good approximation at high energy), we have

$$\boxed{\frac{1}{f} \approx \frac{1}{(H\rho)^2} \int H_y^2 dz - \frac{1}{H\rho} \frac{\partial}{\partial x} \int H_y dz,} \quad (2)$$

where, in the same spirit, the edge term has become $[1/(H\rho)^2] \int H_y^2 dz$. Suppose now that $\int H_y dz$ is constant to within a fraction ϵ inside some aperture $-x_0 < x < +x_0$, and is symmetrical about the midline $x = 0$. Then if we write $I_0 = HL$, this means that

$$\int H_y dz \approx I_0 \left[1 - \epsilon \left(\frac{x}{x_0} \right)^2 \right]$$

and therefore

$$\frac{1}{f} = \left(\frac{1}{f} \right)_{\text{edge}} + \frac{1}{H\rho} \left(I_0 \frac{2\epsilon x}{x_0} \right).$$

At $x = \pm x_0$,

$$\frac{1}{f} \approx \left(\frac{1}{f} \right)_{\text{edge}} \pm \frac{L}{\rho} \left(\frac{2\epsilon}{x_0} \right).$$

The crude approximation $(1/f)_{\text{edge}} \approx L/\rho^2$ enables us to write

$$\frac{1}{f} \approx \frac{L}{\rho^2} \left(1 \pm \frac{2\rho\epsilon}{x_0} \right). \quad (3)$$

We would ordinarily describe a magnet with $\epsilon = 1\%$ over the central 8 in., say, as being rather good. Equation (3) says that for such a magnet, at 1 BeV/c and 17 kG ($\rho = 196 \text{ cm} = 77 \text{ in.}$),

$$\begin{aligned}
\frac{1}{f} &= \left(\frac{1}{f}\right)_{\text{edge}} \left(1 \pm \frac{2 \times 77 \text{ in.} \times 0.01}{4 \text{ in.}}\right) \\
&= \left(\frac{1}{f}\right)_{\text{edge}} (1 \pm 0.38) \\
&= 0.62 \left(\frac{1}{f}\right)_{\text{edge}} \quad \text{or} \quad 1.38 \left(\frac{1}{f}\right)_{\text{edge}} .
\end{aligned}$$

In other words, the focal length differs by more than a factor of two in going from one side of the aperture to the other!

This extreme sensitivity to the first term of Eq. (1), (which we may identify as gradient or n-value focussing) is what usually causes difficulty with bending magnets in beams employing good optics.

Bending magnets, to be useful optical elements, must have their gradients carefully controlled. In the usual case this means making $\int H_y dz$ exceedingly flat.

3. VERTICAL MOTION IN AN IDEAL QUADRUPOLE

Equation (1) applies here also. If we approximate $\cos \beta = \cos \alpha = 1$, $\sin \alpha = 0$, we get

$$\ddot{y} = \frac{1}{H\rho} y \left| \frac{\partial H_y}{\partial x} \right|_{y=0} .$$

This leads to the usual quadrupole formula

$$\frac{1}{f} = \omega \sin \omega L ; \quad \omega^2 = -\frac{1}{H\rho} \frac{\partial H_y}{\partial x} .$$

If, instead, we keep the next higher terms in an expansion about $\alpha = \beta = 0$, we have

$$\ddot{y} = \frac{1}{H\rho} \left[\left(1 - \frac{a^2 + \beta^2}{2} \right) \frac{\partial H_y}{\partial x} - a \frac{\partial H_y}{\partial z} \right] y,$$

$$\Delta \dot{y} = (\Delta \dot{y})_{\text{quad}} \left(1 - \frac{1}{2} \langle a^2 + \beta^2 \rangle \right) - \frac{1}{H\rho} \int a y \left| \frac{\partial H_y}{\partial z} \right|_{y=0} dz.$$

Since $\partial H_y / \partial z$ is small except at the ends, and both a and y are approximately constant through the end region, we have

$$\begin{aligned} \Delta \dot{y} &\approx (\Delta \dot{y})_{\text{quad}} \left(1 - \frac{1}{2} \langle a^2 + \beta^2 \rangle \right) - \frac{1}{H\rho} (a_1 y_1 H_{y_1} - a_2 y_2 H_{y_2}) \\ &\approx (\Delta \dot{y})_{\text{quad}} \left(1 - \frac{1}{2} \langle \theta^2 \rangle \right) - \frac{G}{H\rho} (a_1 x_1 y_1 - a_2 x_2 y_2), \end{aligned}$$

where G is the gradient $\partial H_y / \partial x$.

The last term is obviously the edge focussing (in the y direction) due to entry at a finite angle a (in the xz plane) into a field H_y . In order to get a rough approximation to it, take the thin-lens limit. Then for $x \gg y$ (so that $H_y \approx H$), $y_1 = y_2 = y$, $H_1 = H_2 = H$. Hence

$$\Delta \dot{y} \approx (\Delta \dot{y})_{\text{quad}} + \frac{(a_2 - a_1)}{\rho} y,$$

$$a_1 - a_2 \approx \frac{L}{\rho},$$

$$\Delta \dot{y} \approx (\Delta \dot{y})_{\text{quad}} - \frac{yL}{\rho},$$

$$\frac{1}{f} \approx \left(\frac{1}{f} \right)_{\text{quad}} + \frac{L}{\rho}.$$

Approximating L/ρ^2 by $[1/(H\rho)^2] \int H^2 dz$ then leads to

$$\frac{1}{f_y} \approx \frac{-1}{H\rho} \frac{\partial}{\partial x} \int H_y dz + \frac{1}{(H\rho)^2} \int H_y^2 dz, \quad (4)$$

which is valid for a thin lens only.

This form, although strictly applicable only to a thin lens, correctly shows the main features of the focussing. The first (gradient) term describes the quadrupole focussing. The second term is the edge focussing; it represents an aberration, since $\int H_y^2 dz \approx x^2$, whereas $1/f$ is supposed to be constant. The aberration is seen to be unavoidable, since $\int H_y^2 dz$ is positive definite. Interestingly enough, the same formula applies to both bending magnets [Eq. (2)] and quadrupoles [Eq. (4)]. In the former the edge term is the desired one, and the gradient term is an unwanted aberration; in the latter, the main focussing is due to the gradient, and the edge focussing represents an unwanted aberration. Equation (4) is useful because, while it is only approximate, it relates the essential features of the focussing to easily measured parameters of the magnet.

The edge-focussing term δf in quadrupoles is usually quite small, but may not always be negligible. A measure of its significance is the quotient

$$\frac{\delta f}{f} \approx \frac{1}{(H\rho)^2} \frac{\int H^2 dz}{(1/H\rho)(\partial/\partial x) \int H dz} \approx \frac{1}{H\rho} \frac{H^2 L}{HL/x}, \quad (5)$$

$$\frac{\delta f}{f} \approx \frac{x}{\rho}.$$

For $x = 10$ cm and $H = 10$ kG, $pc = 3$ BeV ($\rho = 1000$ cm),

$$\frac{\delta f}{f} = \frac{x}{\rho} = \frac{10 \text{ cm}}{1000 \text{ cm}} = 1\%.$$

Since the effect is always positive, it strengthens the focussing elements and weakens the defocussing ones, and thus is of such a sign as to always change the over-all focal length of a doublet or triplet by the largest possible amount.

The other correction term, $\frac{1}{2} \langle \theta^2 \rangle$ is usually considerably smaller, although not entirely negligible. To see this, set $\frac{1}{2} \langle \theta^2 \rangle \approx (1/6) (L/\rho)^2$, which is what it is for normal entry (and which in turn is the worst case). Then

$$\frac{\delta_1^f}{\delta_2^f} \approx \frac{\theta^2/2}{x/\rho} = \frac{L^2}{6\rho^2} \left(\frac{\rho}{x} \right) = \frac{L^2}{6\rho x} \approx \frac{L}{6f} \ll 1.$$

In a quadrupole which is well constructed $\int H_y dz$, where the integral includes the end regions of the quadrupole, will be strictly linear over a good part of the aperture, because of the 4-fold symmetry. What we have shown is that such an ideal quadrupole has an aberration because of the edge focussing at entry and exit.

4. HORIZONTAL MOTION

All the above discussion was for vertical motion.

For horizontal motion we have, for both bending magnets and quadrupoles, the exact relation

$$\begin{aligned} \ddot{x} &= \frac{1}{H\rho} [(\sin \beta) H_z - (\cos \beta \cos \alpha) H_y] \\ &= \frac{1}{H\rho} \left[(\sin \beta) y \left| \frac{\partial H_z}{\partial y} \right|_{y=0} - (\cos \beta \cos \alpha) \left(\left| H_y \right|_{y=0} + y \left| \frac{\partial H_y}{\partial y} \right|_{y=0} \right) \right]. \end{aligned}$$

But

$$\frac{\partial H}{\partial y} = - \frac{\partial H}{\partial x} - \frac{\partial H}{\partial z}$$

since $\vec{\nabla} \cdot \vec{H} = 0$; and since $H_x = H_z = 0$ at $y = 0$, the last term drops out.

Hence

$$\ddot{x} = \frac{1}{H\rho} \left[(\sin \beta) y \left| \frac{\partial H}{\partial z} \right|_{y=0} - \cos \beta \cos \alpha |H_y|_{y=0} \right]. \quad (6)$$

In the case of a bending magnet, the second term is just the bend and gives

$$\Delta \dot{x} = - \frac{1}{H\rho} \int |H_y|_{y=0} dz.$$

Hence

$$- \frac{d(\Delta \dot{x})}{dx} = \frac{1}{f_x} = + \frac{1}{H\rho} \frac{\partial}{\partial x} \int |H_y|_{y=0} dz. \quad (7)$$

The focal length is just due to the gradient term, and is equal and opposite to the gradient term in the vertical motion, as expected.

The first term is an aberration, an x deflection proportional to $y \sin \beta$. It can be written

$$\Delta \dot{x} = (\Delta \dot{x})_{\text{bend}} + \frac{1}{H\rho} \int (\sin \beta) y \left| \frac{\partial H}{\partial z} \right|_{y=0} ds.$$

When the integral is carried out by use of the expression

$ds = dz/(\cos \beta \cos \alpha)$, and the approximation that $\partial H_y / \partial z$ is large only at the ends, this becomes

$$\Delta \dot{x} = (\Delta \dot{x})_{\text{bend}} + \frac{1}{H\rho} \left(\frac{H_1 y_1 \tan \beta_1}{\cos \alpha_1} - \frac{H_2 y_2 \tan \beta_2}{\cos \alpha_2} \right). \quad (8)$$

For a point source, the extra term at each face is an x deflection proportional to y^2 and to the field. This seems to be of no great interest.

In the case of a quadrupole,

$$\left| \frac{\partial H}{\partial y} \right|_{y=0} = Gx,$$

where G is the gradient. Thus we have

$$\begin{aligned} \ddot{x} &= \frac{1}{H\rho} \left[(\sin \beta) y \left| \frac{\partial H}{\partial z} \right|_{y=0} - (\cos \beta \cos \alpha) x \left| \frac{\partial H}{\partial x} \right|_{x=0} \right] \\ &= \frac{1}{H\rho} \left[(\sin \beta) xy \frac{\partial}{\partial z} \left| \frac{\partial H}{\partial x} \right|_{x=0} - (\cos \beta \cos \alpha) x \left| \frac{\partial H}{\partial x} \right|_{x=0} \right]. \end{aligned}$$

The second term leads to the familiar quadrupole formula (with a $1 - \frac{1}{2} \langle \theta^2 \rangle$ correction) as before. The first term is the edge-focussing aberration

$$\Delta \dot{x} \approx (\Delta \dot{x})_{\text{quad}} + \frac{G}{H\rho} [\beta_1 x_1 y_1 - \beta_2 x_2 y_2]. \quad (9)$$

The similar term in the vertical focussing is obtained from this by interchanging x with y and α with β , and changing the sign of G , as it should be.

It is instructive to obtain the thin-lens limit for both the x and y edge focussing in order to see whether the effect can be compensated for by correcting magnets. In the thin-lens limit,

$$\Delta \dot{x} \approx (\Delta \dot{x})_{\text{quad}} + \frac{G}{H\rho} (\beta_1 - \beta_2) xy,$$

$$\Delta \dot{y} \approx (\Delta \dot{y})_{\text{quad}} - \frac{G}{H\rho} (\alpha_1 - \alpha_2) xy.$$

But

$$\beta_1 - \beta_2 \approx - \frac{H_x L}{H\rho} = - GyL/H\rho,$$

$$\alpha_1 - \alpha_2 \approx \frac{H_y L}{H\rho} = GxL/H\rho.$$

Therefore

$$\Delta \dot{x} \approx (\Delta \dot{x})_{\text{quad}} - \frac{G^2 L}{(H\rho)^2} xy^2, \quad (10)$$

$$\Delta \dot{y} \approx (\Delta \dot{y})_{\text{quad}} - \frac{G^2 L}{(H\rho)^2} x^2 y,$$

and

$$\Delta \left(\frac{1}{f_x} \right) = + \frac{G^2 L y^2}{(H\rho)^2}, \quad (11)$$

$$\Delta \left(\frac{1}{f_y} \right) = + \frac{G^2 L x^2}{(H\rho)^2},$$

The horizontal focussing has, naturally enough, an aberration of the same form as the vertical.

5. CORRECTION FOR THE EDGE FOCUSING OF QUADRUPOLES

Exact compensation for the edge focussing seems not to be possible, at least by means of a transverse magnetic field. The reason for this is that the desired compensating field has a nonzero curl, as may be seen from Eq. (10).

If instead we expand the correcting field in multipoles, we find that the dipole, quadrupole, and sextupole terms have no effect, but one of the octupole terms,

$$\phi_8 = \frac{1}{3} (x^3 y - xy^3)$$

has some beneficial effect. This produces a force

$$F_x \sim -H_{8y} = -\frac{x^3}{3} + xy^2,$$

$$F_y \sim H_{8x} = x^2y - \frac{y^3}{3}.$$

Hence

(12)

$$\Delta\left(\frac{1}{f_x}\right) \approx -\frac{x^2}{3} + y^2,$$

$$\Delta\left(\frac{1}{f_y}\right) \approx x^2 - \frac{y^2}{3}.$$

Comparison with Eq. (11) shows that compensation can be achieved in the y motion only for $x \gg y$, and at the same time in the x motion only for $y \gg x$.

Since these are just the regions in which the effect is worst, the partial compensation may be quite worthwhile.