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XENON-135 INDUCED INSTABILITIES OF THE NEUTRON FLUX
IN NUCLEAR REACTORS

MASTER

Gary Marlin Sandquist

PHILLIPS
PETROLEUM
COMPANY

ATOMIC ENERGY DIVISION

NATIONAL REACTOR TESTING STATION
US ATOMIC ENERGY COMMISSION

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**XENON-135 INDUCED INSTABILITIES OF THE NEUTRON FLUX
IN NUCLEAR REACTORS***

by

Gary Marlin Sandquist

* Based on a thesis submitted to the Department of Mechanical Engineering at the University of Utah on June, 1964, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

**PHILLIPS
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ABSTRACT

An extensive analytic and computer investigation is made of several systems of kinetic equations which describe to varying degrees of accuracy the space and time behavior of the neutron flux, the xenon-135, and the iodine-135 in nuclear reactors. Criteria for stability of the neutron flux under the influence of xenon-135 are established, and those factors which significantly affect stability are found and examined.

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SUMMARY

Evidence that the fission by-product, xenon-135, with its enormous absorption cross section for neutrons can induce instabilities in the neutron flux distribution in large, thermal reactors has been both analytically and experimentally established. Although the characteristic time associated with the instabilities is generally in the order of hours, and thus no rapid transient problem is posed, the instabilities can occur in the higher flux harmonics, as well as the fundamental mode, and may result in an oscillating power distribution even though the total power is held constant. Therefore, because of the obviously undesirable consequences of uncontrolled power oscillations that are sustained or increase in time, it is important that adequate and sufficient criteria be established to determine the stability of a given reactor system against xenon induced instabilities.

This report presents an extensive analytic and numerical (by computer means) investigation of several systems of kinetic equations which describe to varying degrees of accuracy the behavior of the neutron flux, the xenon, and the iodine in nuclear reactors. Furthermore, those parameters which effect the study of xenon induced flux instabilities are found and examined. Significantly, it is found that one important measure of stability is the ratio of fast to thermal flux at equilibrium.

The kinetic equations which account for the flux, xenon, and iodine in a space independent reactor model are examined by various techniques which include linearization by first order perturbation approximations, high flux level approximation, phase plane analysis, time derivative analysis, and digital and analog computer simulation. The criteria for stability are determined and corroborated by computer simulation and analytic means. It is deduced that, in general, the use of space independent kinetics is highly restrictive, predicting a threshold value for instability of the flux of 1.5×10^9 , and unwarranted in examining most physical reactor systems for xenon induced flux instabilities.

The "P₁" approximations to the neutron transport equation are extended to account for xenon dependence, and approximate solutions are given for the

stationary flux distribution. The flux distribution is similar to that found from elementary diffusion theory but is dependent upon the flux level. The " P_1 " approximations and the diffusion equation are also examined for compatibility, and the expressions for the material buckling and the diffusion coefficient are found and compared.

The time and space dependent kinetic equations for one energy group of neutrons are examined by the following techniques: First, the resulting linear kinetic equations determined by first order perturbation techniques are subjected to "modal analysis" and the criteria for stability found. For the further assumption of high or "flat" flux distributions the kinetic equations are examined by root locus methods. Secondly, sufficient conditions for specific mode stability are developed by imposing certain requirements upon the equation's time derivatives. Thirdly, for the assumption of high flux levels the time and space solutions are determined. Finally, computer simulation of the kinetic equations is rendered.

Several of the techniques, including computer simulation used in examining the one energy group, time and space dependent kinetic equations, are employed to investigate the kinetic equations for two and more energy groups of neutrons.

Employing the stability criteria previously established and simulating the space and time dependent kinetic equations for one and two energy groups of neutrons the stability of the axial power distribution of the Advanced Test Reactor is examined. On the basis of present design parameters the axial power distribution is deemed stable. However, the need for additional experimental data and improved computer simulation is indicated. Also, a possible method for experimental simulation and examination of xenon induced flux instabilities employing the Advanced Test Reactor Critical Facility is given.

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XENON-135 INDUCED INSTABILITIES OF THE NEUTRON FLUX IN NUCLEAR REACTORS

I - INTRODUCTION

STATEMENT OF THE PROBLEM

Before the advent of the first critically sustained fission reaction in 1941 (43)*, considerable concern was expressed over man's ability to control and stably maintain such a potentially hazardous reaction. However, in the light of the present existence of a multitude of such sustained fission reaction systems, popularly called reactors, it might appear that earlier concern about control and stability was unwarranted. Nevertheless, in general, reactor systems are not fully and inherently stable, and their control and operation is not trivial. Indeed, the determination, analysis, and evaluation of the control operation and associated control system and the consequences of a controlled or uncontrolled event which might prove hazardous or damaging are vital.

There are many factors and influences which affect reactor control and stability. This report is devoted to the specific study of the kinetic behavior of a thermal nuclear reactor under the influence of the isotope, xenon-135, which occurs in a reactor as a by-product of fission.

HISTORICAL AND LITERATURE REVIEW

The significance of xenon-135 as a potent reactor "poison" and a formidable factor in reactor control was established soon after the first Hanford Reactors became operational. Indeed, for a short time the discovery of a decrease of steady state reactivity with increasing reactor power threatened the success of the entire Hanford project (43, 48). It was only after intensive and brilliant scientific sleuthing and analysis by E. Fermi, J. H. Wheeler, and their associates at the Hanford site that the source of reactivity loss with increasing power and the inability to return the reactor to criticality shortly after shutdown were traced to the unstable isotope xenon-135, and that it was found possible to compensate for the reactivity loss.

* Numbers in parentheses refer to the corresponding reference given in the List of References.

The discovery of this poisoning effect of xenon-135 on the then so-called thermal piles fostered an interest among many as to the effect of xenon (all references to "xenon" in this thesis will imply the isotope xenon-135) with its enormous absorption cross section for thermal neutrons (known then to be about three million barns) on the stability of the thermal flux. Investigations were made on various occasions by F. L. Friedman, H. Soodak, G. P. Nordheim, L. W. Nordheim, and others. One of these early investigations was by K. O. Donelian and J. R. Menke entitled "Xenon Instability Periods as a Function of Flux in Thermal Piles" in the secret document (now declassified) MON-P-379, U. S. Atomic Energy Commission, dated September 10, 1947. The authors' purpose for the investigation was, as stated in the introduction, to examine the dependence of the xenon period on flux level with a view towards defining practical flux domains for future piles as determined by xenon.

Later, the effects of xenon on the spacial stability of the thermal flux, particularly the first harmonic, were first indicated with the detection of local hot spots at unusual positions in the Hanford Reactors. That xenon induced oscillations were actually possible was experimentally established at Savannah River, and a report to that effect was given by R. Haefner (17) of the Savannah River Laboratory in the journal, Nuclear Science and Technology.

With the existence of xenon induced flux oscillations verified, a vast amount of literature on the general problem of xenon and its influence on the kinetic behavior of reactors has evolved. A. G. Ward (47) made an analysis of a linearized mathematical model for an unreflected slab reactor with a flat flux and uniformly loaded core. A. F. Henry and J. D. Germann (18) investigated the influence of temperature and the amplitude and frequency characteristics of the flux oscillations. D. Randall and D. S. St. John (38) employed "modal analysis" to investigate the linearized equations and extended their treatment to general reactor configurations and flux distributions. Nonuniform formation and depletion of xenon and two group diffusion theory were studied by R. S. Wick (50). Later, a study of the space independent, time behavior of a reactor controlled solely by xenon was made by J. Chernick (5), and in a succeeding report by J. Chernick, G. S. Lellouche, and W. Wollman (6) the study was extended to account for temperature effects. The effects of xenon and temperature with spacial dependence for cylindrical geometry were investigated by G. S. Lellouche (26). An analog computer simulation of the xenon instability problem was made on the basis of linearized equations by R. M. Pearce (36), while a one group,

two dimensional neutron flux distribution for a xenon dependent buckling was simulated on a digital computer for the CANDU Reactor by W. M. Barss (2).

Besides the authors mentioned many others have contributed to the repertory of information on the influences of xenon on reactor kinetics. Of particular interest are the contributions of T. A. Welton, V. M. Popov, H. B. Smets, E. P. Gyftopoulos, J. Devooght, J. Lewins, J. G. Margolis, S. Kaplan, G. L. Gyorey, L. M. Shotkin, F. H. Abernathy, K. Mochizuki, A. Takeda, M. A. Schultz, B. H. Noordhoff, and others.

In the light of this vast literature available, it would be satisfying to write that presently the kinetic behavior of thermal reactors under the influence of xenon is well known and adequately described by analytic means. However, such is not the case (4, 49). Indeed, there exist many areas of understanding of the influence of xenon on reactor kinetics which are grey and perhaps black. The reasons for this "ignorance" are manifold and are typical of the difficulties which arise in any sophisticated study of reactor kinetics. The prime source of difficulty encountered in the xenon problem is that the set of equations normally imposed to describe the kinetic behavior of the flux and xenon in a reactor system is nonlinear, and the possibility of obtaining analytic solutions of the generalized problem seems remote. The nature of the nonlinearity in the describing equations is, specifically, (neglecting temperature effects, iodine absorption, and imposing the "standard" approximations and simplifications) the occurrence of the term accounting for the rate of thermal neutron absorption by xenon in both the xenon equation and the equation(s) for the neutron flux.

Because of the nonlinear nature of the describing equations, the bulk of analytic treatment has been restricted to the analysis of the behavior of some linear approximation, usually formulated via first order perturbation theory, of the original equations. The efficacy of analytic study based on linear approximations to the describing equations is mathematically known and demonstratable (13, 15, 28, 30). However, for practical application, it often appears that the only way to obtain actual, quantitative knowledge of the limitations inherent in the linearized analysis is to obtain solutions for the nonlinear equations and compare results found from linear analysis. Unfortunately, this is again not always practical or even possible. Of course, for a given situation an analytic linearized study can be compared with numerical results obtained from the nonlinear describing equations found by computer simulation.

DESCRIPTION AND SIMPLE EXAMPLE OF XENON INDUCED INSTABILITY

Perhaps the most interesting xenon induced instability is that of xenon induced flux oscillations. To physically describe the mechanics of such an oscillation consider a certain region of a large, high flux, thermal reactor in which equilibrium values of the flux, xenon, and iodine have been established. Then further suppose that the flux is perturbed for some reason (i.e., either slightly increased or decreased) within the region. This perturbation (considered positive for this example) increases the absorption of xenon. Also, an increase in the production of xenon from prompt fission occurs; but the total output or destruction of xenon is greater than the total input or production, and the resulting effect is that the net concentration of xenon within the region decreases. Now a decrease of the xenon concentration from its equilibrium value results in an increase of neutron multiplication. Thus, if no reduction in the multiplication occurs due to other factors (e.g., movement of a control rod, a negative multiplication feedback due to temperature increase, etc.), then the flux is increased, and the original flux perturbation begins to grow. The flux continues to increase until the production of xenon via the decay of iodine, which is increasing with the increasing flux, surpasses the destruction rate of xenon and returns the xenon to its equilibrium value. At this point the oscillation of the flux is at a maximum. The decay of iodine continues to produce more xenon, and the net concentration of the xenon within the region increases from its equilibrium value. Now an increase in the xenon concentration from its equilibrium value results in a decrease of neutron multiplication. So the flux within the region begins to decrease. The decreasing flux decreases the destruction of xenon by neutron absorption, and the xenon is further increased. The flux continues to decrease because of the decrease in neutron multiplication until the reduced production of xenon from the decay of iodine returns the xenon to its equilibrium value. The flux then assumes a minimum value. The cycle now repeats itself producing an oscillation of the flux. The amplitude of the flux oscillation will either grow in time, be sustained, or decrease in time depending upon the net regenerative changes which occur in the neutron multiplication.

Figure 1 gives a pictorial representation of the behavior of a simplified, representative set of space independent kinetic equations for high flux levels under a xenon induced flux oscillation of the fundamental mode. Figure 2 gives a typical analog computer plot of the simulated time response of the flux, xenon, and iodine exemplifying a xenon induced flux oscillation.

Figure 1. Behavior of the Space Independent Kinetic Equations for a Typical Flux Oscillation Induced by Xenon.

Simplified Equations

$$\frac{d\phi}{dt} = \rho_0 (1-x) \phi = \dot{\phi} \quad \phi \text{ is the normalized flux}$$

$$\frac{dx}{dt} = \lambda_0 (\gamma_x \phi + \gamma_i i - x\phi) = \dot{x} \quad x \text{ is the normalized xenon-135 concentration}$$

$$\frac{di}{dt} = \lambda_i (\phi - i) = \dot{i} \quad i \text{ is the normalized iodine-135 concentration}$$

Note: See Chapter II for complete definition of terms

<p><u>1 - Equilibrium</u></p> <p>$\phi = 1 \quad \dot{\phi} = 0$ $x = 1 \quad \dot{x} = 0$ $i = 1 \quad \dot{i} = 0$</p>	<p><u>2 - Perturbed Flux</u></p> <p>$\phi > 1 \quad \dot{\phi} > 0$ $x = 1 \quad \dot{x} < 0$ $i = 1 \quad \dot{i} > 0$</p>
<p><u>3 - Flux Increasing</u></p> <p>$\phi > 1 \quad \dot{\phi} > 0$ $x < 1 \quad \dot{x} > 0$ $i > 1 \quad \dot{i} > 0$</p>	<p><u>4 - Maximum Flux</u></p> <p>$\phi > 1 \quad \dot{\phi} = 0$ $x = 1 \quad \dot{x} > 0$ $i > 1 \quad \dot{i} > 0$</p>
<p><u>5 - Flux Decreasing</u></p> <p>$\phi > 1 \quad \dot{\phi} < 0$ $x > 1 \quad \dot{x} > 0$ $i > 1 \quad \dot{i} < 0$</p>	<p><u>6 - Flux Through Equilibrium</u></p> <p>$\phi = 1 \quad \dot{\phi} < 0$ $x > 1 \quad \dot{x} > 0$ $i > 1 \quad \dot{i} < 0$</p>
<p><u>7 - Flux Decreasing</u></p> <p>$\phi < 1 \quad \dot{\phi} < 0$ $x > 1 \quad \dot{x} < 0$ $i < 1 \quad \dot{i} < 0$</p>	<p><u>8 - Minimum Flux</u></p> <p>$\phi < 1 \quad \dot{\phi} = 0$ $x = 1 \quad \dot{x} < 0$ $i < 1 \quad \dot{i} < 0$</p>
<p><u>9 - Flux Increasing</u></p> <p>$\phi < 1 \quad \dot{\phi} > 0$ $x < 1 \quad \dot{x} < 0$ $i < 1 \quad \dot{i} > 0$</p>	<p><u>10 - Flux Through Equilibrium</u></p> <p>$\phi = 1 \quad \dot{\phi} > 0$ $x < 1 \quad \dot{x} > 0$ $i < 1 \quad \dot{i} > 0$</p>

Figure 2. Typical Analog Computer Simulation of a Space Independent Xenon Induced High Flux Oscillation.

In principle, it may be said that xenon induced flux instabilities are attributable to the large absorption cross section of xenon for thermal neutrons and the time delay inherent in the production of xenon from the decay of iodine.

SCOPE

Within the scope of this introduction it is appropriate to define several pertinent terms from the nomenclature of reactor kinetics. A steady state or equilibrium condition of a reactor system is defined as some combination of values of the dependent variables of the system for which all of the dependent variables of the system remain constant in time. This equilibrium condition is said to be stable if a small departure from equilibrium results in a return to equilibrium. Otherwise, the equilibrium condition is unstable. Furthermore, "linear stability" or "stability in the small" implies that the system is stable if the departure of any of the dependent variables from equilibrium is arbitrarily small, while "nonlinear stability" or "stability in the large" implies that the departure from equilibrium is sufficiently large to make nonlinear effects important. Quoting from T. A. Welton (49, p. 377), "Nonlinear stability implies linear stability and linear instability implies nonlinear instability. Linear stability can be associated with either nonlinear stability or instability and nonlinear instability similarly permits no conclusion on linear stability." A precise mathematical definition of "stability" will be given later.

Aside from external control (40), the most significant factors acting to stabilize the neutron flux within a nuclear reactor are the loss of neutrons from

the fissionable core by leakage and the existence of a small fraction of the total neutron population, most of which are released during fission, being released by certain fission products some time after fission has occurred. Thus, the "delayed neutrons" serve to considerably increase the effective neutron "lifetime" or "regeneration time" and result in appreciably slowing down the kinetic response of the neutron flux.

Also, certain other factors, such as those temperature effects which tend to oppose changes in neutron multiplication, can be stabilizing. However, this report will be chiefly concerned with determining the kinetic behavior of the neutron flux as a result of the interaction of neutron leakage and delayed neutrons with the xenon-135 concentration, which appears as a consequence of fission and tends to destabilize the flux.

A brief outline of this study is as follows: In Chapter I an introduction to the study of xenon-135 influence on reactor kinetics is given. A historical and literature review provides a background and "state of the art" description of the problem. Then, a simple example of a xenon induced flux oscillation is presented with an account of the "physical mechanisms" responsible for producing the oscillation. Finally, the scope of the report is presented.

In Chapter II the space independent kinetic behavior of a reactor model under the influence of xenon is examined. Since leakage is essentially a spacial phenomenon, this chapter examines the kinetic behavior as determined by the delayed neutrons and the xenon dependence. The "linear stability criterion" is determined by imposing first order perturbation theory upon the pertinent equations. For the assumption of high flux level the kinetic equations are examined, and several important relationships are formulated. Also, the applicability of the linear stability criterion to the nonlinear kinetic equations is examined by imposing phase plane analysis of the kinetic equations at high flux levels. Finally, analog and digital computer simulation of the kinetic equations is made and the time response graphically presented.

Chapter III contains an examination and comparison of diffusion theory and transport theory with xenon dependence. The dependence of the diffusion coefficient with xenon concentration and the expressions for the material buckling that result from diffusion theory and transport theory with xenon dependence are determined. Finally, the equilibrium distribution of the flux for

a bare critical slab model using the P_1 approximations to the set of coupled transport equations is found.

The space averaged kinetics using two group diffusion theory is examined in Chapter IV with attention to possible xenon induced flux harmonic instability. The criterion for "stability in the small" (i.e., linear stability) is established.

Chapter V contains an examination of the space and time dependent, mono-energy group kinetics with xenon dependence. The chapter begins with the presentation of the pertinent kinetic equations and an equilibrium analysis. Linear approximations of the kinetic equations are determined by imposing first order perturbation theory. These resulting "linear equations" are subjected to modal analysis, and the criteria for stability are established. For the assumption of high flux level or flat flux distribution the linear Laplace Transformed kinetic equations are investigated employing root locus techniques and the behavior of the time solutions studied. Next in the chapter, sufficient conditions for stability are developed by imposing certain requirements upon the flux derivative for positive or negative changes from equilibrium in the flux. Then, for the assumption of high flux levels, it is found possible to solve the kinetic equations and determine the characteristic roots from the characteristic equation, which determine the nature of the time behavior. The complete time and space solution for a high flux, cylindrical core reactor model is found along with the buckling expression used in the stability relationship previously found. The stability criteria for a cylindrical, a rectangular parallelepiped, and a spherical core reactor model are given. Finally, to complete the chapter, the space and time dependent, mono-energy group kinetic equations are simulated on an IBM 7040 Digital Computer.

Chapter VI presents an examination of the xenon dependent reactor kinetic equations employing multi-energy group, diffusion theory to describe the space and time behavior of the flux. The treatment of Randall and St. John is extended to two group diffusion theory, and the stability criteria are established. Comparison of the stability criteria is made between the extended Randall and St. John treatment and the space averaged kinetic treatment given in Chapter IV. Under the assumption of high flux levels, the multi-group kinetic equations are examined. The stability criteria at high flux levels for two and three energy group kinetics are determined, and the extension to multi-group analysis is indicated. Next in the chapter, the two energy group kinetic equations are simulated in time and

one dimension on an IBM 7040 Digital Computer. Graphical presentation of the resulting computer simulation is given for typical parameter values. Finally, sufficient conditions for stability are developed for the two energy group kinetic equations.

In Chapter VII an application of the previously found stability criteria is made to the Advanced Test Reactor, which is currently under construction for the U. S. Atomic Energy Commission and will be operated by Phillips Petroleum Company at the National Reactor Testing Station at Idaho. Specifically, the axial stability of the neutron flux is studied under the influence of xenon induced instabilities. Results of experimental investigation of the axial flux distribution found from a core "mock-up" of the Advanced Test Reactor (ATR) are employed in the digital computer simulation (developed in Chapter VI) of the two energy group kinetic equations to examine the stability of the axial flux. The results found from computer simulation are then compared to those results found from the previously determined analytic stability criteria.

In Chapter VIII the findings and conclusions of the report are given.

II - SPACE INDEPENDENT REACTOR KINETICS WITH XENON DEPENDENCE

The study of reactor stability is primarily concerned with determining whether the reactor power and its spacial distribution are sufficiently steady and stable to permit operation within the scope of the reactor design and its control system.

INTRODUCTION

For familiarization with the particular problem of xenon-135 induced instabilities of the neutron flux, which may exist in certain thermal reactors, consider a reactor model in which the neutron flux's spacial dependence is averaged over the reactor core at each point in time. Thus, the behavior of this average flux becomes a function of time only since the spacial dependence is eliminated. Characterization of the spacially distributed flux by some appropriate average value is often a reasonable description of the fundamental mode of the flux, and many of the results and conclusions found from the time behavior of this average flux may be successfully alluded to the more realistic spacially dependent flux.

In addition to the previous assumption that the flux is spacially averaged at each point in time, it is further assumed that the time behavior of this average flux is described by one group diffusion theory. Then the equations which describe the time behavior of the spacially averaged thermal flux, the xenon-135 concentration, and the iodine-135 concentration are the following:

$$\frac{1}{v} \frac{d\phi}{dt} = (v\Sigma_f - \Sigma_a - \sigma_x X)\phi, \quad (2-1.1)$$

$$\frac{dX}{dt} = \bar{\gamma}_x \Sigma_f \phi + \lambda_i I - \sigma_x X \phi - \lambda_x X, \quad (2-1.2)$$

$$\frac{dI}{dt} = \bar{\gamma}_i \Sigma_f \phi - \lambda_i I, \quad (2-1.3)$$

where

$\phi(t)$ is the spacially averaged neutron flux (nts/cm²sec),
 $X(t)$ is the concentration of xenon-135 (atoms/cm³),
 $I(t)$ is the concentration of iodine-135 (atoms/cm³),
 v is the average neutron velocity (cm/sec),
 ν is the total average neutron yield per fuel fussion,
 Σ_f is the macroscopic cross section for fission (1/cm),
 Σ_a^* is the total effective macroscopic cross section for absorption with leakage excluding xenon (1/cm),
 σ_x is the microscopic absorption cross section for xenon-135 (cm²),
 $\bar{\gamma}_x$ is the average fission yield of xenon-135,
 $\bar{\gamma}_i$ is the average fission yield of iodine-135 and tellurium-135,
 λ_x is the decay constant for xenon-135 decaying to cesium-135 (1/sec),
 λ_i is the decay constant for iodine-135 decaying to xenon-135 (1/sec).

The xenon-135 concentration in nuclear reactors results chiefly from the production of the fission products tellurium-135, iodine-135, and the direct fission yield of xenon-135. These fission products undergo the following decay scheme (24).

* Σ_a may be defined as $\Sigma_a = \Sigma_a'(1 + M^2 B^2)$ where Σ_a' is the macroscopic absorption cross section (excluding xenon), M^2 is the migration area, and B^2 is the geometric buckling.

Since tellurium-135 decays rapidly and cesium-135 is relatively stable, the decay scheme of chief interest is that of iodine decay to xenon. Thus, it will be assumed that since the decay time for tellurium is negligible compared to the 6.7 hr life of iodine, the fission yield of tellurium appears as a prompt fission yield of iodine. Also, the isomeric transition of excited xenon to 9.2 hr ground state xenon will be neglected, and the assumption made that all iodine decays to 9.2 hr xenon. See Chernick (5) for a discussion of the effect of neglecting this isomeric state of excited xenon.

Thus, equations (2-1) describing the reactor model consider only two sources of xenon; that of prompt iodine production through fission and its subsequent decay into xenon, and the prompt fission yield of xenon. Also, the absorption of neutrons by iodine-135 is neglected since its absorption cross section is small (10), and delayed neutrons are not considered except for their effect upon the average neutron lifetime (5, 14, 20).

Possible nontrivial steady state or stationary values exist for the dependent variables of equations (2-1) when

$$\phi_o = \frac{\lambda_x (\nu \Sigma_f - \Sigma_a)}{\sigma_x (\bar{\gamma} \Sigma_f + \Sigma_a - \nu \Sigma_f)} = \frac{\lambda_x X_o}{\bar{\gamma} \Sigma_f - \sigma_x X_o}, \quad (2-2.1)$$

$$X_o = \frac{\nu \Sigma_f - \Sigma_a}{\sigma_x} = \frac{\bar{\gamma} \Sigma_f \phi_o}{\sigma_x \phi_o + \lambda_x}, \quad (2-2.2)$$

$$I_o = \frac{\bar{\gamma}_i \Sigma_f \lambda_x (\nu \Sigma_f - \Sigma_a)}{\lambda_i \sigma_x (\bar{\gamma} \Sigma_f + \Sigma_a - \nu \Sigma_f)} = \frac{\bar{\gamma}_i \Sigma_f \phi_o}{\lambda_i}, \quad (2-2.3)$$

where

$$\bar{\gamma} = \bar{\gamma}_i + \bar{\gamma}_x.$$

Physically realistic fluxes (i.e., positive flux values) occur when

$$\nu \Sigma_f > \Sigma_a > (\nu - \bar{\gamma}) \Sigma_f$$

$$1 < k_{\text{eff}} < \frac{1}{1 - \frac{\lambda}{\nu}}$$

where

$$k_{\text{eff}} = \frac{\nu \Sigma_f}{\Sigma_a}$$

For convenience, the dependent variables will be redefined as follows so they will be dimensionless and assume their steady value when equal to unity.

$$\phi(t) = \phi_0 \phi(t) \quad (2-3.1)$$

$$X(t) = X_0 x(t) \quad (2-3.2)$$

$$I(t) = I_0 i(t) \quad (2-3.3)$$

Equations (2-1) become under the transformation of variables given in equations (2-3) the following:

$$\frac{d\phi}{dt} = \rho_0 (1 - x) \phi, \quad (2-4.1)$$

$$\frac{dx}{dt} = (\lambda_0 + \lambda_x)(\gamma_x \phi + \gamma_i i) - (\lambda_0 \phi + \lambda_x) x, \quad (2-4.2)$$

$$\frac{di}{dt} = \lambda_i (\phi - i), \quad (2-4.3)$$

where

$$\rho_0 = \nu \sigma_x X_0 = \frac{1}{\ell} \left(\frac{\nu \Sigma_f}{\Sigma_a} - 1 \right) = \frac{1}{\ell} \left[\frac{1}{\frac{\nu}{\lambda} \left(1 + \frac{\lambda_x}{\lambda_0} \right) - 1} \right], \quad (2-4.4)$$

$\ell = 1/\nu \Sigma_a$ (the effective neutron lifetime excluding xenon),

$$\lambda_o = \sigma_x \phi_o,$$

$$\gamma_x = \frac{\bar{\gamma}_x}{\bar{\gamma}},$$

$$\gamma_i = \frac{\bar{\gamma}_i}{\bar{\gamma}},$$

$$\gamma_i + \gamma_x = 1.$$

A nonlinear integral equation for flux may be obtained from equations (2-4) by the following: From equation (2-4.1) solving for x gives

$$x = 1 - \frac{1}{\rho_o} \frac{d \ln \phi}{dt} \quad (2-5)$$

and equation (2-4.3) may be integrated, assuming as initial condition that $i(o) = 1$, to give

$$i = \lambda_i \int_0^t e^{\lambda_i(\xi-t)} \phi(\xi) d\xi + e^{-\lambda_i t}.$$

Now substituting these equations for the iodine and xenon into equation (2-4.2) gives

$$\begin{aligned} -\frac{1}{\rho_o} \frac{d^2 \ln \phi}{dt^2} &= (\lambda_o + \lambda_x) \left[\gamma_x \phi + \gamma_i \lambda_i \int_0^t e^{\lambda_i(\xi-t)} \phi d\xi + \gamma_i e^{-\lambda_i t} \right] \\ &- (\lambda_o \phi + \lambda_x) \left(1 - \frac{1}{\rho_o} \frac{d \ln \phi}{dt} \right) \end{aligned}$$

which may be integrated twice over the interval of time, 0 to t , to yield

$$-\frac{1}{\rho_o} \ln \phi = \int_0^t f(t, \xi) \phi(\xi) d\xi + \frac{\lambda_x}{\rho_o} \int_0^t \ln \phi(\xi) d\xi + g(t) \quad (2-6)$$

where

$$f(t, \xi) = \lambda_x(t - \xi) - \gamma_i \frac{(\lambda_o + \lambda_x)}{\lambda_i} \left[1 - e^{-\lambda_i(t - \xi)} \right] + \frac{\lambda_o}{\rho_o},$$

$$g(t) = \frac{\gamma_i}{\lambda_i} (\lambda_o + \lambda_x) \left[\frac{1}{\lambda_i} (e^{-\lambda_i t} - 1) + t \right] - \frac{\lambda_o}{\rho_o} t - \frac{\lambda_x}{2} t^2 + \frac{t}{\rho_o} \left\{ \frac{d \ln \phi}{dt} \right\}_{t=0}$$

Note in equation (2-6) that a possible approximation technique would be to expand the logarithm term as follows:

$$\ln \phi(t) \approx \phi(t) - 1$$

which is valid for ϕ values sufficiently close to unity. With this expansion introduced into equation (2-6), a linear equation, specifically a Volterra Integral Equation of the Second Kind (19), results.

APPROXIMATION BY FIRST ORDER PERTURBATION THEORY

As equations (2-4) and (2-6) stand, they are difficult if not impossible to solve analytically. However, approximation techniques exist which permit quasi-solutions to be found.

As an approximation technique, a first order perturbation solution around steady state can be determined by letting

$$\phi(t) = 1 + \delta\phi(t), \quad (2-7.1)$$

$$x(t) = 1 + \delta x(t), \quad (2-7.2)$$

$$i(t) = 1 + \delta i(t), \quad (2-7.3)$$

and substituting equations (2-7) into equations (2-4) and neglecting products of perturbations as being of second order gives

$$\frac{d\delta\phi}{dt} = -\rho_o \delta x, \quad (2-8.1)$$

$$\frac{d\delta x}{dt} = (\lambda_o + \lambda_x)(\gamma_x \delta\phi + \gamma_i \delta i - \delta x) - \lambda_o \delta\phi, \quad (2-8.2)$$

$$\frac{d\delta i}{dt} = \lambda_i (\delta\phi - \delta i). \quad (2-8.3)$$

Equations (2-8) are now linear and amenable to Laplace Transformation techniques which yield in matrix form

$$\begin{bmatrix} s & \rho_o & 0 \\ \gamma_i \lambda_o - \gamma_x \lambda_x & s + \lambda_o + \lambda_x & -\gamma_i (\lambda_o + \lambda_x) \\ -\lambda_i & 0 & s + \lambda_i \end{bmatrix} \begin{bmatrix} \delta\bar{\phi}(s) \\ \delta\bar{x}(s) \\ \delta\bar{i}(s) \end{bmatrix} = 0 \quad (2-9)$$

where initial values are assumed to be, at zero time, the following:

$$\delta\phi(0) = \delta x(0) = \delta i(0) = 0.$$

Now matrix equation (2-9) possesses a nontrivial solution for the dependent variables if and only if the coefficient matrix is singular (44), and singularity is satisfied only if the determinant of the coefficient matrix vanishes. Expanding the coefficient determinant of equation (2-9) gives

$$0 = s^3 + A_1 s^2 + A_2 s + A_3 \quad (2-10)$$

where

$$A_1 = \lambda_o + \lambda_x + \lambda_i,$$

$$A_2 = \lambda_i (\lambda_o + \lambda_x) - \rho_o (\gamma_i \lambda_o - \gamma_x \lambda_x),$$

$$A_3 = \rho_o \lambda_i \lambda_x.$$

The three roots of the characteristic equation (2-10), which are, in general, complex, determine the nature of the time behavior of the perturbed flux, xenon, and iodine. For these solutions to be bounded for all positive time, it is necessary that the real part of each of the roots of equation (2-10) be negative. This condition may be examined without explicitly finding the roots by employing the Routh-Hurwitz Stability Criteria (25, 51) which states that for any polynomial

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n = 0$$

the real part of each root is negative if and only if each determinant, D_i , is positive where

$$D_0 = a_0 > 0,$$

$$D_1 = a_1 > 0,$$

$$D_2 = \begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix} > 0,$$

$$D_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_2 & a_1 & a_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & \dots & \dots & \dots & a_n \end{vmatrix} > 0.$$

Thus, for boundness of the time solutions of equations (2-8), it is necessary that the following four conditions be satisfied.

Condition I - $A_1 > 0$

So

$$\lambda_0 + \lambda_x + \lambda_i > 0.$$

But this is always satisfied for physically real fluxes.

Condition II - $A_2 > 0$

So

$$\lambda_i + \rho_0 \frac{\lambda_x}{\lambda_0 + \lambda_x} > \gamma_i \rho_0.$$

Condition III - $A_3 > 0$

$$\rho_o \lambda_i \lambda_x > 0.$$

But this is always satisfied for $\rho_o > 0$ (i.e., for $\phi_o > 0$).

Condition IV - $A_1 A_2 > A_3$

So

$$(\lambda_o + \lambda_x + \lambda_i) \lambda_i + \gamma_x \rho_o \lambda_x > \gamma_i \rho_o (\lambda_o + \lambda_i)$$

and rearranging

$$\lambda_i + \rho_o \frac{\lambda_x}{\lambda_o + \lambda_x + \lambda_i} > \gamma_i \rho_o.$$

Now the only conditions which are not trivially satisfied are conditions II and IV. However, it is obvious that if condition IV is satisfied, then so is condition II. Thus, the requirement for boundness of solutions of equations (2-8) is condition IV. Significantly, " ρ_o " is the only parameter, in general, which can be varied in this reactor model. The decay constants " λ_x " and " λ_i " and the fission yields for a given fuel " γ_i ", " γ_x ", are natural constants not subject to any significant variation. Thus, the value of " ρ_o " which provides stability in the perturbation sense is

$$\rho_o < \frac{\lambda_i}{\gamma_i - \frac{\lambda_x}{\lambda_o + \lambda_x + \lambda_i}}. \quad (2-11)$$

A graph for inequality (2-11) is given in Figure 3 for a set of typical reactor parameters.

If a xenon free effective multiplication factor is defined as

$$k_{eff} = \frac{v \Sigma_f}{\Sigma_a}$$

then ρ_o becomes

$$\rho_o = \frac{k_{eff} - 1}{\beta}$$

and the requirement for stability is then

Figure 3. Linear Stability Criterion for Space Independent Kinetics.

$$k_{\text{eff}}^{-1} < \frac{\lambda_i \ell}{\gamma_i - \frac{\lambda_x}{\lambda_0 + \lambda_x + \lambda_i}}. \quad (2-12)$$

It can be seen that the reactor is stable for all positive values of the flux if

$$k_{\text{eff}}^{-1} \leq \frac{\lambda_i \ell}{\gamma_i}. \quad (2-13)$$

However, both stability requirements (2-12) and (2-13) are extremely restrictive and cannot always be satisfied in general. Thus, one may conclude that the model under investigation is inherently unstable in the small perturbation sense, and external control is required to maintain stability. The fact that the fundamental mode of a high flux thermal reactor can be unstable due to the influence of xenon and that external control is required should not be surprising, however. The control systems of most reactors are specifically designed to control the fundamental mode of the flux, which is essentially a measure of the power output of the reactor.

For the situation in which no external control upon ρ_0 is imposed then ρ_0 is given by

$$\rho_0 = \frac{1}{\ell} \left[\frac{1}{\frac{\nu}{\bar{\gamma}} \left(1 + \frac{\lambda_x}{\lambda_0} \right) - 1} \right] \quad (2-14)$$

and combining equations (2-14) and stability requirement (2-11) gives

$$\gamma_i \lambda_o^2 + \lambda_o (\gamma_i \lambda_i - \gamma_x \lambda_x) < \ell \lambda_i \left\{ \left(\frac{\nu}{\bar{\gamma}} - 1 \right) \lambda_o^2 + \left[\left(\frac{\nu}{\bar{\gamma}} - 1 \right) \lambda_x + \lambda_i + \frac{\nu}{\bar{\gamma}} \lambda_x \right] \lambda_o + \frac{\nu}{\bar{\gamma}} \lambda_x (\lambda_x + \lambda_i) \right\}. \quad (2-15)$$

In general, it is found that the effective neutron lifetime, ℓ , is less than one second so inequality (2-15) may be well approximated by the following:

$$\lambda_o < \frac{\nu \ell \lambda_i \lambda_x (\lambda_x + \lambda_i)}{(\bar{\gamma}_i \lambda_i - \bar{\gamma}_x \lambda_x)} \quad (2-16.1)$$

or

$$\phi_o < \frac{\nu \ell \lambda_i \lambda_x (\lambda_x + \lambda_i)}{\sigma_x (\bar{\gamma}_i \lambda_i - \bar{\gamma}_x \lambda_x)}. \quad (2-16.2)$$

Expressions (2-16) give the value of the flux level below which the reactor model under investigation is stable in the small perturbation sense without any external control being imposed. For an effective neutron lifetime of 0.1 sec, and the other parameters the same as those given in Figure (3), the threshold flux level is 1.48×10^9 *.

HIGH FLUX APPROXIMATION

Another approximation technique, which is quite satisfactory at high flux levels ($\phi_o > 10^{13}$), is to neglect the decay of xenon compared to xenon burnout (i.e., assume $\lambda_x X \ll \sigma_x X \phi$). Then effectively setting $\lambda_x = 0$, equations (2-4) become

$$\frac{d\phi}{dt} = \rho_o (1-x) \phi, \quad (2-17.1)$$

$$\frac{dx}{dt} = \lambda_o (\gamma_x \phi + \gamma_i i - x\phi), \quad (2-17.2)$$

* All flux values given in this report will be understood to be in the standard units (neutron/centimeter squared-second).

$$\frac{di}{dt} = \lambda_i (\phi - i). \quad (2-17.3)$$

where nontrivial stationary values are (see equations (2-2))

ϕ_o is arbitrary but large,

$$X_o = \frac{\bar{\gamma} \Sigma_f}{\sigma_x},$$

$$I_o = \frac{\bar{\gamma}_i \Sigma_f \phi_o}{\lambda_i},$$

and

$$\rho_o = v \sigma_x X_o = v \bar{\gamma} \Sigma_f = \frac{\bar{\gamma} \Sigma_f}{\ell \Sigma_a} = \frac{\bar{\gamma}}{\ell(v - \bar{\gamma})}.$$

To provide a basis for later comparison, a first order perturbation analysis of equations (2-17) leads to the following matrix equation for the Laplace Transformed dependent variables.

$$\begin{bmatrix} s & \rho_o & 0 \\ \gamma_i \lambda_o & s + \lambda_o & -\gamma_i \lambda_i \\ -\lambda_i & 0 & s + \lambda_i \end{bmatrix} \begin{bmatrix} \delta \bar{\phi}(s) \\ \delta \bar{x}(s) \\ \delta \bar{i}(s) \end{bmatrix} = 0$$

Now as before for nontrivial solutions the determinant of the coefficient matrix must vanish so

$$s \left[s^2 + (\lambda_o + \lambda_i) s + \lambda_o \lambda_i - \gamma_i \rho_o \lambda_o \right] = 0 \quad (2-18.1)$$

and the roots of this polynomial are

$$s_1 = 0, \quad (2-18.2)$$

$$s_2 = -\frac{(\lambda_o + \lambda_i)}{2} + \sqrt{\left(\frac{\lambda_o - \lambda_i}{2}\right)^2 + \gamma_i \rho_o \lambda_o}, \quad (2-18.3)$$

$$s_3 = -\frac{(\lambda_o + \lambda_i)}{2} - \sqrt{\left(\frac{\lambda_o - \lambda_i}{2}\right)^2 + \gamma_i \rho_o \lambda_o}. \quad (2-18.4)$$

The general solution for $\delta\phi(t)$ in time becomes

$$\delta\phi(t) = C_1 + C_2 e^{s_2 t} + C_3 e^{s_3 t} \quad (2-19.1)$$

where the C's are arbitrary constants.

The perturbed flux in equation (2-19.1) will become unbounded as $t \rightarrow \infty$ unless $C_2 = 0$ or $s_2 < 0$.

Now $s_2 < 0$ if the following is satisfied.

$$\lambda_i > \gamma_i \rho_o = \frac{\bar{\gamma} \Sigma_f}{\ell \Sigma_a} = \frac{\bar{\gamma}}{\ell(v - \bar{\gamma})} \quad (2-19.2)$$

This requirement for stability is the same as that in expression (2-11) if λ_x is set to zero.

A singular solution for $\delta\phi(t)$ for the initial conditions that

$$\begin{aligned} \delta\phi(0) &= \delta\phi_o, \\ \delta x(0) &= \delta x_o, \\ \delta i(0) &= \delta i_o, \end{aligned}$$

is the following:

$$\begin{aligned} \delta\phi(t) &= \frac{1}{s_2 s_3} \left[\lambda_o \lambda_i \delta\phi_o - \gamma_i \lambda_o \rho_o \delta i_o - \lambda_i \rho_o \delta x_o \right] \\ &+ \frac{1}{s_2 (s_2 - s_3)} \left[(s_2 + \lambda_o)(s_2 + \lambda_i) \delta\phi_o - \gamma_i \lambda_o \rho_o \delta i_o - (s_2 + \lambda_i) \rho_o \delta x_o \right] e^{s_2 t} \\ &+ \frac{1}{s_3 (s_3 - s_2)} \left[(s_3 + \lambda_o)(s_3 + \lambda_i) \delta\phi_o - \gamma_i \lambda_o \rho_o \delta i_o - (s_3 + \lambda_i) \rho_o \delta x_o \right] e^{s_3 t} \end{aligned}$$

For the condition that

$$\lambda_i = \gamma_i \rho_o$$

the roots of the characteristic equation become

$$s_1 = 0 \text{ (double root)}$$

and

$$s_2 = -(\lambda_o + \lambda_i)$$

and the general solution for $\delta\phi(t)$ is then

$$\delta\phi(t) = D_1 + D_2 t + D_3 e^{-(\lambda_o + \lambda_i) t}$$

which is obviously unbounded as $t \rightarrow \infty$ (unless, of course, $D_2 = 0$).

Now returning to equations (2-17), if equation (2-17.3) is multiplied by " γ_i " and then all of equations (2-17) are added together forming the following equality

$$\begin{aligned} \frac{1}{\rho_o} \frac{d\phi}{dt} - (1 - x) \phi &= \frac{1}{\lambda_o} \frac{dx}{dt} - \gamma_x \phi - \gamma_i i + x\phi \\ &+ \frac{\gamma_i}{\lambda_i} \frac{di}{dt} - \gamma_i \phi + \gamma_i i \end{aligned} \quad (2-20)$$

and making the obvious cancellations, equation (2-20) becomes

$$\frac{1}{\rho_o} \frac{d\phi}{dt} = \frac{1}{\lambda_o} \frac{dx}{dt} + \frac{\gamma_i}{\lambda_i} \frac{di}{dt} \quad (2-21)$$

which can be integrated in time to give

$$\frac{1}{\rho_o} (\phi - 1) = \frac{1}{\lambda_o} (x - 1) + \frac{\gamma_i}{\lambda_i} (i - 1) \quad (2-22)$$

where initial conditions are assumed to be at zero time

$$\phi(0) = x(0) = i(0) = 1. \quad (2-23)$$

Now for convenience the dependent variables will be defined to reflect their departure from stationary values as follows: (This variable transformation is not

to be misconstrued with the assumptions of perturbation theory employed earlier in which the perturbations around stationary values were assumed to be small and products of perturbations were neglected.)

$$\phi(t) = 1 + \delta\phi(t) \quad (2-23.1)$$

$$x(t) = 1 + \delta x(t) \quad (2-23.2)$$

$$i(t) = 1 + \delta i(t) \quad (2-23.3)$$

Thus, when $\delta\phi = \delta x = \delta i = 0$, stationary or steady state conditions exist. Equations (2-21) and (2-22) become under the transformation of variables in equations(2-23) as follows:

$$\frac{1}{\rho_0} d\delta\phi = \frac{1}{\lambda_0} d\delta x + \frac{\gamma_1}{\lambda_1} d\delta i \quad (2-24)$$

and

$$\frac{1}{\rho_0} \delta\phi = \frac{1}{\lambda_0} \delta x + \frac{\gamma_1}{\lambda_1} \delta i. \quad (2-25)$$

Now equation (2-25) plots as a flat plane in three dimensional space as shown in Figure 4.

The time solution of equation (2-25) with initial conditions given by equation (2-23) must lie on the plane in Figure 4. Any solutions of equation (2-21) with given initial conditions will lie on planes parallel to the plane in Figure 4.

Figure 4. Phase Space of the Space Independent Kinetic Equations at High Flux.

If the transformations of variables found in equations (2-23) are imposed upon equations (2-17) there results

$$\frac{d\delta\phi}{dt} = -\rho_0 \delta x (1 + \delta\phi), \quad (2-26.1)$$

$$\frac{d\delta x}{dt} = \lambda_0 (\gamma_1 \delta i - \gamma_1 \delta\phi - \delta x - \delta\phi \delta x), \quad (2-26.2)$$

$$\frac{d\delta i}{dt} = \lambda_1 (\delta\phi - \delta i). \quad (2-26.3)$$

Furthermore, equation (2-25) may be employed in equations (2-26) to obtain the following set of coupled equations.

$$\frac{d\delta\phi}{dt} = -\rho_o \delta x (1 + \delta\phi), \quad (2-27.1)$$

$$\frac{d\delta x}{dt} = \lambda_i \lambda_o \left[\left(\frac{1}{\rho_o} - \frac{\gamma_i}{\lambda_i} \right) \delta\phi - \left(\frac{1}{\lambda_o} + \frac{1}{\lambda_i} \right) \delta x - \frac{1}{\lambda_i} \delta x \delta\phi \right], \quad (2-27.2)$$

$$\frac{d\delta\phi}{d\delta x} = -\frac{\rho_o}{\lambda_o \lambda_i} \left[\frac{\delta x (1 + \delta\phi)}{\left(\frac{1}{\rho_o} - \frac{\gamma_i}{\lambda_i} \right) \delta\phi - \left(\frac{1}{\lambda_o} + \frac{1}{\lambda_i} \right) \delta x - \frac{1}{\lambda_i} \delta x \delta\phi} \right], \quad (2-27.3)$$

and

$$\frac{d\delta\phi}{dt} = -\rho_o \lambda_o \left(\frac{1}{\rho_o} \delta\phi - \frac{\gamma_i}{\lambda_i} \delta i \right) (1 + \delta\phi), \quad (2-28.1)$$

$$\frac{d\delta i}{dt} = \lambda_i (\delta\phi - \delta i), \quad (2-28.2)$$

$$\frac{d\delta\phi}{d\delta i} = -\frac{\rho_o \lambda_o}{\lambda_i} \frac{(1 + \delta\phi) \left(\frac{1}{\rho_o} \delta\phi - \frac{\gamma_i}{\lambda_i} \delta i \right)}{(\delta\phi - \delta i)}, \quad (2-28.3)$$

and

$$\frac{d\delta x}{dt} = \lambda_o \rho_o \left[\left(\frac{\gamma_i}{\rho_o} - \frac{\gamma_i^2}{\lambda_i} \right) \delta i - \delta x \left(\frac{\gamma_i}{\lambda_o} + \frac{1}{\rho_o} + \frac{1}{\lambda_o} \delta x + \frac{\gamma_i}{\lambda_i} \delta i \right) \right], \quad (2-29.1)$$

$$\frac{d\delta i}{dt} = \lambda_i \rho_o \left[\frac{1}{\lambda_o} \delta x + \left(\frac{\gamma_i}{\lambda_i} - \frac{1}{\rho_o} \right) \delta i \right], \quad (2-29.2)$$

$$\frac{d\delta x}{d\delta i} = \frac{\lambda_o}{\lambda_i} \left[\frac{\left(\frac{\gamma_i}{\rho_o} - \frac{\gamma_i^2}{\lambda_i} \right) \delta i - \delta x \left(\frac{\gamma_i}{\lambda_o} + \frac{1}{\rho_o} + \frac{1}{\lambda_o} \delta x + \frac{\gamma_i}{\lambda_i} \delta i \right)}{\frac{1}{\lambda_o} \delta x + \left(\frac{\gamma_i}{\lambda_i} - \frac{1}{\rho_o} \right) \delta i} \right]. \quad (2-29.3)$$

It would be very desirable to obtain solutions to any one of equations (2-27), (2-28), and (2-29) as they stand. Unfortunately, these equations are nonlinear and difficult to solve, in general, without some further approximation technique being employed. However, equations (2-27) through (2-29) are amenable to phase plane analysis and this will be accomplished later.

An interesting approximation technique can be imposed if, in equations (2-26) δx and δi are eliminated to yield the following equation in $\delta\phi$.

$$\frac{d\delta\phi}{dt} = \lambda_o (1 + \delta\phi) \left[\gamma_i \rho_o \int_0^t e^{\lambda_i(\xi-t)} \delta\phi(\xi) d\xi - \delta\phi \right] \quad (2-30)$$

Integrating equation (2-30), assuming $\delta\phi(0) = 0$, the following nonlinear integral equation results. (This same result would have been found by setting $\lambda_x = 0$ in equation (2-6).)

$$\text{Ln}(1+\delta\phi) = \lambda_o \int_0^t \left[\frac{\gamma_i \rho_o}{\lambda_i} \left(1 - e^{-\lambda_i(t-\xi)} \right) - 1 \right] \delta\phi(\xi) d\xi \quad (2-31)$$

Now for $-1 < \delta\phi \leq 1$ the logarithm term can be expanded in a power series, so equation (2-31) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \delta\phi^n}{n} = \lambda_o \int_0^t \left[\frac{\gamma_i \rho_o}{\lambda_i} \left(1 - e^{-\lambda_i(t-\xi)} \right) - 1 \right] \delta\phi(\xi) d\xi. \quad (2-32)$$

An alternate integral equation which can be developed from equation (2-30) is

$$\delta\phi = \lambda_o \int_0^t \left[\frac{\gamma_i \rho_o}{\lambda_i} \left(1 - e^{-\lambda_i(t-\xi)} \right) - \delta\phi - 1 + \gamma_i \rho_o \int_0^{\xi} e^{\lambda_i(\eta-\xi)} \delta\phi d\eta \right] \delta\phi d\xi.$$

However, returning to equation (2-32), for small values of $|\delta\phi|$ only the first term of the expansion of $\text{Ln}(1 + \delta\phi)$ need be retained, and thus a Volterra Integral Equation of the Second Kind results (19), viz.,

$$\delta\phi = \lambda_0 \int_0^t \left[\frac{\gamma_i \rho_c}{\lambda_i} \left(1 - e^{-\lambda_i(t-\xi)} \right) - 1 \right] \delta\phi(\xi) d\xi. \quad (2-33)$$

Equation (2-33) is readily amenable to Laplace Transformation if the Convolution Theorem for Laplace Transforms (51) is employed. The Convolution Theorem states that

$$Lf(t) \quad Lg(t) = L \int_0^t f(t-w) g(w) dw$$

where $Lf(t) = f(s)$ denotes the Laplace transform of $f(t)$. Thus, equation (2-33) becomes

$$\delta\bar{\phi}(s) = \left(\frac{\gamma_i \rho_0}{\lambda_i} - 1 \right) \frac{\lambda_0}{s} \delta\bar{\phi}(s) - \frac{\gamma_i \rho_0}{\lambda_i} \frac{\lambda_0}{(s+\lambda_i)} \delta\bar{\phi}(s)$$

and the characteristic equation is

$$0 = s^2 + (\lambda_i + \lambda_0)s + (\lambda_i - \gamma_i \rho_0)\lambda_0$$

Again imposing the Routh-Hurwitz stability criteria, the requirement for stable solutions of $\delta\phi(t)$ is the following:

$$\lambda_i > \gamma_i \rho_0.$$

This is the same requirement for stability that was found from perturbation theory which produced stability requirement (2-19.2). It is thus obvious that the two apparently different linearization techniques imposed on the same system of equations lead to the same requirement for stable solutions.

Returning now to equations (2-27), (2-28), and (2-29) it is possible to obtain useful stability relationships under various conditions. Indeed, if each group of

coupled equations is considered separately, then for equations (2-27) the following assumptions lead to useful results.

If $\gamma_i \rho_0 = \lambda_i$ then from equation (2-27.3)

$$\frac{d\delta\phi}{d\delta x} = \frac{\lambda_i (1 + \delta\phi)}{\gamma_i [\lambda_i + \lambda_0 (1 + \delta\phi)]} \quad (2-34)$$

and the solution becomes under the initial condition that at zero time $\delta\phi(0) = \delta x(0) = 0$

$$\delta x = \gamma_i \frac{\lambda_0}{\lambda_i} \delta\phi + \gamma_i \ln (1 + \delta\phi). \quad (2-35)$$

Using equation (2-35) to eliminate δx in equation (2-27.1) gives

$$\frac{d\delta\phi}{dt} = - (1 + \delta\phi) \left[\lambda_i \ln (1 + \delta\phi) + \lambda_0 \delta\phi \right]. \quad (2-36)$$

Now equation (2-36) satisfies the following conditions:

1. For $\delta\phi > 0$, $\frac{d\delta\phi}{dt} < 0$;
2. For $\delta\phi = 0$, $\frac{d\delta\phi}{dt} = 0$;
3. For $-1 < \delta\phi < 0$, $\frac{d\delta\phi}{dt} > 0$;
4. For $\delta\phi \rightarrow -1$, $\frac{d\delta\phi}{dt} \rightarrow 0$.

Thus, for the conditions that $\gamma_i \rho_0 = \lambda_i$ and $\delta\phi(0) = \delta x(0) = \delta i(0) = 0$ at zero time, the equilibrium value of the flux is a stable point; and, regardless of the size of a change in $\delta\phi$ from equilibrium, the flux will return to its equilibrium value. This, also, is readily verified from the phase plane plot of equation (2-27.3) to be made later.

Equation (2-36) can be reduced to quadratures as indicated.

$$\int_0^{\delta\phi} \frac{d\delta\phi}{(1 + \delta\phi) [\lambda_i \ln (1 + \delta\phi) + \lambda_0 \delta\phi]} = -t$$

but integration is involved requiring the expansion of the integrand into a uniformly convergent power series and then termwise integration.

If $|\delta\phi| \ll 1$ then $\delta\phi$ may be neglected in the numerator and $\delta\phi\delta x$ in the denominator of equation (2-27.3) which gives

$$\frac{d\delta\phi}{d\delta x} = \frac{\rho_o}{\lambda_o \lambda_i} \left[\frac{\delta x}{\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_o}\right) \delta x - \left(\frac{1}{\rho_o} - \frac{\gamma_i}{\lambda_i}\right) \delta\phi} \right]. \quad (2-37)$$

Now equation (2-37) is a homogeneous equation and may be solved by letting

$$\delta\phi = u\delta x.$$

So equation (2-37) becomes

$$\delta x \frac{du}{d\delta x} = \frac{\rho_o}{\lambda_o \lambda_i} \left[\frac{1}{\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_o}\right) - \left(\frac{1}{\rho_o} - \frac{\gamma_i}{\lambda_i}\right) u} \right] - u$$

and can be integrated to yield

$$\int \frac{\beta_o - \gamma_o u}{\alpha_o - \beta_o u + \gamma_o u^2} du = \ln \delta x + \text{const.} \quad (2-38)$$

where

$$\alpha_o = \frac{\rho_o}{\lambda_o \lambda_i},$$

$$\beta_o = \frac{1}{\lambda_i} + \frac{1}{\lambda_o},$$

$$\gamma_o = \frac{1}{\rho_o} + \frac{\gamma_i}{\lambda_i}.$$

To determine the form of the integral in equation (2-38) it is necessary to examine the sign of a quantity, q_o , defined as

$$q_o = \frac{4\rho_o}{\lambda_o \lambda_i} \left(\frac{1}{\rho_o} - \frac{\gamma_i}{\lambda_i} \right) - \left(\frac{1}{\lambda_i} + \frac{1}{\lambda_o} \right)^2.$$

For $q_0 > 0$, that is when

$$2 > \frac{1}{\lambda_i} (4\gamma_i \rho_0 + \lambda_0) + \frac{\lambda_i}{\lambda_0}$$

then equation (2-38) has the solution in terms of $\delta\phi$ and δx

$$\frac{2\beta_0}{\sqrt{q_0}} \tan^{-1} \left[\frac{2\gamma_0 \delta\phi - \beta_0 \delta x}{\sqrt{q_0} \delta x} \right] = \ln \left[\alpha_0 \delta x^2 - \beta_0 \delta x \delta\phi + \gamma_0 \delta\phi^2 \right] + \text{const.} \quad (2-39)$$

And when $q_0 < 0$, that is when

$$2 < \frac{1}{\lambda_i} (4\gamma_i \rho_0 + \lambda_0) + \frac{\lambda_i}{\lambda_0}$$

then equation (2-38) has the solution

$$\left[\frac{2\gamma_0 \delta\phi - (\beta_0 + \sqrt{-q_0}) \delta x}{2\gamma_0 \delta\phi - (\beta_0 - \sqrt{-q_0}) \delta x} \right]^{\frac{\beta_0}{\sqrt{-q_0}}} = (\alpha_0 \delta x^2 - \beta_0 \delta x \delta\phi + \gamma_0 \delta\phi^2) (\text{const.}). \quad (2-40)$$

Both equations (2-39) and (2-40) are rather complicated functions, but they could be employed for phase plane analysis for the assumption, of course, that $|\delta\phi| \ll 1$. They will not be employed for phase plane analysis, however, because the assumption that $|\delta\phi| \ll 1$ leads to time solutions that are similar to those that were found from perturbation theory in equations (2-18). This is evident if $\delta\phi$ is neglected in equation (2-27.1) and $\delta\phi \delta x$ in equation (2-27.2), and the equations are then combined to give

$$\frac{d^2 \delta\phi}{dt^2} + (\lambda_0 + \lambda_i) \frac{d\delta\phi}{dt} + \lambda_0 (\lambda_i - \gamma_i \rho_0) \delta\phi = 0 \quad (2-41)$$

which possesses the following characteristic equation in the Laplace variable "s".

$$s^2 + (\lambda_0 + \lambda_i)s + \lambda_0 (\lambda_i - \gamma_i \rho_0) = 0 \quad (2-42)$$

Equation (2-41) leads to the condition that

$$\lambda_i > \gamma_i \rho_0$$

for stability which is the same as that found from equation (2-19.2). The roots of equation (2-42) are

$$s = -\left(\frac{\lambda_o + \lambda_i}{2}\right) \pm \sqrt{\left(\frac{\lambda_o - \lambda_i}{2}\right)^2 + \gamma_i \rho_o \lambda_o}$$

which duplicate the significant roots of equations (2-18).

Equations (2-28) will now be studied under the following assumptions.

If $\lambda_i = \gamma_i \rho_o$ then equation (2-28.3) becomes

$$\frac{d\delta\phi}{d\delta i} = -\frac{\lambda_o}{\lambda_i} (1 + \delta\phi)$$

and integration produces for the initial condition that at zero time $\delta\phi(0) = \delta i(0) = 0$.

$$\delta\phi = e^{-\frac{\lambda_o}{\lambda_i} \delta i} - 1. \quad (2-43)$$

Introduction of equation (2-43) into equation (2-28.2) yields

$$\frac{d\delta i}{dt} = \lambda_i \left(e^{-\frac{\lambda_o}{\lambda_i} \delta i} - 1 - \delta i \right) \quad (2-44)$$

and it is seen that equation (2-44) satisfies the following conditions.

1. For $\delta i > 0$, $\frac{d\delta i}{dt} < 0$;
2. For $\delta i = 0$, $\frac{d\delta i}{dt} = 0$;
3. For $\delta i < 0$, $\frac{d\delta i}{dt} > 0$.

Which again, as was true of $\delta\phi$ for the assumption that $\gamma_i \rho_o = \lambda_i$, implies that the equilibrium value of δi is a stable point subject to the initial condition that

$$\delta\phi(0) = \delta x(0) = \delta i(0) = 0.$$

Equation (2-44) can be reduced to quadratures as follows, but performing the integration is involved.

$$\int_0^{\delta i} \frac{d\delta i}{e^{-\lambda_o/\lambda_i \delta i} - 1 - \delta i} = \lambda_i t$$

If $|\delta\phi| \ll 1$ then $\delta\phi$ can be neglected in the numerator of equation (2-28.3) yielding

$$\frac{d\delta\phi}{d\delta i} = - \frac{\rho_o \lambda_o}{\lambda_i} \frac{\left(\frac{1}{\rho_o} \delta\phi - \frac{\gamma_i}{\lambda_i} \delta i\right)}{(\delta\phi - \delta i)}$$

which has as a solution

$$(\beta_1 \delta i^2 - \epsilon_1 \delta i \delta\phi - \delta\phi^2) = (\text{const.}) \left[\frac{2\delta\phi + (\epsilon_1 - \sqrt{-q_1} \delta i)}{2\delta\phi + (\epsilon_1 + \sqrt{-q_1} \delta i)} \right]^{\frac{2 + \epsilon_1}{\sqrt{-q_1}}} \quad (2-45)$$

where

$$\alpha_1 = \frac{\rho_o \lambda_o}{\lambda_i},$$

$$\beta_1 = \frac{\alpha_1 \gamma_i}{\lambda_i},$$

$$\epsilon_1 = \frac{\alpha_1}{\rho_o} - 1,$$

$$q_1 = - \frac{4\gamma_i \rho_o \lambda_o}{\lambda_i^2} - \left(\frac{\lambda_o}{\lambda_i} - 1\right)^2.$$

Equation (2-45) is only of passing interest, however, since it has been shown that the assumption $|\delta\phi| \ll 1$ leads to a stability requirement the same as that found from perturbation theory. (See equation (2-19.2).)

Finally, examination of equations (2-29) under the following assumptions is profitable.

Again if $\lambda_i = \gamma_i \rho_o$ then equation (2-29.3) reduces to

$$\frac{d\delta x}{d\delta i} = -\gamma_i \frac{\lambda_o}{\lambda_i} \left(\frac{\lambda_o}{\lambda_i} + 1 \right) - \frac{\lambda_o}{\lambda_i} \delta x - \frac{\gamma_i \lambda_o^2}{\lambda_i^2} \delta i$$

which can be solved employing the appropriate integrating factor to give under the initial condition that $\delta x(0) = \delta i(0) = 0$ the following:

$$\delta x = \frac{\gamma_i \lambda_o}{\lambda_i} \left(e^{-\lambda_o/\lambda_i \delta i} - 1 - \delta i \right) \quad (2-46)$$

Equation (2-46) could have been obtained directly from equations (2-43) and (2-25).

If $\gamma_i \rho_o \gg \lambda_i$ and $\gamma_i \rho_o \gg \lambda_o$ (conditions which perturbation theory indicated led to instability), so that $\frac{1}{\rho_o} \ll \frac{\gamma_i}{\lambda_i}$ and $\frac{1}{\rho_o} \ll \frac{\gamma_i}{\lambda_o}$; since $\gamma_i \approx 1$, then, $\frac{1}{\rho_o}$ may be neglected in the numerator and denominator of equation (2-29.3) which then gives

$$\frac{d\delta x}{d\delta i} = -\frac{\lambda_o}{\lambda_i} (\gamma_i + \delta x)$$

which can be integrated, assuming as initial conditions $\delta x(0) = \delta i(0) = 0$, to give

$$\ln \left(1 + \frac{1}{\gamma_i} \delta x \right) = -\frac{\lambda_o}{\lambda_i} \delta i$$

or in terms of δx

$$\delta x = \gamma_i \left(e^{-\lambda_o/\lambda_i \delta i} - 1 \right). \quad (2-47)$$

Equation (2-47) may be combined with equation (2-25) to give

$$\delta \phi = \frac{\gamma_i \rho_o}{\lambda_o} \left(e^{-\lambda_o/\lambda_i \delta i} - 1 + \frac{\lambda_o}{\lambda_i} \delta i \right). \quad (2-48)$$

Now substituting equation (2-47) into equation (2-29.2) gives

$$\frac{d\delta i}{dt} = \gamma_i \rho_o \frac{\lambda_i}{\lambda_o} \left(e^{-\lambda_o/\lambda_i \delta i} - 1 + \frac{\lambda_o}{\lambda_i} \delta i \right) \quad (2-49)$$

where $\frac{1}{\gamma_i \rho_o}$ has been neglected in the last term because of the assumption that $\frac{1}{\rho_o} \ll \frac{\gamma_i}{\lambda_i}$. Equation (2-49) reveals the following:

1. For $\delta i > 0$, $\frac{d\delta i}{dt} > 0$;
2. For $\delta i = 0$, $\frac{d\delta i}{dt} = 0$;
3. For $\delta i < 0$, $\frac{d\delta i}{dt} > 0$.

Thus, for $\gamma_i \rho_o \gg \lambda_i$ and $\gamma_i \rho_o \gg \lambda_o$, the general solution for δi becomes unbounded for $t \rightarrow \infty$. That this is actually the case will now be shown by determining the general solution of equation (2-49).

If equation (2-49) is reduced to quadratures then

$$\int \frac{d\delta i}{(e^{-\lambda_o/\lambda_i \delta i} - 1 + \frac{\lambda_o}{\lambda_i} \delta i)} = \gamma_i \rho_o \frac{\lambda_o}{\lambda_i} t + \text{const.} \quad (2-50)$$

Again performing the integration of the integral term in equation (2-50) is somewhat involved. To make the integrand more tractable let

$$\frac{d\delta i}{e^{-\lambda_o/\lambda_i \delta i} - 1 + \frac{\lambda_o}{\lambda_i} \delta i} = \frac{\lambda_i}{\lambda_o} \frac{dz}{z^2 \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(n+2)!}} \quad (2-51)$$

where

$$z = \frac{\lambda_o}{\lambda_i} \delta i.$$

Now assume the following expansion exists.

$$\frac{1}{\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(n+2)!}} = \sum_{j=0}^{\infty} a_j z^j. \quad (2-52)$$

where the coefficients "a_j" are found by satisfying the following for all z's.

$$1 = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^n a_j z^{n+j}}{(n+2)!}$$

The first few coefficients are found to be

$$a_0 = 2!,$$

$$a_1 = \frac{(2!)^2}{3!},$$

$$a_2 = \frac{(2!)^3}{(3!)^2} - \frac{(2!)^2}{4!},$$

$$a_3 = \frac{(2!)^4}{(3!)^3} - \frac{2(2!)^3}{4!3!} + \frac{(2!)^2}{5!},$$

$$a_4 = \frac{(2!)^5}{(3!)^4} - \frac{3(2!)^4}{4!(3!)^2} + \left(\frac{2}{5!3!} + \frac{1}{4!} \right) (2!)^3 - \frac{(2!)^2}{6!},$$

$$a_5 = \frac{(2!)^6}{(3!)^5} - \frac{4(2!)^5}{4!(3!)^3} + \frac{1}{3!} \left(\frac{2}{5!3!} + \frac{2}{(4!)^2} + \frac{1}{4!} \right) (2!)^3 - \left(\frac{2}{6!3!} + \frac{2}{5!4!} \right) (2!)^3 + \frac{(2!)^2}{7!}.$$

A recursion relationship for the a's is given by the following for n > 1.

$$a_n = 2! \sum_{j=1}^n \frac{(-1)^{j+1} a_{n-j}}{(j+2)!}$$

Now utilizing equations (2-51) and (2-52) for the integrand of equation (2-50) and assuming uniform convergence of the series so that term by term integration is permitted (44), then δi has the following solution in time.

$$-2 \frac{\lambda_i}{\lambda_o} \frac{1}{\delta i} + \frac{2}{3} \ln \left(\frac{\lambda_o}{\lambda_i} \delta i \right) + \sum_{j=2}^{\infty} \frac{a_j}{j-1} \left(\frac{\lambda_o}{\lambda_i} \delta i \right)^{j-1} = \gamma_i \rho_o \left(\frac{\lambda_o}{\lambda_i} \right)^2 t + \text{const.} \quad (2-53)$$

As $t \rightarrow \infty$ in equation (2-53) only $\delta i \rightarrow \infty$ will satisfy the equation. Thus, when $\gamma_i \rho_o \gg \lambda_i$, δi becomes unbounded in time and, thus, unstable as was shown before.

Next consider the assumption that

$$\frac{1}{\rho_o} + \frac{\gamma_i}{\lambda_o} \gg \frac{1}{\lambda_o} |\delta x| + \frac{\gamma_i}{\lambda_i} |\delta i|$$

then for $|\delta x| \leq 1$ and $|\delta i| \leq 1$ and since $\gamma_i \approx 1$ the assumption is valid if $\lambda_i \gg \gamma_i \rho_o$. (Again, recall that for this condition perturbation analysis indicates stability.) Imposing the above assumption, equations (2-29.1) and (2-29.2) become

$$\frac{d\delta x}{dt} = \alpha \delta i - \beta \delta x, \quad (2-54.1)$$

$$\frac{d\delta i}{dt} = \gamma \delta x + \epsilon \delta i, \quad (2-54.2)$$

where

$$\alpha = \gamma_i \rho_o \lambda_o \left(\frac{1}{\rho_o} - \frac{\gamma_i}{\lambda_i} \right),$$

$$\beta = \lambda_o \rho_o \left(\frac{\gamma_i}{\lambda_o} + \frac{1}{\rho_o} \right),$$

$$\gamma = \frac{\lambda_i \rho_o}{\lambda_o},$$

$$\epsilon = \lambda_i \rho_o \left(\frac{\gamma_i}{\lambda_i} - \frac{1}{\rho_o} \right).$$

Eliminating δx from equations (2-54.1) and (2-54.2) gives

$$\frac{d^2 \delta x}{dt^2} + (\beta - \epsilon) \frac{d\delta x}{dt} - (\beta\epsilon + \alpha\gamma) \delta x = 0,$$

and Laplace Transformation techniques lead to the following characteristics equation roots.

$$s = \left(\frac{\lambda_o + \lambda_i}{2} \right) \pm \sqrt{\left(\frac{\lambda_o - \lambda_i}{2} \right)^2 + \gamma_i \rho_o \lambda_o}$$

These roots duplicate those found from perturbation theory and the same stability requirement results. However, it has been assumed that $\lambda_i \gg \gamma_i \rho_o$ so that contingent upon the assumptions made, the flux, xenon, and iodine are all stable (i.e., bounded for all time).

Returning to equations (2-17) it is possible to obtain analytic solutions for those flux levels such that

$$\frac{dx}{dt} \approx 0.$$

Then equations (2-17) become

$$\frac{d\phi}{dt} = \rho_o (1 - x) \phi, \quad (2-55.1)$$

$$x\phi = \gamma_x \phi + \gamma_i i, \quad (2-55.2)$$

$$\frac{di}{dt} = \lambda_i (\phi - i), \quad (2-55.3)$$

and solutions for equations (2-55) are as follows:

$$\phi(t) = \left(\frac{\gamma_i \rho_o i_o - \lambda_i \phi_o}{\gamma_i \rho_o - \lambda_i} \right) \left(1 - e^{(\gamma_i \rho_o - \lambda_i)t} \right) + \phi_o e^{(\gamma_i \rho_o - \lambda_i)t} \quad (2-56.1)$$

$$x(t) = \gamma_x + \frac{\lambda_i}{\rho_o} \left[\frac{\rho_o}{\gamma_i \rho_o - \lambda_i} (1 - e^{(\gamma_i \rho_o - \lambda_i)t}) + \frac{\rho_o \phi_o}{\gamma_i \rho_o i_o - \lambda_i \phi_o} e^{(\gamma_i \rho_o - \lambda_i)t} \right]^{-1} \quad (2-56.2)$$

$$i(t) = i_o + \frac{\lambda_i}{\gamma_i \rho_o - \lambda_i} (i_o - \phi_o) \left(1 - e^{(\gamma_i \rho_o - \lambda_i)t} \right), \quad (2-56.3)$$

where

$$\phi(0) = \phi_o,$$

$$x(0) = \gamma_x + \gamma_i \frac{i_o}{\phi_o},$$

$$i(0) = i_o.$$

It is obvious from equations (2-56) that the requirement for stability is that

$$\lambda_i > \gamma_i \rho_o.$$

For the conditions that $\gamma_i \rho_o \gg \lambda_i$ then equations (2-56) are well approximated by the following:

$$\phi(t) = i_o + (\phi_o - i_o) e^{\gamma_i \rho_o t} \quad (2-57.1)$$

$$x(t) = \gamma_x + \gamma_i \frac{i_o}{i_o + (\phi_o - i_o) e^{\gamma_i \rho_o t}} \quad (2-57.2)$$

$$i(t) = i_o + \frac{\lambda_i}{\gamma_i \rho_o} (\phi_o - i_o) (e^{\gamma_i \rho_o t} - 1) \quad (2-57.3)$$

Figure 5 indicates the fate of the dependent variables as $t \rightarrow \infty$ for the conditions given.

Figure 5. Asymptotic Values of the Dependent Variables at High Flux.

Condition	$\phi(\infty)$	$x(\infty)$	$i(\infty)$
$\phi_o > i_o$	∞	γ_x	∞
$\phi_o = i_o$	ϕ_o	1	i_o
$\phi_o < i_o$	0^*	0^*	0^*

* Note: Mathematically the dependent variables approach negative infinity; however, negative values have no physical significance and the reactor system consequently shuts down.

Computer simulation of the kinetics equations (equations (2-4)) to be rendered later in this chapter gives excellent agreement with the analytic solutions given by equations (2-57) for large flux values when $\gamma_i \rho_o \gg \lambda_i$.

ALTERNATE STABILITY CRITERION

An interesting alternate approach for high flux levels which leads to the same stability requirement as that found from perturbation theory can be adduced by considering the following equations which govern the reactor model under consideration. (These equations have all been derived previously.)

$$\frac{1}{\rho_o} (\delta\phi - \Delta_o) = \frac{1}{\lambda_o} \delta x + \frac{\gamma_i}{\lambda_i} (\delta i - \Delta_o) \quad (2-58)$$

$$\frac{d\delta\phi}{dt} = -\rho_o \delta x (1 + \delta\phi) \quad (2-26.1)$$

$$\frac{d\delta x}{dt} = \lambda_o (\gamma_i \delta i - \gamma_i \delta\phi - \delta x - \delta x \delta\phi) \quad (2-26.2)$$

$$\frac{d\delta i}{dt} = \lambda_i (\delta\phi - \delta i) \quad (2-26.3)$$

Equation (2-58) differs from equation (2-25) because a more general initial condition has been considered. It is apparent that equation (2-26.1) reaches steady state value when either $\delta x = 0$ or $\delta\phi = -1$. However, for $\delta\phi = -1$ the flux is zero, and thus this is a trivial condition. So assuming that as an equilibrium value $\delta x = 0$ then equations (2-58) and (2-26) possess steady state or stationary values when $\delta\phi = \delta i = \Delta_o$ and it is apparent that $\delta\phi$ and δi do not have to assume a zero value for equilibrium steady state to exist. Actually, this arbitrariness of the flux and iodine values at steady state does not exist when λ_x is not neglected in equation (2-4.2), as was evident from equations (2-2). However, to satisfy the model under immediate consideration equation (2-58) should be employed.

Now suppose that at some time $\delta\phi$ is perturbed to a value greater than its equilibrium value Δ_o , that is $\delta\phi > \Delta_o$. Then for this positive perturbation not to grow in time it is necessary that $\frac{d\delta\phi}{dt} < 0$. This is satisfied only if $\delta x > 0$ ($\delta\phi < -1$ is not physically realistic). Since equation (2-58) is assumed to be valid for all time, the requirement that $\delta x > 0$ implies the following from equation (2-58):

$$\frac{1}{\rho_o} (\delta\phi - \Delta_o) > \frac{\gamma_i}{\lambda_i} (\delta i - \Delta_o) \quad (2-59)$$

or

$$\delta i < \frac{\lambda_i}{\gamma_i \rho_o} (\delta \phi - \Delta_o) + \Delta_o. \quad (2-60)$$

Also from equation (2-27), if $\frac{d\delta\phi}{dt} < 0$, then $\delta\phi$ is decreasing, and to insure that $\delta x > 0$ then δi should decrease also. Now δi will decrease only if $\frac{d\delta i}{dt} < 0$ which implies from equation (2-26.3) that

$$\delta i > \delta \phi. \quad (2-61)$$

Now inequalities (2-60) and (2-61) can be simultaneously satisfied if and only if the following is satisfied.

$$\gamma_i \rho_o (\delta \phi - \Delta_o) < \lambda_i (\delta \phi - \Delta_o) \quad (2-62)$$

Since $\delta \phi \neq \Delta_o$, otherwise the flux would be at its equilibrium value, inequality (2-62) leads to the following stability requirement which is the same as that found from perturbation theory. (Note, however, no assumption as to the size of the perturbation of the flux has been made in developing this stability requirement.)

$$\lambda_i > \gamma_i \rho_o \quad (2-63)$$

It can be shown also that if the flux perturbation is negative, that is $\delta \phi < 0$, then the same inequality (2-63) is required if the flux is to return to its equilibrium value.

PHASE PLANE ANALYSIS FOR HIGH FLUX

Before undertaking the phase plane analysis of equations (2-27), (2-28), and (2-29) it is advisable to render a precise mathematical statement to define what is meant when a system of equations is said to possess a stable equilibrium point. This statement is taken from page 13 of the Theory of Oscillations by A. A. Andronow and C. E. Chaikin (1) which will serve as a general reference for this section. "A state of equilibrium is stable whenever given any region ϵ containing it (the equilibrium state) there is another $\delta(\epsilon)$ in ϵ such that any motion starting in the region δ remains in the region ϵ . The state of equilibrium

is unstable whenever given any region ϵ containing it each region in ϵ containing it contains a point M such that a motion starting at M leaves ϵ ." Mathematically then a solution $Z = (z_1, z_2, z_3, \dots, z_n)$ of a system of equations is said to be stable if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon)$ such that any other solution $Y = (y_1, y_2, y_3, \dots, y_n)$ of the same system of equations, for which $\|Z - Y\| \leq \delta$ at $t = t_0$, satisfies the further inequality $\|Z - Y\| \leq \epsilon$ for all $t \geq t_0$ (3). (Note: $\|A\|$ is defined as the "norm" of a matrix A and is equal to $\|A\| = \sum_{i,j} |a_{ij}|$. Physically

the above mathematical statement implies that any finite displacement around an equilibrium point remains finite.

With a precise definition of mathematical stability given, the phase plane analysis of equations (2-27), (2-28), and (2-29) may now be made. Since the model under investigation is spacially independent and thus meant to be informative only, an exhaustive phase plane analysis will not be rendered. Also, since equations (2-23) and (2-25) are the basis of equations (2-27), (2-28), and (2-29), the assumptions made in their formulation must be recognized. In particular, it is evident that equation (2-25) vanishes when any two of the dependent variables vanish.

Generally speaking, equations (2-27), (2-28), and (2-29) each constitute a dynamical system with two degrees of freedom. (Note that equation (2-25) determines a given dependent variables as a function of the other two dependent variables and thus serves as a constraint imposed upon the system.) Furthermore, since equations are autonomous (not explicit functions of time), equations (2-27), (2-28), and (2-29) are possible. Thus, for example, in equation (2-28.3) $\delta\phi$ may be considered a function of δi only, although both depend implicitly upon time. Now $\delta\phi$ and δi may be considered as coordinates of a space of two dimensions called a "phase space". To each state of the system there corresponds a point, say M , with coordinates $(\delta\phi_0, \delta i_0)$ in the space, and as time varies the point M describes a curve called a path, which describes the time history of the system. The totality of all such paths may be described as the phase portrait of the system (42). It represents all the possible time histories of the system; any one of which is determined by a single state M_0 (21). Geometrically speaking, this merely means that through every ordinary point of the phase plane passes one, and only one, integral curve defined by the equation. (The preceeding is from a theorem of Cauchy.)

A special situation arises for a point such that both the numerator and denominator of the phase plane derivatives ($\frac{d\delta\phi}{d\delta x} \rightarrow \frac{0}{0}$) vanish. Such a point is called a singular point of the system, and it is evident that a singular point is a point of equilibrium (21,7).

Phase plane solutions of equations (2-27) through (2-29) are presented in Figures 6 through 14 with the results given in the table of phase plane results for space independent kinetics at high flux (Figure 15).

$$\frac{d\delta\phi}{d\delta x} = \frac{1}{2} \frac{\delta x (1 + \delta\phi)}{(2 + \delta\phi)\delta x - \delta\phi} \quad \frac{\gamma_1 \rho_0}{\lambda_1} = \frac{1}{2}, \gamma_1 = 1, \frac{\lambda_0}{\lambda_1} = 1$$

$$\frac{d\delta\phi}{dt} = \rho_0 \delta x (1 + \delta\phi) \quad \frac{d\delta x}{dt} = -\lambda_0 [(2 + \delta\phi)\delta x - \delta\phi]$$

PPCo - B - 5338

Figure 6. Phase Plane of Flux and Xenon Response at High Flux for $\gamma_1 \rho_0 = \lambda_1/2$.

$$\frac{d\delta\phi}{d\delta x} = \frac{1 + \delta\phi}{2 + \delta\phi}, \frac{\lambda_1}{\lambda_0} = 1, \gamma_1 \rho_0 = \lambda_1, \gamma_1 = 1, \frac{d\delta\phi}{dt} = -\lambda_1 (1 + \delta\phi) \delta x$$

$$\frac{d\delta x}{dt} = -\lambda_0 (2 + \delta\phi) \delta x \quad \delta x = \delta\phi + \ln(1 + \delta\phi) + K$$

PPCo - B - 5339

Figure 7. Phase Plane of Flux and Xenon Response at High Flux for $\gamma_1 \rho_0 = \lambda_1$.

$$\frac{d\delta\phi}{d\delta x} = \frac{2\delta x(1+\delta\phi)}{\frac{1}{2}\delta\phi + (2+\delta\phi)\delta x} \quad \frac{\gamma_i \rho_0}{\lambda_i} = 2, \gamma_i = 1, \frac{\lambda_0}{\lambda_i} = 1$$

$$\frac{d\delta\phi}{dt} = -\rho_0 \delta x (1+\delta\phi) \quad \frac{d\delta x}{dt} = -\lambda \left[\frac{1}{2} \delta\phi + (2+\delta\phi)\delta x \right]$$

PPCo - B - 5340

Figure 8. Phase Plane of Flux and Xenon Response at High Flux for $\gamma_i \rho_0 = 2\lambda_i$.

$$\frac{d\delta\phi}{d\delta i} = \frac{(1+\delta\phi)(1/2\delta i - \delta\phi)}{\delta\phi - \delta i}, \quad \frac{\lambda_0}{\lambda_i} = 1, \frac{\gamma_i \rho_0}{\lambda_i} = \frac{1}{2}$$

$$\frac{d\delta i}{dt} = \lambda_i (\delta\phi - \delta i) \quad \frac{d\delta\phi}{dt} = \lambda_i (1+\delta\phi)(1/2\delta i - \delta\phi)$$

PPCo - B - 5341

Figure 9. Phase Plane of Flux and Iodine Response at High Flux for $\gamma_i \rho_0 = \lambda_i/2$.

$$\frac{d\delta\phi}{d\delta i} = -(1+\delta\phi), \quad \delta\phi = \lambda e^{-\delta i} - 1, \quad \frac{\lambda_0}{\lambda_i} = 1, \gamma_i \rho_0 = \lambda_i$$

$$\frac{d\delta\phi}{dt} = \lambda_0 (\delta i - \delta\phi)(1+\delta\phi) \quad \frac{d\delta i}{dt} = \lambda_i (\delta\phi - \delta i)$$

PPCo - B - 5342

Figure 10. Phase Plane of Flux and Iodine Response at High Flux for $\gamma_i \rho_0 = \lambda_i$.

$$\frac{d\delta\phi}{d\delta i} = \frac{(1+\delta\phi)(2\delta i - \delta\phi)}{(\delta\phi - \delta i)}, \quad \frac{\lambda_0}{\lambda_i} = 1, \frac{\gamma_i \rho_0}{\lambda_i} = 2$$

$$\frac{d\delta\phi}{dt} = \lambda_0 (2\delta i - \delta\phi)(1+\delta\phi) \quad \frac{d\delta i}{dt} = (\delta\phi - \delta i)$$

PPCo - B - 5343

Figure 11. Phase Plane of Flux and Iodine Response at High Flux for $\gamma_i \rho_0 = 2\lambda_i$.

$$\frac{d\delta x}{d\delta i} = \frac{\delta i - \delta x(2 + \delta x + \delta i)}{\delta x - \delta i} \quad \frac{\gamma_i \rho_0}{\lambda_i} = \frac{1}{2}, \quad \frac{\lambda_0}{\lambda_i} = 1, \gamma_i = 1$$

$$\frac{d\delta x}{dt} = \rho_0 [\delta i - \delta x(2 + \delta x + \delta i)] \quad \frac{d\delta i}{dt} = \rho_0 [\delta x - \delta i] \quad \text{PPCo-B-5344}$$

Figure 12. Phase Plane of Xenon and Iodine Response at High Flux for $\gamma_i \rho_0 = \lambda_i/2$.

$$\frac{d\delta x}{d\delta i} = -(2 + \delta x + \delta i) \quad \frac{\lambda_0}{\lambda_i} = 1, \gamma_i = 1, \gamma_i \rho_0 = \lambda_i$$

$$\frac{d\delta x}{dt} = -\rho_0 \delta x(2 + \delta x + \delta i), \quad \frac{d\delta i}{dt} = \rho_0 \delta x, \quad \delta x = \kappa e^{-\delta i - 1 - \delta i} \quad \text{PPCo-B-5345}$$

Figure 13. Phase Plane of Xenon and Iodine Response at High Flux for $\gamma_i \rho_0 = \lambda_i$.

$$\frac{d\delta x}{d\delta i} = -\frac{\delta i + 2\delta x(2 + \delta x + \delta i)}{2\delta x + \delta i}, \quad \lambda_0 = \lambda_i, \frac{\gamma_i \rho_0}{\lambda_i} = 2, \gamma_i = 1$$

$$\frac{d\delta x}{dt} = -\rho_0 \left[\frac{1}{2} \delta i + \delta x(2 + \delta x + \delta i) \right] \quad \frac{d\delta i}{dt} = \rho_0 (\delta x + 1/2 \delta i) \quad \text{PPCo-B-5346}$$

Figure 14. Phase Plane of Xenon and Iodine Response at High Flux for $\gamma_i \rho_0 = 2\lambda_i$.

Figure 15. Phase Plane Results for Space Independent Kinetics at High Flux.

Figure	Describing Equation	Remarks
6	$\frac{d\delta\phi}{d\delta x} = \frac{1}{2} \frac{\delta x (1 + \delta\phi)}{(2 + \delta\phi) \delta x - \delta\phi} , \gamma_i \rho_o = \frac{\lambda_i}{2}$	stable
7	$\frac{d\delta\phi}{d\delta x} = \frac{1 + \delta\phi}{2 + \delta\phi} , \gamma_i \rho_o = \lambda_i$	stable*
8	$\frac{d\delta\phi}{d\delta x} = \frac{2\delta x (1 + \delta\phi)}{\delta\phi/2 + (2 + \delta\phi) \delta x} , \gamma_i \rho_o = 2\lambda_i$	unstable
9	$\frac{d\delta\phi}{d\delta i} = \frac{(1 + \delta\phi)(\delta i/2 - \delta\phi)}{\delta\phi - \delta i} , \gamma_i \rho_o = \frac{\lambda_i}{2}$	stable
10	$\frac{d\delta\phi}{d\delta i} = - (1 + \delta\phi) , \gamma_i \rho_o = \lambda_i$	stable*
11	$\frac{d\delta\phi}{d\delta i} = \frac{(1 + \delta\phi)(2\delta i - \delta\phi)}{\delta\phi - \delta i} , \gamma_i \rho_o = 2\lambda_i$	unstable
12	$\frac{d\delta x}{d\delta i} = \frac{\delta i - \delta x(2 + \delta x + \delta i)}{\delta x - \delta i} , \gamma_i \rho_o = \frac{\lambda_i}{2}$	stable**
13	$\frac{d\delta x}{d\delta i} = - (2 + \delta x + \delta i) , \gamma_i \rho_o = \lambda_i$	stable*
14	$\frac{d\delta x}{d\delta i} = - \frac{\delta i + 2\delta x (2 + \delta x + \delta i)}{2\delta x + \delta i} , \gamma_i \rho_o = 2\lambda_i$	unstable

Note: The following values, $\lambda_o = \lambda_i$ and $\gamma_i = 1$, were used for all of the phase plane figures. $\lambda_o^i = \lambda_i$ implies a flux level of $\approx 10^{13}$. However, reasonably higher flux values do not change the character of the phase plane figures significantly, and the appropriate remarks made corresponding to attendant conditions are unchanged. Variable range is $-1 \leq (\delta\phi, \delta x, \delta i) \leq 1$

* Although a finite perturbation results in the system acquiring a new equilibrium state, this is still deemed stable.

** A large negative perturbation in the xenon or iodine will result in shutting down the reactor model.

ANALOG AND DIGITAL COMPUTER SIMULATION

To corroborate the results found from the analytic investigation in the previous section and to further examine the space independent kinetic behavior of the reactor model under consideration, equations (2-4) were programmed both for simulation on an electronic analog computer and simulation on an IBM 7040 Digital Computer. The complete formulation of the analog and digital computer programs is given in Appendix B. The purpose in simulation by both analog and digital means was to ascertain the advantages and disadvantages of each and to provide a means of checking results.

The analog program was simulated on the general purpose, electronic analog computer facility of the Atomic Energy Division of Phillips Petroleum Company. The setup employed electronic multipliers and high gain operational amplifiers used as inverters, integrators, and summers. Output results were graphically presented by an "X-Y" plotter. Some comments pertinent to the analog simulation of equations (2-4) were as follows:

1. High precision components were necessary especially at high flux levels to obtain accurate output.
2. At high flux levels the wide range of associated parameters leads to scaling difficulties and high gain feedback problems.
3. Variation of parameters and initial conditions is easily and rapidly implemented.
4. Output results are immediate and graphically portrayed in a continuous fashion. Because of the nature of the output, appropriate time scaling is easily implemented.

The digital program (Computer program 40089) was simulated on the IBM 7040 Digital Computer of the Atomic Energy Division of Phillips Petroleum Company. The computer program consisted of a main FORTRAN Source program and two subroutines. The first subroutine, called "DER", is a FORTRAN program written to provide the computer with the system of first order differential equations and to compute and store the derivatives. The second subroutine, called "DEQ", which uses Gill's variation of the Runge-Kutta Method, integrates, stepwise, a given system of simultaneous first order differential equations. The main FORTRAN source program sets the general tenure of the problem by providing internal and external control, the initial conditions, the time spacing, etc.

Some comments and observations pertinent to the IBM 7040 simulation of equations (2-4) were as follows:

1. Prior knowledge of the appropriate time scaling is necessary.
2. "Experimenting" with the equations, specifically, determining the effect of a large number of different parameter and initial condition settings, is somewhat expensive and time consuming.
3. Output results are presented numerically. Thus, a very high degree of accuracy of solution data is possible.
4. It is possible to accurately study very large changes in the dependent variables without overloading the computer.
5. The digital program is very general and permits investigation of a wide range of parameters and initial conditions.

The bulk of the output given by computer simulation of equations (2-4) is given in graphical form in Appendix B. However, certain figures of significant interest are presented here.

Figures 16 and 17 present typical kinetic responses as determined by the digital computer arising from initial positive flux perturbations for a flux of 2×10^{14} with no external control. Note that the dependent variables appear to be approximately exponential in character. Figure 18 gives the kinetic response of the flux found from analog computer simulation for various initial flux perturbations for a flux of 2×10^{14} with no external control. In Figure 19 the space independent response of the flux, xenon, and iodine to an initial 10 percent perturbation in the 2×10^{16} flux is given. It is significant to observe that the perturbed flux assumes a value twice its equilibrium value in only 26 seconds. Such a short period of time could cause concern if the reactor control system were manually operated. Figure 20 gives the kinetic response resulting from a negative perturbation in the 2×10^{16} flux with no external control. Observe that the flux is essentially zero within 56 seconds of the initial 10 percent perturbation, and the xenon is rapidly increasing to its peak value. The iodine response is not given. The condition in which an immediate step control response is made, corresponding to a value of $\rho_0 = 3.1 \times 10^{-5}$, for a 15 percent positive perturbation of the 2×10^{14} flux is given in Figure 21. Several significant observations can be deduced from Figure 21. First, certain control operations can lead to long period oscillations of the flux. Furthermore even though the oscillation is damped, the initial overshoot of the flux from its initial perturbed condition carries the flux to nearly 180 percent of its equilibrium

Figure 16. Digital Simulation of 2×10^{14} Flux.

Figure 17. Digital Simulation of 2×10^{14} Flux.

Figure 18. Analog Simulation of 2×10^{14} Flux.

Figure 19. Digital Simulation of 2×10^{16} Flux.

Figure 20. Digital Simulation of 2×10^{16} Flux.

Figure 21. Analog Simulation of 2×10^{14} Flux.

value. Secondly, with respect to the time behavior character of the dependent variables, it is interesting to observe that although the flux and iodine oscillations are essentially sinusoidal in character, the xenon oscillation possesses a "saw tooth" or "ramp" character, which is probably attributable to the xenon burnout (the disappearance of xenon due to neutron absorption, which is significant at high flux levels).

It was shown earlier in this chapter that the threshold flux value which results from the linearized, space independent kinetic equations (2-8) for xenon induced instability is about 1.48×10^9 for typical parameter values. That this flux value is essentially the correct value as determined by the more correct, nonlinear, space independent kinetic equations (2-4) is evidenced by Figures 22 and 23. Figures 22 and 23 present the output of an analog computer simulation of equations (2-4) for $\phi_0 \leq 10^{11}$. The value of flux given in parentheses to the right of each value of ρ_0 is that value of flux determined by equation (2-4.4) for a given ρ_0 , which is valid if no external control is imposed. Figure 22 gives the time behavior of the flux which results from an initial 50 percent positive perturbation for various values of ρ_0 . Figure 23 gives the time response of the flux due to an initial 50 percent positive perturbation, for $\rho_0 = 5.435 \times 10^{-5}$. It is evident that this value of ρ_0 leads to sustained flux oscillations and thus determines the value of the threshold flux. A value of $\rho_0 = 5.435 \times 10^{-5}$ corresponds to a flux value of 1.48×10^9 which is in agreement with the analytic investigation.

Figure 22. Analog Simulation for Threshold Flux.

Figure 23. Analog Simulation for Threshold Flux.

CONCLUSIONS

In the preceding chapter the space independent kinetics of a nuclear reactor model under the influence of xenon-135 has been briefly examined for stability. Analysis of the nonlinear kinetic equations for small changes of the dependent variables (via first order perturbation approximations) gave the result that the reactor model was stable for reactivity changes less than the order of the product of the effective neutron lifetime times the decay constant for iodine-135. (See equation (2-12) for the precise stability requirement.) The reactivity of a reactor is defined as $(k_{\text{eff}} - 1)/k_{\text{eff}}$ where k_{eff} is the effective neutron multiplication factor for the reactor.

For larger reactivity changes, first order perturbation approximations indicated instability. Nonlinear analysis of the kinetic equations for high equilibrium flux levels led essentially to the same restriction on reactivity as perturbation results revealed, and this restriction was further corroborated by phase plane analysis and digital and analog computer simulation of the nonlinear kinetic equations. Interestingly, although first order perturbation approximations of the kinetic equations did not give, in general, the correct time behavior, which was determined by more sophisticated techniques, of the reactor model, perturbation approximations did indicate "substantially" the correct stability criterion as evidenced by the other analysis schemes.

It is important to note that although the occurrence of sustained oscillatory behavior of the flux was characterized by a real time of hours or even days, for

low values of the equilibrium flux or small values of ρ_0 , the typical real time behavior of the fundamental mode of the flux for a high flux level reactor model is in the order of minutes or less. Thus, although there is sufficient time for a servo-type automatic control system to respond to a xenon induced flux instability, the necessary response time might be somewhat demanding upon manual reaction and control implementation, especially at very high flux levels.

One may reasonably conclude that the reactor model described by equations (2-4) is, in general, unstable without external control when the equilibrium flux level is greater than approximately 1.5×10^9 . This is true since the restriction upon permissible reactivity changes is very prohibitive and cannot be reasonably satisfied. The stability criteria found in this chapter are useful and permit application to some physical reactor systems. However, it is important to recognize the applicability and limitations in employing equations (2-4) to describe a physical system. Since equations (2-4) are space independent, they cannot account for changes in leakage in time and space; and because leakage is a stabilizing factor, the results found here with respect to stability may be overly restrictive. Also, many additional factors may affect the stability of a physical reactor system, and when coupled with the influence of xenon these factors may further reduce or even enhance the stability of the physical reactor system.

III. - EXAMINATION AND COMPARISON WITH XENON DEPENDENCE OF DIFFUSION THEORY AND TRANSPORT THEORY

In principle, any precise attempt to describe mathematically the time and space behavior of neutrons within a reactor should begin with the neutron transport equation. However, availability of solutions to the transport equation, in general, is limited, and as is obvious from the literature most attempts for solutions when xenon feedback is considered to be significant are restricted to simple diffusion theory.

To determine to some extent how valid diffusion theory is for the investigation of xenon induced flux instabilities, the first two members (which are often referred to as the P_1 approximations) of the infinite set of coupled equations, which satisfy the one dimensional, time dependent transport equation, will be considered. No attempt will be made to account for the energy dependence of the neutrons. However, if diffusion theory is reasonably accurate for the xenon instability problem, then multigroup diffusion theory can be employed to account for the energy dependence of the parameters and dependent variables.

From Chapter 7 of Reactor Analysis by Meghreblian and Holmes, (33, p. 350) the equations to be examined are

$$-\frac{1}{v} \frac{\partial \phi_0}{\partial t} - \Sigma_a \phi_0 - \sigma_x X \phi_0 + S_0 = \frac{\partial \phi_1}{\partial r}, \quad (3-1.1)$$

$$-\frac{1}{v} \frac{\partial \phi_1}{\partial t} - \Sigma_a \phi_1 - \sigma_x X \phi_1 - (1 - \bar{\mu}_0) \Sigma_s \phi_1 = \frac{1}{3} \frac{\partial \phi_0}{\partial r}, \quad (3-1.2)$$

where

$\phi_0(r, t)$ is the first term of the Legendre series expansion for the vector flux (physically, ϕ_0 corresponds to the total track length of all neutrons),

$\phi_1(r, t)$ is the second term of the Legendre series expansion for the vector flux (physically, ϕ_1 corresponds to the net neutron current in the "r" direction),

$S_0(r, t)$ is the first term of the Legendre series expansion for the neutron source term,

$\bar{\mu}_0$ is the average value of the cosine of the scattering angle,
 Σ_s is the macroscopic scattering cross section for neutrons,
the other parameters have been previously defined in Chapter II.
Now assume that the source term S_0 in equation (3-1.1) is given by $S_0 = v\Sigma_f$
where the terms have been previously defined.

The steady state or equilibrium values of equations (3-1) must satisfy the following:

$$[v\Sigma_f - \Sigma_a - \sigma_x X_0] \phi_0 = \frac{d\phi_1}{dr}, \quad (3-2.1)$$

$$[\Sigma_a + \sigma_x X + (1 - \bar{\mu}_0)\Sigma_s] \phi_1 = -\frac{1}{\beta} \frac{d\phi_0}{dr}, \quad (3-2.2)$$

where

$$\phi_0(r) = \Phi_0(r, t_{eq}),$$

$$\phi_1(r) = \Phi_1(r, t_{eq}),$$

The equations which describe the xenon and iodine concentrations in time are the following (all parameters have been previously defined in Chapter II):

$$\frac{\partial X}{\partial t} = \bar{\gamma}_x \Sigma_f \Phi_0 + \lambda_1 I - \sigma_x X \Phi_0 - \lambda_x X,$$

$$\frac{\partial I}{\partial t} = \bar{\gamma}_i \Sigma_f \Phi_0 - \lambda_1 I,$$

and the steady state value of the xenon concentration satisfies

$$X_0 = \frac{\bar{\gamma}_x \Sigma_f \Phi_0}{\sigma_x \Phi_0 + \lambda_x}, \quad \bar{\gamma} = \bar{\gamma}_x + \bar{\gamma}_i. \quad (3-3)$$

The dependent variable ϕ_1 can be eliminated from equations (3-2) to give

$$D \frac{d^2 \phi_0}{dr^2} + (v\Sigma_f - \Sigma_a - \sigma_x X_0) \phi_0 = \beta D^2 \sigma_x \frac{d\phi_0}{dr} \frac{dX_0}{dr} \quad (3-4)$$

where

$$D = \frac{1}{\beta (\Sigma_a + \sigma_x X_0 + \Sigma_s - \bar{\mu}_0 \Sigma_s)} = \frac{1}{\beta (\Sigma_t - \bar{\mu}_0 \Sigma_s)} \quad (3-5)$$

and

$$\Sigma_t = \Sigma_a + \sigma_x X_o + \Sigma_s$$

The "D" defined by equation (3-5) is the usual diffusion coefficient found from transport theory (10, 33, 48).

Now the form of the diffusion equation generally employed to study xenon is

$$\nabla^2 \phi + (v\Sigma_f - \Sigma_a - \sigma_x X) \phi = 0 \quad (3-6)$$

where the diffusion coefficient is considered space independent. Hence, to reconcile equations (3-6) and (3-4), it is necessary to assume that not only is the diffusion coefficient space independent, but that the last term in equation (3-4), viz.,

$$3D^2 \sigma_x \frac{d\phi_o}{dr} \frac{dX_o}{dr} = 3D^2 \sigma_x \frac{\bar{\gamma} \Sigma_f \lambda_x}{(\sigma_x \phi_o + \lambda_x)^2} \left(\frac{d\phi_o}{dr} \right)^2 \quad (3-7)$$

is zero or at least negligible compared to the other terms in the equation. It is interesting to observe that expression (3-7) vanishes if the gradient of either the flux or xenon concentration is zero.

To examine the assumption that the diffusion coefficient in equation (3-4) is space independent, substitute equation (3-3) for X_o into equation (3-4) which gives

$$D = \frac{1}{3[\Sigma_a + \Sigma_s(1 - \bar{\mu}_o) + \bar{\gamma} \Sigma_f / (1 + \frac{\lambda_x}{\sigma_x \phi_o})]} \quad (3-8)$$

Now for very small values of the flux ($\phi_o \ll 10^{13}$) "D" becomes

$$D_o = \frac{1}{3[\Sigma_a + \Sigma_s(1 - \bar{\mu}_o)]} \quad (3-9)$$

while for large values of the flux ($\phi_o \gg 10^{13}$) the diffusion coefficient becomes

$$D_\infty = \frac{1}{3[\Sigma_a + \Sigma_s(1 - \bar{\mu}_o) + \bar{\gamma} \Sigma_f]} \quad (3-10)$$

It is apparent that the diffusion coefficient decreases as the xenon concentration increases. Figure 24 displays graphically how the equilibrium xenon concentration (normalized variable) depends on the flux level.

The decreasing of the diffusion coefficient with increasing xenon concentration is reasonable on physical grounds since the diffusion coefficient is a measure of the neutron's ability to diffuse through material space; this ability is obviously decreased as the probability of neutron capture by xenon or any other absorber is increased.

A useful approximate expression for calculating the diffusion coefficient given the parameters of the system and the flux level is the following:

$$D \approx D_0 - 3D_0^2 \bar{\gamma} \Sigma_f \frac{\sigma_x \phi_0}{\sigma_x \phi_0 + \lambda_x} \quad (3-11)$$

where D_0 (see equation (3-9)) is the value of the diffusion coefficient for the system with no xenon present. Introduction of typical numerical values into equation (3-11) reveals, however, that the difference between "D" and " D_0 " is generally small (in the order of a percent); and the diffusion coefficient in equation (3-6) may be considered spacially independent for most practical applications.

To investigate the significance of the term in expression (3-7) and to gain some insight into the nature of the flux distribution given by equations (3-2) the following transformations of dependent variables will be effected.

$$w_0(r) = \sigma_x \phi_0(r) \quad (3-12.1)$$

$$w_1(r) = \sigma_x \phi_1(r) \quad (3-12.2)$$

Imposing the transformations in equation (3-12) upon equations (3-2) gives

$$\left(A - C \frac{w_0}{w_0 + \lambda_x} \right) w_0 = \frac{dw_1}{dr}, \quad (3-13.1)$$

Figure 24. Equilibrium Xenon-135 Concentration Versus Flux Level.

$$- (B + 3C \frac{w_o}{w_o + \lambda_x}) w_1 = \frac{dw_o}{dr}, \quad (3-13.2)$$

where

$$A = v\Sigma_f - \Sigma_a,$$

$$B = 3 \left[(1 - \bar{\mu}_o) \Sigma_s + \Sigma_a \right],$$

$$C = \bar{\gamma}\Sigma_f.$$

If equation (3-13.1) is divided by equation (3-13.2) there results

$$- \frac{(D'w_o + F) w_o}{(E w_o + G) w_1} = \frac{dw_1}{dw_o} \quad (3-14)$$

where

$$D' = A - C = v\Sigma_f - \Sigma_a - \bar{\gamma}\Sigma_f,$$

$$E = B + 3C = 3 \left[\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s + \bar{\gamma}\Sigma_f \right],$$

$$F = A\lambda_x = (v\Sigma_f - \Sigma_a) \lambda_x,$$

$$G = B\lambda_x = 3 \left[\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s \right] \lambda_x.$$

Equation (3-14) becomes when reduced to quadratures

$$2H_1 \int \frac{s^2 ds}{s+1} + H_2 \int \frac{s ds}{s+1} = -w_1^2 + K_1 \text{ (const.)} \quad (3-15)$$

where

$$s = \frac{E}{G} w_o = \frac{[\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s + \bar{\gamma}\Sigma_f] \sigma_x \phi_o}{[\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s] \lambda_x},$$

$$H_1 = \frac{D'G^2}{E^3} = \frac{(v\Sigma_f - \Sigma_a - \bar{\gamma}\Sigma_f) [\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s]^2 \lambda_x^2}{3 [\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s + \bar{\gamma}\Sigma_f]^3},$$

$$H_2 = \frac{2FG}{E^2} = \frac{2(v\Sigma_f - \Sigma_a)[\Sigma_a + (1-\bar{\mu}_0)\Sigma_s]\lambda_x^2}{3[\Sigma_a + (1-\bar{\mu}_0)\Sigma_s + \bar{\gamma}\Sigma_f]^2}.$$

Integration of equation (3-15) gives (K_2 is the integration constant)

$$w_1 = \pm \sqrt{K_2 - H_1 s^2 - H_3 s + H_3 \ln(s+1)} \quad (3-16)$$

where

$$H_3 = H_2 - H_1 = \frac{2[\Sigma_a + (1-\bar{\mu}_0)\Sigma_s][v\Sigma_f + (1-\bar{\mu}_0)\Sigma_s]\bar{\gamma}\Sigma_f\lambda_x^2}{3[\Sigma_a + (1-\bar{\mu}_0)\Sigma_s + \bar{\gamma}\Sigma_f]^3}.$$

Now combining equation (3-16) and equation (3-13.2) and reducing to quadratures yields (for the boundary condition that the flux vanish at $r = 0$)

$$\begin{aligned} -r = & + (\pm 1) \frac{G}{E^2} \int_0^s \frac{ds}{\sqrt{K_2 - H_1 s^2 - H_3 s + H_3 \ln(s+1)}} \\ & + (\pm 1) \left(\frac{\lambda_x}{E} - \frac{G}{E^2} \right) \int_0^s \frac{ds}{(s+1) \sqrt{K_2 - H_1 s^2 - H_3 s + H_3 \ln(s+1)}}. \end{aligned} \quad (3-17)$$

Equation (3-17) defies integration in closed form. Expansion of the integrand leads to the situation of evaluating an infinite series of exponential integrals. However, for low values of the flux ($\phi_0 < 10^{13}$) the logarithm term in the integrand may be approximated as follows:

$$\ln(s+1) \approx s - \frac{s^2}{2}$$

So equation (3-17) becomes

$$- (\pm 1) \sqrt{\frac{H_2}{2}} r = \frac{G}{E^2} \int_0^s \frac{ds}{\sqrt{K_3^2 - s^2}} + \left(\frac{\lambda_x}{E} - \frac{G}{E^2} \right) \int_0^s \frac{ds}{(s+1) \sqrt{K_3^2 - s^2}} \quad (3-18)$$

where

$$K_3^2 = \sqrt{\frac{2}{H_1}} K_2.$$

Integration of equation (3-18) produces (observing that $\sin^{-1}(0) = n\pi$, $n = 0, \pm 1, \dots$)

$$-(\pm 1)\sqrt{\frac{H_2}{2}} r = \frac{G}{E^2} \left[\sin^{-1} \left(\frac{s}{K_3} \right) - n\pi \right] + \left(\frac{\lambda_x}{E} - \frac{G}{E^2} \right) I \quad (3-19)$$

and for $K_3^2 > 1$, the expression for "I" in equation (3-19) is

$$I = \frac{1}{\sqrt{K_3^2 - 1}} \ln \left\{ \frac{\sqrt{(K_3 + 1)(s + K_3)} - \sqrt{(K_3 - 1)(K_3 - s)}}{\sqrt{(K_3 + 1)(s + K_3)} + \sqrt{(K_3 - 1)(K_3 - s)}} \right\} \Bigg|_0^s \quad (3-20)$$

while for $K_3^2 < 1$, then "I" is given by

$$I = \frac{2}{\sqrt{1 - K_3^2}} \tan^{-1} \sqrt{\frac{(1 + K_3)(K_3 + s)}{(1 - K_3)(K_3 - s)}} \Bigg|_0^s \quad (3-21)$$

It is found that the last term in equation (3-19) is determined by the magnitude of the arbitrary integration constant " K_3 ".

For the second boundary condition that the flux and, hence, "s" must vanish at $r = L$, then equation (3-19) gives (obviously the term "I" vanishes for this boundary condition.)

$$s(L) = K_3 \sin \left[-(\pm 1)\sqrt{\frac{H_2}{2}} \frac{E^2}{G} L + n\pi \right] = 0 \quad (3-22)$$

which can only be satisfied nontrivially (i.e., $K_3 \neq 0$) when

$$-(\pm 1)\sqrt{\frac{H_2}{2}} \frac{E^2}{G} L + n\pi = m\pi \quad (3-23)$$

where "m" is an integer or zero. Equation (3-23) also may be written as

$$B_x^2 = \frac{H_2}{2} \frac{E^4}{G^2} = \left(\frac{p\pi}{L} \right)^2 \quad (3-24)$$

where "p" is an integer or zero and B_x^2 is defined as the material buckling. Now equation (3-24) determines criticality (i.e., equality of the material and

geometrical buckling) for the system and must be satisfied if the reactor system described by equations (3-1) is to achieve a critical (nonzero) equilibrium flux. If equation (3-24) is not satisfied, the system is either subcritical or supercritical.

Substituting original parameters into equation (3-24) gives

$$B_x^2 = 3 \left(\nu \Sigma_f - \Sigma_a \right) \left[1 + \frac{\bar{\gamma} \Sigma_f}{\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s} \right] \left[\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s + \bar{\gamma} \Sigma_f \right]. \quad (3-25)$$

Now simple diffusion theory (neglecting the effects of xenon) leads to the following expression for the material buckling.

$$B_\ell^2 = 3 \left(\nu \Sigma_f - \Sigma_a \right) \left[\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s \right] \quad (3-26)$$

It will be shown that the material buckling for criticality to exist that results from equation (3-17) when high flux levels ($\phi_0 > 10^{13}$) are considered is the same as equation (3-25). Diffusion theory, when the diffusion coefficient and absorption cross section are modified to include xenon absorption at high flux, leads to the following for the material buckling.

$$B_h^2 = 3 \left(\nu \Sigma_f - \Sigma_a - \bar{\gamma} \Sigma_f \right) \left[\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s + \bar{\gamma} \Sigma_f \right] \quad (3-27)$$

Ratios of equation (3-25) with equations (3-26) and (3-27) give

$$\frac{B_x^2}{B_\ell^2} = \left[1 + \frac{\gamma \Sigma_f}{(1 - \bar{\mu}_o) \Sigma_s + \Sigma_a} \right]^2 \quad (3-28)$$

and

$$\frac{B_x^2}{B_h^2} = \left[1 + \frac{\bar{\gamma} \Sigma_f}{\nu \Sigma_f - \Sigma_a - \bar{\gamma} \Sigma_f} \right] \left[1 + \frac{\bar{\gamma} \Sigma_f}{\Sigma_a + (1 - \bar{\mu}_o) \Sigma_s + \bar{\gamma} \Sigma_f} \right]. \quad (3-29)$$

Thus, it is obvious that neither equation (3-28) nor equation (3-29) gives the same expression for the material buckling as found from transport theory. Especially significant is the fact that "modifying" the diffusion coefficient and the macroscopic absorption cross section in simple diffusion theory as was done to yield equation (3-27) still does not lead to the more correct expression, equation (3-25), for the material buckling found from transport theory.

Now returning to equation (3-17), under the assumption of large flux values ($\phi_0 > 10^{13}$), the term "Ln(s+1)" will be neglected compared to the "s" term. So there results

$$-(\pm 1)r = \frac{G}{E^2} \int_0^s \frac{ds}{\sqrt{K_2 - H_1 s^2 - H_3 s}} + \left(\frac{\lambda_x}{E} - \frac{G}{E^2} \right) \int_0^s \frac{ds}{(s+1) \sqrt{K_2 - H_1 s^2 - H_3 s}} \quad (3-30)$$

Equation (3-30) may be integrated to give

$$-(\pm 1)r = \frac{G}{E^2} \frac{1}{\sqrt{H_3}} \left[\sin^{-1} \left(\frac{2H_3 s + H_1}{\sqrt{H_1^2 + 4K_2 H_3}} \right) - \sin^{-1} \left(\frac{H_1}{\sqrt{H_1^2 + 4K_2 H_3}} \right) \right] + J \quad (3-31.1)$$

where for $K_2 + H_3 > H_1$, "J" is given by

$$J = \frac{\left(\frac{G}{E^2} - \frac{\lambda_x}{E} \right)}{\sqrt{K_2 + H_3 - H_1}} \ln \left[\frac{\sqrt{K_2 - H_3 s - H_1 s^2} + \sqrt{K_2 + H_3 - H_1}}{(s+1)} + \frac{2H_1 - H_3}{2\sqrt{K_2 + H_3 - H_1}} \right] \Bigg|_0^s \quad (3-31.2)$$

and when $K_2 + H_3 < H_1$, "J" becomes

$$J = \frac{\left(\frac{\lambda_x}{E} - \frac{G}{E^2} \right)}{\sqrt{H_1 - H_3 - K_2}} \sin^{-1} \left[\frac{s(2H_1 - H_3) + 2(K_2 + H_3 - H_1)}{(s+1)\sqrt{H_3^2 + 4H_1 K_2}} \right] \Bigg|_0^s \quad (3-31.3)$$

Equation (3-31.1) may be written as

$$s = \frac{\sqrt{H_1^2 + 4K_2 H_3}}{2H_3} \sin \left[-(\pm 1) \frac{E^2}{G} \sqrt{H_3} r + \sin^{-1} \left(\frac{H_1}{\sqrt{H_1^2 + 4K_2 H_3}} \right) + J \right] - \frac{H_1}{2H_3} \quad (3-32)$$

and imposing the boundary condition that the flux and thus "s" vanish at $r = L$ gives (note that the term "J" vanishes when $s = 0$)

$$\sin(A_1 L + \sin^{-1} A_2) - A_2 = 0 \quad (3-33)$$

where

$$A_1 = -\frac{E^2}{G} \sqrt{H_3} (\pm 1),$$

$$A_2 = \frac{H_1}{\sqrt{H_1^2 + 4K_2 H_3}}.$$

Imposing the appropriate trigonometric identities and rearranging, equation (3-33) may be written as

$$\sin A_1 L (\sin A_1 L - 2A_2 \sqrt{1 - A_2^2}) = 0$$

and solutions are found to be

$$\sin A_1 L = 0$$

so $A_1 L = n\pi$ (where $n = 0, \pm 1, \pm 2, \dots$) and

$$\sin A_1 L = 2A_2 \sqrt{1 - A_2^2}.$$

The first solution leads to the following expression for the buckling

$$B_x^2 = \frac{H_2}{2} \frac{E_4}{G^2} = \left(\frac{p\pi}{L}\right)^2, \quad p = 0, \pm 1, \pm 2, \dots$$

which is the same as that found in equation (3-24) and (3-25).

It is now possible to express the general solution for the flux distribution at steady state, "critical", operation found from transport theory. For flux values such that $\phi_0 < 10^{13}$ then the general solution for the flux distribution in a slab is approximately the following:

$$\phi_0(r) = \frac{\lambda_x [\Sigma_a + (1 - \bar{\mu}_0) \Sigma_s]}{\bar{\alpha}_x [\Sigma_a + (1 - \bar{\mu}_0) \Sigma_s + \bar{\gamma} \Sigma_f]} \sum_{m=-\infty}^{\infty} A_m \sin \left\{ \frac{m\pi r}{L} - I_m(\phi_0(r)) \right\} \quad (3-34)$$

where for $A_m^2 < 1$

$$I_m(\phi_o(r)) = \frac{2\alpha}{\sqrt{1 - A_m^2}} \left\{ \text{Tan}^{-1} \sqrt{\frac{(1+A_m)}{(1-A_m)}} \left[1 + \frac{2(1+\alpha)\sigma_x \phi_o(r)}{A_m \lambda_x - (1+\alpha)\sigma_x \phi_o(r)} \right] \right. \\ \left. - \text{Tan}^{-1} \sqrt{\frac{(1+A_m)}{(1-A_m)}} \right\}$$

and for $A_m^2 > 1$

$$I_m(\phi_o(r)) = \frac{\alpha}{\sqrt{A_m^2 - 1}} \left\{ \text{Ln} \left[1 - \frac{2\sqrt{(A_m - 1)(A_m \lambda_x - \alpha \sigma_x \phi_o(r))}}{\sqrt{(A_m + 1)(A_m \lambda_x + \alpha \sigma_x \phi_o(r))} + \sqrt{(A_m - 1)(A_m \lambda_x - \alpha \sigma_x \phi_o(r))}} \right] \right. \\ \left. - \text{Ln} \left[1 - \frac{2\sqrt{A_m - 1}}{\sqrt{A_m + 1} + \sqrt{A_m - 1}} \right] \right\}$$

where

$$\alpha = \frac{\bar{\gamma} \Sigma_f}{\Sigma_a + (1 - \mu_o) \Sigma_o}$$

The arbitrary constants, " A_m 's", are found from satisfying the value for the observed flux level. It is interesting to observe that equation (3-34) reduces to the familiar expression for the equilibrium flux distribution found from diffusion theory when $\bar{\gamma}$ approaches zero. In equation (3-34) the term " I_m " involves the dependent variable $\phi_o(r)$ and thus is not expressed explicitly in terms of the dependent variable. Indeed, it appears impossibly complicated to obtain a complete, closed, analytic expression explicitly for $\phi_o(r)$. An expansion of the I_m term and then a reversion of the series would be required. However, equation (3-34) can be suitably approximated by analytical means or programmed for machine calculation. Many interesting conclusions may be gained from equation (3-34), perhaps the most significant is that the flux is not distributed in a simple sinusoidal manner and that the flux distribution is a complicated function of the power level when the influence of xenon is considered.

For flux values such that $\phi_0 > 10^{13}$, the general solution for the equilibrium flux distribution in a slab is given approximately by the following:

$$\phi_0(r) = \sum_{n=-\infty}^{\infty} K_n \sin \left\{ \frac{n\pi r}{L} + \sin^{-1} \frac{\delta}{K_n} + J \right\} - \delta \quad (3-35)$$

where

$$\delta = \frac{\lambda_x \left[1 - \frac{\bar{\gamma} \Sigma_f}{v \Sigma_f - \Sigma_a} \right]}{12 \sigma_x \left[1 + \frac{\bar{\gamma} \Sigma_f}{\Sigma_a + (1-\mu_0) \Sigma_s} \right] \left[\frac{1}{v \Sigma_f - \Sigma_a} + \frac{1}{\Sigma_a + (1-\mu_0) \Sigma_s} \right]},$$

$$K_n = \frac{G}{2 \sigma_x E H_3} \sqrt{H_1^2 + 4 K_2 H_3},$$

and the expression for the term "J" is given by either equation (3-31.2) or (3-31.3) as appropriate.

IV - SPACE AVERAGED KINETICS FOR XENON INDUCED FLUX HARMONIC INSTABILITY

In general, xenon induced flux instabilities, particularly flux oscillations, (i.e., if they exist in a reactor) are truly a spacial problem. Indeed, it is possible, in principle, for the flux to oscillate from one region to another even though the total reactor power remains constant (27, 31, 34). Thus, any accurate approach to the determination of possible xenon induced instabilities, especially of the first and higher harmonics of the flux, should consider spacial dependence.

As an approach to a time and space description of the flux, xenon, and iodine, consider the following reactor model (see Figure 25).

The reactor model core is assumed to be divided into two similar regions. It is not necessary to restrict the analysis to simple bifurcation of the core, however, and any system and number of divisions could be imposed in principle. Two energy groups of neutrons, a thermal and a fast group, will be considered in each region. The dependent variables (viz., the flux, xenon, and iodine) and the system parameters in a given region are assumed to be spacially averaged over that region. Thus, the dependent variables become functions only of time. Furthermore, it will be assumed that the power over the reactor model is maintained at some constant value. (See (18, 20) for a discussion of this assumption.)

Now a thermal neutron balance statement in region (a), for example, is as follows:

Figure 25. Reactor Model with Space Averaged Regions.

The rate of change of ϕ_a = Leakage of ϕ_a out of the reactor from region (a) + net coupling leakage from region (a) to (b) + absorption of ϕ_a in region (a) + slowing down of fast flux θ_a to thermal flux ϕ_a in region (a).

The mathematical formulation of the above typical neutron balance gives the following for the thermal and fast flux in region (a).

$$\frac{1}{v_o} \frac{d\phi_a}{dt} = - \frac{D_o}{A_o} \phi_a - \frac{D_o}{A_l} (\phi_a - \phi_b) - \Sigma'_{ao} \phi_a + \Sigma_s \theta_a, \quad (4-1.1)$$

$$\frac{1}{v_l} \frac{d\theta_a}{dt} = - \frac{D_l}{A_o} \theta_a - \frac{D_l}{A_l} (\theta_a - \theta_b) - \Sigma_{al} \theta_a + v\Sigma_f \phi_a, \quad (4-1.2)$$

where for region (a)

v_o the average thermal neutron velocity (cm/sec),

v_l the average fast neutron velocity (cm/sec),

$\phi_a(t)$ the thermal neutron flux (nts/cm²sec),

$\theta_a(t)$ the fast neutron flux (nts/cm²sec),

D_o the diffusion coefficient for thermal neutrons (cm),

D_l the diffusion coefficient for fast neutrons (cm),

A_o the effective area for leakage of neutrons out of region (a) (cm²),

A_l the effective coupling area for neutron leakage between regions (a) and (b) (cm²),

Σ'_{ao} the total macroscopic absorption cross section for thermal neutrons (1/cm)

Σ_{al} the total macroscopic removal cross section for fast neutrons (1/cm),

Σ_s the total effective macroscopic slowing down cross section for fast neutrons (1/cm),

$v\Sigma_f$ the average yield of fast neutrons from thermal fission (1/cm).

Now the total absorption of thermal neutrons in region (a) is that due to xenon-135 plus all other absorbers. So $\Sigma'_{ao}(t)$ may be written as

$$\Sigma'_{ao} = \Sigma_{ao} + \sigma_x X_a(t) \quad (4-2)$$

and the absorption of fast neutrons by xenon is neglected. The assumption of constant power will be defined as follows:

$$\phi_a(t) + \phi_b(t) = 2\phi_o \quad (4-3.1)$$

$$\theta_a(t) + \theta_b(t) = 2\theta_o \quad (4-3.2)$$

Equations (4-3) are assumed to hold for all time. If equations (4-2) and (4-3) are substituted into equations (4-1), then there results

$$\frac{1}{v_o} \frac{d\phi_a}{dt} = - \left(\frac{D_o}{A_o} + \frac{2D_o}{A_1} + \Sigma_{ao} \right) \phi_a - \sigma_x X_a \phi_a + \Sigma_s \theta_a + \frac{2D_o}{A_1} \phi_o, \quad (4-4.1)$$

$$\frac{1}{v_1} \frac{d\theta_a}{dt} = - \left(\frac{D_1}{A_o} + \frac{2D_1}{A_1} + \Sigma_{a1} \right) \theta_a + v \Sigma_f \phi_a + \frac{2D_1}{A_1} \theta_o. \quad (4-4.2)$$

The subscript denoting region (a) will be dropped under the further assumption that the same type of equations describe the flux behavior in region (b) as well as region (a), when ϕ_a is replaced by ϕ_b and θ_a is replaced by θ_b . Thus, the equations which describe the time behavior of the thermal and fast flux and xenon and iodine concentrations in either region are equations (4-4) and the following:

$$\frac{dX}{dt} = \bar{\gamma}_x \Sigma_f \phi + \lambda_1 I - \sigma_x X \phi - \lambda_x X, \quad (4-5.1)$$

$$\frac{dI}{dt} = \bar{\gamma}_i \Sigma_f \phi - \lambda_i I, \quad (4-5.2)$$

where the parameters of equations (4-5) have been previously defined in Chapter II.

The following transformations of dependent variables

$$w_o(t) = \sigma_x \phi(t),$$

$$w_1(t) = \sigma_x \theta(t),$$

$$x(t) = (\sigma_x / \bar{\gamma} \Sigma_f) X(t),$$

$$i(t) = (\sigma_x / \bar{\gamma} \Sigma_f) I(t),$$

where

$$\bar{\gamma}_x + \bar{\gamma}_i = \bar{\gamma}$$

transform equations (4-4) and (4-5) to give

$$\frac{dw_o}{dt} = -a_1 w_o - a_2 x w_o + a_3 w_1 + a_4 \bar{w}_o, \quad (4-6.1)$$

$$\frac{dw_1}{dt} = -b_1 w_1 + b_2 w_o + b_3 \bar{w}_1, \quad (4-6.2)$$

$$\frac{dx}{dt} = \gamma_x w_o + \lambda_i i - x w_o - \lambda_x x, \quad (4-6.3)$$

$$\frac{di}{dt} = \gamma_i w_o - \lambda_i i, \quad (4-6.4)$$

where

$$a_1 = \left(\frac{D_o}{A_o} + \frac{2D_o}{A_1} + \Sigma_{ao} \right) v_o,$$

$$a_2 = \bar{\gamma} \Sigma_f v_o,$$

$$a_3 = \Sigma_s v_o,$$

$$a_4 = \frac{2D_o v_o}{A_1},$$

$$b_1 = \left(\frac{D_1}{A_o} + \frac{2D_1}{A_1} + \Sigma_{a1} \right) v_1,$$

$$b_2 = v \Sigma_f v_1,$$

$$b_3 = \frac{2D_1 v_1}{A_1},$$

$$\gamma_i = \frac{\bar{\gamma}_i}{\bar{\gamma}},$$

$$\gamma_x = \frac{\bar{\gamma}_x}{\bar{\gamma}},$$

$$\bar{w}_0 = w_0(0),$$

$$\bar{w}_1 = w_1(0).$$

Equations (4-6) have nontrivial stationary values when

$$w_0(0) = \bar{w}_0 = \frac{\lambda_x(a_4 + \frac{a_3 b_2}{b_1 - b_3} - a_1)}{a_1 + a_2 - a_4 - \frac{a_3 b_2}{b_1 - b_3}},$$

$$w_1(0) = \bar{w}_1 = \frac{b_2}{b_1 - b_3} \bar{w}_0,$$

$$x(0) = x_0 = \frac{\bar{w}_0}{\bar{w}_0 + \lambda_x},$$

$$i(0) = i_0 = \frac{\gamma_i \bar{w}_0}{\lambda_i}.$$

Useful stability criteria for the model under study may be found if first order perturbation theory is employed to linearize equations (4-6). Thus the dependent variables are expanded as small perturbations from their respective equilibrium values as follows:

$$w_0(t) = \bar{w}_0 + \delta w_0(t),$$

$$w_1(t) = \bar{w}_1 + \delta w_1(t),$$

$$x(t) = x_0 + \delta x(t),$$

$$i(t) = i_0 + \delta i(t),$$

and substituted into equations (4-6). After neglecting perturbation products and subtracting out stationary values, the matrix equation of the Laplace Transformed dependent variables gives

$$\begin{bmatrix} s + a_1 + a_2 x_o & -a_3 & a_2 \bar{w}_o & 0 \\ -b_2 & s + b_1 & 0 & 0 \\ x_o - \gamma_x & 0 & s + \bar{w}_o + \lambda_x & -\lambda_i \\ -\gamma_i & 0 & 0 & s + \lambda_i \end{bmatrix} \begin{bmatrix} \delta \bar{w}_o(s) \\ \delta \bar{w}_1(s) \\ \delta \bar{x}(s) \\ \delta \bar{i}(s) \end{bmatrix} = 0. \quad (4-7)$$

Again for nontrivial solutions to exist the determinant of the coefficient matrix must vanish. The polynomial equation which results from expanding the determinant is

$$0 = s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4 \quad (4-8)$$

where

$$A_1 = a_1 + a_2 x_o + \bar{w}_o + \lambda_x + \lambda_i + b_1,$$

$$A_2 = (a_1 + a_2 x_o)(\bar{w}_o + \lambda_x) + b_1 \lambda_i - a_2 \bar{w}_o (x_o - \gamma_x) - a_3 b_2,$$

$$\begin{aligned} A_3 = & \gamma_i \lambda_i a_2 \bar{w}_o + b_1 \lambda_i (a_1 + a_2 x_o + \bar{w}_o + \lambda_x) \\ & + (b_1 + \lambda_i)(a_1 + a_2 x_o)(\bar{w}_o + \lambda_x) - a_2 \bar{w}_o (x_o - \gamma_x)(b_1 + \lambda_i) \\ & - a_3 b_2 (\bar{w}_o + \lambda_x + \lambda_i), \end{aligned}$$

$$\begin{aligned} A_4 = & \gamma_i \lambda_i a_2 b_1 \bar{w}_o + b_1 \lambda_i (a_1 + a_2 x_o)(\bar{w}_o + \lambda_x) \\ & - \lambda_i a_2 b_1 \bar{w}_o (x_o - \gamma_x) - a_3 b_2 \lambda_i (\bar{w}_o + \lambda_x). \end{aligned}$$

For stability it is necessary that all roots of equation (4-8) have negative real parts which is true if and only if

$$\text{I} \quad - \quad A_1 > 0,$$

$$\text{II} \quad - \quad A_2 > 0,$$

$$\text{III} \quad - \quad A_3 > 0,$$

$$\text{IV} \quad - \quad A_4 > 0,$$

$$\text{V} \quad - \quad A_1 A_2 > A_3,$$

$$\text{VI} \quad - \quad A_1 A_2 A_3 > A_1^2 A_4 + A_3^2.$$

It is obvious that investigation of these six requirements on the parameters of equations (4-6) would be a lengthy task to implement. However considerable simplification results if $\frac{1}{v_0} \frac{d\phi}{dt} \approx \frac{1}{v_1} \frac{d\theta}{dt} \approx 0$. The justification of this approximation rests upon two observations.

First: The reactor power is maintained constant, and perturbations from steady state are assumed to be small.

Second: The behavior in time of xenon induced oscillations of the flux is slow, and the effects of delayed neutrons upon the time behavior of the flux may be, in general, neglected (20).

Under the above assumption, the coefficient determinant of equation (4-7) then becomes

$$\begin{vmatrix} a_1 + a_2 x_0 & -a_3 & a_2 \bar{w}_0 & 0 \\ -b_2 & b_1 & 0 & 0 \\ x_0 - \gamma_x & 0 & s + \bar{w}_0 + \lambda_x & -\lambda_i \\ -\gamma_i & 0 & 0 & s + \lambda_i \end{vmatrix} = 0.$$

And the expanded determinant is given by

$$0 = B_0 s^2 + B_1 s + B_2$$

where

$$B_0 = (a_1 + a_2 x_0) b_1 - a_3 b_2,$$

$$B_1 = (\bar{w}_0 + \lambda_x + \lambda_i) [b_1(a_1 + a_2 x_0) - a_3 b_2] - a_2 \bar{w}_0 b_1 (x_0 - \gamma_x),$$

$$B_2 = a_2 b_1 \lambda_i \bar{w}_0 (1 - x_0) + (\bar{w}_0 + \lambda_x) \lambda_i [b_1(a_1 + a_2 x_0) - a_3 b_2].$$

Now for stability it is necessary that the following conditions be satisfied:

$$I \quad B_0 > 0$$

which implies

$$(a_1 + a_2 x_0) b_1 > a_3 b_2.$$

$$II \quad B_1 > 0$$

then

$$(\bar{w}_0 + \lambda_x + \lambda_i) b_1 (a_1 + a_2 x_0) + \gamma_x a_2 b_1 \bar{w}_0 > a_3 b_2 (\bar{w}_0 + \lambda_x + \lambda_i) + a_2 b_1 \bar{w}_0 x_0.$$

$$III \quad B_2 > 0$$

and

$$a_2 b_1 \bar{w}_0 + (\bar{w}_0 + \lambda_x) (a_1 + a_2 x_0) b_1 > (\bar{w}_0 + \lambda_x) a_3 b_2 + a_2 b_1 \bar{w}_0 x_0.$$

Now the definition of equilibrium xenon was

$$x_0 = \frac{\bar{w}_0}{\bar{w}_0 + \lambda_x}$$

and since $x_0 \leq 1$ for $w_0 \geq 0$, so condition III is satisfied if condition I is satisfied.

It can be shown that if

$$\bar{w}_0 < \frac{\gamma_x}{\gamma_i} \lambda_x$$

then only condition I must be satisfied for stability. If

$$\bar{w}_0 > \frac{\gamma_x}{\gamma_i} \lambda_x$$

then only condition II must be satisfied.

Now returning to the stationary values given in equations, it is found that

$$x_0 = \frac{\bar{w}_0}{\bar{w}_0 + \lambda_x} = \frac{1}{a_2} \left(\frac{a_3 b_2}{b_1 - b_2} + a_4 - a_1 \right)$$

and substituting this into condition I yields

$$b_1 + \frac{a_3 b_2 b_3}{a_4 b_1} > b_3.$$

Recalling the definitions for the a's and b's, one obtains

$$\frac{D_1}{A_0} + \Sigma_{a1} + \frac{v \Sigma_f \Sigma_s D_1}{D_0 \left(\frac{D_1}{A_0} + \frac{2D_1}{A_1} + \Sigma_{a1} \right)} > 0$$

which is obviously satisfied.

Thus, condition I is satisfied for all values of flux and only condition II must be satisfied for perturbation stability. It was found that condition II was satisfied

and thus the reactor model is stable if $\bar{w}_0 \leq \frac{\gamma_x}{\gamma_i} \lambda_x \left(\phi_0 \leq 3 \times 10^{11} \right)$, which

is in agreement with other studies of xenon induced instabilities (26, 38, 41).

Condition II, may be written as follows:

$$a_1 b_1 + a_2 b_1 \frac{\bar{w}_0}{\bar{w}_0 + \lambda_x + \lambda_i} \left(\gamma_x + \frac{\lambda_x + \lambda_i}{\bar{w}_0 + \lambda_x} \right) > a_3 b_2$$

or

$$\left[\frac{D_0}{A_0} + \frac{2D_0}{A_1} + \Sigma_{a0} + \bar{\gamma} \Sigma_f \frac{\bar{w}_0}{\bar{w}_0 + \lambda_x} \left(\gamma_x + \frac{\lambda_x + \lambda_i}{\bar{w}_0 + \lambda_x} \right) \right] \left(\frac{D_1}{A_0} + \frac{2D_1}{A_1} + \Sigma_{a1} \right) > v \Sigma_f \Sigma_s.$$

Condition II may be written in several different forms in addition to those above. Indeed, if one lets $\Sigma_s = \Sigma_{a1}$ and incorporates any absorption of the

fast flux that may occur within the resonance escape probability contained in the infinite multiplication factor which will be defined as $\nu \Sigma_f = k_\infty \Sigma_{a0}$ then condition II becomes

$$\left[1 + B^2 L_0^2 + \frac{\bar{\gamma} \Sigma_f}{\Sigma_{a0}} \frac{\bar{w}_0}{\bar{w}_0 + \lambda_x} \left(\gamma_x + \frac{\lambda_x + \lambda_i}{\bar{w}_0 + \lambda_x} \right) \right] \left[1 + B^2 L_1^2 \right] > k_\infty$$

where

$$L_0^2 = \frac{D_0}{\Sigma_{a0}} \quad \text{the diffusion length squared for thermal neutrons,}$$

$$L_1^2 = \frac{D_1}{\Sigma_{a1}} \quad \text{the effective slowing down length squared for fast neutrons,}$$

$$B^2 = \frac{1}{A_0} + \frac{2}{A_1} \quad \text{the geometrical buckling for the model.}$$

V - SPACE AND TIME DEPENDENT, MONO-ENERGY GROUP KINETICS WITH XENON DEPENDENCE

In this chapter the problem of xenon induced flux instabilities will be examined employing one energy group, time and space dependent diffusion theory.

INTRODUCTION AND EQUILIBRIUM ANALYSIS

It has been previously shown in Chapter VI that diffusion theory does not give results identical with the more correct P_1 approximations to the transport equation when xenon is considered. However, in many practical cases, since xenon exhibits its "large" cross section (see Appendix A) chiefly at thermal energies and the angular dependence of the flux has little effect on reactor instabilities induced by xenon, diffusion theory is adequate and simple to employ.

Furthermore, it is assumed that the reactor model under investigation has no strong isolated neutron sources or sinks which would violate the assumptions of diffusion theory. Indeed, the fuel distribution is assumed to be approximately uniform over the reactor model core. Also, the more precise description of the flux at boundaries and interfaces given by transport theory is deemed unnecessary for determining overall stability of the flux.

Consider then that the following equations describe the flux, xenon-135 and iodine-135, in space and time for a given model.

$$\frac{1}{v} \frac{\partial \phi(\bar{r}, t)}{\partial t} = D \nabla^2 \phi + (v \Sigma_f - \Sigma_a - \sigma_x X) \phi, \quad (5-1.1)$$

$$\frac{\partial X(\bar{r}, t)}{\partial t} = \bar{\gamma}_x \Sigma_f \phi + \lambda_i I - \sigma_x X \phi - \lambda_x X, \quad (5-1.2)$$

$$\frac{\partial I(\bar{r}, t)}{\partial t} = \bar{\gamma}_i \Sigma_f \phi - \lambda_i I, \quad (5-1.3)$$

where the parameters and variables have the same definitions as have been previously defined, and $\bar{r}$ denotes the generalized space variables, and ∇^2 is the usual Laplacian Operator (25).

Now at steady state or equilibrium the nontrivial, stationary values for the dependent variables are given by

$$D \nabla^2 \phi_o + \left(v \Sigma_f - \Sigma_a \right) \phi_o - \bar{\gamma} \Sigma_f \phi_o \left(\frac{\sigma_x \phi_o}{\sigma_x \phi_o + \lambda_x} \right) = 0, \quad (5-2.1)$$

$$X_o = \frac{\bar{\gamma} \Sigma_f \phi_o}{\sigma_x \phi_o + \lambda_x}, \quad (5-2.2)$$

$$I_o = \frac{\bar{\gamma} \Sigma_f \phi_o}{\lambda_i}, \quad (5-2.3)$$

and letting $\lambda_o = \sigma_x \phi_o$ equation (5-2.1) for the equilibrium flux becomes

$$\nabla^2 \lambda_o + \left(\frac{v \Sigma_f - \Sigma_a}{D} \right) \lambda_o - \frac{\bar{\gamma} \Sigma_f}{D} \left(\frac{\lambda_o^2}{\lambda_o + \lambda_x} \right) = 0. \quad (5-3)$$

Equation (5-2.1) can also be written as

$$\nabla^2 \lambda_o + \frac{1}{M^2} \left[\left(\frac{v \Sigma_f}{\Sigma_a + \bar{\gamma} \Sigma_f} - 1 \right) \lambda_o + \left\{ \frac{(v + \bar{\gamma}) \Sigma_f}{\Sigma_a + \bar{\gamma} \Sigma_f} - 1 \right\} \lambda_x \right] \frac{\lambda_o}{\lambda_o + \lambda_x} = 0 \quad (5-4.1)$$

where

$$M^2 \rightarrow \frac{D}{\Sigma_a + \bar{\gamma} \Sigma_f}. \quad (\bar{\gamma} \Sigma_f \text{ represents the absorption due to equilibrium xenon at high flux levels.}) \quad (5-4.2)$$

The parameter M^2 (the migration length squared) is not strictly equal to the above expression, indeed, $D/(\Sigma_a + \bar{\gamma} \Sigma_f)$ is actually the thermal diffusion length squared for neutrons at high flux levels. However, considerable improvement results in the description of the one energy group flux if the migration length is employed rather than the diffusion length only. For a general commentary on this matter see Meghrebian and Holms (33) and Wick (50). Thus, whenever it appears advantageous, the substitution in expression (5-4.2) will be made.

Actually, it is not necessary to expend great effort attempting to solve either equation (5-3) or (5-4.1) since the equilibrium distribution for the flux

resulting from the P_1 approximations of the transport equation has been found previously (see Chapter III). However, in the flux region of particular interest ($\phi_0 > 10^{13}$) several approximation techniques exist which permit simple, closed, and analytic solutions for equations (5-3) and (5-4.1).

One approximation technique is to consider the expanded form of the following expression for $\lambda_0 > \lambda_x$

$$\frac{\lambda_0}{\lambda_0 + \lambda_x} = \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{\lambda_x}{\lambda_0} \right\}^n \quad (5-5)$$

and if only the first two terms are retained then equations (5-4) become

$$\nabla^2 \lambda_0 + \frac{1}{M^2} \left(\frac{v\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f} - 1 \right) \lambda_0 + \frac{1}{M^2} \left(\frac{\bar{\gamma}\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f} \right) \lambda_x = 0 \quad (5-6)$$

which is simply the time independent wave equation with a nonhomogeneous term. However, equation (5-6) is not valid at the reactor boundaries where the flux must vanish. Indeed, equation (5-5) is not valid for $\lambda_0 \leq \lambda_x$; but, if the flux level is sufficiently high, the general solution for equation (5-6) for a slab reactor with the boundary conditions that $\lambda_0(0) = \lambda_0(L) = 0$ is reasonably given by

$$\phi_0(r) = \sum_{n=0}^{\infty} A_n \sin B_n r - \frac{\bar{\gamma} \Sigma_f \lambda_x}{\sigma_x (v\Sigma_f - \Sigma_a - \bar{\gamma}\Sigma_f)} [1 - \delta(r) - \delta(r-L)]$$

where

$$B_n^2 = \left(\frac{n\pi}{L} \right)^2 = \frac{1}{M^2} \left(\frac{v\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f} - 1 \right),$$

A_n are arbitrary constants found by assigning the flux,

$$\delta(r) \text{ (Dirac delta function)} = \begin{cases} 0, & r \neq 0 \\ 1, & r = 0. \end{cases}$$

For criticality the fundamental mode gives

$$B_1^2 = \left(\frac{\pi}{L} \right)^2 = \left(\frac{v\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f} - 1 \right)$$

or more familiarly

$$\frac{k_{\infty}}{1 + M^2 B_1^2} = 1$$

where

$$k_{\infty} = \frac{v \Sigma_f}{\Sigma_a + \bar{\gamma} \Sigma_f}$$

and is defined to be the infinite multiplication factor for the system at high flux ($\phi_0 > 10^{13}$), and $1/(1+M^2 B_1^2)$ is the nonleakage probability for neutrons.

Another approximation technique is to consider from equation (5-6) the following term (called "T" for convenience).

$$T = \frac{1}{\lambda_0 + \lambda_x} \left[\left(\frac{v \Sigma_f}{\Sigma_a + \bar{\gamma} \Sigma_f} - 1 \right) \lambda_0 + \left\{ \frac{(v + \bar{\gamma}) \Sigma_f}{\Sigma_a + \bar{\gamma} \Sigma_f} - 1 \right\} \lambda_x \right]$$

which can be written as follows.

$$T = \frac{k_{\infty} - 1}{1 + \lambda_x / \lambda_0} \left[1 + \left(1 + \frac{\bar{\gamma}}{v} \frac{k_{\infty}}{k_{\infty} - 1} \right) \frac{\lambda_x}{\lambda_0} \right]$$

For $\lambda_0 > \lambda_x$ the denominator of "T" can be expanded to give

$$T = (k_{\infty} - 1) \left\{ 1 + \frac{\bar{\gamma}}{v} \frac{k_{\infty}}{k_{\infty} - 1} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\lambda_x}{\lambda_0} \right)^n \right\}. \quad (5-7)$$

To determine somewhat the rate of convergence of equation (5-7) consider the following typical values:

$$\bar{\gamma} = 0.063$$

$$v = 2.47 \text{ (value for U-235)}$$

$$k_{\infty} \approx 1.1 \text{ (assumes total leakage of approximately 10% or greater)}$$

$$\lambda_0 / \lambda_x = 45 \text{ (implies } \phi_0 \approx 3 \times 10^{14} \text{)}$$

Thus "T" becomes numerically

$$T = (k_{\infty} - 1) \left\{ 1 + 0.00566 - 0.000126 + 0.0000028 - \dots \right\}$$

which may be reasonably approximated as

$$T \approx k_{\infty} - 1$$

and then equation (5-4.1) becomes

$$\nabla^2 \lambda_o + \left(\frac{k_\infty - 1}{M^2} \right) \lambda_o = 0. \quad (5-8)$$

Equation (5-8) gives the following general solution for a slab reactor with the boundary conditions that $\phi_o(0) = \phi_o(L) = 0$.

$$\phi_o(r) = \sum_n A_n \sin B_n r$$

where as before

$$B_n^2 = \left(\frac{n\pi}{L} \right)^2, \quad n = 0, \pm 1, \pm 2, \dots$$

and

$$B_1^2 = \left(\frac{\pi}{L} \right)^2 = \frac{1}{M^2} (k_\infty - 1)$$

and for a uniform cylindrical reactor model, equation (5-8) gives

$$\phi_o(r, z) = \sum_{n,m} C_{n,m} \sin \left(\frac{n\pi r}{L} \right) J_o \left(\frac{mr}{r_o} \right)$$

where

$$B_{n,m}^2 = \left(\frac{n\pi}{L} \right)^2 + \left(\frac{m}{r_o} \right)^2$$

and "m" is given by the roots of $J_o(m) = 0$.

Now returning to equations (5-1) considerable convenience results if the dependent variables of equations (5-1.1), (5-1.2), and (5-1.3) are transformed into the following unitless, normalized variables:

$$\phi(\bar{r}, t) = \phi_o(\bar{r}) \phi(\bar{r}, t) \quad (5-9.1)$$

$$X(\bar{r}, t) = X_o(\bar{r}) x(\bar{r}, t) \quad (5-9.2)$$

$$I(\bar{r}, t) = I_o(\bar{r}) i(\bar{r}, t) \quad (5-9.3)$$

If the above transformation of variables is imposed upon equations (5-1) there results

$$\tau_e \frac{\partial \phi}{\partial t} = \frac{1}{\rho_o} \nabla^2 \phi + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \phi + (1 - x) \phi \frac{\lambda_o}{\lambda_o + \lambda_x} \quad (5-10.1)$$

$$\frac{\partial x}{\partial t} = (\lambda_o + \lambda_x)(\gamma_x \phi + \gamma_i i) - (\lambda_o \phi + \lambda_x)x, \quad (5-10.2)$$

$$\frac{\partial i}{\partial t} = \lambda_i (\phi - i), \quad (5-10.3)$$

where

$$\rho_o = \frac{\bar{\gamma}\Sigma_f}{D} \rightarrow \frac{1}{M^2} \left\{ \frac{\bar{\gamma}\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f} \right\} = \frac{\bar{\gamma}}{v} \frac{k_\infty}{M^2}, \quad (5-10.4)$$

$$\tau_e = \frac{1}{v\bar{\gamma}\Sigma_f} = \ell \left\{ \frac{\Sigma_a + \bar{\gamma}\Sigma_f}{\bar{\gamma}\Sigma_f} \right\} = \ell \frac{v}{\bar{\gamma}k_\infty}, \quad (5-10.5)$$

$\ell = \frac{1}{v(\Sigma_a + \bar{\gamma}\Sigma_f)}$ is the effective neutron lifetime for high flux levels,

$$\lambda_o(\bar{r}) = \sigma_x \phi_o(\bar{r}), \quad (5-10.6)$$

$\nabla\phi$ = gradient of ϕ

$k_\infty = \frac{v\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f}$ is the infinite multiplication factor for high flux levels.

It is possible to eliminate $x(t)$ and $i(t)$ from equations (5-10) by the following procedure. If equation (5-10.3) is solved for $i(t)$ using as an initial condition that $i(0) = 1$, one obtains

$$i(t) = \lambda_i \int_0^t \phi(\xi) e^{\lambda_i(\xi-t)} d\xi + e^{-\lambda_i t} \quad (5-11)$$

Equation (5-11) for $i(t)$ may be substituted into equation (5-10.2) and the following equation found for $x(t)$ for the initial condition that $x(0) = 1$.

$$\begin{aligned} x(t) = & \frac{\gamma(\lambda_o + \lambda_x)}{\lambda_o + \lambda_x} \int_0^t \phi(\xi) \exp \left\{ \lambda_o \int_0^\xi \phi d\xi - \lambda_o \int_0^t \phi dt + \lambda_x(\xi-t) \right\} d\xi \\ & + \gamma_i \lambda_i (\lambda_o + \lambda_x) \int_0^t \int_0^\xi \phi(\eta) \exp \left\{ \lambda_o \int_0^\xi \phi d\xi - \lambda_o \int_0^t \phi dt + \lambda_x(\xi-t) + \lambda_i(\eta-\xi) \right\} d\eta d\xi \\ & + \gamma_i (\lambda_o + \lambda_x) \int_0^t \exp \left\{ \lambda_o \int_0^\xi \phi d\xi - \lambda_o \int_0^t \phi dt + \lambda_x(\xi-t) - \lambda_i \xi \right\} d\xi \end{aligned}$$

$$+ \exp \left\{ -\lambda_0 \int \phi dt - \lambda_x t \right\} \quad (5-12)$$

And finally equation (5-12) may be substituted into equation (5-10.1) to give

$$\begin{aligned} \tau_e \frac{\partial \phi}{\partial t} = & \frac{1}{\rho_0} \nabla^2 \phi + \frac{2}{\rho_0} \frac{\nabla \lambda_0}{\lambda_0} \nabla \phi + \frac{\lambda_0}{\lambda_0 + \lambda_x} \left[1 - \exp \left(-\lambda_0 \int \phi dt - \lambda_x t \right) \right. \\ & - \gamma_x (\lambda_0 + \lambda_x) \int_0^t \phi(\xi) \exp \left\{ \lambda_0 \int \phi d\xi - \lambda_0 \int \phi dt + \lambda_x (\xi - t) \right\} d\xi \\ & - \gamma_i \lambda_i (\lambda_0 + \lambda_x) \int_0^t \int_0^\xi \phi(\eta) \exp \left\{ \lambda_0 \int \phi d\xi - \lambda_0 \int \phi dt + \lambda_x (\xi - t) + \lambda_i (\eta - \xi) \right\} d\eta d\xi \\ & \left. - \gamma_i (\lambda_0 + \lambda_x) \int_0^t \exp \left\{ \lambda_0 \int \phi d\xi - \lambda_0 \int \phi dt + \lambda_x (\xi - t) - \lambda_i \xi \right\} d\xi \right]. \end{aligned} \quad (5-13)$$

If one recalls that $\lambda_0(\bar{r})$ is a function of space and observes the manner in which the dependent variable occurs in equation (5-13), its severe complexity and nonlinearity become obvious. Indeed, the possibility of securing an analytic solution to this nonlinear, integral, partial differential equation seems extremely remote. Of course, equation (5-13) could be placed in quadratures to eliminate the time derivative term but no significant simplification results. Indeed, equation (5-13) has been given in the particular form presented because one approximation technique, which has been employed by many others (20, 38, 50), is to neglect the term " $\tau_e \frac{\partial \phi}{\partial t}$ " as being small compared to the other terms in the equation.

LINEAR KINETIC EQUATIONS DETERMINED BY FIRST ORDER PERTURBATION THEORY

In an attempt to discover the nature of the solutions of equations (5-10) a reasonable beginning is to make the equations linear by imposing perturbation analysis. Thus, if the dependent variables are expressed as departures from their equilibrium values as follows.

$$\phi(\bar{r}, t) = 1 + \delta\phi(\bar{r}, t) \quad (5-14.1)$$

$$x(\bar{r}, t) = 1 + \delta x(\bar{r}, t) \quad (5-14.2)$$

$$i(\bar{r}, t) = 1 + \delta i(\bar{r}, t) \quad (5-14.3)$$

Inserting equations (5-14) into equations (5-10) gives

$$\tau_e \frac{\partial \delta\phi}{\partial t} = \frac{1}{\rho_o} \nabla^2 \delta\phi + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta\phi - \frac{\lambda_o}{\lambda_o + \lambda_x} \delta x (1 + \delta\phi), \quad (5-15.1)$$

$$\frac{\partial \delta x}{\partial t} = (\lambda_o + \lambda_x)(\gamma_x \delta\phi + \gamma_i \delta i - \delta x) - \lambda_o (1 + \delta x) \delta\phi, \quad (5-15.2)$$

$$\frac{\partial \delta i}{\partial t} = \lambda_i (\delta\phi - \delta i). \quad (5-15.3)$$

Now the approximation (and thus the error) which is imposed is that the term $\delta x \delta\phi$ is of second order and thus may be neglected when compared to the terms δx and $\delta\phi$. It is precisely this assumption of neglecting perturbation products in equations (5-15) which incurs the error implicit in perturbation analysis. If the perturbation products in equations (5-15) are not neglected, then equations (5-15) are just as valid as equations (5-10). This should be kept in mind since equations (5-15) will prove useful for a later analysis. Returning then to equations (5-15) and neglecting $\delta x \delta\phi$ terms, there results

$$\tau_e \frac{\partial \delta\phi}{\partial t} = \frac{1}{\rho_o} \nabla^2 \delta\phi + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta\phi - \frac{\lambda_o}{\lambda_o + \lambda_x} \delta x, \quad (5-16.1)$$

$$\frac{\partial \delta x}{\partial t} = (\lambda_o + \lambda_x)(\gamma_x \delta\phi + \gamma_i \delta i - \delta x) - \lambda_o \delta\phi, \quad (5-16.2)$$

$$\frac{\partial \delta i}{\partial t} = \lambda_i (\delta\phi - \delta i). \quad (5-16.3)$$

Since equations (5-16) are now linear, their time dependence may be removed by Laplace transformation techniques. Then for the initial conditions that $\delta\phi = \delta x = \delta i = 0$ at $t = 0$ one obtains

$$\delta \bar{i}(s) = \left[\frac{\lambda_i}{s + \lambda_i} \right] \delta \bar{\phi}(s) \quad (5-17)$$

where the term in brackets is the "transfer function" for iodine.

Substituting equation (5-17) for $\delta i(s)$ into the transform of equation (5-16.2) gives

$$\delta \bar{x}(s) = \left[\frac{s(\gamma_x \lambda_x - \gamma_i \lambda_o) + \lambda_i \lambda_x}{(s + \lambda_i)(s + \lambda_o + \lambda_x)} \right] \delta \bar{\phi}(s) \quad (5-18)$$

where the term in brackets is the "transfer function" for xenon.

Finally, equation (5-18) may be substituted into the transform of equation (5-16.1) to give

$$\nabla^2 \delta \bar{\phi}(s) + 2 \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta \bar{\phi}(s) + \rho_o \left[\frac{\lambda_o}{\lambda_o + \lambda_x} \left\{ \frac{(\gamma_i \lambda_o - \gamma_x \lambda_x)s - \lambda_i \lambda_x}{(s + \lambda_o + \lambda_x)(s + \lambda_i)} \right\} - \tau_e s \right] \delta \bar{\phi} = 0, \quad (5-19)$$

It would be satisfying to imply that equation (5-19) could be readily solved. However, it must be recalled that $\lambda_o(\bar{r})$, which is proportional to the equilibrium flux, is a function of space; thus, the coefficients of $\delta \bar{\phi}(s)$ and $\nabla \delta \bar{\phi}$ are spatially dependent.

Modal Analysis of the Linear Kinetic Equations

A technique to obtain quasi-solutions for the spatially dependent, perturbed flux as given by equation (5-19) can be devolved from a more general mathematical technique called modal analysis. The general application of modal analysis to reactor problems has been discussed by Kaplan (23) and others. Briefly, modal analysis entails the expansion of the dependent variables of interest into some appropriate, infinite power series of eigen functions which possess the property of being complete and orthogonal over the volume under study. In general, in reactor application, a term of the expanded series may be approximately associated with an appropriate spacial mode of the physical flux. This association is reflected by the series coefficient and the eigen value corresponding to its respective term in the series.

Equation (5-19) may be written as follows

$$\nabla^2 \delta \bar{\phi} + 2 \frac{\nabla \lambda_o}{\lambda_o} \delta \bar{\phi} + G(\bar{r}, s) \delta \bar{\phi} = 0 \quad (5-20)$$

where

$$G(\bar{r}, s) = \rho_0 \left[\frac{\lambda_0(\bar{r})}{\lambda_0(\bar{r}) + \lambda_x} \left\{ \frac{(\gamma_i \lambda_0(\bar{r}) - \gamma_x \lambda_x) s - \lambda_i \lambda_x}{(s + \lambda_0(\bar{r}) + \lambda_x)(s + \lambda_i)} \right\} - \tau_e s \right].$$

Now $G(\bar{r}, s)$ is a spacially dependent function, and if the perturbed flux $\delta\phi(\bar{r}, s)$ is expanded in an appropriate series of orthogonal eigen functions it is found that because of the spacial dependence of G the coefficients of the respective terms in the series are coupled to each other. The physical counterpart of this coupling of the series coefficients is the modal coupling of the flux. Several stability studies (38, 50) have neglected the effect of modal interaction (or for the equivalent mathematical counterpart the coupling of the coefficients of the series terms), and generally satisfactory stability criteria have been found. However, in some instances modal coupling can be significant, and the error imparted to the stability criteria is not always conservative (16, 26).

It is possible to obtain some quantitative knowledge of the technique of modal analysis and the effects and importance of modal coupling. To exemplify the analysis, the model to be investigated will be assumed to have an equilibrium distribution of the flux along the dimension of interest which is reasonably sinusoidal. Then $\lambda_0(\bar{r})$ may be expressed as follows:

$$\lambda_0(r) = \alpha \lambda_x \sin B_1 r \quad (5-21)$$

where

$\alpha \approx$ a unitless parameter which is a function of the remaining spacial coordinates and is determined by the equilibrium flux distribution,

$$B_1 = \frac{\pi}{L}.$$

The expression for $G(r, s)$ becomes

$$G(r, s) = a_1 \left[\frac{\sin B_1 r (\sin B_1 r - a_2)}{(\sin B_1 r + a_3)(\sin B_1 r + a_4)} \right] - a_5 \quad (5-22)$$

where

$$a_1 = \gamma_i \rho_0 \frac{s}{s + \lambda_i},$$

$$a_2 = \frac{\gamma_x}{\gamma_1 \alpha} + \frac{\lambda_1}{\gamma_1 \alpha s},$$

$$a_3 = \frac{1}{\alpha},$$

$$a_4 = \frac{1}{\alpha} + \frac{s}{\alpha \lambda_x},$$

$$a_5 = \rho_o \tau_e s.$$

In general, the higher the mode, the greater is the damping due to leakage. Thus, the fundamental mode is the most unstable. So if adequate analysis indicates that the fundamental mode is stable then the higher modes (the second, third, fourth, etc. mode are often referred to as the first, second, third, etc. harmonic, respectively) are also stable. Since the fundamental mode and the first few harmonics contribute most to the flux distribution the series expansion for the Laplace transformed flux will be truncated as follows:

$$\delta\phi(r, s) = \sum_{n=1}^4 A_n(s) \sin nB_1 r \quad (5-23)$$

Combining equations (5-20) and (5-23) leads to the following equation.

$$\begin{aligned} & \sum_{n=1}^4 A_n(s) (C_n \sin^2 B_1 r + D_n \sin B_1 r + E_n) \sin nB_1 r = 0 \\ & + \sum_{n=1}^4 2n A_n(s) B_1^2 \cos B_1 r \left[\sin^2 B_1 r + a_6 \sin B_1 r + a_7 \right] \cos nB_1 r = 0 \end{aligned} \quad (5-24)$$

where

$$C_n = a_1 - B_n^2,$$

$$D_n = -a_1 a_2 - (a_3 + a_4) B_n^2,$$

$$E_n = -a_3 a_4 B_n^2,$$

$$B_n^2 = \left(\frac{n\pi}{L}\right)^2,$$

$$a_6 = a_3 + a_4,$$

$$a_7 = a_3 a_4,$$

and the term "a₅" has been set to zero.

Now equation (5-24) contains four undetermined coefficients, the A's, and it is necessary to determine four linearly independent equations to uniquely determine these coefficients and thus establish the characteristic roots (the s's) which will evince the appropriate linear stability criteria.

Although there are many ways of establishing four linearly independent equations from equation (5-24), the following technique will be employed for convenience here. If each term in equation (5-24) is multiplied by the term, $\sin B_1 r$, and the result is integrated in r from 0 to L , there results

$$\begin{aligned}
0 = A_1 & \left\{ C_1 \int_0^\pi \sin^5 \theta \, d\theta + D_1 \int_0^\pi \sin^4 \theta \, d\theta + E_1 \int_0^\pi \sin^3 \theta \, d\theta \right. \\
& + 2B_1^2 \left[\int_0^\pi \sin^3 \theta \cos^2 \theta \, d\theta + a_6 \int_0^\pi \sin^2 \theta \cos^2 \theta \, d\theta \right. \\
& \left. \left. + a_7 \int_0^\pi \sin \theta \cos^2 \theta \, d\theta \right] \right\} \\
& + A_3 \left\{ 4 \left[C_3 \int_0^\pi \sin^5 \theta \cos^2 \theta \, d\theta + D_3 \int_0^\pi \sin^4 \theta \cos^2 \theta \, d\theta \right. \right. \\
& \left. \left. + E_3 \int_0^\pi \sin^3 \theta \cos^2 \theta \, d\theta \right] \right. \\
& - \left[C_3 \int_0^\pi \sin^5 \theta \, d\theta + D_3 \int_0^\pi \sin^4 \theta \, d\theta + E_3 \int_0^\pi \sin^3 \theta \, d\theta \right] \\
& + 6B_1^2 \left[4 \left(\int_0^\pi \sin^3 \theta \cos^4 \theta \, d\theta + a_6 \int_0^\pi \sin^2 \theta \cos^4 \theta \, d\theta + a_7 \int_0^\pi \sin \theta \cos^4 \theta \, d\theta \right) \right. \\
& \left. \left. - 3 \left(\int_0^\pi \sin^3 \theta \cos^2 \theta \, d\theta + a_6 \int_0^\pi \sin^2 \theta \cos^2 \theta \, d\theta + a_7 \int_0^\pi \sin \theta \cos^2 \theta \, d\theta \right) \right] \right\}
\end{aligned}$$

where the following trigonometric relations have been employed.

$$\sin 2 B_1 r = 2 \sin B_1 r \cos B_1 r, \quad (5-26.1)$$

$$\sin 3 B_1 r = \sin B_1 r (3 - 4 \sin^2 B_1 r), \quad (5-26.2)$$

$$\sin 4 B_1 r = 4 \cos B_1 r \sin B_1 r (2 \cos^2 B_1 r - 1), \quad (5-26.3)$$

$$\cos 2 B_1 r = 2 \cos^2 B_1 r - 1 \quad (5-26.4)$$

$$\cos 3 B_1 r = \cos B_1 r (4 \cos^2 B_1 r - 3) \quad (5-26.5)$$

$$\cos 4 B_1 r = 8 \cos^4 B_1 r - 8 \cos^2 B_1 r + 1 \quad (5-26.6)$$

and the following recursion relationship is useful (11).

$$\int_0^\pi \sin^\nu \theta \cos^\eta \theta d\theta = [1 + (-1)^\eta] \frac{\Gamma(\frac{\nu+1}{2}) \Gamma(\frac{\eta+1}{2})}{2 \Gamma(\frac{\nu+\eta}{2} + 1)} \quad (\nu, \eta \geq -1)$$

where

$$\Gamma(t) = \int_0^\infty e^{-\xi} \xi^{t-1} d\xi$$

denotes the familiar gamma function (25).

Completion of the integration of equation (5-25) yields the following. (Notice that the coefficients of A_2 and A_4 vanish which indicates the mutual independence between even and odd flux modes under linear analysis.)

$$M_1 A_1 + M_3 A_3 = 0 \quad (5-27)$$

where

$$M_1 = \frac{16}{15} C_1 + \frac{3\pi}{8} D_1 + \frac{4}{3} E_1 + 2B_1^2 \left(\frac{4}{15} + \frac{\pi}{8} a_6 + \frac{2}{3} a_7 \right),$$

$$M_2 = -\frac{16}{35} C_3 - \frac{\pi}{8} D_3 - \frac{4}{15} E_3 - 6B_1^2 \left(\frac{12}{35} + \frac{\pi}{8} a_6 + \frac{2}{5} a_7 \right).$$

To establish another linearly independent relationship, equation (5-24) may be multiplied by the term, $\sin 3 B_1 r$, and the result integrated in r from 0 to L to give

$$N_1 A_1 + N_3 A_3 = 0 \quad (5-28)$$

where

$$N_1 = -\frac{48}{105} C_1 - \frac{\pi}{8} D_1 - \frac{4}{15} E_1 + 2B_1^2 \left(\frac{4}{21} + \frac{\pi}{8} a_6 + \frac{14}{15} a_7 \right),$$

$$N_3 = \frac{208}{315} C_3 + \frac{\pi}{4} D_3 + \frac{36}{35} E_3 - 6B_1^2 \left(\frac{4}{315} + \frac{6}{35} a_7 \right).$$

To establish a third linearly independent equation, equation (5-24) is multiplied by $\sin 2B_1$ and then integrated in r from 0 to L to give

$$P_2 A_2 + P_4 A_4 = 0 \quad (5-29)$$

where

$$P_2 = \frac{16}{105} C_2 + \frac{\pi}{16} D_2 + \frac{4}{15} E_2 + 4B_1^2 \left(-\frac{2}{105} + \frac{1}{15} a_7 \right),$$

$$P_4 = -\frac{32}{315} C_4 - \frac{\pi}{32} D_4 - \frac{8}{105} E_4 - 8B_1^2 \left(\frac{22}{315} + \frac{\pi}{32} a_6 + \frac{13}{105} a_7 \right).$$

Finally, to establish a fourth linearly independent equation, if equation (5-24) is multiplied by $\sin 4B_1 r$ and the result is integrated in r from 0 to L one obtains

$$Q_2 A_2 + Q_4 A_4 = 0 \quad (5-30)$$

where

$$Q_2 = -\frac{128}{315} C_1 - \frac{\pi}{8} D_2 - \frac{32}{105} E_2 + 4B_1^2 \left(\frac{16}{63} + \frac{\pi}{8} a_6 + \frac{88}{105} a_7 \right),$$

$$Q_4 = \frac{256}{385} C_4 + \frac{\pi}{4} D_4 + \frac{64}{63} E_4 + 8B_1^2 \left(-\frac{16}{3465} + \frac{8}{63} a_7 \right).$$

Now equations (5-27), (5-28), (5-29), and (5-30) lead to the following matrix equation,

$$\begin{bmatrix} M_1 & 0 & M_3 & 0 \\ N_1 & 0 & N_3 & 0 \\ 0 & P_2 & 0 & P_4 \\ 0 & Q_2 & 0 & Q_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \quad (5-31)$$

For nontrivial values for the A 's it is necessary, as usual, that the determinant of the coefficient matrix of equation (5-31) vanish. Thus, the characteristic

equation becomes

$$(P_4 Q_2 - P_2 Q_4) (M_1 N_3 - M_3 N_1) = 0. \quad (5-32)$$

And equation (5-32) is satisfied if

$$P_4 Q_2 - P_2 Q_4 = 0 \quad (5-33)$$

or

$$M_1 N_3 - M_3 N_1 = 0. \quad (5-34)$$

Equation (5-33) leads to the following polynomial equation in "s".

$$F_0 s^4 + F_1 s^3 + F_2 s^2 + F_3 s + F_4 = 0 \quad (5-35)$$

where

$$F_0 = \frac{1}{\alpha^2 \lambda_x^2} \left\{ \left(\frac{\pi}{16} + \frac{1}{5\alpha} \right) \left(\frac{\pi}{4} + \frac{20}{21\alpha} \right) - \left(\frac{\pi}{64} + \frac{1}{70\alpha} \right) \left(\frac{\pi}{4} + \frac{8}{7\alpha} \right) \right\}, \quad (5-35.1)$$

$$\begin{aligned} F_1 = \frac{1}{\alpha \lambda_x} & \left\{ \left[\frac{\pi}{4} + \frac{20}{21\alpha} \right] \left[\frac{16}{105} \beta_2 + \frac{\pi}{16\alpha} \gamma_2 + \frac{9}{35\alpha^2} \lambda_{ix} + \frac{2}{105} \right] \right. \\ & + \left[\frac{\pi}{16} + \frac{1}{5\alpha} \right] \left[\frac{256}{385} \beta_4 + \frac{\pi}{4\alpha} \gamma_4 + \frac{20}{21\alpha^2} \lambda_{ix} + \frac{8}{3465} \right] \\ & - \left[\frac{\pi}{64} + \frac{1}{70\alpha} \right] \left[\frac{128}{315} \beta_2 + \frac{\pi}{8\alpha} \gamma_2 + \frac{\pi}{8\alpha} (\lambda_{ix} + 1) + \frac{8}{7\alpha^2} \lambda_{ix} + \frac{16}{63} \right] \\ & \left. - \left[\frac{\pi}{4} + \frac{20}{21\alpha} \right] \left[\frac{32}{315} \beta_4 + \frac{\pi}{32\alpha} \gamma_4 - \frac{\pi}{64\alpha} (\lambda_{ix} + 1) + \frac{1}{70\alpha^2} \lambda_{ix} - \frac{11}{315} \right] \right\}, \\ F_2 = \frac{\lambda_1}{\alpha \lambda_x} & \left\{ \left[\frac{\pi}{4} + \frac{20}{21\alpha} \right] \left[\frac{6}{35} + \frac{\pi}{16\alpha} \delta_2 + \frac{1}{5\alpha^2} \right] + \left[\frac{\pi}{16} + \frac{1}{5\alpha} \right] \times \right. \\ & \left[\frac{2312}{3465} + \frac{\pi}{4\alpha} \delta_4 + \frac{20}{21\alpha^2} \right] \\ & - \left[\frac{\pi}{64} + \frac{29}{210\alpha} \right] \left[\frac{208}{945} + \frac{\pi}{8\alpha} \delta_2 + \frac{\pi}{4\alpha} + \frac{8}{7\alpha^2} \right] - \left[\frac{\pi}{4} + \frac{8}{7\alpha} \right] \times \end{aligned} \quad (5-35.2)$$

$$\begin{aligned}
& \left[\frac{1}{15} + \frac{\pi}{32\alpha} \delta_4 - \frac{\pi}{32\alpha} + \frac{1}{70\alpha^2} \right] \\
& + \left[\frac{16}{105} \beta_2 + \frac{\pi}{16\alpha} \gamma_2 + \frac{1}{5\alpha^2} \lambda_{ix} + \frac{2}{105} \right] \left[\frac{256}{385} \beta_4 + \frac{\pi}{4\alpha} \gamma_4 \right. \\
& \left. + \frac{20}{21\alpha^2} \lambda_{ix} + \frac{8}{3465} \right] \\
& - \left[\frac{128}{315} \beta_2 + \frac{\pi}{8\alpha} \gamma_2 + \frac{\pi}{8\alpha} (\lambda_{ix} + 1) + \frac{8}{7\alpha^2} \lambda_{ix} + \frac{16}{63} \right] \cdot \left[\frac{32}{315} \beta_4 + \frac{\pi}{32\alpha} \gamma_4 \right. \\
& \left. - \frac{\pi}{64\alpha} (\lambda_{ix} + 1) + \frac{1}{70\alpha^2} \lambda_{ix} - \frac{11}{315} \right],
\end{aligned}$$

$$\begin{aligned}
F_3 = \lambda_i \Bigg\{ & \left[\frac{16}{105} \beta_2 + \frac{\pi}{16\alpha} \gamma_2 + \frac{1}{5\alpha^2} \lambda_{ix} + \frac{2}{105} \right] \left[\frac{2312}{3465} + \frac{\pi}{4\alpha} \delta_4 + \frac{20}{21\alpha^2} \right] \\
& + \left[\frac{256}{385} \beta_4 + \frac{\pi}{4\alpha} \gamma_2 + \frac{20}{21\alpha^2} \lambda_{ix} + \frac{8}{3465} \right] \left[\frac{6}{35} + \frac{\pi}{16\alpha} \delta_2 + \frac{1}{5\alpha^2} \right] \\
& - \left[\frac{128}{315} \beta_2 + \frac{\pi}{8\alpha} \gamma_2 + \frac{\pi}{8\alpha} (\lambda_{ix} + 1) + \frac{8}{7\alpha^2} \lambda_{ix} + \frac{16}{63} \right] \times \\
& \left[\frac{1}{15} + \frac{\pi}{32\alpha} (\delta_4 - 1) + \frac{1}{70\alpha^2} \right] \\
& - \left[\frac{32}{315} \beta_4 + \frac{\pi}{32\alpha} \gamma_4 - \frac{\pi}{64\alpha} (\lambda_{ix} + 1) + \frac{29}{105\alpha^2} \lambda_{ix} - \frac{11}{315} \right] \times \\
& \left[\frac{208}{315} + \frac{\pi}{8\alpha} \delta_2 + \frac{\pi}{4\alpha} + \frac{8}{7\alpha^2} \right] \Bigg\},
\end{aligned}$$

$$\begin{aligned}
F_4 = \lambda_i^2 \Bigg\{ & \left[\frac{6}{35} + \frac{\pi}{16\alpha} \delta_2 + \frac{1}{5\alpha^2} \right] \left[\frac{2312}{3465} + \frac{\pi}{4\alpha} \delta_4 + \frac{20}{21\alpha^2} \right] \\
& - \left[\frac{208}{315} + \frac{\pi}{8\alpha} \delta_2 + \frac{\pi}{4\alpha} + \frac{8}{7\alpha^2} \right] \cdot \left[\frac{1}{15} + \frac{\pi}{32\alpha} \delta_4 + \frac{\pi}{32\alpha} + \frac{1}{70\alpha^2} \right] \Bigg\}.
\end{aligned}$$

Equation (5-34) leads to the following polynomial equation in "s".

$$H_0 s^4 + H_1 s^3 + H_2 s^2 + H_3 s + H_4 = 0 \quad (5-36)$$

where

$$H_0 = \frac{1}{\alpha^2 \lambda_x} \left\{ \frac{\pi}{8} \left(\frac{\pi}{4} + \frac{8}{7\alpha} \right) - \frac{\pi}{24} \left(\frac{3\pi}{8} + \frac{32}{15\alpha} \right) \right\},$$

$$\begin{aligned} H_1 = \frac{1}{\alpha \lambda_x} & \left\{ \left[\frac{\pi}{4} + \frac{8}{7\alpha} \right] \left[\frac{16}{15} \beta_2 + \frac{3\pi}{8\alpha} \gamma_1 - \frac{\pi}{4\alpha} (\lambda_{ix} + 1) - \frac{8}{15} \right] \right. \\ & + \frac{\pi}{8} \left[\frac{208}{315} \beta_3 + \frac{\pi}{4\alpha} \gamma_3 + \frac{8}{7\alpha} \lambda_{ix} + \frac{8}{945} \right] \\ & - \frac{\pi}{24} \left[\frac{48}{105} \beta_1 + \frac{\pi}{8\alpha} \gamma_1 + \frac{\pi}{4\alpha} (\lambda_{ix} + 1) + \frac{32}{15\alpha^2} \lambda_{ix} + \frac{8}{21} \right] \\ & \left. - \left[\frac{3\pi}{8} + \frac{32}{15\alpha} \right] \left[\frac{16}{35} \beta_3 + \frac{\pi}{8\alpha} \gamma_3 - \frac{\pi}{12\alpha} (\lambda_{ix} + 1) - \frac{8}{35} \right] \right\}, \end{aligned}$$

$$\begin{aligned} H_2 = \frac{\lambda_1}{\alpha \lambda_x} & \left\{ \left[\frac{\pi}{4} + \frac{8}{7\alpha} \right] \left[\frac{8}{15} + \frac{3\pi}{8\alpha} \delta_1 \right] + \frac{\pi}{8} \left[\frac{632}{945} + \frac{\pi}{4\alpha} \delta_3 + \frac{8}{7\alpha^2} \right] \right. \\ & - \frac{\pi}{24} \left[\frac{88}{105} + \frac{\pi}{8\alpha} \delta_1 + \frac{\pi}{2\alpha} + \frac{32}{15\alpha^2} \right] - \left[\frac{3\pi}{8} + \frac{32}{15\alpha} \right] \left[\frac{8}{35} + \frac{\pi}{8\alpha} \delta_3 - \frac{\pi}{6\alpha} \right] \Big\} \\ & + \left[\frac{16}{15} \beta_1 + \frac{3\pi}{8\alpha} \gamma_1 - \frac{\pi}{4\alpha} (\lambda_{ix} + 1) - \frac{8}{15} \right] \cdot \left[\frac{208}{315} \beta_3 + \frac{\pi}{4\alpha} \gamma_3 \right. \\ & + \frac{8}{7\alpha^2} \lambda_{ix} + \frac{8}{945} \Big] \\ & - \left[\frac{48}{105} \beta_1 + \frac{\pi}{8\alpha} \gamma_1 + \frac{\pi}{4\alpha} (\lambda_{ix} + 1) + \frac{32}{15\alpha^2} \lambda_{ix} + \frac{8}{21} \right] \cdot \left[\frac{16}{35} \beta_3 + \frac{\pi}{8\alpha} \gamma_3 \right. \\ & \left. - \frac{\pi}{12\alpha} (\lambda_{ix} + 1) - \frac{24}{105} \right] \Big\}, \end{aligned}$$

$$\begin{aligned}
H_3 = \lambda_1 \left\{ \left[\frac{16}{15} \beta_1 + \frac{3\pi}{8\alpha} \gamma_1 - \frac{\pi}{4\alpha} (\lambda_{ix} + 1) - \frac{8}{15} \right] \cdot \left[\frac{632}{945} + \frac{\pi}{4\alpha} \delta_3 + \frac{8}{7\alpha^2} \right] \right. \\
+ \left[\frac{208}{315} \beta_3 + \frac{\pi}{4\alpha} \gamma_3 + \frac{8}{7\alpha^2} \lambda_{ix} + \frac{8}{945} \right] \cdot \left[\frac{8}{15} + \frac{3\pi}{8\alpha} \delta_1 - \frac{\pi}{2\alpha} \right] \\
- \left[\frac{48}{105} \beta_1 + \frac{\pi}{8\alpha} \gamma_1 + \frac{\pi}{4\alpha} (\lambda_{ix} + 1) + \frac{32}{15\alpha^2} \lambda_{ix} + \frac{8}{21} \right] \cdot \left[\frac{8}{35} \right. \\
+ \left. \frac{\pi}{8\alpha} \delta_3 - \frac{\pi}{6\alpha} \right] \\
- \left[\frac{16}{35} \beta_3 + \frac{\pi}{8\alpha} \gamma_3 - \frac{\pi}{12\alpha} (\lambda_{ix} + 1) - \frac{24}{105} \right] \cdot \left[\frac{88}{105} + \frac{\pi}{8\alpha} \delta_1 \right. \\
+ \left. \frac{\pi}{2\alpha} + \frac{32}{15\alpha^2} \right] \left. \right\} ,
\end{aligned}$$

$$\begin{aligned}
H_4 = \lambda_1^2 \left\{ \left[\frac{8}{15} + \frac{3\pi}{8\alpha} \delta_1 - \frac{\pi}{2\alpha} \right] \cdot \left[\frac{632}{945} + \frac{\pi}{4\alpha} \delta_3 + \frac{8}{7\alpha^2} \right] \right. \\
- \left[\frac{88}{105} + \frac{\pi}{8\alpha} \delta_1 + \frac{\pi}{2\alpha} + \frac{32}{15\alpha^2} \right] \cdot \left[\frac{8}{35} + \frac{\pi}{8\alpha} \delta_3 - \frac{\pi}{6\alpha} \right] \left. \right\} .
\end{aligned}$$

The following terms have been used to simplify somewhat the coefficient terms in equations (5-35) and (5-36).

$$\beta_n = 1 - \frac{\gamma_1 \rho_0}{B_n^2}$$

$$\gamma_n = \frac{\gamma_x \rho_0}{B_n^2} + 2 + \frac{\lambda_1}{\lambda_x}$$

$$\delta_n = \frac{\rho_0}{B_n^2} + 2$$

$$\lambda_{ix} = 1 + \frac{\lambda_1}{\lambda_x}$$

$$B_n^2 = \left(\frac{n\pi}{L}\right)^2$$

$$\rho_o = \frac{1}{M^2} \left(\frac{\bar{\gamma}\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f} \right)$$

Now the following inequalities must all be satisfied for stability of the perturbed flux as given by equation (5-24).

$$F_i > 0 \quad i = 0, 1, 2, 3, 4 \quad (5-37.1)$$

$$H_i > 0 \quad i = 0, 1, 2, 3, 4 \quad (5-37.2)$$

$$F_1 F_2 > F_0 F_3 \quad (5-37.3)$$

$$H_1 H_2 > H_0 H_3 \quad (5-37.4)$$

$$F_1 F_2 F_3 > F_1^2 F_4 + F_0 F_3^2 \quad (5-37.5)$$

$$H_1 H_2 H_3 > H_1^2 H_4 + H_0 H_3^2 \quad (5-37.6)$$

It is obvious from the length of the expressions for the F_i 's and G_i 's that implementing the stability requirements above is essentially a numerical procedure. However, for a given flux level (which fixes α), and effective reactor core length, L , and knowledge of ρ_o , the above requirements are easily examined.

Stability and Root Locus Study of Linear Kinetic Equations for High or Flat Flux Distribution

For the condition that the term within the brackets of equation (5-19) may be assumed approximately constant over the reactor volume (e.g., if the equilibrium flux is approximately flat over the reactor volume or if the flux level is very high) and the flux gradient term is small then equation (5-19) may be solved as approximating the time independent wave equation. Thus, since the wave equation gives

$$\nabla^2 \delta\phi = -B^2 \delta\phi \quad (5-38)$$

where B^2 is the geometrical buckling of the reactor system, then the characteristic equation becomes

$$\frac{\lambda_o}{\lambda_o + \lambda_x} \left[(\gamma_i \lambda_o - \gamma_x \lambda_x) s - \lambda_i \lambda_x \right] - \tau_e s (s + \lambda_o + \lambda_x) (s + \lambda_i) - \frac{B^2}{\rho_o} (s + \lambda_o + \lambda_x) (s + \lambda_i) = 0. \quad (5-39)$$

And again for stability, the Routh-Hurwitz criteria leads to the following requirement for stability.

$$\begin{aligned} \tau_e \lambda_i (\lambda_o + \lambda_x) + \frac{B^2}{\rho_o} (\lambda_o + \lambda_x + \lambda_i) + \frac{\lambda_o \lambda_x}{\lambda_o + \lambda_x} &> \gamma_i \lambda_o \\ &+ \frac{\tau_e \lambda_i}{\tau_e (\lambda_o + \lambda_x + \lambda_i) + \frac{B^2}{\rho_o}} \left[\frac{B^2}{\rho_o} (\lambda_o + \lambda_x) + \frac{\lambda_o \lambda_x}{\lambda_o + \lambda_x} \right] \end{aligned} \quad (5-40)$$

which also may be written as follows:

$$\begin{aligned} \left[\tau_e^2 \lambda_i (\lambda_o + \lambda_x) + \left(\frac{B^2}{\rho_o} \right)^2 + \tau_e \frac{B^2}{\rho_o} (\lambda_o + \lambda_x + \lambda_i) \right] (\lambda_o + \lambda_x + \lambda_i) &> \\ \lambda_o \left[\tau_e + \frac{B^2}{\rho_o (\lambda_o + \lambda_x)} \right] (\gamma_i \lambda_o - \gamma_i \lambda_x) \end{aligned} \quad (5-41)$$

It is obvious that expression (5-41) is always satisfied if $\gamma_i \lambda_o \leq \gamma_x \lambda_x$ which implies a flux level threshold ($\phi_o = 3 \times 10^{11}$) below which the flux is stable (for small perturbations at least) against xenon induced instabilities(41).

It is possible under certain added assumptions to easily determine, in a qualitative fashion at least, the nature of the time solutions for the flux from equation (5-38). The assumptions to be made are as follows:

1. The equilibrium flux level which is given by the term, λ_o , is sufficiently large that λ_x may be neglected when compared to λ_o . (See Chapter II for expansion and verification of this assumption.)
2. The term τ_e is set to zero. (This assumption will be discussed later in this chapter.)

Imposing the above assumptions, equation (5-39) then becomes

$$B^2 s^2 + [B^2(\lambda_o + \lambda_i) - \gamma_i \rho_o \lambda_o] s + \rho_o \lambda_i \lambda_x + B^2 \lambda_o \lambda_i = 0 \quad (5-42)$$

and for stability of the time solutions the material buckling must satisfy the following.

$$B^2 > \gamma_i \rho_o \frac{\lambda_o}{\lambda_o + \lambda_i} \quad (5-43)$$

and for the assumption $\lambda_o > \lambda_x$ and since numerically $\lambda_i \approx \lambda_x$ then to good approximation the requirement for high flux stability is

$$B^2 > \gamma_i \rho_o. \quad (5-44)$$

Now to effect a root locus study (12, 46), which is a technique that will permit graphical representation of the Laplace Transformed time behavior of the flux for a given equilibrium flux level, equation (5-42) is expressed as the following transfer function for the flux.

$$T(s) = [s^2 + s\lambda_i + \frac{1}{\gamma_i} (1 - \alpha)\lambda_i \lambda_x] + \lambda_o [\alpha s + \lambda_i] = 0 \quad (5-45)$$

where

$$\alpha = 1 - \frac{\gamma_i \rho_o}{B^2}.$$

The first bracketed term of $T(s)$ is defined as the poles of the transfer function, which are as follows:

$$s = -\frac{\lambda_i}{2} \pm \sqrt{\left(\frac{\lambda_i}{2}\right)^2 - \frac{1}{\gamma_i} (1 - \alpha)\lambda_i \lambda_x} \quad (5-46)$$

The second bracketed term of $T(s)$ is defined as the zero of the transfer function which is the following:

$$s = -\frac{\lambda_i}{\alpha} \quad (5-47)$$

Considering $T(s)$ as the transfer function for the flux then a root locus is a plot of the location of the closed-loop poles in the complex polar plane as a function of the gain factor, λ_o , which is proportional to the equilibrium distribution of the thermal flux. By observing the motion of the closed-loop poles as

A root locus plot of the transfer function, $T(s)$, for the perturbed flux, using λ_0 as the gain factor, is given in Figures 26, 27, 28, and 29 for $\alpha = 1$,

0.1, 0, and -0.1, respectively. Figure 26 for $\alpha = 1$ represents the transformed time response of the reactor model for infinite neutron leakage. Though an infinite leakage reactor is not physically realistic, Figure 26 does represent a limiting case and gives the transformed time response of the model for the greatest degree of stability possible. Figure 27 represents the time response for $\alpha = 0.1$. Since all of the root loci lie in the left hand plane, the reactor model is stable. For $\alpha = 0.1$ the model exhibits its most stable performance at a flux value of $\phi_0 = 4 \times 10^{15}$. The reactor model is stable when $\alpha < 0$ for

Figure 28. Root Locus Study for $\alpha = 0.0$.

Figure 29. Root Locus Study for $\alpha = -0.1$.

all values of flux and unstable when $\alpha < 0$ for sufficiently large values of the flux. Figure 28 represents the transformed time response for the special case when $\alpha = 0$. It is evident from Figure 28 that the model is stable for all values of flux, but the frequency of the damped oscillations increases with increasing flux level. For any value of $\alpha < 0$ there exists some range of values of the flux for which the root loci cross the imaginary axis into the right half of the complex plane, and thus for these values of flux the model becomes unstable. For the value $\alpha = -0.1$ Figure 29 gives the transformed time response. Apparently for flux values such that $\phi_o = 10^{14}$ the model exhibits pure sinusoidal oscillations, while for $\phi_o > 10^{14}$ the oscillations become exponentially divergent.

SUFFICIENT CONDITIONS FOR FLUX STABILITY

It is possible to obtain a sufficient (though not always necessary) condition for stability of the solutions of equations (5-10) and (5-15) by examining the derivative of the flux for a given perturbation scheme. (This analysis makes the obvious assumption that dependent variables in equations (5-10) and (5-15) are continuous in time and possess first derivatives in time which are continuous.) To examine this stability requirement, a reactor model which is described by equations (5-10) and (5-15) will be considered. The equations describing the flux and perturbed flux are repeated here.

$$\tau_e \frac{\partial \phi}{\partial t} = \frac{1}{\rho_o} \nabla^2 \phi + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \phi + \frac{\lambda_o}{\lambda_o + \lambda_x} (1 - x) \phi \quad (5-10.1)$$

$$\tau_e \frac{\partial \delta \phi}{\partial t} = \frac{1}{\rho_o} \nabla^2 \delta \phi + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta \phi - \frac{\lambda_o}{\lambda_o + \lambda_x} (1 + \delta \phi) \delta x \quad (5-15.1)$$

Now suppose that the flux within some region of the reactor model at some period in time is perturbed positively from its equilibrium value such that

$$\phi \rightarrow \phi^+ = 1 + \delta \phi^+$$

where

$$\delta \phi^+ > 0.$$

For this positive perturbation not to increase in time it is necessary that

$$\frac{\partial \phi^+}{\partial t} < 0 \quad (5-48)$$

until ϕ^+ again returns to its equilibrium value of unity. Now inequality (5-48) is satisfied if

$$\tau_e \frac{\partial \phi^+}{\partial t} = \frac{1}{\rho_o} \nabla^2 \phi^+ + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \phi^+ + \frac{\lambda_o}{\lambda_o + \lambda_x} (1 - x) \phi^+ < 0 \quad (5-49)$$

or in terms of equation (5-15.1)

$$-\frac{1}{\rho_o} \nabla^2 \delta \phi^+ > \frac{\lambda_o}{\lambda_o + \lambda_x} (1 + \delta \phi^+) \delta x + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta \phi^+ \quad (5-50)$$

Since the leakage term, $\nabla^2 \delta \phi^+$, is intrinsically negative (i.e., $\nabla^2 \delta \phi^+ < 0$) the first term of expression is always greater than zero. The second term

$$\frac{\lambda_o}{\lambda_o + \lambda_x} (1 + \delta \phi^+) \delta x$$

depends for its sign upon the sign of δx .

Now consider that the flux in some region of the reactor model at some period in time is perturbed negatively from its equilibrium value so that

$$\phi \rightarrow \phi^- = 1 - \delta \phi^-$$

where

$$\delta \phi^- > 0.$$

Then for this negative perturbation not to further decrease in time it is necessary that

$$\frac{\partial \phi^-}{\partial t} > 0 \quad (5-51)$$

until ϕ^- returns to its equilibrium value of unity. Inequality (5-51) is satisfied if

$$\tau_e \frac{\partial \phi^-}{\partial t} = \frac{1}{\rho_o} \nabla^2 \phi^- + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \phi^- + \frac{\lambda_o}{\lambda_o + \lambda_x} (1 - x) \phi^- > 0 \quad (5-52)$$

or again in terms of equation (5-15.1)

$$-\frac{1}{\rho_o} \nabla^2 \delta \phi^- > \frac{\lambda_o}{\lambda_o + \lambda_x} (1 - \delta \phi^-) \delta x + \frac{2}{\rho_o} \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta \phi^- \quad (5-53)$$

Again, since the leakage term is intrinsically negative, the first term of expression (5-53) is always positive. The second term

$$-\frac{\lambda_o}{\lambda_o + \lambda_x} (1 - \delta \phi^-) \delta x$$

is positive if $\delta x < 0$, zero if $\delta \phi^- = 1$ or $\delta x = 0$, and negative if $\delta x > 0$.

Now the sufficient condition for stability for either positive or negative perturbations of the flux is that both inequalities (5-50) and (5-53) be satisfied

for all appropriate values of the flux and xenon. Thus, in a certain region of the reactor model if $\phi > 1$ ($\delta\phi^+ > 0$), it is sufficient that

$$\left| \nabla^2 \delta\phi^+ \right| > - \frac{\lambda_o \rho_o}{\lambda_o + \lambda_x} (1 + \delta\phi^+) \delta x + 2 \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta\phi^+ \quad (5-54.1)$$

and if $\phi < 1$ ($\delta\phi^- > 0$) then

$$\left| \nabla^2 \delta\phi^- \right| > \frac{\lambda_o \rho_o}{\lambda_o + \lambda_x} (1 - \delta\phi^-) \delta x + 2 \frac{\nabla \lambda_o}{\lambda_o} \nabla \delta\phi^- \quad (5-54.2)$$

for stability of the flux in that region.

To provide a simple example of how expressions (5-54.1) and (5-54.2) might be employed to investigate the stability of a reactor model, suppose that the flux at some instant in time (say $t = 0$) is perturbed along one dimension of a reactor model such that

$$\delta\phi(r, t) = \epsilon_\phi(t) \sin \frac{2\pi r}{L} \quad (5-55)$$

where

$$\epsilon_\phi(0) = \epsilon_{\phi 0} > 0.$$

Now for stability, expressions (5-54.1) and (5-54.2) provide the following sufficient conditions for stability for $0 \leq r \leq L/2$

$$\left(\frac{2\pi}{L}\right)^2 \epsilon_\phi \sin \frac{2\pi r}{L} > \frac{\lambda_o(r) \rho_o}{\lambda_o(r) + \lambda_x} (1 + \epsilon_\phi \sin \frac{2\pi r}{L}) (-\delta x) + \frac{\nabla \lambda_o}{\lambda_o} \left(\frac{4\pi}{L}\right) \epsilon_\phi \cos \frac{2\pi r}{L} \quad (5-56.1)$$

and for $L/2 \leq r \leq L$

$$\left(\frac{2\pi}{L}\right)^2 \epsilon_\phi \left(-\sin \frac{2\pi r}{L}\right) > \frac{\lambda_o(r) \rho_o}{\lambda_o(r) + \lambda_x} (1 + \epsilon_\phi \sin \frac{2\pi r}{L}) \delta x - \frac{\nabla \lambda_o}{\lambda_o} \left(\frac{4\pi}{L}\right) \epsilon_\phi \cos \frac{2\pi r}{L} \quad (5-56.2)$$

If it is then conservatively assumed that

$$\delta x = - \epsilon_x \sin \frac{2\pi r}{L} \quad (5-56.3)$$

where

$$|\delta x| \leq \epsilon_x$$

for $0 \leq r \leq L$ and all $t \geq 0$ (physically, ϵ_x , is the maximum absolute value expected to occur for the resulting xenon perturbation), and further assumed that the equilibrium flux distribution satisfies the following:

$$\lambda_o(r) = \alpha \lambda_x \sin \frac{\pi r}{L} \quad (5-57)$$

then inequality (5-56.1) is satisfied for $0 \leq r \leq L/2$ if

$$\left(\frac{2\pi}{L}\right)^2 \epsilon_\phi \sin \frac{2\pi r}{L} > \frac{\alpha \rho_o \sin \frac{\pi r}{L}}{1 + \alpha \sin \frac{\pi r}{L}} (1 + \epsilon_\phi \sin \frac{2\pi r}{L}) \epsilon_x \sin \frac{2\pi r}{L} + \left(\frac{2\pi}{L}\right)^2 \frac{\cos \frac{\pi r}{L}}{\sin \frac{\pi r}{L}} \epsilon_\phi \cos \frac{2\pi r}{L} \quad (5-58.1)$$

and inequality (5-56.2) for $L/2 \leq r \leq L$ if

$$\begin{aligned} \left(\frac{2\pi}{L}\right)^2 \epsilon_\phi \left(-\sin \frac{2\pi r}{L}\right) &> \frac{\alpha \rho_o \sin \frac{\pi r}{L}}{1 + \alpha \sin \frac{\pi r}{L}} (1 + \epsilon_\phi \sin \frac{2\pi r}{L}) \epsilon_x \left(-\sin \frac{2\pi r}{L}\right) \\ &\quad - \left(\frac{2\pi}{L}\right)^2 \frac{\cos \frac{\pi r}{L}}{\sin \frac{\pi r}{L}} \epsilon_\phi \cos \frac{2\pi r}{L} \end{aligned} \quad (5-58.2)$$

Since all terms in expressions (5-58) maintain the same sign within each respective range of definition of r (i.e., for $0 \leq r \leq L/2$ and $L/2 \leq r \leq L$), then (5-58) may be integrated over each respective range. Integration of expression (5-58.1) from 0 to $L/2$ gives

$$\left(\frac{2\pi}{L}\right)^2 \epsilon_\phi \left(\alpha + \frac{2}{3}\right) > \alpha \rho_o \epsilon_x \left(1 + \epsilon_\phi \frac{\pi}{4}\right) \quad (5-59.1)$$

and similarly integration of expression (5-58.2) from $L/2$ to L gives

$$\left(\frac{2\pi}{L}\right)^2 \epsilon_\phi \left(\alpha + \frac{2}{3}\right) > \alpha \rho_o \epsilon_x \left(1 - \epsilon_\phi \frac{\pi}{4}\right) \quad (5-59.2)$$

It is interesting to observe that inequality (5-59.2) is apparently more easily satisfied than inequality (5-59.1). This condition implies that for modal perturbations of the kind assumed in equations (5-55) and (5-56.3) the reactor model possesses greater stability against the negative perturbation in the flux. Thus, a sufficient condition for stability of the model against the perturbation assumed is the following:

$$\left(\frac{2\pi}{L}\right)^2 > \rho_o \frac{\epsilon_x}{\epsilon_\phi} \left(1 + \epsilon_\phi \frac{\pi}{4}\right) \left\{ \frac{1}{1 + \frac{2}{3\alpha}} \right\} \quad (5-60)$$

where

$$\rho_0 = \frac{1}{M^2} \left(\frac{\bar{\gamma}\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f} \right) = \frac{\bar{\gamma}k_\infty}{vM^2}. \quad (5-61)$$

If inequality (5-60) is satisfied, the stability of the reactor model against a pure first harmonic perturbation in the flux is assured. However, as inequality (5-60) stands it is necessary to make some estimate on the magnitude of both ϵ_ϕ and ϵ_x . One possible means of obtaining an estimate of ϵ_x is to assume that if an extremum in time for δx exists, it is given by the condition that

$$\frac{\partial \delta x}{\partial t} = 0. \quad (5-62)$$

Imposing equation (5-62) upon equation (5-15.2) gives (assuming, of course, that the extremum exists)

$$\delta x_e = \frac{(\lambda_0 + \lambda_x)(\gamma_x \delta \phi + \gamma_i \delta i) - \lambda_0 \delta \phi}{\lambda_0 (1 + \delta \phi) + \lambda_x}. \quad (5-63)$$

Now for those regions such that $\phi > 1$, it can be shown, in general, that only if $\delta x < 0$ can an instability occur. Thus for $\phi > 1$ and $\delta x < 0$ it is necessary that

$$\lambda_0 \delta \phi > (\lambda_0 + \lambda_x)(\gamma_x \delta \phi + \gamma_i \delta i). \quad (5-64)$$

Furthermore since initially $\phi > 1$ it is physically obvious that $\delta i \geq 0$. So if it is conservatively assumed that $\delta i = 0$, which makes δx assume the largest negative value consistent with equation (5-63) and the physical fact that $\delta i \geq 0$ when $\phi > 1$, then inequality (5-64) becomes

$$\gamma_i \lambda_0 \delta \phi > \gamma_x \lambda_x \delta \phi \quad (5-65)$$

and it is evident that for

$$\lambda_0 \leq \frac{\gamma_x}{\gamma_i} \lambda_x \quad (5-66)$$

or

$$\phi_0 \leq 3 \times 10^{11}$$

the reactor model is always stable for positive perturbations in the flux. The same inequality (5-64) results for instability for those regions such that $\phi < 1$. Furthermore, the extremum for δx is given by equation (5-63).

Therefore, if it is assumed that the extremum for δx is conservatively given by

$$\delta x_e = \frac{(\gamma_x \lambda_x - \gamma_i \lambda_o) \delta \phi}{\lambda_o (1 + \delta \phi) + \lambda_x} \quad (5-67)$$

and equations (5-66), and (5-55), and (5-57) are substituted into expressions (5-54.10) and (5-54.2) then there results after integration in r from 0 to $L/2$.

$$\left(\frac{2\pi}{L}\right)^2 > \rho_o \frac{\left[\gamma_i \left(\frac{2}{5} + \frac{32}{105} \epsilon_\phi \right) - \frac{\gamma_x}{\alpha} \left(\frac{1}{2} + \frac{\pi}{8} \epsilon_\phi \right) \right]}{\frac{7}{15} + \frac{12}{35} \epsilon_\phi + \frac{1}{\alpha} \left(1 + \frac{\pi}{8} \epsilon_\phi \right) + \frac{1}{3\alpha^2}} \quad (5-68.1)$$

Integration in r of the same expression from $L/2$ to L yields

$$\left(\frac{2\pi}{L}\right)^2 > \rho_o \frac{\left[\gamma_i \left(\frac{2}{5} - \frac{32}{105} \epsilon_\phi \right) - \frac{\gamma_x}{\alpha} \left(\frac{1}{2} - \frac{\pi}{8} \epsilon_\phi \right) \right]}{\frac{7}{15} - \frac{12}{35} \epsilon_\phi + \frac{1}{\alpha} \left(1 - \frac{\pi}{8} \epsilon_\phi \right) + \frac{1}{3\alpha^2}} \quad (5-68.2)$$

For all flux levels such that $\alpha > \frac{4\gamma_x}{3\gamma_i}$ (i.e., $\phi_o > 4 \times 10^{11}$) inequality

(5-68.1) is the pertinent sufficient condition for stability. It is significant to note that for large values of the flux (i.e., large α) inequality (5-68.1) becomes

$$\left(\frac{2\pi}{L}\right)^2 > \gamma_i \rho_o \frac{42 + 32 \epsilon_\phi}{49 + 36 \epsilon_\phi} \longrightarrow \frac{6}{7} \gamma_i \rho_o \quad (\text{for small } \epsilon_\phi) \quad (5-69)$$

which is in essential agreement with expression (5-44) previously determined as stability criteria for the first harmonic for large flux values.

HIGH FLUX STABILITY ANALYSIS

It is possible to solve equations (5-1) under the assumption of high flux levels. Indeed, for those flux levels such that $\sigma_x \phi \gg \lambda_x$ (i.e., $\phi_o \gg 10^{13}$) it is possible to neglect the decay out of xenon compared to the burnout of xenon, and then equations (5-1) reduce to the following

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = D \nabla^2 \phi + (v \Sigma_f - \Sigma_a - \sigma_x X) \phi \quad (5-70.1)$$

$$\frac{\partial X}{\partial t} = \bar{\gamma}_x \Sigma_f \phi + \lambda_i I - \sigma_x X \phi \quad (5-70.2)$$

$$\frac{\partial I}{\partial t} = \bar{\gamma}_i \Sigma_f \phi - \lambda_i I \quad (5-70.3)$$

Adding equations (5-70) together properly and making the obvious cancellations one obtains

$$M^2 \nabla^2 \phi + (k_\infty - 1) \phi + \frac{\partial}{\partial t} \left[\frac{X + I}{\Sigma_a + \bar{\gamma} \Sigma_f} - \ell \phi \right] \quad (5-71)$$

where

$$M^2 \rightarrow \frac{D}{\Sigma_a + \bar{\gamma}\Sigma_f}$$

$$k_\infty = \frac{v\Sigma_f}{\Sigma_a + \bar{\gamma}\Sigma_f}$$

$$\ell = \frac{1}{v(\Sigma_a + \bar{\gamma}\Sigma_f)}$$

Now to examine the magnitude of the terms within brackets in equation (5-71) it is convenient to transform, momentarily, the dependent variables into the following equilibrium unit variables

$$\phi(\bar{r}, t) = \phi_o(\bar{r})\phi(\bar{r}, t)$$

$$X(\bar{r}, t) = X_o(\bar{r})x(\bar{r}, t) = \frac{\bar{\gamma}\Sigma_f \phi_o x}{\sigma_x \phi_o + \lambda_x} \rightarrow \frac{\bar{\gamma}\Sigma_x}{\sigma_x} (\phi_o \gg 10^{13})$$

$$I(\bar{r}, t) = I_o(\bar{r})i(\bar{r}, t) = \frac{\bar{\gamma}_i \Sigma_f \phi_o}{\lambda_i} i$$

where ϕ_o is defined as the equilibrium flux distribution. The bracketed term of equation (5-71) may now be written as

$$\frac{\bar{\gamma}k_\infty}{v\sigma_x} \frac{\partial}{\partial t} \left[x + \gamma_i \frac{\lambda_o}{\lambda_i} i - \frac{v\ell}{\bar{\gamma}k_\infty} \lambda_o \phi \right] \quad (5-72)$$

where

$$\lambda_o = \sigma_x \phi_o$$

$$\gamma_i = \frac{\bar{\gamma}_i}{\bar{\gamma}}$$

At equilibrium $x = i = \phi = 1$, so for reasonable reactor operation in the neighborhood of equilibrium if

$$\gamma_i \lambda_o \gg \lambda_i \quad (\phi_o \gg 10^{13})$$

then one may neglect the xenon concentration compared to the iodine concentration and equation (5-72) becomes in original variables

$$M^2 \nabla^2 \phi + (k_\infty - 1)\phi + \frac{\partial}{\partial t} \left[\frac{I}{\Sigma_a + \bar{\gamma}\Sigma_f} - \ell\phi \right] \quad (5-73)$$

Equations (5-73) and (5-70.3) are now linear and may be solved by assuming solutions of the form.

$$\phi = e^{mt} \theta(\bar{r}) \quad (5-74.1)$$

$$I = e^{mt} I'(\bar{r}) \quad (5-74.2)$$

After elimination of I' one obtains the following stationary wave equation for θ .

$$M^2 \nabla^2 \theta + \left\{ k_{\infty} - 1 + \frac{\bar{\gamma}_i}{v} \frac{k_{\infty} m}{m + \lambda_i} - \ell m \right\} \theta = 0 \quad (5-75.1)$$

The solution for equation (5-75.1) is given by

$$\nabla^2 \theta = -B^2 \theta \quad (5-75.2)$$

where B^2 is defined as the geometrical buckling.

Equations (5-75.1) and (5-75.2) may now be combined to give the following characteristic equation in m

$$\ell^* m^2 + \left[\frac{M^2(B^2 - B_o^2)}{1 + M^2 B_o^2} + \ell^* \lambda_i - \frac{\bar{\gamma}_i}{v} \right] m + \left[\frac{M^2(B^2 - B_o^2)}{1 + M^2 B_o^2} \right] \lambda_i = 0 \quad (5-76)$$

where

$$k_{\infty} = 1 + M^2 B_o^2$$

B_o^2 = the critical or stationary value of the geometrical buckling

$$\ell^* = \frac{\ell}{1 + M^2 B_o^2}$$

B^2 = the value of the geometrical buckling corresponding to the particular flux mode under investigation.

Equation (5-76) possesses the following roots

$$\left. \begin{matrix} m_1 \\ m_2 \end{matrix} \right\} = - \frac{1}{2\ell^*} \left[\frac{M^2(B^2 - B_o^2)}{1 + M^2 B_o^2} + \ell^* \lambda_i - \frac{\bar{\gamma}_i}{v} \right] \pm \frac{1}{2\ell^*} \sqrt{\left[\frac{M^2(B^2 - B_o^2)}{1 + M^2 B_o^2} + \ell^* \lambda_i - \frac{\bar{\gamma}_i}{v} \right]^2 - \frac{4\ell^* \lambda_i M^2(B^2 - B_o^2)}{1 + M^2 B_o^2}} \quad (5-77)$$

The requirement for stability which follows from equation (5-77) is the following.

$$\frac{M^2(B^2 - B_o^2)}{1 + M^2 B_o^2} + \ell^* \lambda_i > \frac{\bar{\gamma}_i}{v} \quad (5-78)$$

For stability of the fundamental mode (i.e., $B^2 = B_0^2$) then it is necessary that (see Chapter II also)

$$\ell^* \lambda_i > \frac{\bar{\gamma}_i}{v}$$

The time solution for the flux becomes from equation (5-74.1)

$$\phi(\bar{r}, t) = (e^{\frac{m_1 t}{\lambda_i}} + C_1 e^{\frac{m_2 t}{\lambda_i}}) \theta(\bar{r}) \quad (5-79)$$

where $\theta(r)$ gives the spacial dependence of the flux over the reactor.

When

$$\frac{M^2(B^2 - B_0^2)}{1 + M^2 B_0^2} + \ell^* \lambda_i = \frac{\bar{\gamma}_i}{v}$$

then equation (5-74.1) is purely sinusoidal in time and becomes

$$\phi(\bar{r}, t) = \sin \left\{ \left(\sqrt{\frac{\lambda_i}{\ell^*} \left(\frac{\bar{\gamma}_i}{v} - \ell^* \lambda_i \right)} + C_1 \right) t \right\} \theta(\bar{r})$$

with a period given by

$$T = 2\pi \sqrt{\frac{\ell^*}{\lambda_i} \left[\frac{1}{\frac{\bar{\gamma}_i}{v} - \ell^* \lambda_i} \right]}$$

If ℓ^* is set to zero in equation (5-76) then one obtains

$$\frac{m}{\lambda_i} = \frac{M^2(B^2 - B_0^2)}{\frac{\bar{\gamma}_i}{v} (1 + M^2 B_0^2) - M^2(B^2 - B_0^2)} \quad (5-80)$$

and the requirement for stability is that

$$M^2(B^2 - B_0^2) > \frac{\bar{\gamma}_i}{v} (1 + M^2 B_0^2) \quad (5-81)$$

A plot of equation (5-80) is given in Figure 30 for $\frac{\bar{\gamma}_i}{v} \left(\frac{1}{M^2} + B_0^2 \right)$ held constant and in Figure 31 for $(B^2 - B_0^2)$ held constant.

Figure 30. Characteristic Root (for Time Solution) for High Flux Level with $\gamma_1 \rho_0$ Constant.

Figure 31. Characteristic Root (for Time Solution) for High Flux Level with B^2 Constant.

To exemplify the procedure for determining the spacial dependence of the flux over the reactor, a uniform cylindrical reactor model will be examined. From equation (5-75.2) the stationary wave equation becomes for uniform cylindrical geometry

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} + B^2 \theta = 0 \quad (5-82)$$

where "r" is the radial independent variable and "z" the axial independent variable. If the function $U(r,z)$ is assumed to be separable as follows.

$$\theta(r, z) = R(r) Z(z)$$

then equation (5-82) becomes

$$\left[\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) \right] + \left[\frac{1}{Z} \frac{d^2 Z}{dz^2} \right] + B^2 = 0. \quad (5-83)$$

Now the first term in brackets in equation (5-83) is a function of r only while the second term in brackets in the same equation is a function of z only; thus, equation (5-83) can be satisfied generally only if each bracketed term is equal to some constant. If the first term is set equal to $-\alpha^2$ to satisfy boundary conditions to be imposed later then there results

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \alpha^2 R = 0. \quad (5-84)$$

Similarly, if the second term of equation (5-85) is set equal to $-\beta^2$ (thus, $B^2 = \alpha^2 + \beta^2$) then

$$\frac{d^2 Z}{dz^2} + \beta^2 Z = 0. \quad (5-85)$$

To put equation (5-84) into the form of Bessel's differential equation let $v = \alpha r$ then

$$v^2 \frac{d^2 R}{dv^2} + v \frac{dR}{dv} + v^2 R = 0. \quad (5-86)$$

Bessel's equation of order "n" is (25)

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0.$$

Thus, equation (5-86) is Bessel's equation of zero order (i.e., provided $\alpha^2 > 0$), and the general solution is found to be the following in terms of the radial variable r . (See (51) for nomenclature.)

$$R(r) = A J_0(\alpha r) + C Y_0(\alpha r)$$

The following boundary conditions pertinent to the radial dependence of $\theta(r, z)$ are assumed.

1. $\theta(r, z)$ must be bounded for $r = 0$ and all z at zero time. (This does not imply that $\theta(0, z)$ is bounded for all time, however.)

2. $\theta(r, z)$ must vanish for $r = r_0$ (the extrapolated radius at which the flux vanishes) for all z and all time.

Boundary condition (1) is satisfied only if $C = 0$ since $Y(0)$ is unbounded, and boundary condition (2) leads to the requirement that either $A = 0$ (which yields a zero solution for the flux) or

$$J_0(\alpha r_0) = 0.$$

Thus, αr_0 must be a root of the J_0 Bessel function. The first root of J_0 is approximately

$$\alpha r_0 \approx 2.405.$$

For convenience, define $k_j = \alpha r_o$ as the j th root that satisfies

$$J_o(k_j) = 0.$$

Returning now to equation (5-86) the general solution for the axial dependence is

$$Z(z) = D \sin \beta z + E \cos \beta z \quad (5-87)$$

and imposing the boundary conditions that

$$1. \theta(r, 0) = 0,$$

$$2. \theta(r, L) = 0,$$

where L is the extrapolated axial length for which the flux vanishes for all time leads to the requirement that $E = 0$ and that

$$\sin \beta L = 0. \quad (5-88)$$

The roots of equation (5-88) require that $\beta = \frac{n\pi}{L}$ where n is an integer or zero.

Both α and β have been determined so that they satisfy the boundary conditions imposed upon the uniform cylindrical reactor model under investigation. Thus, the material buckling B^2 satisfies the following equation.

$$B^2 = \left(\frac{k_j}{r_o}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \quad (5-89)$$

and the solution for the flux for a high flux, uniform, cylindrical reactor is given by

$$\begin{aligned} \phi(r, z, t) = & \sum_{j,n}^{\infty} A_{j,n} J_o(k_j \frac{r}{r_o}) \sin(\frac{n\pi z}{L}) e^{m_1 t} \\ & + \sum_{j,n}^{\infty} C_{j,n} J_o(k_j \frac{r}{r_o}) \sin(\frac{n\pi z}{L}) e^{m_2 t}. \end{aligned}$$

In order to evaluate the arbitrary constants $A_{j,n}$ and $C_{j,n}$ consider the following initial conditions.

$$1. \phi(r, z, 0) = \phi_o(r, z)$$

$$2. \phi(r, z, t_o) = f(r, z, t_o)$$

where $f(r, z)$ is any given arbitrary perturbation imposed upon the flux distribution at some given time t_o .

Now initial condition (1) produces the requirement that for all $0 \leq r \leq r_0$ and $0 \leq z \leq L$ the following be satisfied.

$$\sum_{j,n}^{\infty} (A_{j,n} + C_{j,n}) J_0(k_j \frac{r}{r_0}) \sin \frac{n\pi z}{L} = \phi_0(r, z) \quad (5-90)$$

To find one determining expression for the constants $A_{j,n}$ and $C_{j,n}$ multiply equation (5-91) by the following:

$$k_\ell \frac{r}{r_0} J_0(k_\ell \frac{r}{r_0}) \sin \frac{m\pi z}{L} \quad (5-91)$$

and integrate the resulting expression over the reactor as follows. (The operations of summation and integration may be interchanged since the pertinent infinite series are uniformly convergent over the range of values assumed by the independent spacial variables (25,51).)

$$\begin{aligned} \sum_{j,n}^{\infty} (A_{j,n} + C_{j,n}) \int_0^{r_0} k_\ell \frac{r}{r_0} J_0(k_\ell \frac{r}{r_0}) J_0(k_j \frac{r}{r_0}) d(k_\ell \frac{r}{r_0}) \int_0^L \sin(\frac{n\pi z}{L}) \sin(\frac{m\pi z}{L}) d(\frac{\pi z}{L}) \\ - \int_0^L \int_0^{r_0} \phi_0(r, z) k_\ell \frac{r}{r_0} J_0(k_\ell \frac{r}{r_0}) \sin(\frac{m\pi z}{L}) d(k_\ell \frac{r}{r_0}) d(\frac{\pi z}{L}) = 0. \end{aligned} \quad (5-92)$$

Now because of orthogonality one finds that

$$\begin{aligned} \int_0^{r_0} k_\ell \frac{r}{r_0} J_0(k_\ell \frac{r}{r_0}) J_0(k_j \frac{r}{r_0}) d(k_\ell \frac{r}{r_0}) = \begin{cases} 0, & \ell \neq j \\ \frac{1}{2} k_\ell^2 J_1^2(k_\ell), & \ell = j, \end{cases} \\ \int_0^L \sin(\frac{n\pi z}{L}) \sin(\frac{m\pi z}{L}) d(\frac{\pi z}{L}) = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n, \end{cases} \end{aligned}$$

and the definite integral in the second term of equation will be defined as

$$F_{\ell m} = \int_0^\pi \int_0^{k_\ell} \phi_0 \eta J_0(\eta) \sin m\xi d\eta d\xi.$$

Substituting the values for these integrals into equation (5-92) then there results

$$A_{\ell, m} + C_{\ell, m} = \frac{4 F_{\ell m}}{\pi k_{\ell}^2 J_1^2(k_{\ell})} \quad (5-93)$$

The second initial condition produces the requirement that

$$\begin{aligned} f(r, z, t_0) = & \sum_{j,n}^{\infty} A_{j,n} J_0(k_j \frac{r}{r_0}) \sin \frac{n\pi z}{L} e^{m_1 t_0} \\ & + \sum_{j,n}^{\infty} C_{j,n} J_0(k_j \frac{r}{r_0}) \sin \frac{n\pi z}{L} e^{m_2 t_0} \end{aligned} \quad (5-94)$$

Again multiplying equation (5-94) by equation (5-91) and integrating over the reactor volume leads to the following equation.

$$\begin{aligned} A_{\ell, m} e^{m_1 t_0} \\ + C_{\ell, m} e^{m_2 t_0} = & \frac{4}{k_{\ell}^2 J_1^2(k_{\ell})} \int_0^L \int_0^{r_0} f(r, z, t_0) J_0(k_{\ell} \frac{r}{r_0}) \sin \frac{m\pi z}{L} d(k_{\ell} \frac{r}{r_0}) d(\frac{\pi z}{L}). \end{aligned} \quad (5-95)$$

Equations (5-93) and (5-95) may be combined to give the following expressions for $A_{\ell, m}$ and $C_{\ell, m}$.

$$A_{\ell, m} = \frac{4}{\pi k_{\ell}^2 J_1^2(k_{\ell}) (e^{m_2 t_0} - e^{m_1 t_0})} \left\{ F_{\ell m} e^{m_2 t_0} - G_{\ell m} \right\} \quad (5-96)$$

and

$$C_{\ell, m} = \frac{4}{\pi k_{\ell}^2 J_1^2(k_{\ell}) (e^{m_2 t_0} - e^{m_1 t_0})} \left\{ G_{\ell m} - F_{\ell m} e^{m_1 t_0} \right\} \quad (5-97)$$

where

$$G_{\ell m} = \int_0^L \int_0^{r_0} f(r, z, t_0) J_0(k_{\ell} \frac{r}{r_0}) \sin(\frac{m\pi z}{L}) d(k_{\ell} \frac{r}{r_0}) d(\frac{\pi z}{L}).$$

It is now possible to introduce equation (5-89) found for the material buckling into the stability requirement expressed in equation (5-78) and give the following requirement for stability for the high flux, cylindrical reactor model under investigation.

$$M^2 \left[\left(\frac{k_j}{r_0} \right)^2 + \left(\frac{n\pi}{L} \right)^2 - \left\{ \left(\frac{2405}{r_0} \right)^2 + \left(\frac{\pi}{L} \right)^2 \right\} \left(1 - \ell^* \lambda_i + \frac{\bar{\gamma}_i}{v} \right) \right] + \ell^* \lambda_i > \frac{\bar{\gamma}_i}{v} \quad (5-98)$$

where "n" is an integer corresponding to the appropriate axial mode of the flux and where " $k_j = \alpha r_0$ ", and " k_j " is the j'th root of $J_0(k_j) = 0$, and "j" is an integer corresponding to the appropriate radial mode of the flux.

Similarly for a high flux, rectangular parallelepiped reactor model the stability requirement is

$$M^2 \left[\left(\frac{n\pi}{L_1} \right)^2 + \left(\frac{m\pi}{L_2} \right)^2 + \left(\frac{p\pi}{L_3} \right)^2 - \left\{ \left(\frac{\pi}{L_1} \right)^2 + \left(\frac{\pi}{L_2} \right)^2 + \left(\frac{\pi}{L_3} \right)^2 \right\} \left(1 - \ell^* \lambda_i + \frac{\bar{\gamma}_i}{v} \right) \right] + \ell^* \lambda_i > \frac{\bar{\gamma}_i}{v} \quad (5-99)$$

where "n, m, and p" are integers, and L_1 , L_2 , and L_3 are the appropriate extrapolated edge dimensions at which the flux vanishes.

Finally, for a high flux, spherical reactor model the stability requirement becomes

$$M^2 \left[\left(\frac{n\pi}{R} \right)^2 - \left(\frac{\pi}{R} \right)^2 \left(1 - \ell^* \lambda_i + \frac{\bar{\gamma}_i}{v} \right) \right] + \ell^* \lambda_i > \frac{\bar{\gamma}_i}{v} \quad (5-100)$$

where "n" is an integer, and "R" is the extrapolated radius at which the flux vanishes.

COMPUTER SIMULATION

Equations (5-10) were simulated on an IBM 7040, Digital Computer to permit a numerical analysis of the time behavior of a given reactor model described by equations (5-10) under the influence of xenon (see 2, 8, 36, 37). The full formulation of the source FORTRAN program and the pertinent subroutines are given in Appendix B. The digital computer simulation of the space dependent, mono-energy group kinetic equations comprises a program philosophy similar to that

program given in Chapter II to study the space independent kinetics. Essentially, the computer simulation of equations (5-10) is composed of a main FORTRAN source program which accepts input data (i.e., equation parameters, initial conditions, space and time mesh, error control limits, etc.), provides internal and external control, calls upon the subroutine to integrate the system of differential equations contained in subroutine "DER" using subroutine "DEQ", and provides the desired output data.

The subroutine, DER, provides the computer with the system of first order differential equations in time (which are written as difference equations in space) and permits computation and storage of the derivatives. The subroutine, DEQ, is essentially an optimized version for the IBM 7040 of a variation on the classic Runge-Kutta method for integrating, stepwise, a given system of simultaneous first order differential equations. Appendix B contains more specific information on the nature of subroutines, "DER" and "DEQ".

Equations(5-10) were simulated on the IBM 7040 Digital Computer for one dimensional study of a given reactor model for two general conditions. Computer Program "40090" simulates equations (5-10) as they appear. Computer Program "40.104" simulates equations (5-10) under the assumption that the flux derivative vanishes. This assumption of neglecting the kinetic response due to delayed neutrons when compared to characteristic time for xenon induced changes has been previously examined in Chapter IV and the present chapter. The reason for imposing this assumption of setting the flux derivative to zero in Program 40.104 is that computer convergence of the time solutions is improved, and the time required for simulation is greatly reduced. Since it is desirable to examine the kinetic response for considerable lengths of time, it is desirable from a computer time standpoint that the time increments be of reasonable magnitude. However, as was found in Program 40090, the presence of the flux derivative requires small time increments (much less than a second for typical parameter values) for convergence and thus renders any attempt to examine the kinetic behavior for long intervals of time by means of digital computer simulation both time consuming and expensive.

Since the flux in Program 40.104 is not explicitly dependent upon time, the value of the xenon at a given time step uniquely determines the flux value. The flux equation when written as a difference equation for the spacial dependence assumes the form of a tri-diagonal matrix equation, which is amenable to solution

by the Gauss reduction procedure (39). For a full description of this solution scheme see Appendix B.

It may be recalled that equations (5-10) were determined as normalized, equilibrium transformations of equations (5-1), i.e., the dependent variables in equations (5-10) assume the value of unity when the condition of steady state or equilibrium occurs. Thus, it is very convenient to impose as initial conditions for the computer simulation of equations (5-10) any perturbation scheme and distribution desired. Of course, the time steps for integration and the density of the space mesh chosen must be compatible with the initial conditions and parameter values so as to insure an "acceptable" computer simulation. For Computer Program 40090, since there are three linearly independent differential equations in time, the initial values of the flux, xenon, and iodine are supplied at each mesh point. In Computer Program 40.104, however, it is assumed that the flux derivative vanishes, and thus equations (5-10) reduce to two linearly independent differential equations in time and one algebraic equation (the diffusion equation). Thus, it is possible to define arbitrarily at each mesh point the values of two of the three independent variables only.

Because it is the flux which is controlled in nuclear reactors, and the magnitude and distribution of the xenon and iodine concentrations are determined by this flux, Computer Program 40.104 is written to accept as initial conditions the values of the flux and iodine at each mesh point. The resulting initial value of the xenon is determined by the computer at each mesh point.

The output in graphical form obtained from the simulation of equations (5-10) by Computer Program 40.104 is presented in the following figures for typical values of the parameters and effective core lengths and initial perturbations as given. However, because the kinetic behavior of the Advanced Test Reactor (ATR) is the topic of study in Chapter VII, the parameters and core length used in the computer simulation were selected to characterize the ATR.

Figures 32.1 and 32.2 give the response in time of the flux and xenon for a neutron migration length equal to that of a fission neutron in water. (This is a conservative estimate of the migration length of neutrons in the fuel elements of the ATR.) The time behavior over one half of the core is displayed for an initial, two percent, first harmonic perturbation in the flux. The equilibrium flux is considered to be the usual cosine form which is the assumption made for all the graphical data presented in this Chapter (see Appendix A). However, it is important to note that none of the computer programs contained

Figure 32.1. Flux Response for +2% Perturbation (in the Flux) Over the Half Core ($\rho_0 = 6.363 \times 10^{-4}$, $\phi_0 = 2 \times 10^{16}$).

Figure 32.2. Xenon Response.

in this thesis are restricted to simple cosine distributions for the equilibrium flux. Returning to Figures 32 it is obvious that the time response is stable, and within 126 seconds the two percent perturbation in the flux has decreased to approximately 0.1 percent. Also, the computer simulation reveals that no overshooting of the flux or xenon occurs, and the response of the model is overdamped.

That the model simulated in Figures 32 is stable for larger initial perturbations is evidenced by Figures 33.1, 33.2 and 33.3. The initial perturbation is made in the flux and has a peak value of about 16.5 percent. The iodine and xenon response to this flux perturbation also is given, and it is again apparent that the model is stable and actually overdamped.

To display the time response of the model considered in the two previous Figures 32 and 33, when the iodine is perturbed is the purpose of Figures 34 and 35. Figures 34.1, 34.2, and 34.3 give the response of the flux, xenon, and iodine over a half core length for a positive, triangular perturbation in the iodine of magnitude of approximately 80 percent. As a result of this initial iodine perturbation, the flux becomes a minimum at about 18 percent below equilibrium in 150 minutes and then returns to equilibrium. It is again apparent that the kinetic variables of the model are stable in time. Similarly, Figures 35.1, 35.2, and 35.3 give the response of the flux, xenon, and iodine for a negative, triangular perturbation in the iodine.

Figure 33.1. Flux Response for +16.5% Perturbation (in the Flux) Over the Half Core ($\rho_0 = 6.363 \times 10^{-4}$, $\phi_0 = 2 \times 10^{16}$).

Figure 33.2. Xenon Response.

Figure 33.3. Iodine Response.

In an effort to corroborate the analytical stability criteria developed in this chapter for one group kinetics, the following figures determined by the digital computer from Computer Program 40.104 are presented. (See expressions (5-37), (5-39), (5-45), (5-68), and (5-78).) The threshold values of ρ_0 given by the stability criteria in Chapter V range from 2.35×10^{-3} to 2.70×10^{-3} for the ATR model parameters (which are those given in Appendix C). Figures 36.1, 36.2, and 36.3 display the time response with $\rho_0 = 2.3 \times 10^{-3}$ (for which

Figure 34.1. Flux Response for +82% Perturbation (in the Iodine) Over the Half Core ($\rho_0 = 6.363 \times 10^{-4}$, $\phi_0 = 2 \times 10^{16}$).

Figure 34.2. Xenon Response.

Figure 34.3. Iodine Response.

Figure 35.1. Flux Response for -81% Perturbation (in the Iodine) Over the Half Core ($\rho_0 = 6.363 \times 10^{-4}$, $\phi_0 = 2 \times 10^{14}$).

Figure 35.2. Xenon Response.

the analytic criteria indicates stability)

for the flux, xenon, and iodine under an initial, positive 16.5 percent perturbation in the flux. Although the time response is underdamped, the model is stable which is in agreement with the analytic stability criteria. Figure 37 presents the time response of the flux for various values of ρ_0 under an initial, positive flux perturbation of approximately 16 percent. It is obvious that the threshold value for ρ_0 resulting from Figure 37 is between 2.7×10^{-3} and 2.8×10^{-3} . This

Figure 35.3. Iodine Response.

Figure 36.1. Flux Response for +16.5% Perturbation (in the Flux) Over the Half Core ($\rho_0 = 2.3 \times 10^{-3}$, $\phi_0 = 2 \times 10^{14}$).

Figure 36.2. Xenon Response.

Figure 36.3. Iodine Response.

is reasonable agreement with the analytic stability criteria, especially when it is recalled that various approximating assumptions were implicit in formulating the analytic stability criteria.

Figures 38, 39, and 40 give the response in time for the unstable condition, $\rho_0 = 2.606 \times 10^{-2}$, over a half core. Within the range of values graphically presented, it appears that the model is exponentially unstable and that no oscillations occur. (Of course, extreme amplitude oscillations might occur, but from a practical, physical reactor standpoint these would be intolerable.)

Figure 41 gives the time response of the fundamental mode of the flux, xenon, and iodine at the center of the core for an initial, positive, 10 percent perturbation in the flux. It is interesting to observe that although the time response is highly underdamped, nevertheless, the response is stable and converges. The pseudo "period" of the

Figure 38.1. Flux Response for a Small Negative Perturbation (in the Flux) Over the Half Core ($\rho_0 = 2.606 \times 10^{-2}$, $\phi_0 = 2 \times 10^{14}$).

oscillations is approximately 7-1/2 hours. Furthermore, while the iodine and flux are reasonably sinusoidal, the xenon "wave" is not symmetrical about its

Figure 37. Flux Response for +16% Perturbation (in the Flux) at the Center of the Half Core for Various Values of ρ_0 ($\phi_0 = 2 \times 10^{14}$).

Figure 38.2. Xenon Response.

Figure 38.3. Iodine Response.

Figure 39.2. Xenon Response.

peak which is undoubtedly due to the nonlinear terms in the kinetics equations. The time behavior of the model in Figure 41 for an initial 10 percent perturbation in the iodine is given in Figure 42. Again the figure displays the time behavior of the fundamental mode of the flux, xenon,

Figure 39.1. Flux Response for +2% Perturbation (in the Flux) Over the Half Core ($\rho_0 = 2.606 \times 10^{-2}$, $\phi_0 = 2 \times 10^{14}$).

Figure 40.1. Flux Response for a Small Positive Perturbation (in the Flux) Over the Half Core ($\rho_0 = 2.606 \times 10^{-2}$, $\phi_0 = 2 \times 10^{14}$).

Figure 40.2. Xenon Response.

Figure 40.3. Iodine Response.

Figure 41. Flux, Xenon, and Iodine Response at the Center of the Core for +10% Perturbation in the Fundamental Mode of the Flux.

Figure 42. Flux, Xenon, and Iodine Response at the Center of the Core for +10% Perturbation in the Fundamental Mode of the Iodine.

and iodine at the center of the core. The "period" of these oscillations is approximately eight hours, but obviously changes with time. It is interesting to note that the "peaks" (maximum regions) of the flux tend to be narrow and pointed while the "minimums" (minimum regions) are broad. It is evident that the oscillations are not symmetric about their equilibrium value. Also, the xenon is out of phase with the flux by 180 degrees, while the iodine is approximately 90 degrees out of phase with the flux. It should be noted that these phase relationships are significantly different from those given by computer simulation of the space independent kinetic equations given in Chapter II (also, see Appendix B). Again, as was evident in Figure 41, the model is underdamped but stable in Figure 42.

VI - SPACE AND TIME DEPENDENT, MULTI-ENERGY GROUP KINETICS WITH XENON DEPENDENCE

One of the first studies of the problem of xenon induced instabilities of the flux which considered space and time dependence was made by Randall and St. John and reported in the publication, Nucleonics, in March of 1958 (38). The investigation employed the technique often referred to as "modal analysis" to determine requirements for stability under small perturbations of the flux, xenon, and iodine.

EXTENSION OF RANDALL AND ST. JOHN TREATMENT

To further explore the nature and use of modal analysis and, more importantly, to secure additional flux stability criteria, the study of Randall and St. John will be examined. The treatment presented here, however, will be extended to two group diffusion theory (rather than one group diffusion theory employed by Randall and St. John) in an attempt to better describe the time and space behavior of the physical neutron flux.

The equations governing the space and time behavior of the flux, and xenon and iodine concentrations are assumed to be the following:

$$\tau_1 \frac{\partial w_1}{\partial t} = \nabla^2 w_1 + B_1 w_1 + C_1 w_2, \quad (6-1.1)$$

$$\tau_2 \frac{\partial w_2}{\partial t} = \nabla^2 w_2 + B_2 w_2 + C_2 w_1, \quad (6-1.2)$$

$$\frac{\partial x}{\partial t} = \gamma_x w_1 + \lambda_i i - x w_1 - \lambda_x x, \quad (6-1.3)$$

$$\frac{\partial i}{\partial t} = \gamma_i w_1 - \lambda_i i, \quad (6-1.4)$$

where

$$w_1(\bar{r}, t) = \sigma_x \phi_1(\bar{r}, t); \quad \phi_1 \text{ is the thermal flux,}$$

$$w_2(\bar{r}, t) = \sigma_x \phi_2(\bar{r}, t); \quad \phi_2 \text{ is the fast flux,}$$

$$x(\bar{r}, t) = \frac{\sigma_x}{\gamma \Sigma_f} X(\bar{r}, t); \quad X \text{ is the xenon concentration,}$$

$$i(\bar{r}, t) = \frac{\sigma_i}{\gamma \Sigma_f} I(\bar{r}, t); \quad I \text{ is the iodine concentration,}$$

$$B_1 = B_1^* + \alpha_x x + \alpha_t w_1,$$

$$B_2 = B_2^* + \beta_t w_2,$$

α_x is the reactivity held by xenon at very high flux,

α_t is the thermal flux temperature coefficient,

β_t is the fast flux temperature coefficient,

and the other parameters have either been defined earlier or have obvious definitions.

Now Randall and St. John assumed that time changes in the flux governed by the delayed neutrons were so rapid compared to any changes in the flux induced by xenon, that the diffusion equation describing the flux could be considered stationary in time. Thus, the interpretation of this assumption that will

be imposed here is that $\tau_1 \frac{\partial w_1}{\partial t} \approx \tau_2 \frac{\partial w_2}{\partial t} \approx 0$.

Furthermore, xenon and iodine production and absorption by the fast flux are neglected. The cross section for xenon-135 as a function of energy is given in Appendix A.

Equations (6-1) are made linear, and time dependence is removed by assuming small perturbations around steady state values for the dependent variables.

$$w_1(\bar{r}, t) = w_{10}(\bar{r}) + \delta w_1(\bar{r}) e^{mt}, \quad (6-2.1)$$

$$w_2(\bar{r}, t) = w_{20}(\bar{r}) + \delta w_2(\bar{r}) e^{mt}, \quad (6-2.2)$$

$$x(\bar{r}, t) = x_0(\bar{r}) + \delta x(\bar{r}) e^{mt}, \quad (6-2.3)$$

$$i(\bar{r}, t) = i_0(\bar{r}) + \delta i(\bar{r}) e^{mt}, \quad (6-2.4)$$

$$B_1 = B_{10} + \alpha_x \delta x + \alpha_t \delta w_1, \quad (6-2.5)$$

$$B_2 = B_{20} + \beta_t \delta w_2, \quad (6-2.6)$$

where

$$B_{10} = B_1^* + \alpha_x x_o + \alpha_t w_{10},$$

$$B_{20} = B_2^* + \beta_t w_{20}.$$

Substituting equations (6-2) into equations (6-1) and neglecting perturbation products then one obtains after eliminating the iodine dependence

$$(\nabla^2 + B_1^* + \alpha_x x_o + 2\alpha_t w_{10}) \delta w_1 + C_1 \delta w_2 + \alpha_x w_{10} \delta x = 0, \quad (6-3.1)$$

$$(\nabla^2 + B_2^* + 2\beta_t w_{20}) \delta w_2 + C_2 \delta w_1 = 0, \quad (6-3.2)$$

$$(m + \lambda_x + w_{10}) \delta x - (\gamma_x + \gamma_i \frac{\lambda_i}{m + \lambda_i} - x_o) \delta w_1 = 0. \quad (6-3.3)$$

Modal analysis is now employed by considering an orthonormal set of functions such that

$$\nabla^2 f_i + k_i^2 f_i = 0 \quad (6-4)$$

where each eigen function f_i is orthogonal over the reactor volume and thus satisfies the following relationship.

$$\iiint f_i f_j dv = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

If the set of eigen functions is complete (3, 25) then it is possible to express the perturbations in the fast and thermal flux and xenon in the following series expansions.

$$\delta w_1(\vec{r}) = \sum_{i=0}^{\infty} a_i f_i(\vec{r}), \quad (6-5.1)$$

$$\delta_{w_2}(\bar{r}) = \sum_{i=0}^{\infty} b_i f_i(\bar{r}), \quad (6-5.2)$$

$$\delta_x(\bar{r}) = \sum_{i=0}^{\infty} c_i f_i(\bar{r}), \quad (6-5.3)$$

where the coefficients in the above series are to be determined. Substituting equations (6-5) into equations (6-3) and multiplying each resulting equation by f_j and integrating over the reactor volume, there results as a consequence of the orthonormal properties of f_i

$$\begin{aligned} & -a_j k_j^2 + B_1 a_j + C_1 b_j + \alpha_x c_j \int_{w_{10}} f_j^2 dv \\ & + \alpha_x \sum_{i \neq j} c_i \int_{w_{10}} f_i f_j dv + a_j \int (\alpha_x x_o + 2\alpha_t w_{10}) f_j^2 dv \\ & + \sum_{i \neq j} a_i \int (\alpha_x x_o + 2\alpha_t w_{10}) f_i f_j dv = 0, \end{aligned} \quad (6-6.1)$$

$$\begin{aligned} & -b_j k_j^2 + R_2 h_j + 2\beta_t h_j \int_{w_{20}} f_j^2 dv + C_2 a_j \\ & + 2\beta_t \sum_{i \neq j} b_i \int_{w_{20}} f_i f_j dv = 0, \end{aligned} \quad (6-6.2)$$

$$\begin{aligned} & (m + \lambda_x) c_i + c_j \int_{w_{10}} f_j^2 dv + \sum_{i \neq j} c_i \int_{w_{10}} f_i f_j dv - (\gamma_x + \gamma_i \frac{\lambda_1}{m + \lambda_i}) a_j \\ & + a_j \int_{x_o} f_j^2 dv + \sum_{i \neq j} a_i \int_{x_o} f_i f_j dv = C. \end{aligned} \quad (6-6.3)$$

The cross terms which contain expressions of the form

$$\int_{w_{10}} f_i f_j dv,$$

$$\int_{w_{20}} f_i f_j dv,$$

and

$$\int_{x_0} f_i f_j dv$$

are neglected by Randall and St. John; so making this same assumption then equations (6-6) can be written in the following matrix form.

$$\begin{bmatrix} (-k_j^2 + B_1^* + \alpha_x \bar{x}_{0jj} + 2\alpha_t \bar{w}_{10jj}) & C_1 & \alpha_x \bar{w}_{10jj} \\ C_2 & (-k_j^2 + B_2^* + 2\beta_t \bar{w}_{20jj}) & 0 \\ (\bar{x}_{0jj} - \frac{\gamma_x m + \lambda_i}{m + \lambda_i}) & 0 & (m + \lambda_x + \bar{w}_{10jj}) \end{bmatrix} \begin{bmatrix} a_j \\ b_j \\ c_j \end{bmatrix} = 0 \quad (6-7)$$

where

$$\bar{w}_{10jj} = \int_{w_{10}} f_j^2 dv,$$

$$\bar{w}_{20jj} = \int_{w_{20}} f_j^2 dv,$$

$$\bar{x}_{0jj} = \int_{x_0} f_j^2 dv.$$

For nontrivial solutions it is necessary that the determinant of the coefficient matrix in equation (6-7) vanish. Expanding the determinant one obtains the following polynomial equation in "m":

$$m^2 + am + b = 0 \quad (6-8)$$

where

$$a = \lambda_i + \lambda_x + \bar{w}_{10jj} - \frac{\alpha_x \bar{w}_{10jj} (\bar{x}_{0jj} - \gamma_x)}{(-k_j^2 + B_1^* + \alpha_x \bar{x}_{0jj} + 2\alpha_t \bar{w}_{10jj})} - \frac{C_1 C_2}{B_2^* + 2\beta_t \bar{w}_{20jj} - k_j^2}, \quad (6-9.1)$$

$$b = \lambda_i \left[\lambda_x + \bar{w}_{10jj} + \frac{\alpha_x \bar{w}_{10jj} (1 - \bar{x}_{ojj})}{(-k_j^2 + B_1^* + \alpha_x \bar{x}_{ojj} + 2\alpha_t \bar{w}_{10jj}) - \frac{C_1 C_2}{B_2^* + 2\beta_t \bar{w}_{10jj} - k_j^2}} \right] \quad (6-9.2)$$

Equation (6-8) is easily solved to give as roots

$$m_1 = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}, \quad m_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}. \quad (6-10)$$

It is interesting to tabulate the following conditions on the roots in equation (6-8) and observe the general time solutions that result for the thermal flux where

$$w_1(\bar{r}, t) = w_{10}(\bar{r}) + \sum_{i=1}^{\infty} a_i f_i(\bar{r}) g(t).$$

It is apparent from Figure 43 and previous discussion that both "a" and "b" must be greater than zero for stability. Also, oscillatory behavior is possible only when $\left(\frac{a}{2}\right)^2 < b$, otherwise time response is exponential.

Consider the requirement that "a" (given by equation (6-9.1)) and "b" (given by equation (6-9.2)) must be greater than zero for the condition of high flux

Figure 43. General Time Solutions for the Extended Randall and St. John Treatment.

Condition Roots		General Equation in Time
$\left(\frac{a}{2}\right)^2 > b$	m_1 and m_2 real	$g(t) = C_1 \exp \left[\left(-\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} \right) t \right] + \exp \left[\left(-\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} \right) t \right]$ (stable if $a > 0$ and $b > 0$)
$\left(\frac{a}{2}\right)^2 = b$	$m_1 = m_2$	$g(t) = C_1 + \exp \left(-\frac{a}{2} t \right)$ (stable if $a > 0$)
$\left(\frac{a}{2}\right)^2 < b$	m_1 and m_2 complex	$g(t) = \exp \left(-\frac{a}{2} t \right) \cos \left(\sqrt{b - \left(\frac{a}{2}\right)^2} t + C_1 \right)$ (stable if $a > 0$)

levels ($\phi_0 > 10^{13}$). Then since $\bar{x}_{0jj} \rightarrow 1$ for large values of flux, it is found that "b" is greater than zero generally, and for the same requirement on "a" it is necessary that

$$1 > \frac{\alpha_x \gamma_i}{(-k_j^2 + B_1^* + \alpha_x + 2\alpha_t \bar{w}_{10jj})} - \frac{C_1 C_2}{B_2^* + 2\beta_\tau \bar{w}_{20jj} - k_j^2}$$

which may be written as follows after recalling parameter definitions.

(6-11)

$$(D_1 k_j^2 + \Sigma_{a1} + \bar{\gamma}_x \Sigma_f - 2\alpha_t D_1 \bar{w}_{10jj})(D_2 k_j^2 + \Sigma_{a2} - 2\beta_\tau D_2 \bar{w}_{20jj}) > \nu \Sigma_f \Sigma_s$$

Now the requirement for stability that was obtained for the two region spacially averaged flux model employing two group diffusion theory in Chapter IV was as follows:

(6-12)

$$\left[\left(\frac{1}{A_0} + \frac{2}{A_1} \right) D_0 + \Sigma_{a0} + \bar{\gamma}_x \Sigma_f \frac{\bar{w}_0}{\bar{w}_0 + \lambda_x} \left(\gamma_x + \frac{\lambda_x + \lambda_1}{\bar{w}_0 + \lambda_x} \right) \right] \left[\left(\frac{1}{A_0} + \frac{2}{A_1} \right) D_1 + \Sigma_{a1} \right] > \nu \Sigma_f \Sigma_s$$

For large values of the flux this condition becomes

$$\left[\left(\frac{1}{A_0} + \frac{2}{A_1} \right) D_0 + \Sigma_{a0} + \bar{\gamma}_x \Sigma_f \right] \left[\left(\frac{1}{A_0} + \frac{2}{A_1} \right) D_1 + \Sigma_{a1} \right] > \nu \Sigma_f \Sigma_s \quad (6-13)$$

Letting $\alpha_l = \beta_l = 0$ (since the model employed to derive expression (6-12) did not include temperature effects) and identifying for $j=2$ (i.e., the geometrical buckling resulting from the first harmonic of the flux)

$$\frac{1}{A_0} + \frac{2}{A_1} \rightarrow k_2^2$$

then the two requirements for stability become the same. Note, of course, that the subscript "o" denotes thermal properties and the subscript "1" denotes fast properties in expression (6-13), while in expression (6-11) subscript "1" denotes thermal and "2" fast properties. (See Chapter IV for the definition of A_0 and A_1 .)

TWO GROUP ANALYSIS FOR HIGH FLUX

An analytic solution is possible for those flux levels such that the following set of equations describe the reactor model for two group diffusion theory.

$$\tau_0 \frac{\partial W_0(\bar{r}, t)}{\partial t} = D_0 \nabla^2 W_0 - K_0 W_0 - L_0 W_0 x + M_0 W_1, \quad (6-14.1)$$

$$\tau_1 \frac{\partial W_1(\bar{r}, t)}{\partial t} = D_1 \nabla^2 W_1 - K_1 W_1 + M_1 W_0, \quad (6-14.2)$$

$$W_0 x(\bar{r}, t) = \gamma_x W_0 + \lambda_1 i, \quad (6-14.3)$$

$$\frac{\partial i(\bar{r}, t)}{\partial t} = \gamma_1 W_0 - \lambda_1 i, \quad (6-14.4)$$

where

$$W_0(\bar{r}, t) = \sigma_x \phi_0(\bar{r}, t); \quad \phi_0 \text{ is the thermal flux,}$$

$$W_1(\bar{r}, t) = \sigma_x \phi_1(\bar{r}, t); \quad \phi_1 \text{ is the fast flux,}$$

$$x(\bar{r}, t) = \frac{\sigma_x}{\gamma \Sigma_f} X(\bar{r}, t); \quad X \text{ is the xenon concentration,}$$

$$i(\bar{r}, t) = \frac{\sigma_x}{\gamma \Sigma_f} I(\bar{r}, t); \quad I \text{ is the iodine concentration,}$$

τ_0, τ_1 the reciprocal thermal and fast neutron velocity,

D_0, D_1 the thermal and fast diffusion coefficients,

K_0, K_1 the thermal and fast removal coefficients,

L_0 the xenon removal coefficient,

M_0, M_1 the thermal and fast input coefficients,

and the other parameters have been defined in previous chapters.

It is assumed in equations (6-14) that the decay out of xenon is negligible compared to xenon burnout by thermal flux (i.e., $\lambda_x X \ll \sigma_x X \phi$) and that the rate of change of xenon is small compared to the production and burnout terms for xenon (i.e., $\frac{\partial X}{\partial t} \ll \gamma_x \Sigma_f \phi + \lambda_1 I - \sigma_x X \phi$). Furthermore, absorption of the fast

flux by xenon and production of xenon by fast fission are neglected. The assumptions above are reasonably valid at high flux levels as has been previously shown in Chapter V.

Since equations (6-14) are linear in the variables $W_0, W_1, W_0 x$, and i and have constant coefficients, the solutions are assumed to be separable into the form.

$$W_0(\bar{r}, t) = w_0(\bar{r}) e^{mt} \quad (6-15.1)$$

$$W_1(\bar{r}, t) = w_1(\bar{r}) e^{mt} \quad (6-15.2)$$

$$W_0(\bar{r}, t)x(r, t) = u(\bar{r}) e^{mt} \quad (6-15.3)$$

$$i(\bar{r}, t) = i'(\bar{r}) e^{mt} \quad (6-15.4)$$

Substitution of equations (6-15) into equations (6-14) gives, after eliminating the dependent variables of u and i'

$$\left[D_0 \nabla^2 - K_0 - \tau_0 m - L_0 \left(\frac{\gamma_x^m + \lambda_1}{m + \lambda_1} \right) \right] w_0 + M_0 w_1 = 0, \quad (6-16.1)$$

$$[D_1 \nabla^2 - K_1 - \tau_1 m] w_1 + M_1 w_0 = 0. \quad (6-16.2)$$

Furthermore, w_1 may be eliminated from equations (6-16) to yield the following bi-harmonic type equation.

$$(\nabla^4 + c_1 \nabla^2 + c_2) w_0 = 0 \quad (6-17)$$

where

$$c_1 = -\frac{1}{D_0} \left[K_0 + \tau_0 m + L_0 \left(\frac{\gamma_x^m + \lambda_1}{m + \lambda_1} \right) \right] - \frac{1}{D_1} [K_1 + \tau_1 m],$$

$$c_2 = \frac{1}{D_0 D_1} \left\{ (K_1 + \tau_1 m) \left[K_0 + \tau_0 m + L_0 \left(\frac{\gamma_x^m + \lambda_1}{m + \lambda_1} \right) \right] - M_0 M_1 \right\}.$$

Now suppose that equation (6-17) has solutions of the form

$$w_0(r) = e^{ar}$$

then one obtains the following characteristic equation for the spacial solutions

$$a^4 + C_1 a^2 + C_2 = 0 \quad (6-18)$$

and the four roots of equation (6-18) are as follows.

$$a_1 = \pm \sqrt{-\frac{C_1}{2} + \sqrt{\left(\frac{C_1}{2}\right)^2 - C_2}}$$

$$a_2 = \pm \sqrt{-\frac{C_1}{2} - \sqrt{\left(\frac{C_1}{2}\right)^2 - C_2}}$$

If all the roots of characteristic equation (6-18) are distinct, which is true if $C_2 \neq 0$ and $C_1^2 \neq 4C_2$, then w_0 has the following general solution.

$$w_0(r) = E_1 e^{a_1 r} + E_2 e^{-a_1 r} + E_3 e^{a_2 r} + E_4 e^{-a_2 r} \quad (6-19)$$

Now both w_0 and w_1 are assumed to satisfy the following boundary conditions for all values of time.

$$w_0(0, t) = w_0(L, t) = 0, \quad (6-20.1)$$

$$w_1(0, t) = w_1(L, t) = 0, \quad (6-20.2)$$

which implies that

$$w_0(0) = w_0(L) = 0. \quad (6-21.1)$$

$$w_1(0) = w_1(L) = 0. \quad (6-21.2)$$

It is now necessary to solve for w_1 from equation (6-16.1) which gives

$$w_1 = -\frac{D_0}{M_0} \nabla^2 w_0 + \frac{A_1}{M_0} w_0$$

where

$$A_1 = K_0 + \tau_0 m + L_0 \left(\frac{\gamma_{xm} + \lambda_1}{m + \lambda_1} \right).$$

Furthermore, since it was assumed that $w_0 = e^{ar}$, then w_1 becomes

$$w_1 = -\frac{D_0}{M_0} \left(a^2 - \frac{A_1}{D_0} \right) w_0. \quad (6-22)$$

Thus, imposing the boundary conditions found in equations (6-21) upon equations (6-22) and (6-19) leads to the following matrix formulation .

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{a_1 L} & -e^{a_1 L} & e^{a_2 L} & -e^{a_2 L} \\ \mu & \mu & \nu & \nu \\ \mu e^{a_1 L} & -\mu e^{a_1 L} & \nu e^{a_2 L} & -\nu e^{a_2 L} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} = 0 \quad (6-23)$$

where

$$\mu = a_1^2 - \frac{A_1}{D_0} ,$$

$$\nu = a_2^2 - \frac{A_1}{D_0} .$$

As usual, it is necessary that the coefficient matrix of the "E's" be singular for nontrivial solutions. The determinant of coefficient matrix is easily expanded by minors to give the following eigen value equation.

$$(e^{-a_1 L} - e^{a_1 L})(e^{-a_2 L} - e^{a_2 L})(\nu - \mu)^2 = 0 \quad (6-24)$$

Equation (6-24) is satisfied for the conditions given below.

1. When $\nu = \mu$ which implies that $a_1^2 = a_2^2$ and, thus, that $C_1^2 = 4C_2$
2. When $e^{2a_1 L} = 1$ which implies that if a_1 is real then $a_1 = 0$ (i.e., $C_2 = 0$), and if a_1 is complex, then letting $a_1 = \alpha + i\beta$ one finds that

$$e^{2a_1 L} = e^{2\alpha L}(\cos 2\beta L + i \sin 2\beta L) = 1$$

and equating real and imaginary parts one finds

$$ie^{2\alpha L} \sin 2\beta L = 0$$

so

$$\beta = \frac{m\pi}{2L}$$

for "m" an integer or zero and

$$e^{2\alpha L} \cos 2\beta L = 1$$

which implies that $\alpha = 0$ and $\beta = \frac{n\pi}{L}$ where "n" is an integer or zero. Therefore, for $e^{2a_1 L} = 1$, one requires that $a_1 = in\pi/L$.

3. When $e^{2a_2 L} = 1$ which implies similarly as condition 2 that $a_2 = ip\pi/L$ where p is an integer or zero (for p = 0 it is necessary that $C_1 = C_2 = 0$ since both C_1 and C_2 are real).

Now that the eigen values have been examined it is possible to express the general solution for the thermal flux for the following conditions on the eigen values:

1. For $a_1^2 = a_2^2 \neq 0$ (i.e., $C_1^2 = 4C_2 \neq 0$) then

$$w_o(r) = E_1 e^{a_1 r} + E_2 e^{-a_1 r} + E_3 e^{a_1 r} + E_4 e^{-a_1 r};$$

2. For $a_1 = 0$ (i.e., $C_2 = 0$) and $a_2 \neq 0$ (i.e., $C_1 \neq 0$) then

$$w_o(r) = E_1 + E_2 r + E_3 e^{a_2 r} + E_4 e^{-a_2 r};$$

3. For $a_2 = 0$ (i.e., $C_1 = C_2 = 0$) then

$$w_o(r) = E_1 + E_2 r + E_3 r^2 + E_4 r^3;$$

4. For $a_1 = in\pi/L$ and $a_2 \neq 0$ and real then

$$w_o(r) = E_1 \cos \frac{n\pi r}{L} + E_2 \sin \frac{n\pi r}{L} + E_3 e^{a_2 r} + E_4 e^{-a_2 r};$$

5. For $a_2 = ip\pi/L$ and $a_1 \neq 0$ and real then

$$w_o(r) = E_1 e^{a_1 r} + E_2 e^{-a_1 r} + E_3 \cos \frac{p\pi r}{L} + E_4 \sin \frac{p\pi r}{L};$$

6. For $a_1 = in\pi/L$ and $a_2 = ip\pi/L$ then

$$w_o(r) = E_1 \cos \frac{n\pi r}{L} + E_2 \sin \frac{n\pi r}{L} + E_3 \cos \frac{p\pi r}{L} + E_4 \sin \frac{p\pi r}{L}.$$

The arbitrary constants (i.e., the E's) corresponding to the various solutions above are not necessarily the same and are used to characterize the general solutions only.

The solutions for the thermal flux which are of prime interest are those which result when

$$a = \pm i \frac{n\pi}{L}, \quad n = 0, \pm 1, \pm 2,$$

then the characteristic equation (6-17) becomes

$$\left(\frac{n\pi}{L}\right)^4 - C_1 \left(\frac{n\pi}{L}\right)^2 + C_2 = 0$$

and recalling the definitions for C_1 and C_2 then "m" must satisfy the following:

$$\begin{aligned} & \left(\frac{n\pi}{L}\right)^4 (m+\lambda_1) + \left(\frac{n\pi}{L}\right)^2 \left\{ \frac{1}{D_o} [(K_o + \tau_o m)(m+\lambda_1) + L_o(\gamma_x m + \lambda_1)] \right. \\ & \left. + \frac{1}{D_1} [(K_1 + \tau_1 m)(m+\lambda_1)] \right\} + \frac{1}{D_o D_1} (K_1 + \tau_1 m) [(K_o + \tau_o m)(m+\lambda_1) \\ & + L_o(\gamma_x m + \lambda_1)] - \frac{1}{D_o D_1} M_o M_1 (m+\lambda_1) = 0. \end{aligned} \quad (6-25)$$

Equation (6-25) is a third degree polynomial equation in "m" possessing three roots. For stability of the time solutions for the thermal and fast flux it is necessary that the three roots of equation (6-25) have negative, real parts.

Under the assumption that $\tau_o = \tau_1 = 0$ (see Chapter V for elaboration on this assumption) then equation (6-25) becomes

$$\begin{aligned} & m \left[D_o D_1 \left(\frac{n\pi}{L}\right)^4 + (K_o D_1 + \gamma_x L_o D_1 + K_1 D_o) \left(\frac{n\pi}{L}\right)^2 + K_o K_1 + \gamma_x L_o K_1 \right. \\ & \left. - M_o M_1 \right] + \lambda_1 \left[D_o D_1 \left(\frac{n\pi}{L}\right)^4 + (K_o D_1 + L_o D_1 + K_1 D_o) \left(\frac{n\pi}{L}\right)^2 \right. \\ & \left. + K_o K_1 + L_o K_1 - M_o M_1 \right] = 0. \end{aligned} \quad (6-26)$$

Now all the parameters in equation (6-26) are positive and since $\gamma_x < 1$ then the requirement for stability of the n^{th} mode is that

$$D_o D_l \left(\frac{n\pi}{L}\right)^4 + \left(\frac{n\pi}{L}\right)^2 (K_o D_l + \gamma_x L_o D_l + K_l D_o) + K_l (K_o + \gamma_x L_o) > M_l M_o$$

and "m" in equation (6-26) is given by

$$m = -\lambda_i \left[1 + \left(\frac{n\pi}{L}\right)^2 \frac{\gamma_{iL_o D_l}}{R} + \frac{\gamma_{iL_o K_l}}{R} \right]$$

where

$$R = D_o D_l \left(\frac{n\pi}{L}\right)^4 + \left(\frac{n\pi}{L}\right)^2 (K_o D_l + \gamma_x L_o D_l + K_l D_o) + K_l (K_o + \gamma_x L_o) - M_l M_o$$

MULTIGROUP ANALYSIS FOR HIGH FLUX

In the previous section a two group diffusion theory model was investigated for stability under the assumption of high flux level. It is possible to extend that analysis to three (or higher) energy group diffusion theory, under the same assumption of high flux level.

The set of equations describing the system for three energy groups is the following:

$$\tau_1 \frac{\partial W_1}{\partial t} = \nabla^2 W_1 + B_1^* W_1 + C_1^* W_2 + D_1^* W_3 + E_1^* xW_1, \quad (6-27.1)$$

$$\tau_2 \frac{\partial W_2}{\partial t} = \nabla^2 W_2 + B_2^* W_2 + C_2^* W_1 + D_2^* W_3, \quad (6-27.2)$$

$$\tau_3 \frac{\partial W_3}{\partial t} = \nabla^2 W_3 + B_3^* W_3 + C_3^* W_1 + D_3^* W_2, \quad (6-27.3)$$

$$xW_1 = \bar{\gamma}_{x1} W_1 + \bar{\gamma}_{x2} W_2 + \bar{\gamma}_{x3} W_3 + \lambda_1 i, \quad (6-27.4)$$

$$\frac{\partial i}{\partial t} = \bar{\gamma}_{i1} W_1 + \bar{\gamma}_{i2} W_2 + \bar{\gamma}_{i3} W_3 - \lambda_1 i, \quad (6-27.5)$$

where

$$W_1(\bar{r}, t) = \sigma_x \phi_1(\bar{r}, t),$$

$$W_2(\bar{r}, t) = \sigma_x \phi_2(\bar{r}, t),$$

$$W_3(\bar{r}, t) = \sigma_x \phi_3(\bar{r}, t).$$

The other parameters are obvious but are left arbitrary so that they may be assigned to conform with a given multigroup analysis.

If solutions are assumed to be of the form

$$W_1(\bar{r}, t) = w_1(\bar{r}) e^{mt},$$

$$W_2(\bar{r}, t) = w_2(\bar{r}) e^{mt},$$

$$W_3(\bar{r}, t) = w_3(\bar{r}) e^{mt},$$

$$x(\bar{r}, t)W_1(\bar{r}, t) = w_1(\bar{r})x(\bar{r}) e^{mt},$$

$$i(\bar{r}, t) = i'(\bar{r}) e^{mt},$$

then equations (6-27) become after elimination of the xenon and iodine dependence

$$\nabla^2 w_1 + B_1 w_1 + C_1 w_2 + D_1 w_3 = 0, \quad (6-28.1)$$

$$\nabla^2 w_2 + B_2 w_2 + C_2 w_1 + D_2 w_3 = 0, \quad (6-28.2)$$

$$\nabla^2 w_3 + B_3 w_3 + C_3 w_1 + D_3 w_2 = 0, \quad (6-28.3)$$

where

$$B_1 = B_1^* + (\gamma_{x_1} + \gamma_{i1} \frac{\lambda_i}{m+\lambda_i}) E_1 - \tau_1 m = \frac{b_1 m + b_2}{m+\lambda_i} - \tau_1 m,$$

$$C_1 = C_1^* + (\gamma_{x2} + \gamma_{i2} \frac{\lambda_i}{m+\lambda_i}) E_1 = \frac{c_1 m + c_2}{m + \lambda_i},$$

$$D_1 = D_1^* + (\gamma_{x3} + \gamma_{i3} \frac{\lambda_i}{m+\lambda_i}) E_1 = \frac{d_1 m + d_2}{m + \lambda_i}.$$

$$B_2 = B_2^* - \tau_2 m,$$

$$B_3 = B_3^* - \tau_3 m.$$

Equations (6-28) constitute a linear, homogeneous set of partial differential equations of the stationary wave equation form. Elimination of any two of the three dependent variables results in a tri-harmonic type equation. Therefore, assuming that

$$\nabla^2 w_i = -\alpha^2 w_i \quad (6-29)$$

for $i = 1, 2$, and 3 and substituting equation (6-29) into equations (6-28) leads to the following requirement for nontrivial solutions.

$$\begin{vmatrix} (B_1 - \alpha^2) & C_1 & D_1 \\ C_2 & (B_2 - \alpha^2) & D_2 \\ C_3 & D_3 & (B_3 - \alpha^2) \end{vmatrix} = 0 \quad (6-30)$$

The coefficients of the determinant above are implicit functions of the characteristic parameter "m". For stability of the time solutions of equations (6-27), the real parts of all the roots of the polynomial equation in "m" that results when the determinant is expanded must be negative.

For the assumptions that $\tau_1 = \tau_2 = \tau_3 = 0$, the requirement for stability is that the terms

$$\begin{aligned} & (b_1 - \alpha^2) \left((B_2 - \alpha^2)(B_3 - \alpha^2) - D_3 D_2 \right) + c_1 (D_2 C_3 - C_2 (B_3 - \alpha^2)) \\ & + d_1 (C_2 D_3 - C_3 (B_2 - \alpha^2)) \end{aligned} \quad (6-31)$$

and

$$\begin{aligned} & (b_2 - \alpha^2 \lambda_1) \left((B_2 - \alpha^2)(B_3 - \alpha^2) - D_3 D_2 \right) + c_2 (D_2 C_3 - C_2 (B_3 - \alpha^2)) \\ & + d_2 (C_2 D_3 - C_3 (B_2 - \alpha^2)) \end{aligned} \quad (6-32)$$

must both be of the same sign.

COMPUTER SIMULATION

To simulate by the digital computer the reactor kinetic equations with xenon and two group diffusion theory, consider the following system of equations:

$$\frac{1}{v_1} \frac{\partial \phi_1}{\partial t} = D_1 \nabla^2 \phi_1 - \Sigma_{a1} \phi_1 - \sigma_x X \phi_1 + \Sigma_s \phi_2, \quad (6-33.1)$$

$$\frac{1}{v_2} \frac{\partial \phi_2}{\partial t} = D_2 \nabla^2 \phi_2 - \Sigma_{a2} \phi_2 - \eta \sigma_x X \phi_2 + \nu \Sigma_f \phi_1, \quad (6-33.2)$$

$$\frac{\partial X}{\partial t} = \bar{\gamma}_x \Sigma_f (\phi_1 + \epsilon_x \phi_2) + \lambda_i I - \sigma_x X (\phi_1 + \eta \phi_2) - \lambda_x X, \quad (6-33.3)$$

$$\frac{\partial I}{\partial t} = \bar{\gamma}_i \Sigma_f (\phi_2 + \epsilon_i \phi_1) - \lambda_i I, \quad (6-33.4)$$

where

$\phi_1(\vec{r}, t)$, $\phi_2(\vec{r}, t)$ - the thermal and fast flux,

$X(\vec{r}, t)$ - the xenon-135 concentration,

$I(\vec{r}, t)$ - the iodine-135 concentration,

D_1, D_2 - the thermal and fast diffusion coefficients,

Σ_{a1}, Σ_{a2} - the thermal and fast macroscopic removal cross sections (excluding xenon, but incorporating any fast or thermal neutron yields from fission, as appropriate),

ν - the average neutron yield per fission,

Σ_f - the macroscopic fission cross section,

Σ_s - the effective macroscopic slowing down cross section,

σ_x - the microscopic absorption cross section (thermal) for xenon,

v_1, v_2 - the average thermal and fast neutron velocities,

η - the fraction of fast to thermal neutrons absorbed by xenon,

ϵ_i - the fraction of iodine production from fast fission,

ϵ_x - the fraction of xenon production from fast fission,

λ_x, λ_i - the xenon and iodine decay constants,

$\bar{\gamma}_x, \bar{\gamma}_i$ - the prompt xenon and iodine yields from thermal fission.

Now at equilibrium, equations (6-33) assume the following form.

$$D_1 \nabla^2 \phi_{10} - \Sigma_{a1} \phi_{10} - \sigma_x X_0 \phi_{10} + \Sigma_s \phi_{20} = 0, \quad (6-34.1)$$

$$D_2 \nabla^2 \phi_{20} - \Sigma_{a2} \phi_{20} - \eta \sigma_x X_o \phi_{20} + \nu \Sigma_f \phi_{10} = 0, \quad (6-34.2)$$

$$X_o = \frac{\bar{\gamma} \Sigma_f \left[\phi_{10} + (\gamma_x \epsilon_x + \gamma_i \epsilon_i) \phi_{20} \right]}{\sigma_x (\phi_{10} + \eta \phi_{20}) + \lambda_x}, \quad (6-34.3)$$

$$I_o = \frac{\bar{\gamma}_i \Sigma_f (\phi_{10} + \epsilon_i \phi_{20})}{\lambda_i}, \quad (6-34.4)$$

and the thermal and fast stationary flux are the spacial solutions of the following equations:

$$D_1 \nabla^2 \lambda_{10} - \Sigma_{a1} \lambda_{10} - \bar{\gamma} \Sigma_f \left\{ \frac{\lambda_{10} + (\gamma_x \epsilon_x + \gamma_i \epsilon_i) \lambda_{20}}{\lambda_{10} + \eta \lambda_{20} + \lambda_x} \right\} \lambda_{10} + \Sigma_s \lambda_{20} = 0, \quad (6-35.1)$$

$$D_2 \nabla^2 \lambda_{20} - \Sigma_{a2} \lambda_{20} - \eta \bar{\gamma} \Sigma_f \left\{ \frac{\lambda_{10} + (\gamma_x \epsilon_x + \gamma_i \epsilon_i) \lambda_{20}}{\lambda_{10} + \eta \lambda_{20} + \lambda_x} \right\} \lambda_{20} + \nu \Sigma_f \lambda_{10} = 0, \quad (6-35.2)$$

where (see Chapter II for notation on the parameter "γ")

$$\phi_{10}(\bar{r}) = \lambda_{10}(\bar{r}) / \sigma_x,$$

$$\phi_{20}(\bar{r}) = \lambda_{20}(\bar{r}) / \nu_x.$$

If equilibrium units for the dependent variables are defined as follows.

$$\phi_1(\bar{r}, t) = \phi_{10}(\bar{r}) \phi_1(\bar{r}, t), \quad (6-36.1)$$

$$\phi_2(\bar{r}, t) = \phi_{20}(\bar{r}) \phi_2(\bar{r}, t), \quad (6-36.2)$$

$$X(\bar{r}, t) = X_o(\bar{r}) x(\bar{r}, t), \quad (6-36.3)$$

$$I(\bar{r}, t) = I_o(\bar{r}) i(\bar{r}, t), \quad (6-36.4)$$

and equations (6-36) are substituted into equations (6-33), there results

$$\frac{1}{v_1} \frac{\partial \phi_1}{\partial t} = D_1 \nabla^2 \phi_1 + 2D_1 \frac{\nabla \lambda_{10}}{\lambda_{10}} \nabla \phi_1 + \bar{\gamma} \Sigma_f M_{12} (1-x) \phi_1 + \Sigma_s \frac{1}{\omega_{12}} (\phi_2 - \phi_1), \quad (6-37.1)$$

$$\frac{1}{v_2} \frac{\partial \phi_2}{\partial t} = D_2 \nabla^2 \phi_2 + 2D_2 \frac{\nabla \lambda_{20}}{\lambda_{20}} \nabla \phi_2 + \bar{\gamma}_{\Sigma_f} \eta M_{12} (1-x) \phi_2 + v_{\Sigma_f} \omega_{12} (\phi_1 - \phi_2), \quad (6-37.2)$$

$$\begin{aligned} \frac{\partial x}{\partial t} = & \frac{\gamma_x}{M_{12}} (\lambda_{10} \phi_1 + \epsilon_x \lambda_{20} \phi_2) + \frac{\gamma_i}{M_{12}} (\lambda_{10} + \epsilon_i \lambda_{20}) i \\ & - (\lambda_{10} \phi_1 + \eta \lambda_{20} \phi_2 + \lambda_x) x, \end{aligned} \quad (6-37.3)$$

$$\frac{\partial i}{\partial t} = \frac{\lambda_i}{\lambda_{10} + \epsilon_i \lambda_{20}} (\lambda_{10} \phi_1 + \epsilon_i \lambda_{20} \phi_2) - \lambda_i i, \quad (6-37.4)$$

where

$$M_{12} = \frac{\lambda_{10} + (\gamma_x \epsilon_x + \gamma_i \epsilon_i) \lambda_{20}}{\lambda_{10} + \eta \lambda_{20} + \lambda_x},$$

$$\omega_{12} = \lambda_{10} / \lambda_{20}.$$

Equations (6-37) serve as the basic kinetic equations for simulation on the IBM 7040 Digital Computer by Computer Program 40.0142. The complete description and formulation of this program along with the solution technique is given in Appendix B. Computer Program 40.0142 simulates equations (6-37) in time and one dimension. The program is similar in philosophy to Program 40.104 given in Chapter V, only both flux derivatives are set to zero.

The axial stability of the flux distribution under the influence of xenon in the Advanced Test Reactor is examined in Chapter VII. Therefore, since Computer Program 40.0142 provides a powerful tool to examine the space and time behavior of the axial fast and thermal flux within the assumptions of the models considered, those models that were simulated by Program 40.0142 possessed parameter values which were characteristic of the Advanced Test Reactor. The graphical results obtained are given in Chapter VII. However, one set of figures is presented here.

Figures 44.1, 44.2, 44.3, and 44.4 give the response in time for the thermal and fast flux, xenon, and iodine over half the core for an initial, positive, 41

Figure 44.1. Thermal Flux Response over the Half Core for $\approx 28\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 44.2. Fast Flux Response.

Figure 44.3. Xenon Response.

Figure 44.4. Iodine Response.

percent perturbation in the fast and thermal flux. The parameters given in these figures are defined in Appendix B. The equilibrium thermal flux (maximum) is 1×10^{14} , while the ratio of thermal to fast flux at equilibrium (ω_{12}) is 0.2. Note the slightly underdamped response and the peculiar distribution of xenon in time.

SUFFICIENT CONDITIONS FOR TWO GROUP FLUX STABILITY

It is possible to obtain sufficient conditions (although not always necessary) for the stability of the general solutions of equations (6-37). The scheme is similar to that employed in Chapter V. (See Chapter V for assumptions.)

To implement the scheme consider that within some region of the reactor model core the thermal flux is perturbed positively so that

$$\phi_1 > 1$$

then for this positive not to increase in time it is sufficient that

$$\frac{\partial \phi_1}{\partial t} < 0 \quad (6-38)$$

until ϕ_1 again returns to its equilibrium value of unity. Now utilizing equation (6-37.1), then inequality (6-38) is satisfied if

$$0 > D_1 \nabla^2 \phi_1 + 2D_1 \frac{\nabla \lambda_{10}}{\lambda_{10}} \nabla \phi_1 + \bar{\gamma} \Sigma_f M_{12} (1-x) \phi_1 + \frac{\Sigma_s}{\omega_{12}} (\phi_2 - \phi_1). \quad (6-39)$$

Furthermore, since $\frac{1}{v_1} \frac{\partial \phi_1}{\partial t} \gg \frac{1}{v_2} \frac{\partial \phi_2}{\partial t}$, then equation (6-37.2) may be well approximated as follows.

$$0 = D_2 \nabla^2 \phi_2 + 2D_2 \frac{\nabla \lambda_{20}}{\lambda_{20}} \nabla \phi_2 + \bar{\gamma} \Sigma_f M_{12} \eta (1-x) \phi_2 + v \Sigma_f \omega_{12} (\phi_1 - \phi_2) \quad (6-40)$$

Expressions (6-39) and (6-40) may be combined to give

$$\begin{aligned} D_1 \left| \nabla^2 \phi_1 \right| + D_2 \frac{\Sigma_s}{v \Sigma_f \omega_{12}} \omega \left| \nabla^2 \phi_2 \right| &> \bar{\gamma} \Sigma_f M_{12} (1-x) \left\{ \phi_1 + \frac{\bar{\gamma} \eta \Sigma_s}{v \Sigma_f \omega_{12}^2} \phi_2 \right\} \\ &+ 2D_1 \frac{\nabla \lambda_{10}}{\lambda_{10}} \nabla \phi_1 + 2D_2 \frac{\Sigma_s}{v \Sigma_f \omega_{12}^2} \frac{\nabla \lambda_{20}}{\lambda_{20}} \nabla \phi_2 \end{aligned} \quad (6-41)$$

where cognizance is made of the fact that the leakage terms are intrinsically negative. A similar inequality results if it is assumed that the thermal flux

is perturbed negatively. Thus, inequality (6-41) constitutes the sufficient condition for stability against any positive perturbation and if satisfied insures that the positive perturbation in the thermal flux will return monotonically to equilibrium.

To provide a working example of how inequality (6-41) may be applied to a high flux reactor model, it is necessary to determine an expression for the extremum of the term $(1 - x)$. If it is assumed that $(1 - x)$ possesses an extremum, x_e , when $\frac{\partial x}{\partial t} = 0$, then there results from equation (6-33.3) the following:

$$M_{12} \delta x_e = \frac{\lambda_{10}(\gamma_x - M_{12}) \delta \phi_1 + \lambda_{20}(\gamma_x \epsilon_x - \eta M_{12}) \delta \phi_2}{\lambda_{10}(1 + \delta \phi_1) + \eta \lambda_{20}(1 + \delta \phi_2) + \lambda_x} \quad (6-42.1)$$

where

$$\phi_1 = 1 + \delta \phi_1, \quad (6-42.2)$$

$$\phi_2 = 1 + \delta \phi_2, \quad (6-42.3)$$

$$x_e = 1 + \delta x_e, \quad (6-42.4)$$

$$i = 1 + \delta i, \quad (6-42.5)$$

and it has been conservatively assumed that $\delta i = 0$.

Now substitution of equations (6-42) into inequality (6-41) gives

$$\begin{aligned} D_1 \left| \nabla^2 \delta \phi_1 \right| + D_2 \frac{\Sigma_s}{v \Sigma_f \omega_{12}^2} \left| \nabla^2 \delta \phi_2 \right| &> \bar{\gamma} \Sigma_f \left\{ (M_{12} - \gamma_x) \delta \phi_1 - \frac{\gamma_x \epsilon_x}{\omega_{12}} \delta \phi_2 \right\} \\ &+ 2D_1 \frac{\nabla \lambda_{10}}{\lambda_{10}} \nabla \delta \phi_1 + 2D_2 \frac{\Sigma_s}{v \Sigma_f \omega_{12}^2} \frac{\nabla \lambda_{20}}{\lambda_{20}} \nabla \delta \phi_2 \end{aligned} \quad (6-43)$$

where the term, η , (which accounts for fast neutron absorption by xenon) has been set to zero, and λ_x has been neglected compared to λ_{10} because of the assumption of a high flux model ($\phi_{10} > 10^{13}$).

Now consider that the perturbations in the thermal and fast flux have the following form:

$$\delta \phi_1 = \xi_1 \sin \frac{2\pi r}{L} \quad (6-44.1)$$

$$\delta \phi_2 = \xi_2 \sin \frac{2\pi r}{L} \quad (6-44.2)$$

Furthermore, under the assumption that η vanishes, then one can solve for ξ_2 from equation (6-40) and using equations (6-44) and (6-40) obtain for inequality (6-43) the following sufficient condition for stability of the model against perturbations of the form given in equations (6-44).

$$\left(\frac{D_1}{\Sigma_s} + \frac{D_2 \psi_2}{v \Sigma_f \omega_{12}} \right) \left(\tan \frac{2\pi r}{L} - \frac{L}{\pi} \frac{\nabla \lambda_{10}}{\lambda_{10}} \right) > \frac{\bar{\gamma} \Sigma_f}{\Sigma_s} (M_{12} - \gamma_x - \gamma_x \epsilon_x \psi_2) \tan \frac{2\pi r}{L} \quad (6-45)$$

where

$$\psi_2 = \frac{v \Sigma_f}{\left(\frac{2\pi}{L} \right)^2 D_2 + v \Sigma_f \omega_{12} - \frac{4\pi}{L} D_2 \frac{\nabla \lambda_{20}}{\lambda_{20}} \cot \frac{2\pi r}{L}}$$

It is significant to note that the sufficient condition given by inequality (6-45) is independent of the magnitude of the initial perturbations. Expression (6-45) may be employed in many ways. It may be imposed to investigate the stability of the reactor at a given point, r , or if knowledge of the equilibrium flux distribution is available then inequality (6-45) may be integrated (using an appropriate weighting function if necessary) over the region of interest.

VII - INVESTIGATION OF THE STABILITY OF THE AXIAL POWER DISTRIBUTION IN THE ADVANCED TEST REACTOR

It is evident from the previous work in this report (and from studies by others) that when the core of a given high flux thermal reactor exceeds a certain size, xenon-135 can cause instability of the power distribution within the core to occur. Although the fundamental mode of the flux is the least stable with respect to xenon, it is the fundamental mode, which essentially determines the power distribution, that is controlled in normal reactor operation. Thus, the instabilities of the flux due to xenon which are of practical interest are those instabilities in the higher modes which are not normally controlled during reactor operation. Furthermore, the higher the mode (or harmonic) the greater is the damping due to leakage, and thus it is generally the first harmonic (or second mode) which is examined for stability.

In several, large, thermal power and production reactors, xenon induced oscillations of the first harmonic of the flux, which gives rise to an oscillating power distribution, have actually been observed and recorded. Appendix A gives the experimental results of an intentionally initiated xenon induced flux oscillation in the Pressurized Water Reactor at the Shippingport Atomic Power Station. (See also (17, 27, 34) .)

In present test reactors the neutron flux levels and core sizes and core coupling have been such that xenon induced flux oscillations either do not occur or have not been a problem. However, the Advanced Test Reactor (ATR) is a significant extension in "test reactors"; and its individual lobes each with different power levels, the high thermal flux level, and the four foot length core all contribute to the need for examining the ATR for possible xenon induced flux oscillations. Furthermore, even if it is determined that any potential flux oscillation is convergent and, thus, stable, it is necessary from a kinetic and control standpoint to ascertain to some quantitative degree the nature of the time response and amount of damping for a given flux disturbance.

The ATR nuclear core is a four foot high, uniform width, vertical fuel annulus shaped in a serpentine fashion between and around nine flux trap regions.

The four flux trap positions on the corners of the square are referred to as inner flux "lobes". The four flux trap positions on the sides of the square are exterior to the fuel annulus. See Figure 45 for a schematic plan view of the ATR core. An extensive physical and nuclear description of the ATR is given in Appendix C.

An oscillation of the power distribution in the core leads to a concentration of power in a smaller region than is the case at equilibrium operation. This concentration of power produces increased heating and thus higher temperatures within the region, which pose the following possible consequences for the ATR.

1. Increased fuel plate temperatures result in increased fuel matrix temperatures which promote irradiation distortion and lead to swelling and other dimensional changes. Also, an increase in the fuel cladding temperatures increases the possibility of blistering and cladding failure. Furthermore, thermal cycling of the temperature due to the oscillating power distribution leads to warping and other stresses and promotes thermal cycling growth.

2. The rate of oxide formation at the cladding-water interface is strongly dependent of the cladding temperature. The rate of oxide formation in the ATR increases roughly by a factor of ten with a temperature rise of 100 degrees Fahrenheit from normal operating temperatures. (See Appendix C.) As the thickness of the oxide layer on the fuel cladding is increased, the thermal conductivity of the cladding is reduced, and the temperature distribution throughout the fuel element increases.

3. Local boiling will occur when the overpower ratio is approximately 1.2, and bulk boiling occurs when the ratio becomes about 1.4 (Appendix C). Boiling leads to gas bubble formation around the fuel plates, and since the gas is not as effective in removing heat from the fuel plates, the temperature in the plate is further increased. Furthermore, thermalization of the neutron population is reduced, and the average "neutron temperature" increases.

Also, significant axial temperature variations from the axial equilibrium temperature distribution in the core can lead to undesirable temperature gradients in regions outside of the fuel regions. For example, significant, nonsymmetrical temperature gradients imposed upon the neck shim housing can lead to serious distortion and alignment problems. Indeed, the neck shim control rods could jam and become inoperable.

Figure 45 Schematic Plan View of ATR Core.

Evidently, there are many problems associated with an uncontrolled oscillating power distribution in the ATR, and, thus, it is evident that adequate experimental and theoretical analysis of the stability of the ATR against xenon induced power oscillations should be rendered. Indeed, it is important that the response of the flux in time and space be reasonably known under any scheme of flux perturbation.

If it is determined that the ATR is unstable or that the power oscillations are significantly underdamped, then the control system of the ATR must include provisions for individual measurement and control of all those regions which can support power oscillations. However, these provisions add to the complexity and cost of the control system, and, thus, it is desirable to resolve the question of xenon induced flux oscillations before the final design of the ATR control system.

During the conceptual design of the ATR, several general types of possible xenon induced flux oscillations were recognized (9).

1. Interlobe oscillations in which the power distribution shifts from one lobe to another. The possible occurrence of this type of oscillation is enhanced by the fact that the power level in each lobe will be intentionally different under normal operating conditions.

2. Horizontal oscillations of the power distribution within a single lobe.

3. Traveling, horizontal oscillations of the power distribution around the serpentine fuel region.

4. Vertical oscillations of the axial power distribution.

The ATR may be susceptible to interlobe oscillations. However, the long time period associated with xenon induced oscillations and the ability to measure and control the power in each lobe provide adequate means to effectively suppress interlobe power oscillations if they do occur.

Previous investigations (9, 32) have indicated that horizontal power oscillations within a single lobe will not occur because of the adequate neutron coupling and small diameter of the lobe. Indeed, considering the dimensions of a "model lobe", the height of the lobe is more than four times as great as the diameter. Thus, it appears reasonable that if a lobe can support flux oscillations they will occur most readily along the axis. Traveling, horizontal oscillations around the fuel annulus, if they occur, should be capable of being measured and controlled by the present ATR measurement and control system

design. However, further investigation of traveling oscillations is necessary.

The chief task of this chapter of the report will be to investigate the axial stability of the ATR against xenon induced flux oscillations by applying the previously derived stability criteria and by digital computer simulation.

The bare height of the core of the ATR is four feet (122 cm). The migration length of fission neutrons in light water (H_2O) is given a value of about 6.4 cm in the literature (10). So the effective height of the ATR core may be taken as approximately 136 cm. (This assumes the sum of two migration lengths each at 7 cm, conservatively, plus the bare core height at 122 cm.) The experimental axial flux distribution over the core of the ATR was determined from the critical experiments conducted by the Babcock and Wilcox Company at Lynchburg, Virginia. (See Appendix C for details.) It was found that the experimental flux distribution could be well approximated by a cosine distribution, and the effective core height was experimentally set at 130 cm.

The most difficult task in preparing a model which adequately simulates axially the ATR core is the determination of the appropriate equation parameters (i.e., the reactor constants which enter into the kinetic equations). The decay constants for iodine and xenon are well known and independent of temperature and pressure. Although the values for the fractional fission yields of xenon-135 and iodine-135 (see Chapter II for the usual assumption made with respect to tellurium-135 are not dependent upon temperature or pressure either, they are not as well known as the decay constants due to experimental difficulties (5). However, the real source of difficulty and possible error is the determination, for each energy group of neutrons, of the appropriate diffusion coefficient, the macroscopic absorption and fission cross sections, the effective macroscopic "slowing down" or "scattering out" cross sections, and the microscopic absorption cross section of xenon for the model.

Usually the task of determining these neutron dependent parameters is given to various specifically formulated computer programs which are reasonably well suited for handling and processing the vast collection of data required and making the many calculations that are necessary. Appendix C gives a tabulation of the pertinent reactor parameters for the ATR determined by several different computer codes. It is obvious that agreement of corresponding parameters between codes is not always good. Recently, the proposed initial fuel loading in

the ATR was increased from 26 to 39 kilograms. A complete analysis by the more sophisticated and time consuming computer codes for the two group parameter values has not been conducted yet for the recently proposed 39 kilogram loading.

In formulating equations (6-37) which are the basis of Computer Program 40.0142 (see Chapter VI), both the fast and thermal removal cross sections are eliminated from the kinetic equations by the transformation of dependent variables given in equations (6-36). Thus, the requirement for knowing the fast and thermal removal cross sections, which are extremely difficult to experimentally determine and subject to considerable error (due in most part to the rapid change of absorption in the neighborhood of resonances) in computer analysis, is obviated. Instead, a knowledge of the fast and thermal flux distribution is required, which is considerably easier to experimentally determine. Indeed, of all the neutron dependent parameters, the values for the macroscopic absorption cross sections are, perhaps, the most difficult to determine. Except for the macroscopic fission cross section, the other parameters are not so sensitive to nuclear resonance phenomena; and the amount and distribution of fuel is usually well known as is the behavior of the absorption cross section with energy for the fuel. Thus, in the formulation of Computer Program 40.0142, perhaps the greatest source of possible error, the macroscopic absorption cross sections, has been replaced by the requirement for knowing the fast and thermal equilibrium flux distributions. Putting the appropriate parameters (see Appendix C) which describe the ATR "model" into the stability criteria contained in Chapters IV, V, and VI leads to the conclusion that the axial power distribution of the ATR model is stable for both the 26 and 39 kilograms fuel loading. Specifically, the stability requirements given by expressions (4-9), (5-37), (5-45), (5-68), (5-78), (6-9), (6-26), and (6-45) are all satisfied for present design values of the nuclear parameters and the effective core height. Extension of the stability criteria given by expressions (6-31) and (6-32) to incorporate four group analysis, using four group constants obtained for a 39 kilogram core loading from a TURBO Computer Program (32), yields a time solution for the flux of the approximate form " $\exp(-0.995 \lambda_1 t)$ ", which is stable. Expression (6-45) is a sufficient (but not necessary) condition developed from two group kinetics for stability against pure, first harmonic perturbations. For the assumed effective core height of 136 cm, expression (6-45) is not satisfied when the

ratio $(1/\omega_{12})$ of the equilibrium, axial, fast to thermal flux is less than approximately 4.4 for the 26 kilograms loading and 6.9 for the 39 kilogram loading (both of these values are applicable to the beginning of core life only). Actually, this ratio $(1/\omega_{12})$ is space dependent; but it is assumed in this study on the basis of experimental evidence (Appendix C) that the equilibrium fast and thermal fluxes both possess cosine distributions axially and have the same value for the effective, extrapolated axial height. Thus, when the ratio $(1/\omega_{12})$ is formed, the space dependence is eliminated. This assumption of a cosine distributed axial fast and thermal flux will be further corroborated in the ATR Critical Facility which achieved criticality in May 1964. The average value of the ratio $(1/\omega_{12})$ over all the fuel has been experimentally determined to be about 5.3 at the beginning of core life for the 26 kilogram loading while no experimental determination has been made for the ratio for the 39 kilogram loading. In general, this ratio for the 39 kilogram loading can be determined in the ATR Critical Facility. Note in Appendix C that experimental analysis also indicates that the ratio of the fast to thermal flux $(1/\omega_{12})$ decreases during the core life to a value of about 4.2 at the end of core life. However, the other parameters also have changed, and the requirement on the ratio $(1/\omega_{12})$ for expression (6-45) to be satisfied is reduced.

Computer Program 40.0142 (see Chapter VI and Appendix B) provides a powerful tool which can be employed to investigate, numerically, the time behavior of the ATR model under various schemes of perturbing the dependent variables and for different sets of parameter values. The graphical results of digital computer simulation of the one energy group, kinetic equations (Computer Program 40090 and 40.104) for the ATR model have been presented in Chapter V and indicate that the design value of the ATR core height provides for a stable power distribution against xenon induced instabilities.

Returning to the computer simulation of the ATR model by Computer Program 40.0142, the following figures present typical output results from the IBM 7040 Digital Computer. Because of the uncertainties and errors involved in establishing adequate equation parameters, a range of values around those given in Appendix C was employed in the Computer simulation of equations (6-37).

Figures 46 through 59 give the response in time for the dependent variables stated at the center of the half core. These figures represent the higher harmonic response which occurs even when the total power output from the core is held approximately constant. It is evident that for an axial fast to thermal equilibrium flux ratio ($1/\omega_{12}$) equal to 10 or greater, the response is slightly underdamped and stable for a wide range of parameters around those values assumed for the ATR model and general initial perturbation conditions. In Figure 52 for a half core height of 110 cm and other parameters as given, the dependent variables display nearly sustained oscillations. Figure 53 graphically portrays the sensitivity of the time response for various values of the ratio of the fast to thermal equilibrium flux. For those parameters given it is apparent that the threshold value for the ratio ($1/\omega_{12}$) is between 4 and 3.5.

Figure 46. Thermal Flux Response at the Center of the Half Core for -58% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.1$).

Figure 49. Thermal Flux Response at the Center of the Half Core for 12% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 50. Thermal Flux, Fast Flux, Xenon, and Iodine Response at the Center of the Half Core for $\approx -7\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.1$).

Figure 51. Thermal Flux Response at the Center of the Half Core for 27% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 52. Thermal Flux Response at the Center of the Half Core for 40% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 53. Thermal Flux Response at the Center of the Half Core for Various Initial Perturbations and Values of ω_{12} .

Figure 54. Thermal Flux, Xenon, and Iodine Response at the Center of the Half Core for $\approx 4\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.1$).

Figure 55. Thermal Flux Response at the Center of the Half Core for $\approx 17\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.1$).

Figure 56. Thermal Flux Response at the Center of the Half Core for 42% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.1$).

In Figure 57 the sensitivity of the ATR model to the ratio $(1/\omega_{12})$ is demonstrated for parameter values pertinent to a 26 kilogram loading. If one recalls that the value of the ratio $(1/\omega_{12})$ required to satisfy sufficient condition (6-45) was 4.4, it is evident from Figure 57 that the critical value does lie between 4 and 5. Indeed, for an initial perturbation of 36 percent in the thermal flux and a ratio value of 5, the thermal flux undershoots to a local minimum of about -18 percent and then overshoots to a local maximum of about 4 percent. Considering the magnitude of the perturbation which is close to the bulk boiling point for the ATR (see Appendix C), the response is satisfactory. Note, however in Figure 57 that the thermal flux response, for a ratio value of 4 and a positive perturbation in the flux of about 8 percent, first undershoots to a -17 percent and then rises monotonically. Obviously for a ratio $(1/\omega_{12})$ of 4 the thermal flux response may be considered unstable.

Figure 57. Thermal Flux Response at the Center of the Half Core for Various Initial Perturbations and Values of ω_{12} .

In Figures 58 and 59 the time response of the thermal flux at the center of the half core is given for various values of the ratio $(1/\omega_{12})$. In Figure 58 for those values which are pertinent to the 39 kilogram fuel loading for the ATR model, it is found that the thermal flux time response is unstable for ratio values $(1/\omega_{12})$ less than or equal to approximately 8.4. The thermal flux response is underdamped but stable for a ratio value of 8.5 or greater. A value of 9 for the ratio is predicted by the simple two group analysis for the

Figure 58. Thermal Flux Response at the Center of the Half Core for Various Initial Conditions and Values of ω_{12} .

Figure 59. Thermal Flux Response at the Center of the Half Core for Various Initial Conditions and Values of ω_{12} .

39 kilogram fuel loading (see Appendix C). However, this value may be somewhat large, and experimental determination of the ratio $(1/\omega_{12})$ should be made for the 39 kilogram loading. Nevertheless, for a value of approximately 8.5 or greater for the ratio of the axial equilibrium fast to thermal flux ratio, the computer simulation of equations (6-37) for the 39 kilogram fuel loaded ATR model indicates that the dependent variables and, specifically, the thermal flux are axially stable in time under any reasonable perturbation condition to be expected in normal operation of the ATR. Thus, on the basis of the analytic and computer investigation and those parameter values given in Appendix C, it is deemed that the axial power distribution in the ATR is stable.

For additional information on the space and time behavior of the dependent variables Figures 60 through 68 give the time response of the dependent variables over the half core for various parameter values and initial conditions.

It is interesting to observe the close similarity in both the magnitude and distribution of the fast and thermal flux. Again, as was found in Chapter V, the xenon exhibits an "unusual" time and space behavior compared to the other dependent variables. Undoubtedly, the unusual xenon behavior is attributable to the nonlinear burnout term which occurs in the xenon kinetic equation.

Figure 60.1. Thermal Flux Response over the Half Core for $\approx 23\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 60.2. Fast Flux Response.

Figure 60.3. Xenon Response.

Figure 60.4. Iodine Response.

Figure 61.1. Thermal Flux Response over the Half Core for $\approx 28\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 61.3. Iodine Response.

Figure 61.2. Xenon Response.

Figure 62.1. Thermal Flux response over the Half Core for 42% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 62.2. Fast Flux Response.

Figure 62.3. Xenon Response.

Figure 62.4. Iodine Response.

Figure 63.1. Thermal Flux Response over the Half Core for 50% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 63.2. Fast Flux Response.

Figure 63.3. Xenon Response.

Figure 63.4. Iodine Response.

Figure 64.1. Thermal Flux Response over the Half Core for $\approx 2\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.1$).

Figure 64.2. Fast Flux Response.

Figure 64.3. Xenon Response.

Figure 64.4. Iodine Response.

Figure 65.1. Thermal Flux Response over the Half Core for 35% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 65.2. Fast Flux Response.

Figure 65.3. Xenon Response.

Figure 65.4. Iodine Response.

Figure 66.1. Thermal Flux Response over the Half Core for 40% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 66.3. Xenon Response.

Figure 66.2. Fast Flux Response.

Figure 66.4. Iodine Response.

Figure 67.1. Thermal Flux Response over the Half Core for $\approx 10\%$ Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.25$).

Figure 67.2. Fast Flux Response.

Figure 67.3. Xenon Response.

Figure 67.4. Iodine Response.

Figure 68.1. Thermal Flux Response over the Half Core for 10% Initial Perturbation in the Thermal Flux ($\omega_{12} = 0.1667$).

Figure 68.2. Fast Flux Response.

Figure 68.3. Xenon Response.

Figure 68.4. Iodine Response.

Figures 69 and 70 give the time and space response of the dependent variables over the whole core for the initial conditions and parameter values indicated. Since the time and space behavior of the normalized fast and thermal flux were found to be very similar, only the thermal flux response is given. Figures 69 and 70 represent the time and space response of the whole core for the condition of no external control on the power output, and it is obvious that the power output is not constant. It is apparent that the response in both figures is unstable in time. In Figures 69 the thermal flux is given an initial, slightly positive, fundamental perturbation. The symmetry of the dependent variable response about the center of the core continues in time. In Figure 69.2 it is interesting to observe the heavy depletion of xenon which occurs at the center of the core because of the increased xenon burnout arising from the increasing thermal flux.

Figure 69.1. Thermal Flux Response over the Whole Core for a Small Fundamental Mode Perturbation in the Thermal Flux ($\omega_{12} = 0.2$).

Figure 69.2. Xenon Response.

In Figures 70 the initial perturbation imposed over the whole core is, substantially, a first harmonic (second mode) disturbance in the thermal flux. It is interesting to observe the modal coupling that occurs. The initial, first harmonic perturbation leads to a decreasing thermal flux which appears to exhibit a fundamental mode depression

Figure 70.1. Thermal Flux Response over the Whole Core for a First Harmonic (Second Mode) Perturbation in the Thermal Flux ($\omega_{12} = 0.1429$).

Figure 69.3. Iodine Response.

Figure 70.2. Xenon Response.

at four hours; and, as the thermal flux increases, it resembles a second harmonic (third mode) at 24-1/2 hours and continues to increase, finally exhibiting a positive, fundamental mode overtone.

It is apparent from the preceding results secured both from analytic stability criteria and digital computer simulation that, on the basis of the pertinent parameters given in Appendix C, the 26 kilogram fuel loading is stable for a fast to thermal flux ratio ($1/\omega_{12}$) of about 4.5 greater; and the 39 kilogram fuel loading is stable for a ratio ($1/\omega_{12}$) of about 7.5 or greater. However, although analytic calculations indicate that the ratio ($1/\omega_{12}$) is somewhat greater than that required for stability, because of the considerable uncertainties in analytically determining the ratio ($1/\omega_{12}$), it appears prudent that a fast and thermal equilibrium flux mapping be conducted in the ATR Critical Facility; and the ratio be experimentally found. Furthermore, in view of the many assumptions which must be made in formulating a "model" of the ATR core for analytic or computer investigation, it is desirable that some scheme of experimental investigation which might be reasonably performed in the ATR Critical Facility be carried out to determine the adequacy of the analytic and computer investigation and provide experimental evidence of the stability of the axial power distribution. A possible scheme to accomplish this experimental investigation might be as follows: If the axial flux distribution in a lobe were perturbed by some poison distribution (perhaps a small first harmonic perturbation) which adequately simulates xenon, then the experimentally determined, perturbed, axial flux distribution (both thermal and fast) could serve as input to the digital computer which would then determine an appropriate time step change in the xenon distribution. This computed distribution of xenon would be simulated by a new distribution in the poison and the procedure repeated until the axial stability of the flux distribution was determined. In this way a more accurate investigation

Figure 70.3. Iodine Response.

could be rendered because the effects of many of the factors unaccounted for in the analog and computer study, viz., the reflector, test samples, control rods, three dimensional effects, neck shim housing, and temperature (assuming that the temperature coefficient of reactivity was known for the 39 kilogram fuel loading) would be manifest in this experimental investigation.

VIII - CONCLUSIONS

As was stated in the Introduction to this report, the chief purpose of this study has been to examine the kinetic interaction of neutron leakage and delayed neutrons (both stabilizing) with xenon-135 induced changes (destabilizing) in the neutron multiplication. From this examination it was envisioned that adequate and practical stability criteria might be deduced.

The general analytic stability criterion was determined to have the functional form

$F(\text{leakage, delayed neutrons}) > G(\text{xenon induced multiplication changes})$
 where "F" and "G" are complicated functions in general. Under certain assumptions it is possible to obtain the explicit form for this stability criterion. For the linearized, space independent kinetic equations, the stability criterion is given by expressions (2-11) and (2-12). Expressions (4-9) and (4-10) give the stability criterion for the two energy group, spaced averaged kinetics model. In Chapter V, where the mono-energy, space and time dependent kinetic equations are examined, expressions (5-35), (5-36), and (5-37) give the stability criteria resulting from linear approximations to the kinetic equations. Sufficient conditions for stability are given by inequalities (5-54), (5-60), and (5-68); while inequalities (5-45), (5-78), and (5-81) give the criteria for high or flat flux distributions. The sufficient condition for stability found from the two energy group kinetic equations is given by inequality (6-45), and other stability criteria deduced from multigroup kinetics is given in Chapter VI.

To provide a qualitative account of the parameters which significantly effect the stability of the flux under xenon induced instabilities, consider the following stability criterion developed in Chapter V for the space and time dependent, one energy group kinetics for high flux levels (see expression).

$$\frac{M^2(B^2 - B_0^2)}{1 + M^2 B_0^2} + \ell^* \lambda_i > \frac{\bar{\gamma}_i}{\nu}$$

where B^2 is the total geometrical buckling corresponding to the mode under examination, and B_0^2 is the total critical buckling. ℓ^* is the effective neutron

lifetime, M^2 is the migration area for neutrons, λ_i is the decay constant for iodine-135, $\bar{\gamma}_i$ is the effective fractional fission yield of iodine-135, and ν is the average neutron release per fission. It is apparent that large migration areas, small reactor dimensions (which result in large B^2), and long effective neutron lifetimes all promote stability, while small migration areas, large reactor dimensions, and short effective neutron lifetimes all promote instability. Furthermore, it was found in Chapter VI and VII that the ratio of the fast to thermal flux in the fuel regions of the core strongly affects stability. Increasing the ratio of fast to thermal flux increases stability. It is interesting to note that when the fundamental mode is examined (i.e., when B_0^2 equals B^2), the resulting inequality is identical to the stability requirement given in expression (2-19) in Chapter II. For the stability criterion to be satisfied for the fundamental mode and no external control, it would be necessary to make the effective neutron lifetime assume a value of about 13.4 minutes or more, which is probably impossible. Evidently then, although both neutron leakage and delayed neutrons exert a stabilizing influence against xenon induced instabilities, neutron leakage is considerably more significant, in general, in controlling xenon instabilities. Thus, establishing stability criteria on the basis of delayed neutrons only as was done in Chapter II is highly restrictive and conservative and appears to be unrealistic and its use unwarranted in examining a physical reactor system for xenon induced instabilities.

The many computer simulations contained in this study provide a means for numerical exhibition of the actual time behavior of the dependent variables. The basic time behavior types for the flux are all graphically displayed. Exponential decay, damped oscillations, sustained oscillations, increasing oscillations, and exponential rise are all found to occur for certain range of values of the pertinent equations parameters. Furthermore, computer simulation provides an important means of corroborating the analytic criteria established. Significantly, the agreement of analytic and computer results was good.

Recommendations of possible ways to extend this study's applicability to physical reactor systems might include consideration of such effects as temperature, reflectors, inhomogeneous cores, and external control.

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APPENDIX A
GENERAL DATA

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Figure 71. PWR Core 1 Seed 2 Reactor Behavior versus Time - Intentional Xenon Oscillations Radial Flux Perturbation Test, 75 Percent Power, 475°F (Average), Group IV Rods.

Figure 72. Total Microscopic Cross Section of Xenon-135 for Neutrons as a Function of Energy (see BNL-325).

Figure 73. PWR-1 Seed 1 Intentional Xenon Power Oscillations.

APPENDIX B
COMPUTER SIMULATION

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ANALOG COMPUTER SIMULATION

Equations (2-4) written for analog simulation are as follows:

$$K_1 \frac{d\phi}{dT} = A \rho_o \left(1 - \frac{K_2 x}{K_2}\right) K_1 \phi$$

$$K_2 \frac{dx}{dT} = A (\lambda_o + \lambda_x) \left(\gamma_x \frac{K_1 \phi}{K_1} + \gamma_i \frac{K_3 i}{K_3}\right) K_2$$

$$- A \left(\lambda_o \frac{K_1 \phi}{K_1} + \lambda_x\right) K_2 x$$

$$K_3 \frac{di}{dT} = A \lambda_i \left(\frac{K_1 \phi}{K_1} - i\right) K_3$$

where

TIME SCALING: $t = AT$

AMPLITUDE SCALING:

$$K_1 \phi = 50 \text{ Volts}$$

$$K_2 x = 50 \text{ Volts}$$

$$K_3 i = 50 \text{ Volts}$$

and the equation variables and parameters are defined in Chapter II. Analog computer "setups" for the parameters indicated are given for the following cases.

CASE I

CASE II

CASE III

CASE IV

COMPUTER PROGRAM 40089

Computer Program 40089 is a machine program which simulates in time equations (2-4) given in Chapter II on an IBM 7040 Digital Computer. The main program and subroutine "DER" are both coded in FORTRAN language. (See (22,35) for a presentation on the nature and use of FORTRAN.) All cards beginning with the symbol "\$" are for control and identification purposes.

The Fortran Formulation of Computer Program 40089 for the IBM 7040 Digital Computer

```
$JOB      P GMS15400-C255 40089                XENON STABILITY EQS,DEC 63
$IBJOB      MAP
$IBFTC MAIN  DECK
C          SPACE INDEPENDENT KINETIC EQS TO STUDY XENON INDUCED INSTABILITIES
C          IN THERMAL REACTORS(IBM 7040 SIMULATION/RUNGE-KUTTA SUBROUTINE)
          DIMENSION Y(3),YP(3),Q(3)
          COMMON RO,XLO,XLX,XLI,GX,GI
          EXTERNAL DER
          N=3
          66 READ(5,101)RO,XLO,XLX,XLI,GX,GI
          101 FORMAT(6E10.3)
          WRITE(6,102)RO,XLO,XLX,XLI,GX,GI
          102 FORMAT(1H0,1P6E10.3)
C          SET INITIAL CONDITIONS
          READ(5,101)Y(1),Y(2),Y(3),H,UPLT,UPLF
          WRITE(6,102)Y(1),Y(2),Y(3),H,UPLT,UPLF
          T=0
          1 CALL DEQ(T,H,N,Y,YP,Q,DER)
          WRITE(6,103)T,Y(1),Y(2),Y(3),YP(1),YP(2),YP(3)
          103 FORMAT(1H ,1P7E15.4)
          IF(T.LE.UPLT.AND.ABS(Y(1)).LT.UPLF) GOTO 1
          CALL DEQSET
          GOTO 66
          END
$IBFTC DER    DECK
C          FORTRAN SOURCE DER
          SUBROUTINE DER(T,N,Y,YP)
          DIMENSION Y(3),YP(3)
          COMMON RO,XLO,XLX,XLI,GX,GI
          YP(1)=RO*Y(1)* (1.-Y(2))
          YP(2)=(XLO+XLX)*(GX*Y(1)+GI*Y(3))-Y(2)*(XLO*Y(1)+XLX)
          YP(3)=XLI*(Y(1)-Y(3))
          RETURN
          END
```

where the program variables and parameters are defined as follows:

$$RO = \rho_o$$

$$XLO = \lambda_o$$

$$XLX = \lambda_x$$

$$XLI = \lambda_i$$

$$GX = \gamma_x$$

$$GI = \gamma_i$$

$$Y(1) = \phi(t), YP(1) = \frac{d\phi}{dt}, \text{ for } 0 \leq t \leq UPLT$$

$$Y(2) = x(t), YP(2) = \frac{dx}{dt}, \text{ for } 0 \leq t \leq UPLT$$

$$Y(3) = i(t), YP(3) = \frac{di}{dt}, \text{ for } 0 \leq t \leq UPLT$$

$$H = t_2 - t_1 = \Delta t \text{ (incremental time step)}$$

UPLT is the upper limit setting for time.

UPLF is the upper limit setting for the flux.

The purpose for the input "UPLF" is to permit an "IF" statement routine to test the magnitude of the flux variable, Y(1), and terminate the processing of that data record when Y(1) exceeds "UPLF". This also prevents the machine from overflowing and dumping.

The input record format is as follows:

A. Parameter Card. In six 10 digit word "E" format.

Word 1. $RO = \rho_o$

Word 2. $XLO = \lambda_o$

Word 3. $XLX = \lambda_x$

Word 4. $XLI = \lambda_i$

Word 5. $GX = \gamma_x$

Word 6. $GI = \gamma_i$

B. Initial Conditions and Control Card. In six 10 digit word floating point "E" format.

Word 1. Value of Y(1) = $\phi(t)$ at $t = 0$. (Note: Y(1) = 1 is the equilibrium value.)

Word 2. Value of Y(2) = $x(t)$ at $t = 0$. (Note: Y(2) = 1 is the equilibrium value.)

Word 3. Value of Y(3) = $i(t)$ at $t = 0$. (Note: Y(3) = 1 is the equilibrium value.)

Word 4. $H = \Delta t$

Word 5. UPLT

Word 6. UPLF

COMPUTER PROGRAM 40090

Computer Program 40090 is a machine program which simulates in time and one dimensional equations (5-10) given in Chapter V on an IBM 7040 digital computer.

The main program, "XNEQS", and subroutine, "DER", are both coded in FORTRAN language. All cards beginning with the symbol "\$" are for control and identification purposes.

The Fortran Formulation of Computer Program 40090 for the IBM 7040 Digital Computer

```
$JOB      P GMS15400-C255 40.0090                XENON STABILITY EQS,DEC 63
$IBJOB      MAP
†IBFTC XNEQS  DECK
C      SIMULATION ON IBM 7040 OF SPACE AND TIME EQS FOR ANALYSIS OF XENON
C      INDUCED FLUX INSTABILITIES VIA SUBROUTINE DEQ
      DIMENSION Y(900),YP(900),Q(900),RIN(300)
      COMMON XL(300),XC,GX,GI,XLX,XLI,NMP,A(300),B(300),M
      EXTERNAL DER
      33 WRITE(6,34)
      34 FORMAT(1H1,30HXENON EQS,G.M.SANDQUIST,NOV.63)
      66 READ(5,67)COMMTS,H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12
      67 FORMAT(13A6)
      68 WRITE(6,69)COMMTS,H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12
      69 FORMAT(1H0,13A6)
      80 READ(5,81)NMP,XA,XB,GX,GI,XLX,XLI,HT,UPLF
      81 FORMAT(1I0,7(E10.6)/E10.6)
      84 WRITE(6,85)NMP,XA,XB,GX,GI,XLX,XLI,HT
      85 FORMAT(1H0,2X,3HNMP,8X,2HXA,14X,2HXB,14X,2HGX,13X,2HGI,13X,3HXLX,
      112X,3HCLI,11X,2HHT//1H ,I6,1P7(E15.5))
C      READ INITIAL CONDITIONS
      102 FORMAT(8F10.6)
      103 READ(5,102)(RIN(J),J=1,NMP)
      108 WRITE(6,109)(RIN(J),J=1,NMP)
      109 FORMAT(1H0,30X,20HINITIAL CONDS NTFLUX//((1H ,1P8E15.5))
      110 DO 111 J=1,NMP
      111 Y(3*J-2)=RIN(J)
      READ(5,102)(RIN(J),J=1,NMP)
      119 WRITE(6,120)(RIN(J),J=1,NMP)
      120 FORMAT(1H0,30X,19HINITIAL CONDS XENON//((1H ,1P8E15.5))
      DO 131 J=1,NMP
      131 Y(3*J-1)=RIN(J)
      READ(5,102)(RIN(J),J=1,NMP)
      WRITE(6,140)(RIN(J),J=1,NMP)
      140 FORMAT(1H0,30X,20HINITIAL CONDS IODINE//((1H ,1P8E15.5))
      DO 151 J=1,NMP
      151 Y(3*J)=RIN(J)
      NEQ=3*NMP
      NMPP1=NMP+1
      FNMP=NMP+1
      M=NEQ-2
      H=HT/FNMP
      HSQ=H*H
      XC=XA/HSQ
```

```

199 READ(5,200)(XL(J),J=1,NMPP1)
200 FORMAT(8E10.5)
201 WRITE(6,202)(XL(J),J=1,NMPP1)
202 FORMAT(1H0,30X,15HINIT VALS OF XL// (1H ,1P8E15.5))
300 READ(5,301)TMIN,DT,TMAX,NPRT
301 FORMAT(3E10.5,I10)
302 WRITE(6,303)TMIN,DT,TMAX,UPLF,NPRT
303 FORMAT(1H0,5X,4HTMIN,17X,2HDT,15X,4HTMAX,12X,4HUPLF,14X,4HNPRT//1P
14E17.5,I12)
310 T=TMIN
A(1)=(XL(2))/(2.*XL(1))+1.
B(1)=XL(1)/(XL(1)+XLX)*XB
DO 266 J=2,NMP
B(3*J-2)=XL(J)/(XL(J)+XLX)*XB
266 A(3*J-2)=(XL(J+1)-XL(J-1))/(2.*XL(J))+1.
CALL TIME(IT1)
NSTEPS=0
350 CALL DEQ(T,DT,NEQ,Y,YP,Q,DER)
DO 360 J=1,NMP
360 IF(ABS(Y(3*J-2)).GE.UPLF) GOTO 450
NSTEPS=NSTEPS+1
IF(NSTEPS.LT.NPRT) GOTO 350
CALL TIME(IT2)
ET=IT2-IT1
ET=ET/60.
400 WRITE(6,401)T
401 FORMAT(1H0,4X,2HMP,10X,4HFLUX,15X,5HXENON,15X,6HIODINE,10X,
15HTIME=,1PE12.5//)
DO 408 J=1,NMP
408 WRITE(6,410)J,Y(3*J-2),Y(3*J-1),Y(3*J)
410 FORMAT(1H ,16,1P3E20.5)
WRITE(6,411)ET
411 FORMAT(1H0,18HELAPSED TIME*****F10.4)
NSTEPS=0
CALL TIME(IT1)
IF(T.LE.TMAX) GOTO 350
450 CALL DEQSET
GOTO 66
END
$IDFTC DER DECK
C
SUBROUTINE DER(T,NEQ,Y,YP)
DIMENSION Y(900),YP(900)
COMMON XL(300),XC,GX,GI,XLX,XLI,NMP,A(300),B(300),M
NEQM5=NEQ-5
YP(1)=XC*(A(1)*Y(4)-2.*Y(1)+2.-A(1)) + (1.-Y(2))*B(1)*Y(1)
YP(2)=(XL(1)+XLX)*(GX*Y(1)+GI*Y(3))-(XL(1)*Y(1)+XLX)*Y(2)
YP(3)=XLI*(Y(1)-Y(3))
JXL=2
DO 10 J=4,NEQM5,3
YP(J)=XC*(A(J)*Y(J+3)-2.*Y(J)+(2.-A(J))*Y(J-3)) + B(J)*(1.-Y(J+1)
1)*Y(J)
YP(J+1)=(XL(JXL)+XLX)*(GX*Y(J)+GI*Y(J+2))-(XL(JXL)*Y(J)+XLX)*Y(J+1
1)
YP(J+2)=XLI*(Y(J)-Y(J+2))
10 JXL=JXL+1
L=NMP
YP(M)=XC*(A(M)-2.*Y(M)+(2.-A(M))*Y(M-3)) + B(M)*(1.-Y(M+1))*Y(M)
YP(M+1)=(XL(L)+XLX)*(GX*Y(M)+GI*Y(M+2))-(XL(L)*Y(M)+XLX)*Y(M+1)
YP(M+2)=XLI*(Y(M)-Y(M+2))
RETURN
END

```

The correspondence between program variables and parameters with those variables and parameters given in equations(5-10) is as follows:

NMP (The number of uniform special mesh points along the dimension under investigation.)

$$XA = \frac{1}{\rho_o \tau_e}$$

$$XB = \frac{1}{\tau_e}$$

$$GX = \gamma_x$$

$$GI = \gamma_i$$

$$XLX = \lambda_x$$

$$XLI = \lambda_i$$

HIT is the total length of the dimension under investigation

UPLF (an upper limit setting for the flux)

$Y(3*J-2) = \phi(r,t)$, $YP(3*J-2) = \frac{\partial \phi}{\partial t}$ for $TMIN \leq t \leq TMAX$ and $0 \leq r \leq HT$ in uniform space steps determined by NMP.

$Y(3*J-1) = X(r,t)$, $YP(3*J-1) = \frac{\partial x}{\partial t}$ for the same range as $Y(3*J-2)$ for the independent variables.

$Y(3*J) = i(r,t)$, $YP(3*J) = \frac{\partial i}{\partial t}$ for the same range as $Y(3*J-2)$ for the independent variables.

$XL(J) = \lambda_o(r) = \sigma_x \phi_o(r)$ for $0 \leq r \leq HT$ where $\phi_o(r)$ is the equilibrium flux distribution.

TMIN. is the initial value for the time variable.

$DT = t_2 - t_1 = \Delta t$ (incremental time step)

TMAX is the terminal value for the time variable.

UPLF is the upper limit setting for the flux.

NPRT is for print control and permits printing of the output data after a given number of stepwise integrations of the kinetic equations.

The differing scheme imposed on the Operators is as follows:

$$\nabla^2 \phi = \frac{1}{h^2} (\phi_{i+1} - 2\phi_i + \phi_{i-1}), \nabla \phi = \frac{1}{2h} (\phi_{i+1} - \phi_{i-1})$$

where h is the distance between mesh points, and the difference scheme is the usual central difference formulation.

The input record format is as follows:

A. Title Card. One title or identification card with any alphanumeric characters punched in columns 1 through 78 as desired.

B. Parameter Card. In eight 10 digit word form. The first word is in fixed point format while the remaining words are in floating point, "E" format.

Word 1. NMP (fixed point number, right adjusted)

Word 2. $XA = \frac{1}{\rho_o \tau_e}$

Word 3. $XB = \frac{1}{\tau_e}$

Word 4. $GX = \gamma_x$

Word 5. $GI = \gamma_i$

Word 6. $XLX = \lambda_x$

Word 7. $XLI = \lambda_i$

Word 8. HT

C. Upper Limit Control Card for Flux. One card with one 10-digit word in floating point "E" format.

Word 1. UPLF

D. Initial Conditions Cards. As many cards as are required to give the dependent variables flux, xenon, and iodine, their initial condition values (at TMIN) at each mesh point in eight 10-digit word form, "F" format.

E. Equilibrium Flux Distribution Cards. As many cards as are required to give the equilibrium flux in the form $\lambda_o(r) = \sigma_x \phi_o(r)$ its value at each mesh point in eight 10-digit word form, "E" format.

F. Time and Control Card. The four 10-digit word form.

Word 1. TMIN - floating point, "E" format.

Word 2. DT - floating point, "E" format.

Word 3. TMAX - floating point, "E" format.

Word 4. NPRT - fixed point number, right adjusted.

COMPUTER PROGRAM 40.104

Computer Program 40.104 is a machine program which simulates in time and one dimension equations (5-10) given in Chapter V on an IBM-7-40 digital computer for the condition that the flux derivative in time vanishes (i.e., effectively, with $\tau_e = 0$). The reasons for setting the flux time derivative to zero are given in Chapter V.

The main program, "XNEQS", and subroutine, "DER", are both coded in FORTRAN language. All cards beginning with the symbol, "\$", are for control and identification purposes and may vary in character with different machines. The flux is determined at each time step (i.e., for a given value of xenon) by a Gauss reduction scheme which operates on the tri-diagonal matrix. (When the equation for the flux is written as a difference equation in space, then for a given value of xenon, the equation for the flux assumes the form of a matrix equation in which the coefficient matrix of the column vector flux is tri-diagonal in form.)

The Fortran Formulation of Computer Program 40.104 for the IBM 7040 Digital Computer

```
$JOB    P GMS15400-C255 40.104                XENON STABILITY EQS, JAN 64, REV
$IBJOB      MAP
$IBFTC XNEQS  DECK
C      SIMULATION CN IBM 7040 OF SPACE AND TIME EQS FOR ANALYSIS OF XENON
C      INDUCED FLUX INSTABILITIES VIA SUBROUTINE DEQ(RUNGE-KUTTA METHOD)
      DIMENSION Y(600),YP(600),Q(600),RIN(300),F(300),A(300),E(300),
      IGXL(300),DXL(300),HXL(300),FXL(300)
      COMMON XL(300),GX,GI,XLX,NMP,XLI,Z(300)
      EXTERNAL DER
33 WRITE(6,34)
34 FORMAT(1H1,38HXENON EQS,G.M.SANDQUIST,DEC.63,REVISED)
66 READ(5,67)H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12,H13
67 FORMAT(13A6)
68 WRITE(6,69)H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12,H13
69 FORMAT(1H0,13A6)
80 READ(5,81)NMP,XA,GX,GI,XLX,XLI,HT
81 FORMAT(110,6(E10.6))
84 WRITE(6,85)NMP,XA,GX,GI,XLX,XLI,HT
85 FORMAT(1H0,5H NMP,8X,2HXA,13X,2HGX,13X,2HGI,13X,3HXLX,13X,3HXL1,1
      13X,2HHT//1H ,I6,1P6(E15.5))
C      READ INITIAL CONDITIONS DEPENDENT VARIABLES
      NEQ=2*NMP
102 FORMAT(8F10.6)
103 READ(5,102)(RIN(J),J=1,NMP)
108 WRITE(6,109)(RIN(J),J=1,NMP)
109 FORMAT(1H0,30X,26HINITIAL CONDITIONS NT FLUX//(1H ,1P8E15.5))
110 DO 111 J=1,NMP
111 Z(J)=RIN(J)
      READ(5,102)(RIN(J),J=1,NMP)
      WRITE(6,140)(RIN(J),J=1,NMP)
```

```

140 FORMAT(1H0,30X,25HINITIAL CONDITIONS IODINE// (1H ,1P8E15.5))
DO 151 J=1,NMP
151 Y(2*J)=RIN(J)
    FNMP=FNMP+1
    H=HT/FNMP
    HSQ=H*H
    NMPP1=NMPP1-1
    NMPP1=NMPP1+1
C    READ EQUILIBRIUM FLUX DISTRIBUTION
199 READ(5,200)(XL(J),J=1,NMPP1)
200 FORMAT(8E10.5)
201 WRITE(6,202)(XL(J),J=1,NMPP1)
202 FORMAT(1H0,30X,20HINITIAL VALUES OF XL// (1H ,1P8E15.5))
    XB=XA*HSQ
    GXL(1)=XL(2)+2.*XL(1)
    DXL(1)=(2.*XL(1)-XL(2))/GXL(1)
    HXL(1)=2.*XL(1)/GXL(1)
    FXL(1)=(XL(1)*XB)/(XL(1)+XLX)
    Y(1)=(Z(2)+DXL(1))/(HXL(1)*FXL(1)*Z(1))-2./FXL(1)+1.
275 DO 280 J=2,NMPP1
    GXL(J)=XL(J+1)+2.*XL(J)-XL(J-1)
    DXL(J)=(XL(J)-XL(J-1)+2.*XL(J)-XL(J+1))/GXL(J)
    HXL(J)=2.*XL(J)/GXL(J)
    FXL(J)=(XL(J)*XB)/(XL(J)+XLX)
280 Y(2*J-1)=(Z(J+1)+DXL(J)*Z(J-1))/(HXL(J)*FXL(J)*Z(J))-2./FXL(J)+1.
    GXL(NMP)=XL(NMPP1)+2.*XL(NMP)-XL(NMPP1)
    DXL(NMP)=(XL(NMPP1)+2.*XL(NMP)-XL(NMPP1))/GXL(NMP)
    HXL(NMP)=2.*XL(NMP)/GXL(NMP)
    FXL(NMP)=(XL(NMP)*XB)/(XL(NMP)+XLX)
    Y(2*NMP-1)=(1.+DXL(NMP)*Z(NMPP1))/(HXL(NMP)*FXL(NMP)*Z(NMP))-2./FX
    L(NMP)+1.
300 READ(5,301)TMIN,DT,TMAX,UPLF,NPRT
301 FORMAT(4E10.5,I10)
302 WRITE(6,303)TMIN,DT,TMAX,UPLF,NPRT
303 FORMAT(1H0,5X,4HTMIN,17X,2HDT,15X,4HTMAX,12X,4HUPLF,14X,4HNPRT//1P
    14E17.5,I12)
310 T=TMIN
    CALL TIME(IT1)
    NSTEPS=NPRT
350 CALL DEQ(T,DT,NEQ,Y,YP,Q,DER)
    DO 3 J=1,NMP
    3 A(J)=HXL(J)*((1.-Y(2*J-1))*FXL(J)-2.)
    E(1)=1./A(1)
    F(1)=-DXL(1)*E(1)
    DO 5 J=2,NMPP1
    E(J)=1./(A(J)-DXL(J)*E(J-1))
    5 F(J)=-DXL(J)*F(J-1)*E(J)
    Z(NMP)=(1.+DXL(NMP)*F(NMPP1))/(E(NMPP1)*DXL(NMP)-A(NMP))
    DO 10 J=1,NMPP1
    I=NMP-J
    Z(I)=-E(I)*Z(I+1)+F(I)
    10 IF(ABS(Z(I)).GE.UPLF) GOTO 450
    NSTEPS=NSTEPS+1
    IF(NSTEPS.LT.NPRT) GOTO 350
    CALL TIME(IT2)
    ET=IT2-IT1
    ET=ET/60.
400 WRITE(6,401)T
401 FORMAT(1H0,4X,2HMP,10X,4HFLUX,15X,5HXENON,15X,6HIODINE,10X,5HTIME=
    1,1PE12.5//)
    DO 408 J=1,NMP

```

```

408 WRITE(6,410)J,Z(J),Y(2*J-1),Y(2*J)
410 FORMAT(1H ,I6,1P3E20.5)
      WRITE(6,411)ET
411 FORMAT(1H0,18HELAPSED TIME*****F10.4)
      NSTEPS=0
      CALL TIME(IT1)
      IF(T.LE.TMAX) GOTO 350
450 CALL DEQSET
500 GOTO 66
      END
*IRFTC DER      DECK
      SUBROUTINE DER(T,NEQ,Y,YP)
      DIMENSION Y(600),YP(600)
      COMMON XL(300),GX,GI,XLX,NMP,XLI,Z(300)
      DO 10 I=1,NMP
      YP(2*I-1)=(XL(I)+XLX)*(GX*Z(I)+GI*Y(2*I))-(XL(I)*Z(I)+XLX)*Y(2*I-1)
10 YP(2*I)=XLI*(Z(I)-Y(2*I))
      RETURN
      END

```

The correspondence between program variables and parameters with those variables and parameters given in equations (5-10) is as follows:

NMP (The number of uniform special mesh points along the dimension under investigation.)

$$XA = \rho_0$$

$$GX = \gamma_x$$

$$GI = \gamma_i$$

$$XLX = \lambda_x$$

$$XLI = \lambda_i$$

HT is the total length of the dimension under investigation

UPLF (an upper limit setting for the flux)

$Z(J) = \phi(r,t)$, for $TMIN \leq t \leq TMAX$ and $0 \leq r \leq HT$ in uniform space steps determined by NMP

$Y(2*J-1) = X(r,t)$, $YP(2*J-1) = \frac{\partial x}{\partial t}$ for the same range as $Z(J)$ for the independent variables

$Y(2*J) = i(r,t)$, $YP(2*J) = \frac{\partial i}{\partial t}$ for the same range as $Z(J)$ for the independent variables

$XL(J) = \lambda_0(r) = \sigma_x \phi_0(r)$ for $0 \leq r \leq HT$ where $\phi_0(r)$ is the equilibrium flux distribution

TMIN is the initial value for the time variable

$DT = t_2 - t_1 = \Delta t$ (incremental time step)

TMAX is the terminal value for the time variable

UPLF is the upper limit setting for the flux

NPRT is the print control and permits printing of the output data after a given number of stepwise integrations of the kinetic equations

The differencing scheme imposed on the Operators is as follows:

$$\nabla^2 \phi = \frac{1}{h^2} (\phi_{i+1} - 2\phi_i + \phi_{i-1}), \quad \nabla \phi = \frac{1}{2h} (\phi_{i+1} - \phi_{i-1})$$

where h is the distance between mesh points, and the difference scheme is the usual central difference formulation.

The input record format is as follows:

A. Title Card. One title or identification card with any alphanumeric characters punched in columns 1 through 78 as desired.

B. Parameter Card. In seven 10-digit word form. The first word is in fixed point format while the remaining words are in floating point, "E" format.

Word 1. NMP (fixed point number, right adjusted)

Word 2. $XA = \rho_o$

Word 3. $GX = \gamma_x$

Word 4. $GI = \gamma_i$

Word 5. $XLX = \lambda_x$

Word 6. $XLI = \lambda_i$

Word 7. HT

C. Initial Conditions Cards. As many cards as are required to give the dependent variables, flux and iodine, their initial condition values (at TMIN) at each mesh point in eight 10-digit word form, "F" format. The xenon at time TMIN is determined by the computer from the initial values of the flux and iodine.

D. Equilibrium Flux Distribution Cards. As many cards as are required to give the equilibrium flux in the form $\lambda_o(r) = \sigma_x \phi_o(r)$ its value at each mesh point in eight 10-digit work form, "E" format.

E. Time and Control Card. In five 10-digit work form.

Word 1. TMIN - floating point, "E" format.

Word 2. DT - floating point, "E" format.

Word 3. TMAX - floating point, "E" format.

Word 4. UPLF - floating point, "E" format.

Word 5. NPRT - fixed point number, right adjusted.

COMPUTER PROGRAM 40.0142

Computer Program 40.0142 is a machine program which simulates in time and one dimensional equations (6-37) given in Chapter VI on an IBM 7040 Digital Computer.

The main program, "XNEQS", and subroutine, "DER", are both coded in FORTRAN language. All cards beginning with the symbol "\$" are for control and identification purposes. The thermal and fast flux are determined at each time step by a Gauss reduction scheme which operates on the pent-diagonal matrix that results from expressing Equations (6-37) as difference equations spacially.

The Fortran Formulation of Computer Program 40.0142 for the IBM 7040 Digital Computer

```
$JOB C GMS15400-C255 40.0142 XENON STABILITY EQS/TWO GROUP
$IBJOB MAP
$IBFTC XNEQS DECK
C SIMULATION ON IBM 7040 OF SPACE AND TIME EQS FOR ANALYSIS OF XENON
C INDUCED FLUX INSTABILITIES VIA SUBROUTINE DEQ(RUNGE-KUTTA METHOD)
C EMPLOYS TWO ENERGY GROUPS OF NEUTRONS (TWO GROUP DIFFUSION THEORY)
  DIMENSION Y(400),YP(400),Q(400),RIN(200),EF(200),RTF(200),
  1 X(400),R(200),S(200),U(200),V(200),XM(200),RET(400),TET(200),
  2 C(200),D(200),A(200),B(200),E(400),F(400),H(400),FET(200)
  COMMON XLX,XLI, ZT(200),ZF(200),NMP,XA(200),XD(200),XC(200),
  1 XD(200),YA(200),YB(200),FT(200)
  EXTERNAL DER
  33 WRITE(6,34)
  34 FORMAT(1H1,48HXENON EQS,G.M.SANDQUIST,JAN.64,(TWO FLUX GROUPS))
  66 READ(5,67)H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12,H13
  67 FORMAT(13A6)
  68 WRITE(6,69)H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12,H13
  69 FORMAT(1H0,13A6)
  80 READ(5,81)NMP,AT,BT,AF,BF,XLX,XLI,GX,GI,EX,EI,FN,HT
  81 FORMAT(110,6E10.6,F10.8/,5(F10.8))
  WRITE(6,85)NMP,AT,BT,AF,BF,XLX,XLI
  85 FORMAT(1H0,4H NMP,5X,2HAT,10X,2HBT,10X,2HAF,10X,2HBF,10X,3HXLX,10X
  1,3HCLI//1H ,15,1P6(E12.3))
  90 WRITE(6,95)GX,GI,EX,EI,FN,HT
  95 FORMAT(1H0,5H GX,8X,2HGI,8X,2HEX,8X,2HEI,8X,2HFN,8X,2HHT//1H ,6F
  110.5)
C READ INITIAL CONDITIONS DEPENDENT VARIABLES
  NEQ=2*NMP
  102 FORMAT(8F10.6)
  READ(5,102)(ZT(J),J=1,NMP)
  WRITE(6,109)(ZT(J),J=1,NMP)
  109 FORMAT(1H0,30X,26HINITIAL CONDITIONS TH FLUX//(1H ,1P8E15.5))
  READ(5,102)(ZF(J),J=1,NMP)
  WRITE(6,119)(ZF(J),J=1,NMP)
  119 FORMAT(1H0,30X,26HINITIAL CONDITIONS FT FLUX//(1H ,1P8E15.5))
  READ(5,102)(RIN(J),J=1,NMP)
  WRITE(6,140)(RIN(J),J=1,NMP)
```

```

140 FORMAT(1H0,30X,25HINITIAL CONDITIONS IODINE// (1H ,1P8E15.5))
DO 151 J=1,NMP
151 Y(2*J)=RIN(J)
C READ EQUILIBRIUM FLUX DISTRIBUTION
NMPPI=NMP+1
198 READ(5,200)(ET(J),J=1,NMPPI)
200 FORMAT(8E10.5)
WRITE(6,202)(ET(J),J=1,NMPPI)
202 FORMAT(1H0,30X,21HEQUIL. VALUES TH FLUX// (1H ,1P8E15.5))
199 READ(5,200)(EF(J),J=1,NMPPI)
206 WRITE(6,208)(EF(J),J=1,NMPPI)
208 FORMAT(1H0,30X,21HEQUIL. VALUES FT FLUX// (1H ,1P8E15.5))
NMPM1=NMP-1
MPD=2*NMP
MPDM1=MPD-1
MPDM2=MPD-2
MPDM3=MPD-3
FNMP=NMP+1
H=HT/FNMP
HSQ=H*H
BBT=BT*HSQ/AT
CDF=BF*HSQ/AF
CT=AT/HSQ
CF=HSQ/AF
TET(1)= 1./(ET(2)+2.*ET(1))
FET(1)= 1./(EF(2)+2.*EF(1))
RET(1)=(2.*ET(1)-ET(2))*TET(1)
RET(2)=(2.*EF(1)-EF(2))*FET(1)
DO 8 J=2,NMP
TET(J)= 1./(ET(J+1)+2.*ET(J)-ET(J-1))
9 FET(J)= 1./(EF(J+1)+2.*EF(J)-EF(J-1))
RET(2*J-1)=(ET(J-1)+2.*ET(J)-ET(J+1))*TET(J)
8 RET(2*J)=(EF(J-1)+2.*EF(J)-EF(J+1))*FET(J)
DO 10 J=1,NMP
RTF(J)=ET(J)/EF(J)
XM(J)=(ET(J)+(GX*EX+GI*EI)*EF(J))/(ET(J)+FN*EF(J)+XLX)
R(J)=ET(J)*TET(J)*BBT*XM(J)*2.
V(J)=ET(J)*TET(J)*(2.+1./(RTF(J)*CT))*2.
U(J)=EF(J)*FET(J)*CDF*XM(J)*2.
S(J)=EF(J)*FFT(J)*(2.+CF*RTF(J))*2.
B(2*J-1)=ET(J)*2.*TET(J)/(CT*RTF(J))
D(2*J)=EF(J)*FET(J)*CF*RTF(J)*2.
XA(J)=GX*ET(J)/XM(J)
XB(J)=EX/RTF(J)
XC(J)=GI*(ET(J)+EI*EF(J))/XM(J)
XD(J)=FN*EF(J)
YA(J)=XLI*RTF(J)/(RTF(J)+EI)
10 YB(J)=EI/RTF(J)
Y(1)=(ZT(2)+RET(1)+B(1)*ZF(1)-V(1)*ZT(1))/(R(1)*ZT(1))+1.
275 DO 280 J=2,NMPM1
280 Y(2*J-1)=(ZT(J+1)+RET(2*J-1)*ZT(J-1)+B(2*J-1)*ZF(J)-V(J)*ZT(J))/(
1 R(J)*ZT(J))+1.
Y(MPDM1)=(1.+RET(MPDM1)*ZT(NMPM1)+B(MPDM1)*ZF(NMP)-V(NMP)*ZT(NMP))
1 /(R(NMP)*ZT(NMP))+1.
300 READ(5,301)TMIN,DT,TMAX,UPLF,NPRT
301 FORMAT(4E10.5,I10)
302 WRITE(6,303)TMIN,DT,TMAX,UPLF,NPRT
303 FORMAT(1H0,5X,4HTMIN,17X,2HDT,15X,4HTMAX,12X,4HUPLF,14X,4HNPRT//1P
14E17.5,I12)

```

```

310 T=TMIN
    NSTEPS=NPRT
    CALL TIME(IT1)
350 CALL DEQ(T,DT,NEQ,Y,YP,Q,DER)
    DO 32 J=1,NMP
        A(2*J-1)= R(J)*(1.-Y(2*J-1))-V(J)
32  C(2*J)= U(J)*(1.-Y(2*J-1))-S(J)
        F(1)=1./A(1)
        E(1)=B(1)*F(1)
        H(1)=-RET(1)*F(1)
        F(2)=1./((C(2)-D(2)*E(1)))
        E(2)=-D(2)*F(1)*F(2)
        H(2)=-((RET(2)+D(2)*H(1))*F(2))
        DO 15 K=3,MPDM3,2
            F(K)=1./((A(K)+RET(K)*(E(K-1)*E(K-2)-F(K-2))))
            E(K)=(B(K)+RET(K)*E(K-2)*F(K-1))*F(K)
            H(K)=(E(K-2)*H(K-1)-H(K-2))*RET(K)*F(K)
            F(K+1)=1./((C(K+1)-RET(K+1)*F(K-1)-(D(K+1)-RET(K+1)*E(K-1))*E(K)))
            E(K+1)=(RET(K+1)*E(K-1)-D(K+1))*F(K)*F(K+1)
15  H(K+1)=((RET(K+1)*E(K-1)-D(K+1))*H(K)-RET(K+1)*H(K-1))*F(K+1)
            E(MPDM1)=(B(MPDM1)+RET(MPDM1)*E(MPDM3)*F(MPDM2))/(A(MPDM1)+RET(MPDM1)*
            (E(MPDM3)*E(MPDM2)-F(MPDM3)))
            H(MPDM1)=(RET(MPDM1)*(E(MPDM3)*H(MPDM2)-H(MPDM3))-1.)/(A(MPDM1)+RET(MPDM1)*
            (E(MPDM3)*E(MPDM2)-F(MPDM3)))
            X(MPD)=(RET(MPD)*E(MPDM2)-D(MPD))*H(MPDM1)-RET(MPD)*H(MPDM2)-1.)/
            (C(MPD)-RET(MPD)*F(MPDM2)+(RET(MPD)*E(MPDM2)-D(MPD))*E(MPDM1))
            X(MPDM1)=H(MPDM1)-E(MPDM1)*X(MPD)
            DO 21 K=1,MPDM2
                I=MPDM1-K
21  X(I)=H(I)-F(I)*X(I+2)-E(I)*X(I+1)
            DO 25 J=1,NMP
                ZT(J)=X(2*J-1)
                IF(ABS(ZT(J)).GE.UPLF) GOTO 450
25  ZF(J)=X(2*J)
60  NSTEPS=NSTEPS+1
        IF(NSTEPS.LT.NPRT) GOTO 350
        CALL TIME(IT2)
        TT=IT2-IT1
        TT=TT/60.
400 WRITE(6,401)T
401 FORMAT(1H0,4X,2HMP,9X,6HT FLUX,14X,6HF FLUX,14X,5HXENON,15X,6HIODI
        1NE,10X,5HTIME=,1PE12.5//)
        DO 408 J=1,NMP
408  WRITE(6,410)J,ZT(J),ZF(J),Y(2*J-1),Y(2*J)
410  FORMAT(1H ,16,4F20.8)
        WRITE(6,411)TT
411  FORMAT(1H0,18HELAPSED TIME*****F10.4)
        NSTEPS=0
        CALL TIME(IT1)
        IF(T.LE.TMAX) GOTO 350
450 CALL DEQSET
500 GOTO 66
    END
$IBFTC DER      DECK
    SUBROUTINE DER(T,NEQ,Y,YP)
    DIMENSION Y(400),YP(400)
    COMMON XI X,XI I,      ZT(200),7F(200),NMP,XA(200),XB(200),XC(200),
    1XD(200),YA(200),YB(200),ET(200)
60  DO 70 J=1,NMP
        YP(2*J-1)=XA(J)*(ZT(J)+XB(J)*ZF(J))+XC(J)*Y(2*J)-(ET(J)*ZT(J)+XD(J)
        1)*ZF(J))*Y(2*J-1)-XLX*Y(2*J-1)
70  YP(2*J)=YA(J)*(YB(J)*ZF(J)+ZT(J))-XLI*Y(2*J)
    RETURN
    END

```

The correspondence between program variables and parameters with those variables and parameters given in equations (6-37) is as follows:

NMP (The number of uniform special mesh points along the dimension under investigation).

$$AT = D_1 / \Sigma_s$$

$$BT = \bar{\gamma} \Sigma_f / \Sigma_s$$

$$AF = D_2 / \nu \Sigma_f$$

$$BF = \bar{\gamma} \eta / \nu$$

$$XLX = \lambda_x$$

$$XLI = \lambda_i$$

$$GX = \gamma_x$$

$$GI = \gamma_i$$

$$EX = \epsilon_x$$

$$EI = \epsilon_i$$

$$FN = \eta$$

HT is the total length of the dimension under investigation.

UPLF is the upper limit setting for the thermal flux.

ZT(J) = $\phi_1(r, t)$, for $TMIN \leq t \leq TMAX$ and $0 \leq r \leq HT$ in uniform steps determined by NMP.

ZF(J) = $\phi_2(r, t)$, for the same range as ZT(J) for the independent variables.

Y(2*J-1) = $x(r, t)$, $YP(2*J-1) = \frac{\partial x}{\partial t}$ for the same range as ZT(J) for the independent variables.

Y(2*J) = $i(r, t)$, $YP(2*J) = \frac{\partial i}{\partial t}$ for the same range as ZF(J) for the independent variables.

ET(J) = $\lambda_{10}(r) = \sigma_x \phi_{10}(r)$, for $0 \leq r \leq HT$ where $\phi_{10}(r)$ is the equilibrium thermal flux.

EF(J) = $\lambda_{20}(r) = \sigma_x \phi_{20}(r)$, for $0 \leq r \leq HT$ where $\phi_{20}(r)$ is the equilibrium fast flux.

TMIN is the initial value for the time variable.

$DT = t_2 - t_1 = \Delta t$ (incremental time step)

TMAX is the terminal value for the time variable.

UPLF is the upper limit setting for the thermal flux.

NPRT is for print control and permits printing of the output data after a given number of stepwise integrations of the kinetic equations.

The differencing scheme imposed on the Operators is as follows:

$$\nabla^2 \phi = \frac{1}{h^2} (\phi_{i+1} - 2\phi_i + \phi_{i-1}), \quad \nabla \phi = \frac{1}{2h} (\phi_{i+1} - \phi_{i-1})$$

where h is the distance between mesh points, and the difference scheme is the usual central difference formulation.

The input record format is as follows:

A. Title Card. One title or identification card with any alphanumeric characters punched in columns 1 through 78 as desired.

B. Parameter Card. In eight 10-digit word form. The first word is in fixed point format while the remaining words are in floating point format.

Word 1. NMP (fixed point number, right adjusted)

Word 2. $AT = \frac{D_1}{\Sigma_s}$, "E" format

Word 3. $BT = \frac{\bar{\gamma} \Sigma_f}{\Sigma_s}$, "F" format

Word 4. $AF = \frac{D_2}{v \Sigma_f}$, "E" format

Word 5. $BF = \frac{\bar{\gamma} \eta}{v}$, "E" format

Word 6. $XLX = \lambda_x$, "E" format

Word 7. $XLI = \lambda_i$, "E" format

Word 8. $GX = \gamma_x$, "F" format

C. Parameter and Control Card. In five 10-digit word form. All words are in floating point, "F" format.

Word 1. $GI = \gamma_i$

Word 2. $EX = \epsilon_x$

Word 3. $EI = \epsilon_i$

Word 4. $FN = \eta$

Word 5. HT

D. Initial Conditions Cards. As many cards as are required to give the dependent variables, thermal and fast flux, and iodine, their initial condition values (at TMIN) at each mesh point in eight 10-digit word ("F" format) form per card.

E. Equilibrium Thermal and Fast Flux Distribution Cards. As many cards as are required to give the equilibrium thermal and fast flux (in the form ET(J) and EF(J)) their value at each mesh point in eight 10-digit word form ("F" format) per card.

F. Time and Control Card. In five 10-digit word form.

Word 1. TMIN - floating point, "E" format

Word 2. DT - floating point, "E" format

Word 3. TMAX - floating point, "E" format

Word 4. UPLF - floating point, "E" format

Word 5. NPRT - fixed point number, right adjusted.

SUBROUTINE "DEQ"

"DEQ" is a MAP-coded subroutine which is called by the FORTRAN statement:

CALL DEQ(X,H,N,Y,YP,Q,DER)

It is recoded version by L. A. Schmittroth, Phillips Petroleum Co., National Reactor Testing Station, Idaho, of SHARE 1205 originally coded by Max Goldstein, New York University, October 1961.

Purpose:

To integrate, stepwise, a set of n simultaneous first order differential equations of the form:

$$y_i' \equiv \frac{dy_i}{dx} = f_i(x, y_1, y_2, \dots, y_n)$$

where

$$i = 1, 2, \dots, n.$$

Method:

Uses Gill's variation of the Runge-Kutta Method. See Gill, S., "A Process for Step-by-Step Integration of Differential Equations in an Automatic Digital Computing Machine," Proc. Camb. Phil. Soc., 47 (1951), pp. 96-108.

Description:

Before calling this subroutine, an initial value for x and an array Y which contains the initial values of $y_1, \dots, y_n$ must be supplied by the programmer.

A subroutine for computing and storing the derivatives y_i' in an array YP must also be supplied by the programmer.

DEQ enters the derivative subroutine; therefore the arguments of this derivative subroutine must conform in number, order, and mode to those which DEQ expects.

The first pass through DEQ produces no integration but returns with the initial values $x, y_1, \dots, y_n$ in Y and the derivatives in YP. Thereafter, for each entry into the subroutine, the program produces

$$x + h, y_1(x + h), \dots, y_n(x + h), y_1'(x + h), \dots, y_n'(x + h).$$

That is, after each entry, the subroutine returns after completing the evaluation of the derivatives and not after only evaluating the new y's.

Usage:

Enter DEQ by:

CALL DEQ(X,H,N,Y,YP,Q,DER)

The symbols shown as arguments may, of course, be replaced by any appropriate variable names; however, the arguments, except for H, must be identical with those of the last given initialization. H may be changed at will by redefining it before each reference to the call.

The symbols represent the following variables:

X Independent variable.

The programmer must supply the initial value before entering DEQ.

H Increment, Δx , may be changed in value before each reference to the call.

N Number of equations (fixed point).

Y Dependent variable block a one dimensional array of dimension $\geq N$.

The programmer must supply initial values of $y_1, y_2, \dots, y_n$ in this block before entering DEQ. This block will contain subsequent values of y.

YP Derivative block a one dimensional array of dimension $\geq N$. The subroutine supplied by the programmer to evaluate the derivatives should store the computed derivatives in this block.

Q Temporary storage, a one dimensional array of dimension $\geq N$ to be used only by DEQ to carry the integration forward. On the first pass, DEQ sets this block to zero. The information in this block is then carried from step to step in order to minimize rounding error.

DER The name of the subroutine written by the programmer to compute and store the derivatives. The FORTRAN rules require that when a subroutine name is to appear in an argument list of a subroutine, the name must appear in an EXTERNAL statement in the calling program.

This subroutine must be coded as a SUBROUTINE, SUBROUTINE DER (X,N,Y,YP), where the symbols have the same meaning as described. The routine should, of course, select x from X, the y's from Y and store the

derivatives in YP. The 4 arguments must appear in this order, none may be omitted.

Notes:

- (a) As the integration is carried forward, the previous values of the variables are destroyed by the program so they should be saved by the programmer if they are needed for future use.
- (b) If at any time one desires to restart the integration either to solve a different set of differential equations or to start with new initial conditions, without reloading the deck, one may use CALL DEQSET (no arguments), before using CALL DEQ (arguments as described) DEQSET is an alternate entry in DEQ.

Macro Assembly Program for Subroutine "DEQ"

```

$IBMAP DEQ      DECK
        ENTRY   DEQ
        ENTRY   DEQSET

*
* *****
* * 7040 VERSION OF SHARE 1205 NU DEQ(PA) BY MAX GOLDSTEIN *
* * GILL VARIATION OF RUNGE-KUTTA. RECODED BY L. SCHMITTROTH *
* * OCT 1963. SEE SHARE DIST. NO. 1205 FOR USAGE. *
* *****
*
DEQ      PZF      **                SAVE
        SXA      EXIT,4
        SXA      EXIT+1,1          INDEX
        SXA      EXIT+2,2          REGISTERS
        AXT      5,2              5 TO IR2
SW       TRA      *+1             SET SW TO TRA TO DEQ2
        CLA      TRA2            AFTER 1ST TIME
        STA      SW              SW NOW HAS TRA DEQ2
        AXT      1,2             1 TO IR2
        LAC      DEQ,4           2S COMPL OF TSL+1 TO IR4
        CLA      2,4             LOC(X)
        STA      NAME+3          TO NAME LIST,
        STA      XP+2            ETC
        STA      XP+3            ETC
        CLA      3,4             LOC(H)
        STA      H               TO H
        STA      XP              ETC
        CLA*     4,4             N
        STO      TEMP            TO TEMP
        STA      DEQ2            TO DEQ2
        CLA      4,4             LOC(N)
        STA      NAME+4          TO NAME LIST
        LXA      TEMP,1          N TRUE TO IR1
        CLA      5,4             LOC(Y1)
        STA      NAME+5          TO NAME LIST
        ADD      TEMP            LOC(Y1+N)
        STA      YN              TO YN

```

	STA	YN+1	ETC
	STA	YN+3	ETC
	STA	YN+4	ETC
	CLA	6,4	LOC(YP1)
	STA	NAME+6	TO LIST FOR NAME
	ADD	TEMP	LOC(YP1+N)
	STA	YPN	TO YPN
	CLA	7,4	LOC(Q1)
	ADD	TEMP	LOC(Q1+N)
	STA	QN	TO QN
	STA	QIJ+5	
	STA	QIJ+6	
	STA	NEWI+1	
	CLA	8,4	LOC (NAME) TO NAME
	STA	NAME	
QN	STZ	**,1	
	TIX	QN,1,1	
NAME	TSL	**	
NUDEQ	TXI	**+6,0,4	
	BCI	1,NUDEQ	
	PZE	**	X
	PZE	**	N
	PZE	**	Y
	PZE	**	YP
TSTIR2	TXL	EXIT,2,1	GOTO EXIT 1ST TIME
DEQ2	AXT	**,1	
NEWI	LDQ	E+5,2	
	FMP	**,1	
	STO	TEMP	
H	LDQ	**	
YPN	FMP	**,1	
	STO	TEMP+1	
	FSB	TEMP	
	STO	TEMP	
	LDQ	TEMP	
	FMP	D+5,2	
YN	UFA	**,1	
	UFS	**,1	
	STO	TEMP	
	FAD	**,1	
	STO	**,1	
QIJ	LDQ	F+5,2	
	FMP	TEMP+1	
	FAD	TEMP	
	FAD	TEMP	
	FAD	TEMP	
	FAD	**,1	
	STO	**,1	
	TIX	NEWI,1,1	
	TXH	XP,2,4	
	TXH	TIX,2,3	
	TXL	TIX,2,2	
XP	LDQ	**	
	FMP	D	
	FAD	**	
	STO	**	
TIX	TIX	NAME,2,1	
EXIT	AXT	**,4	
	AXT	**,1	
	AXT	**,2	
	TRA*	DEQ	
DEQSET	PZE	**	
	CLA	TRA1	
	STA	SW	
	TRA*	DEQSET	
TRA1	PZE	SW+1	

TRA2	PZE	DEQ2
D	DEC	.5,.292893219,1.70710678,.166666667
E	DEC	2.0,1.0,1.0,2.0
F	DEC	-.5,-.292893219,-1.70710678,-.5
TEMP	BSS	2
	END	

SUBROUTINE "TIME"

"TIME" is a MAP-coded (macro assembly program) subroutine which is called by the FORTRAN statement:

CALL TIME(ITJ)

where the term in the argument is arbitrary but should be a fixed point variable.

Purpose:

To determine from the computer clock the time when the "CALL TIME" statement is executed. Its main use is in determining real computer time required to perform some operation of interest.

Description:

Suppose for example that a certain operation is of interest, and it is desirable to know the actual computer time required to perform the operation. If the operation is called "X" for convenience then the following FORTRAN program will determine the computer time required to perform X.

```
CALL TIME (IT1)
OPERATION X
CALL TIME (IT2)
ET = (IT2-IT1)/60.
WRITE(6,1)ET
1 FORMAT(1H0,4HTIME = ,1PE12.5/)
```

The expression (IT2-IT1) is divided by "60." since the subroutine TIME gives the computer clock reading in minutes and it is desired here that the elapsed computer time for operation X be given in seconds. Of course, the proper functioning of the subroutine "TIME" requires, obviously, that the computer clock be in operation when the clock time is called by the subroutine.

Macro Assembly Program for Subroutine "TIME"

```
$IBMAP TIME DECK
      ENTRY TIME
TIME  PZE **
      CLA TIME
      ADD =2
      STA STORE
      CLA 5
STORE STO* **
      TRA* TIME
      END
```

Figure 74. Perturbed Flux Distribution Resulting from an Initial First Harmonic Perturbation.

Figure 75. Flux, Xenon, and Iodine Response for $\rho_0 = 3.1 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and 5% Initial Perturbation in the Flux.

Figure 76. Flux, Xenon, and Iodine Response for $\rho_0 = 3.1 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and 10% Initial Perturbation in the Flux.

Figure 77. Flux, Xenon, and Iodine Response for $\rho_0 = 3.0 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and 20% Initial Perturbation in the Flux.

Figure 78. Flux Response for $\phi_0 = 2 \times 10^{14}$, and Various Values of ρ_0 , and 3% Initial Perturbation in the Flux.

Figure 79. Flux Response for $\phi_0 = 2 \times 10^{14}$, and Various Values of ρ_0 , and 20% Initial Perturbation in the Flux.

Figure 80. Flux Response for $\rho_0 = 2.976 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and $\pm 20\%$ Initial Perturbation in the Flux.

Figure 81. Flux Response for $\rho_0 = 2.976 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and $\pm 10\%$ Initial Perturbation in the Flux.

Figure 82. Flux Response for $\rho_0 = 2.976 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and $\pm 3\%$ Initial Perturbation in the Flux.

Figure 83. Flux Response for $\rho_0 = 3.0 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and 40% Initial Perturbation in the Flux.

Figure 84. Flux Response for $\rho_0 = 3.094 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and -14% Initial Perturbation in the Flux.

Figure 85. Flux Response for $\rho_0 = 3.094 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and 14% Initial Perturbation in the Flux.

Figure 86. Flux Response for $\rho_0 = 3.094 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and -20% Initial Perturbation in the Flux.

Figure 87. Flux Response for $\rho_0 = 3.1 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and 2% Initial Perturbation in the Flux.

Figure 88. Flux, Xenon, and Iodine Response for $\rho_0 = 3.1 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and -2% Initial Perturbation in the Xenon.

Figure 89. Flux Response for $\rho_0 = 3.117 \times 10^{-5}$, $\phi_0 = 2 \times 10^{14}$, and -20% Initial Perturbation in the Flux.

Figure 90. Flux Response for $\phi_0 \leq 2 \times 10^{11}$, and Various Values of ρ_0 , and 50% Initial Perturbation in the Flux.

Figure 91. Flux, Xenon, and Iodine Response for $\rho_0 = 1.5 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and 30% Initial Perturbation in the Flux.

Figure 92. Flux Response for $\rho_0 = 3.950 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbations in the Flux.

Figure 93. Flux Response for $\rho_0 = 5.0 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbation in the Flux.

Figure 94. Flux Response for $\rho_0 = 5.2 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and 10% Initial Perturbation in the Flux.

Figure 95. Flux, Xenon, and Iodine Response for $\rho_0 = 5.2 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and 10% Initial Perturbation in the Flux.

Figure 96. Flux Response for $\rho_0 = 5.2 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and -10% Initial Perturbation in the Flux.

Figure 97. Flux Response for $\rho_0 = 5.435 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and -50% Initial Perturbation in the Flux.

Figure 98. Flux Response for $\rho_0 = 6.0 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbations in the Flux.

Figure 99. Flux Response for $\rho_0 = 6.342 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbations in the Xenon.

Figure 100. Flux Response for $\rho_0 = 6.342 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbations in the Flux.

Figure 101. Xenon Response for $\rho_0 = 6.342 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbations in the Flux.

Figure 102. Iodine Response for $\rho_0 = 6.342 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbations in the Flux.

Figure 103. Flux Response for $\rho_0 = 6.498 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and 10% Initial Perturbation in the Flux.

Figure 104. Flux Response for $\rho_0 = 6.498 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and -10% Initial Perturbation in the Flux.

Figure 105. Flux Response for $\rho_0 = 6.498 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and 20% Initial Perturbation in the Flux.

Figure 106. Flux Response for $\rho_0 = 6.498 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and -20% Initial Perturbation in the Flux.

Figure 107. Flux Response for $\rho_0 = 6.498 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and +2% Initial Perturbation in the Flux.

Figure 108. Flux, Xenon, and Iodine Response for $\rho_0 = 6.5 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and 10% Initial Perturbation in the Flux.

Figure 109. Flux Response for $\rho_0 = 6.498 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and -2% Initial Perturbation in the Flux.

Figure 110. Flux Response for $\rho_0 = 6.98 \times 10^{-5}$, $\phi_0 \leq 2 \times 10^{11}$, and Various Initial Perturbations in the Flux.

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APPENDIX C
ADVANCED TEST REACTOR DATA

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A PHYSICAL AND NUCLEAR DESCRIPTION OF THE ADVANCED TEST REACTOR
UNDER CONSTRUCTION AT THE NATIONAL REACTOR TESTING STATION
IN IDAHO FOR THE UNITED STATES ATOMIC ENERGY COMMISSION.

(The source of the information given here is Advanced Test Reactor - ATR, Preliminary Safety Analysis Report, U. S. Atomic Energy Commission, IDO-24040, dated April 1962. (See also references (9, 29, 32, 45).)

CORE DESCRIPTION

General Core Configuration

The Advanced Test Reactor (ATR) is a four foot high, uniform width, vertical fuel annulus shaped in a serpentine fashion between and around nine flux trap areas located in a 3 x 3 square array. (See Figures 111 through 116.)

The four flux trap positions on the corners of the square are referred to as inner flux trap positions (lobes), since they are enclosed by the fuel annulus. The four flux trap positions on the sides of the square are exterior to the fuel annulus and are referred to as outer flux trap positions. The central flux trap position is referred to as the center flux trap. The plan section of the complete fuel annulus is bounded by the nine flux trap baffles, the four neck shim rod housings, and the inner face of the beryllium reflector blocks. The fuel annulus contains 40 fuel elements. (See Figure 115.)

The outer radius of the fuel annulus for the 24 elements supported by the four inner flux trap baffles is formed by the inner face of the beryllium reflector. This beryllium reflector surrounds the core area and consists of eight segmental blocks 52 inches high. These blocks form a cylinder approximately 51 inches in diameter. Their inner contour follows the fuel annulus around the inner flux traps and the periphery of the outer flux trap baffles. Each of the eight blocks contains two vertical cylindrical cavities to accommodate rotating beryllium control cylinders.

Figure 111. Reactor Vessel Trimetric.

Figure 112. Reactor Core Trimetric.

① OUTER SHIM DRIVE SHAFT

② REACTOR VESSEL

③ THERMAL SHIELD

④ INSTRUMENT THIMBLE

⑤ NECK SHIM ROD HOUSING

⑥ INNER CAPSULE IRRADIATION FACILITY

⑦ OUTER SHIM CONTROL CYLINDER

⑧ OUTER SHIM GEAR BOX

⑨ OUTLET FLOW PIPE

⑩ CORE-REFLECTOR TANK

⑪ INLET FLOW BAFFLE ASSEMBLY

⑫ N-16 SAMPLE TUBE

⑬ OUTER SHIM GEAR BOX SUPPORT PLATE

⑭ OUTER CAPSULE IRRADIATION FACILITY

⑮ SAFETY ROD

⑯ DROP TUBE

⑰ BERYLLIUM REFLECTOR BLOCK

⑱ PRESSURE TUBE & INSULATION CAN ASSEMBLY

⑲ FUEL ELEMENT

⑳ FUEL STORAGE RACK

Figure 113. Reactor Internals Plan View.

① ATTACHMENT TO NECK SHIM HOUSING	⑭ NECK SHIM HOUSING SUPPORT POINT
② GEAR BOX SUPPORT PLATE	⑮ SAFETY ROD ASSEMBLY SUPPORT POINT
③ OUTER CAPSULE IRRADIATION FACILITY (ALUMINUM)	⑯ PRESSURE & INSULATION TUBE ASSEMBLY
④ CANAL LINER	⑰ THERMAL EXPANSION JOINT
⑤ DROP TUBE	⑱ CORE-REFLECTOR TANK (ALUMINUM)
⑥ OUTER REFLECTOR BLOCKS (ALUMINUM)	⑲ FLUX TRAP BAFFLE (ALUMINUM)
⑦ THERMAL SHIELD	⑳ OUTER SHIM DRIVE NOZZLE
⑧ INLET FLOW BAFFLE	㉑ OUTLET FLOW PIPE
⑨ SAFETY ROD CAVITY	㉒ REFLECTOR SUPPORT TANK (ALUMINUM)
⑩ N-16 SAMPLE TUBE PENETRATION	㉓ FLOW DISTRIBUTION TANK (STAINLESS STEEL)
⑪ BERYLLIUM REFLECTOR SUPPORT POINT	㉔ FLUX TRAP COOLANT PLENUM
⑫ FUEL ELEMENT SUPPORT POINT	㉕ SAFETY ROD DRIVE NOZZLE
⑬ OUTER SHIM CONTROL CYLINDER BEARING SUPPORT	㉖ CORE SUPPORT TANK (STAINLESS STEEL)

Figure 114. Reactor Internals Elevation.

① N-16 SAMPLE TUBES	⑩ CORE-REFLECTOR TANK (ALUMINUM)
② OUTER SHIM POISON INSERT (HAFNIUM)	⑪ PRESSURE TUBE
③ FUEL ELEMENTS	⑫ INSULATION CAN
④ OUTER SHIM CONTROL CYLINDER	⑬ SAFETY ROD GUIDE TUBE
⑤ NECK SHIM ROD HOUSING	⑭ SAFETY ROD POISON INSERT (HAFNIUM)
⑥ OUTER SHIM DRIVE SHAFT	⑮ FLUX TRAP BAFFLE
⑦ OUTER SHIM GEAR BOX SUPPORT PLATE	⑯ NECK SHIM CONTROL ROD (HAFNIUM)
⑧ OUTER SHIM GEAR BOX	⑰ SURVEILLANCE SPECIMENS
⑨ INNER CAPSULE IRRADIATION FACILITIES	⑱ BERYLLIUM REFLECTOR BLOCK

Figure 115. Core Plan.

① NECK SHIM ROD HOUSING ORIFICE PLATE	⑪ PRESSURE TUBE VIBRATION TIES
② NECK SHIM ROD HOUSING	⑫ PRESSURE & INSULATION TUBE ASSEMBLY
③ OUTER SHIM GEAR BOX	⑬ SAFETY ROD ASSEMBLY (UP POSITION)
④ OUTER SHIM CONTROL CYLINDER ORIFICE PLATE	⑭ OUTER SHIM GEAR BOX SUPPORT PLATE
⑤ SAFETY ROD ASSEMBLY (DOWN POSITION)	⑮ BERYLLIUM REFLECTOR BLOCKS
⑥ INNER CAPSULE IRRADIATION FACILITY	⑯ CORE-REFLECTOR TANK
⑦ OUTER SHIM CONTROL CYLINDER	⑰ N-16 SAMPLE TUBE CAVITY
⑧ OUTER SHIM CONTROL CYLINDER THRUST BEARING	⑱ FUEL ELEMENT
⑨ POSITION INDICATOR LINE	⑲ FLUX TRAP BAFFLE
⑩ FLOW DISTRIBUTION TANK	⑳ REFLECTOR SUPPORT TANK
	㉑ FUEL ELEMENT SIDE PLATE SUPPORT

Figure 116. Core Elevation.

Fuel Elements

The geometric arrangement of the fuel in the core requires a circular section fuel element. Each element consists of 19 parallel curved fuel plates mechanically attached to two side plates to form a 45-degree sector of a right circular cylinder. Each fuel plate is 49.5 inches long with an active fuel length of 48 inches (see Figure 117). Forty of these fuel elements are required for a complete core loading.

The two outermost fuel plates of each element are 0.080 inch thick with 0.020 inch of fuel and 0.030 inch of cladding. The 17 inner fuel plates are 0.050 inch thick, with 0.020 inch of fuel and 0.015 inch of cladding. The nominal coolant channel width between plates is 0.078 inch.

The fuel is a dispersion of U_3O_8 in an X-8001 aluminum matrix, the X-8001 material being used for its bonding properties. The fuel matrix contains B_4C as a burnable poison. The fuel matrix is clad with an aluminum alloy by using conventional picture frame techniques. Two cladding alloys, X-8001 and 6061, are under consideration.

In general, the irradiation experience on the $Al-U_3O_8$ system confirms the suitability of the fuel for research reactors and provides some confidence that the system can be used successfully in the high performance ATR. Since the maximum fuel temperature in the ATR is greater than 300°F and since the ATR fuel will have a fission density exceeding any of the existing irradiation experience, an irradiation test program is necessary to confirm the adequacy of the $Al-U_3O_8-B_4C$ composition for the anticipated thermal conditions and burnup level.

The mechanical properties and corrosion behavior under heat throughput conditions of the two potential cladding alloys are under investigation at ORNL. The rate of oxide formation is strongly dependent on plate surface temperature as shown in Figure 118. Thus, a reduction in surface temperature produces a corresponding reduction in film thickness. Since the aluminum oxide corrosion film is a thermal barrier, increases in the corrosion film thickness result in higher metal temperatures. The higher clad temperatures will result in a weaker fuel plate, and the higher matrix temperatures increase the probability of irradiation distortion of the fuel plate, such as swelling or blistering.

The present fuel element design criteria do not allow for axial or radial variations in either fuel loading or boron poison content. If the corrosion rates

Figure 117. Fuel Element Trimetric.

for the proposed uniformly loaded element are excessive due to the high design temperatures, nonuniform fuel and poison concentrations can be used. Preliminary calculations show that substantial reductions (more than 100°F) in plate metal

Figure 118. Rate of Oxide Accumulation Versus Oxide-Water Temperature.

temperature can be achieved, without compromising reactor performance, by using nonuniform loading or plate spacing. Several other design modifications, including changes to reduce clad structural requirements, can be made if necessary.

Reflector and Moderator

Light water is used as the core moderator and coolant. Eight 52-inch-long metallic beryllium blocks are used as a reflector. Each beryllium block contains two movable beryllium drums (Figure 115).

A complete analytical program is under way to establish the final coolant hole pattern in the beryllium, the clearance between the beryllium cylinder and the cylinder bore, the fuel annulus clearances, and the clearance between beryllium blocks. This program will include consideration of radiation induced growth, thermal distortion, and fabrication tolerances.

Several beryllium reflector arrangements are being studied. Preliminary stress and distortion data indicate that the eight-block arrangement shown in Figure 114 will result in combined thermal and growth stresses below 17,500 psi for 300 days of irradiation. The maximum calculated temperature in the beryllium for the present cooling hole pattern as shown in Figure 115 is 250°F.

Control and Safety Elements

Two shim systems and one safety system are used in the ATR. The safety system consists of five safety rods which can be located in any five of the eight flux traps exclusive of the center position. Sixteen rotating beryllium outer shim control cylinders with attached hafnium poison inserts are arranged as shown in Figure 115. The second shim system consists of seven small diameter neck shim rods located in each of four neck shim rod housings.

The neck shim rod element consists of a 48-inch hafnium rod $\frac{3}{8}$ inch in diameter, and a 48-inch aluminum follower rod $\frac{1}{4}$ inch in diameter. The aluminum and hafnium rod are joined by a riveted joint. This assembly is attached mechanically to a flexible drive cable.

NUCLEAR CHARACTERISTICS

Static Nuclear Characteristics

The ATR core design is based on extensive preliminary analysis and experimental work performed by Phillips Petroleum Company. A detailed analytical program is under way to fix the final design parameters, and an extensive critical experiment program will be performed to confirm the detailed design. An operating critical experiment facility (ATRC), located adjacent to the reactor, will be available to mock up actual core conditions to determine reactivities, flux distributions, power distributions, and shim control conditions for each core cycle.

The ATR is designed to operate at 250 MW for 17 days. The philosophy is to achieve satisfactory operation with a minimum fuel loading. A 40-50-60 MW power split in the fuel regions was selected as the maximum unbalance in power to be programmed for normal operation. This means that 40 MW can be generated in each of two opposite lobes, 60 MW in each of the remaining lobes, and 50 MW in the center region. The center region consists of the eight fuel elements closest to the central experiment, and each lobe consists of eight fuel elements arranged around one of the inner flux traps. (See Figure 115.) Initial calculations indicated that this reference power split can be held for 17 days with a core loading of approximately 26 kg* of U-235. Design calculations to date have been used mainly to establish the nuclear properties of the reactor system. Final values of core parameters, such as core loading, burnable poison content in the fuel, and shim control element positions, will be confirmed or adjusted with the aid of the experimental program.

Reactivity Balance

Figure 119 shows a composite curve indicating the relative reactivity contribution of each reactor parameter during core life. The curve designated " k_{excess} with no movable control" represents the expected excess neutron multiplication factor with the neck rods and the rotating reflector cylinders

* Recently the core loading has been increased to 39 kg of U-235. However, the analysis of this new loading is not complete, and thus is not given here.

Figure 119. Reactivity Composite for Core Life.

fully withdrawn under hot operating conditions. Data for the composite curve were taken from single lobe and two dimensional calculations. There is an additional reactivity contribution inherent in the composite curve due to the

shifting of lobe importance during core life. The differences in fuel burnup, experiment burnup, fission product build-up, and B-10 depletion between lobes have an effect on total core reactivity which is not accounted for in single lobe calculational representations of the reactor, but are properly accounted for in the multi-lobe two dimensional representation. These important effects have not been isolated, but are included with depletion effects in the area of the composite curve.

Reactivity losses during the core cycle are attributed to xenon, samarium, stable fission products, experiment burnup, and fuel burnup. The high flux level in the ATR results in a nearly saturated ($\Delta k \approx 5.0$ percent) equilibrium xenon concentration. A reactivity gain appears in the depletion of the boron burnable poison.

Following a complete shutdown, the xenon worth builds up. At the beginning of life, xenon override can be accomplished for about two hours after shutdown. The time in which restart must be accomplished is reduced toward the end of core life. Anything less than a full shutdown would have a correspondingly longer time available for start-up.

Movable Control

Two sources of shim control are used in the ATR. The neck regions between the lobes contain small cylindrical poison rods to control (or shim) some of the excess reactivity of the system and to control the center region power level. Rotating outer shim control cylinders in the reflector, acting in conjunction with the neck shim rods, provide the poison to accommodate power shifts and to control the remainder of the excess reactivity.

In order to avoid vertical power peaking, a neck shim rod will be either all in or all out of the core. The small changes in reactivity introduced by this operational mode are easily compensated by outer shim cylinder motion.

Symmetrical shimming operations, such as withdrawing neck shim rods (one at a time) in opposite lobes, simplify the problem of maintaining a preset power distribution. The worth of a single neck shim rod does not depend on whether it is pulled by itself or as part of a symmetrical pattern. Asymmetrical neck shim programming could be used if desired.

The rotating outer shim control cylinders have a total reactivity worth approximately equal to that of the neck rods. Since the outer shim cylinders are connected mechanically in pairs, two always rotate together (one cylinder cannot be rotated by itself). Satisfactory operation is achieved without the added complexity of individual outer shim cylinder operation. Each lobe has two pairs of outer shim cylinders or a total of four cylinders in each lobe.

Combined neck shim rod and outer shim cylinder worth is greater than the sum of the individual worths. When the outer shim cylinders are rotated inward, the importance of the neck region is greater than when the cylinders are rotated outward. Conversely, when the neck shim rods are in, the outer shim cylinders assume a higher importance which enhances their worth.

Safety Rods

Five safety rods are used in the ATR. The safety rods are inserted three inches into the core during operation to obtain a higher initial differential worth for rapid shutdown. Partial insertion results in only a minor increase in axial power peaking in the fuel regions; the resulting axial maximum-to-average power ratios ($P/\bar{P}$) are tabulated in Figure 120 for three positions of initial insertion. A shutdown margin analysis with two stuck safety rods has demonstrated the high degree of neutronic coupling between lobes in this reactor.

Figure 120. Safety Rod Insertion and Axial $P/\bar{P}$ Ratios.

<u>Insertion Position, Inches</u>	<u>Fuel Region</u>	<u>Experiment Region</u>
0	1.41	1.43
3	1.43	1.47
6	1.47	1.53

The large amount of reactivity required by the movable control system to maintain the 40-50-60-MW power split shows that there is a very large coupling effect between lobes, and the safety rod analysis firmly emphasizes this fact. The accident studies included a case in which positive reactivity was inserted into one lobe by voiding one A-1 flux trap. The least favorable rod interaction and region importance effects are obtained when three safety rods are inserted in and adjacent to the opposite lobe to cause a shutdown. In this case, the three

rods held the perturbed core subcritical. It is apparent therefore that the ATR lobes are highly coupled, and one lobe cannot in any sense operate independently. Figure 121 shows the change in flux across the reactor with the three safety rods inserted around one lobe.

Figure 121. Effects of Voided A-1 Flux Trap and Safety Rods on Thermal Flux Distribution.

<u>Reactor Conditions</u>	$\bar{\phi}_{vt}/\bar{\phi}_{sr}^*$
No voiding, no rods inserted	1.000
100% voiding, no rods inserted	0.762
50% voiding, no rods inserted	1.064
Trap voided, 3 safety rods inserted in the opposite lobe	7.46

* $\bar{\phi}_{vt}$ - Average flux in lobe containing voided trap.
 $\bar{\phi}_{sr}$ - Average flux in opposite lobe around which the three safety rods are inserted.

Shutdown margins depend on the rotating cylinder and neck shim rod behavior. Although the safety rods can shut the reactor down at any time without help from the movable shim control systems, this will not be done. Immediately following a scram, the shim control systems begin to insert negative reactivity. Within a few minutes, the full hold-down of the shim systems will be effective.

Axial Flux Distribution Determined by Critical Experiments

The axial flux distribution was measured in the Critical Experiments for the ATR conducted by Babcock and Wilcox Company at Lynchburg, Virginia and reported in reference (29). The axial flux distribution was experimentally determined by attaching a 48 inch long, 1 inch wide, 0.025 inch thick polyethylene strip to the surface of a central fuel plate. This strip had depressions milled into the face to accept the 0.482-inch diameter gold foils on 2-inch centers.

The data, shown in Figure 122, were normalized to the average axial flux as determined by averaging the activity of the equally spaced foils. The axial buckling and end reflector savings were determined from the data which were fitted by least squares technique to a cosine using a computer program for that purpose. The equation determined was

Figure 122. Axial Gold Activity in Fuel Element.

$$Y = A \cos B(x-C)$$

where

$$A = 1.433 \pm 0.007,$$

$$B = 0.02339 \pm 0.00008 \text{ cm}^{-1},$$

$$C = 60.30 \pm 0.08 \text{ cm}.$$

Thus, the reflector savings is approximately 6.16 cm on each end of the core having a displacement of about 0.7 cm from the geometric center.

Neutron Flux Distribution

ATR operation at 250 MW with the reference 40-50-60 power split produces the following fluxes in the fuel elements:

Axially averaged fast flux over all fuel elements	$1.3 \times 10^{15} \text{ n/cm}^2\text{-sec}$
Axially averaged thermal flux over all fuel elements	0.24×10^{15} (at 0 days) 0.31×10^{15} (at 17 days)
Maximum fast flux in any fuel element	2.3×10^{15}
Maximum thermal flux in any fuel element	0.68×10^{15} (at 0 days) 0.82×10^{15} (at 17 days)

Figures 123 and 124 show the approximate flux distributions for the beginning and end of life. Axially averaged fluxes are presented for each fuel element and experiment. The magnitude and radial location of the thermal flux peaks also are designated. The hot spot does not move during core life. The magnitude of the thermal flux peak increases during life, with maximum power generation decreases as shown in Figure 124 and 125. This is the result of greater than average fuel depletion.

The unusual geometry and power distributions in the ATR do not necessarily produce severe flux gradients in the system. Figure 125 is a plot of epithermal and thermal fluxes along a diagonal from the center of the reactor through the exterior experiment. The curve corresponds to a 45 degree diagonal in Figure 123. The fluxes are axially averaged values. The detailed shape of the curve varies somewhat with core life, but remains similar to the one shown in Figure 125.

Neutrons with an energy less than 0.625 eV are considered thermal neutrons. At the beginning of life, 90 percent of all fissions are caused by thermal neutrons.

Figure 123. Epithermal and Thermal Average Fluxes - Beginning of Life.

This increases to 92 percent of all fissions by the end of the core cycle. However, the mean energy of all neutrons assumed to cause fission is 9138 eV. This number is obtained by multiplying the number of neutrons causing fission in each energy group by the energy of that group, by summing the overall groups, and by dividing by the number of neutrons.

Figure 124. Epithermal and Thermal Average Fluxes - End of Life.

Figure 125. Epithermal and Thermal Flux Profile.

Temperature Coefficients of Reactivity

Two dimensional calculations by Phillips Petroleum Company indicated a core temperature coefficient of $-1.45 \times 10^{-4} \Delta k / ^\circ F$. Good agreement on the sign and the approximate magnitude of coefficients was reported between single lobe calculations and two dimensional work.

The single lobe one dimensional model has been used in preliminary calculations to determine the basic response of the reactor regions to the temperature change. Figure 126 lists the various region coefficients obtained with the single lobe geometry. The fuel region coefficient agrees with previous Phillips Petroleum Company analysis. The influence of the movable control cylinders appears in several ways. As the hafnium sections approach the fuel, the fuel region temperature coefficient becomes more negative. Also, the positive reflector contribution is reduced as the cylinders are inserted. The reflector coefficient is expected to be more negative because the single lobe model overestimates reflector importance.

Figure 126. Temperature Coefficients
Calculated for ATR Single Lobe Geometry.

<u>Region</u>	<u>Temperature, °F</u>	<u>Distance of the Hafnium Control Ring</u>	<u>$\Delta k / ^\circ F \times 10^4$</u>
Fuel	163	No control	-1.5
Fuel	163	2 inches	-1.8
Fuel	163	1 inch	-2.0
Reflector	187	No control	+0.51
Reflector	187	1 inch	Near Zero
Flux trap, 70% H ₂ O 30% Al	145	No control	+0.19
Flux trap, 100% H ₂ O	145	No control	+0.28
Experiment, 50% H ₂ O	600 (moderator)		
50% metal	400 (neutron)	No control	+0.07

A value of $-1 \times 10^{-4} \Delta k/^{\circ}\text{F}$ is the nominal value for the fuel region coefficient. Because of the Phillips Petroleum Company results and the single lobe calculations described previously, this represents a conservative, but reasonable value for use at this time. An overall reactor coefficient has not been computed because of its limited usefulness. The temperature rise in the fuel would be expected to lead those of the other regions if a sudden power increase occurs.

Void Coefficient of Reactivity

Void coefficients have been computed for various sections of the reactor by using the single lobe model. The results summarized in Figure 127 show that the ATR has a very strong negative void coefficient.

Figure 127. Void Coefficients - ATR Single Lobe Geometry Results.

<u>Region</u>	<u>Void Fraction *</u>	<u>Distance of the Hafnium Control Ring</u>	<u>Void Coefficient **</u>
Fuel	0.10	No control	-0.167
Fuel	0.10	1 inch	-0.309
Reflector	0.10	No control	+0.024
Reflector	0.10	1 inch	Near zero
Flux trap, 70% H ₂ O 30% Al	0.10	No control	+0.029
Flux trap, 100% H ₂ O	0.10	No control	+0.033
Experimental loop, 50% H ₂ O 50% metal	0.10	No control	+0.014

$$* \text{ Void fraction} = \frac{\text{Volume Water Removed}}{\text{Initial Volume Water}}$$

$$** \text{ Void coefficient} = \frac{\Delta k_{\text{eff}}}{\text{Void Fraction}}$$

Removal of water from nonfuel regions in the core and reflector assembly produces a positive reactivity contribution. This is due to the neutron absorption properties of water in an overmoderated region such as the beryllium reflector or flux trap. The overall void coefficient is negative, however, because of the very strong negative coefficient in the fuel region. A sizeable void addition is not likely to occur in the reflector without a corresponding void addition in the adjacent fuel regions. The overall void coefficient is less negative when the movable reflector cylinders are all the way out. When the cylinders are used to effect a power shift (poison sections partially inserted), the importance is shifted from the reflector to the fuel, and the void coefficient is more negative.

Void reactivity calculations have been made for the hot, clean condition. Since the temperature deficit in the ATR is small, there is very little change in system response to void addition as a function of coolant temperature. Voids will have the same reactivity worth under saturation conditions as under normal operating conditions, since the nuclear properties of the core do not change significantly over this pressure and temperature range.

Power Level and Fuel Loading Variations

All design work is based on a core life of 17 days at 250 MW. The fuel loading is being determined as the minimum loading which will meet these criteria. All 40 fuel elements are assumed to be replaced at the end of each cycle. Current design work does not include fuel rearrangement, the use of partially spent fuel elements, and spiking of the core with fresh fuel, or any operational modes other than the one specified.

ATR General Nuclear Parameters

Prompt neutron lifetime - Start of life - 53 microseconds
- End of life - 58 microseconds

Delay fractions

<u>Group</u>	<u>β_i</u>	<u>τ_i</u>
1	0.000211	55.72 seconds
2	0.001401	22.72
3	0.001255	6.22
4	0.002527	2.30
5	0.000737	0.61
6	<u>0.000267</u>	0.23
	0.006398	
Ratio of effective delay fraction to delay fraction		1.14
Thermal utilization in fuel region		0.823
Fast fission factor		1.000
Age of fission neutron in fuel, cm ²		52.6
Age of fission neutron in reflector, cm ²		50.5
Thermal diffusion length in fuel, cm		1.13
Thermal diffusion coefficient in fuel		0.244
Thermal diffusion length in reflector, cm		11.5
Thermal diffusion coefficient in reflector		0.360
Resonance escape probability in fuel		0.845

KINETIC NUCLEAR CHARACTERISTICS

The ATR uses three types of control elements and the burnable poison in the fuel elements for reactivity control. Except for burnable poison, the negative reactivity provided by these means contributes to the kinetic and transient behavior of the reactor. The total reactivity worth of all control rods is approximately 35 percent $\Delta k/k$. A maximum reactivity addition rate of 0.1 percent $\Delta k/k$ per second may be attained if all control elements are withdrawn simultaneously. However, this is not allowed under any operating conditions because of interlocks.

Safety Rods

Five safety rods have a total of 15 percent $\Delta k/k$ when fully inserted to three-quarters of the core length. Four of these rods have a withdrawal rate of 2.4 inches per minute. The fifth safety rod is used for fast recovery scram as well as for a full scram. Its withdrawal speed is 4.8 inches per minute corresponding to a 0.017 percent $\Delta k/k$ per second maximum reactivity rate. Normally, all safety rods are withdrawn in a single group of five. The maximum reactivity addition rate of this group is 0.05 percent $\Delta k/k$ per second and 15 minutes are required for withdrawal at a constant speed.

Neck Shim Control Rods

Twenty eight neck shim rods provide a total reactivity worth of 8.7 percent $\Delta k/k$. These rods are used primarily for lifetime shimmiing and, to a lesser extent, for flux shifting. The fixed speed of these rods is 3.2 inches per minute. The maximum reactivity rate of a single neck shim rod is 0.0014 percent $\Delta k/k$ per second. For all neck shims simultaneously in motion, the maximum reactivity rate is 0.020 percent $\Delta k/k$ per second.

Outer Shim Control Cylinders

The outer shim control cylinders serve as shim and regulating elements. Approximately 8.6 percent $\Delta k/k$ is provided by these eight pairs of cylinders for use primarily in flux shifting and, to a lesser degree, in shimmiing to compensate for fuel depletion, lifetime fission product buildup, and xenon buildup.

Shimming speed is 0.033 rpm. For one pair of cylinders, this speed corresponds to a maximum reactivity rate of 0.0028 percent $\Delta k/k$ per second. The total eight pairs operating as a group have a maximum reactivity capability of 0.023 percent $\Delta k/k$ per second.

The maximum reactivity rates are summarized in Figure 128.

Figure 128. Maximum Reactivity Addition Rates.

<u>Type of Rods</u>	<u>% $\Delta k/k$/second</u>
4 safety rods	0.033
Fast recovery rod	0.017
28 neck shim rods	0.020
8 pairs of control cylinders	0.023
Rod insertion	<u>0.003</u>
Total	0.096*

* The total listed is for ganged operation of all rods. This condition is not permitted by interlocks.

Power Regulation

Automatic power regulation requires a faster reactivity rate than is provided in the shims in order to achieve the power regulation requirements of $\pm 1/2$ percent of full power. One duty of the outer shim control cylinders is to operate as regulating rods. The selected maximum regulating rate is adjustable from zero to approximately 0.3 percent $\Delta k/k$ per second. The cylinder speed, within this range, is proportional to power error during normal operation. Provisions are available to the reactor operator to group any or all of the cylinder pairs as the regulating group. The cylinders, when assigned to regulation, are driven by separate drive motors, and positive stops are provided to limit the cylinder travel. Under no condition will the total regulating stroke worth exceed the delayed neutron fraction.

Safety Action Response

The time required for emergency shutdown by the safety rod elements is determined by delay times associated with the input devices, the safety element release time, and the time required for rod travel. The fastest scram inputs are electronic signals from the nuclear instrumentation initiating the electronic scram. All other scram inputs cause the full scram by relay action from a recorder, a sensor, or manual input. The delay associated with the nuclear instrumentation channel for an electronic scram is less than one msec and the scram rod release time is 25 msec or less. A relay scram requires 15 msec for the relay action, thus a total of 40 msec is required to initiate rod action after a relay scram signal is received. Ramp and step reactivity additions have been investigated to determine the relative importance of various parameters on reactor behavior in selecting the scram trip levels and the safety rod requirements. It was concluded that ramp reactivity additions from rod motion at any power level do not result in transients as severe as those associated with rapid accidents (assumed to be step) despite the fact that large total amounts of Δk are available in the rods.

The safety rod characteristics provide an adequate reactor shutdown mechanism with the upper scram trip level at 130 percent full power. This prevents the core from experiencing any damage caused by the maximum reactivity insertion postulated for a reactor accident. The larger moderator coefficient has the greater influence in holding down the peak power at higher values of $\Delta k/k$.

Although the total Δk of the safety rods inserted 36 inches into the core is not as great as if they were inserted in full length of the core (15 percent $\Delta k/k$ versus 19 percent $\Delta k/k$ for a 48-inch insertion), the rod worth curve versus position is identical for either design. Therefore, the kinetic response of the rods is not compromised by the limited 36-inch stroke.

The nominal moderator temperature coefficient of the fuel region is $-1.0 \times 10^{-4} \Delta k/^{\circ}\text{F}$, and the average temperature rise across the fuel elements at full power is 65°F . The increase in coolant average temperature from zero to full power is approximately 33°F . The resulting reactivity deficit is 0.33 percent $\Delta k/k$. Therefore, the amount of negative reactivity required to drive the reactor from full power to subcritical is slightly more than 0.33 percent

$\Delta k/k$. This corresponds to an insertion of the five safety rods for only four inches. This distance will be traveled in 66 msec and the full 36-inch stroke in 342 msec.

The effect of scram delay time on the response of the ATR to step reactivity addition has been evaluated for various initial safety rod positions. Decreasing the total electronic scram delay time by a factor of two from 32 to 16 msec, for an initial rod insertion of three inches increases the amount of stepped Δk required to produce a fixed amount of integrated energy by only 10 percent (relative capability increases from 90 to 100 percent).

THERMAL AND HYDRAULIC CHARACTERISTICS

Flow Path

The reactor is cooled by pressurized light water entering the reactor vessel through two 24-inch nozzles located near the bottom of the reactor vessel. Approximately 95 percent of the entering water flows up along the thermal shields. The remainder is diverted into the large space between the core reflector tank and the inlet flow baffle. This water maintains a cool heat sink to permit natural circulation cooling for the irradiated fuel elements stored in this area.

The two entering flow paths meet in a large plenum above the core internals. Then, they reverse and flow down into the fuel elements and other internals. Figure 129 lists the most probable division of coolant flow down through the reactor internals.

Figure 129. Reactor Flow Distribution.

<u>Reactor Region</u>	<u>Flow Rate, gpm</u>
Fuel elements	25,800
Reflector blocks	1,930
Reflector control cylinders	9,550
Neck region	270
Flux traps	5,800
Inner capsule irradiation facilities	1,000
Between reflector and tank	650
Seal leakage	500
Outer capsule irradiation facility	<u>2,160</u>
Total	47,660

Core Heat Transfer

Figure 130 lists several parameters regarding the heat transfer performance of the core. Of principal design concern is the hot channel performance. This channel is one which might experience the highest power density while at the same time be subject to all possible adverse combinations of manufacturing and operational tolerance conditions. To predict the worst situation, a list of

Figure 130. ATR Thermal and Hydraulic Parameters.

Reactor power, MW	250	
Maximum power density, MW/liter	216	
Reactor inlet temperature, °F	130	
Cover average outlet temperature, °F	195	
Saturation temperature at hot spot, °F	400	
	<u>Average</u>	<u>Maximum (Hot Spot)</u>
Plate temperature, °F (beginning of life)		
Surface	203	408
Fuel clad interface	209	426
Fuel center	211	432
Core minimum inlet pressure, psig	300	
Average core coolant velocity, fps	44.5	
Core flow rate, gpm	25,800	
Flow rate for internals cooling, gpm	21,860	
Total reactor flow rate, gpm	47,660	
Core heat transfer area, ft ²	1,390	
Average heat flux, Btu/hour - ft ²	0.57×10^6	
Maximum heat flux (including F_q'')	1.92×10^6	
Power distribution at beginning of life		
Horizontal peaking factor	2.0	
Axial (vertical) peaking factor	1.4	
Fraction of power generated in fuel plates	0.93	
Fraction of reactor power picked up by core coolant	0.97	
Burnout* (hot channel)		
Safety factor of 100% power	2.57	
Ratio of burnout power to design power	1.38	
Ratio of local boiling power in hot channel to design power	1.18	

* Using 60% of Bernath's correlation

calculated hot channel factors is given in Figure 131, listed in columns F_{θ} , $F_{q''}$, and $F_{\Delta T}$ as they affect coolant film temperature rise, heat flux, and coolant bulk temperature rise, respectively.

Many factors listed in Figure 131 are estimates based upon MTR and ETR experience. The ATR development program will produce data which may be applied more specifically to this reactor. While the listed operational factors do not depend on the development program, the manufacturing tolerances and calculational factors listed in Figure 131 will be affected.

Figure 131. ATR Hot Channel Factors.

<u>Manufacturing Tolerances</u>	<u>F_{θ}</u>	<u>$F_{q''}$</u>	<u>$F_{\Delta T}$</u>
Channel geometry	1.041	1.0	1.155
Surface roughness	1.015	1.0	1.018
Fuel content			
Local	1.10	1.10	1.0
Channel	1.0	1.0	1.04
Fuel heat transfer area	1.04	1.04	1.0
Fuel eccentricity	1.0	1.0	1.0
Sub-products	1.209	1.144	1.223
<u>Calculational Factors</u>			
Film coefficient correlation	1.25	1.0	1.0
Plenum distribution	1.04	1.0	1.05
Channel velocity distribution	1.04	1.0	1.05
Sub-products	1.352	1.0	1.103
<u>Operational Factors</u>			
Power measurement and regulation	1.05	1.05	1.05
Flow control	1.016	1.0	1.02
Inlet temperature regulation			+3 F
Sub-products	1.067	1.05	1.071
Overall products	1.744	1.201	1.445

Consequently, most of the uncertainties involved in the calculation of the hot channel performance will be guided strongly by results from the ATR research and development programs.

Figure 132 shows the bulk coolant and fuel plate surface temperatures in the hot channel obtained by using the hot channel factors listed in Figure 130.

Figure 132. Temperature Versus Distance from Entrance to Hot Channel.

Flow Stability and Burnout

Hot channel burnout can be caused by steam blanketing of a fuel plate in a local boiling region or by vapor binding of the channel initiated by boiling.

In the first case, the surface heat flux can be increased until steam bubbles form so rapidly that they coalesce and steam blanket the fuel plate surface instead of collapsing in the subcooled bulk coolant. The local boiling burnout heat flux has been predicted for the ATR by the extended method of Bernath with a 60 percent safety factor to allow for error in the correlation. This method predicts an approximate burnout power level of 140 percent of rated power.

If the power level of the ATR is raised sufficiently, bulk boiling will occur in the hot channel. Since the flow is governed by the total pressure drop for all of the parallel channels in the fuel element, the increased resistance due to two phase flow causes a flow reduction in the hot channel. The reduction in flow causes more steam to form and autocatalytic vapor binding of the channel results. The nature of this effect is illustrated in Figure 133. The calculated power level at which net boiling begins in the hot channel is 138 percent of rated power. The maximum heat flux corresponding to this power level is 2.65×10^6 Btu/hour-ft².

Summary

The reactor is designed to operate with an overpower ratio at burnout of 138 percent of full power even in the most adverse situation where all the hot channel factors are combined at the point of maximum heat flux. A number of uncertainties exist, and estimates of their effect are based on past experience with the MTR and ETR. The ATR research and development program is planned and is being carried out to give answers to these uncertainties and solutions to the development programs. The studies that have been made and the programs that are being pursued supply confidence that a fuel element will be developed which will be satisfactory for heat transfer and fluid flow.

Figure 133. Flow Rate as Function of Overpower Ratio.

Figure 134. Two Energy Group
 Constants for ATR Axial Power Stability Study
 (Reference numbers given in parenthesis)
 Fuel Loading

Constant	26 k.g. (45)	26 k.g. (32)	39 k.g.+	39 k.g.*
D_1 (cm)	0.26395	0.2807	0.3591	0.3568
D_2 (cm)	1.71458	1.5436	1.41397	1.41397
Σ_{a1} (cm ⁻¹)	0.1598	0.157	0.2007	0.2008
Σ_{a2} (cm ⁻¹)	0.00422	0.00476	0.00779	0.00779
Σ_r (cm ⁻¹)	0.02600	0.02442	0.02259	0.02259
$v\Sigma_{f1}$ (cm ⁻¹)	0.26810	0.267	0.3494	0.3528
$v\Sigma_{f2}$ (cm ⁻¹)	0.00563	0.00644	0.00405	0.00405
σ_x (cm ²)	2.668×10^{-18}	2.5×10^{-18}	1.75×10^{-18}	2.67×10^{-18}
$\bar{\gamma}_x$	0.003	0.003	0.003	0.003
$\bar{\gamma}_i$	0.061	0.064	0.06	0.061
λ_x (sec ⁻¹)	2.093×10^{-5}	$2.1^+ \times 10^{-5}$	2.11×10^{-5}	2.093×10^{-5}
λ_i (sec ⁻¹)	2.882×10^{-5}	2.9×10^{-5}	2.88×10^{-5}	2.882×10^{-5}
B^2 (1/cm ²)	0.0106	0.0116	0.00876	0.00901
ω_{12}	0.151	0.152	0.111	0.111
$1/\omega_{12}$	6.62	6.57	9.01	9.01
AT (cm ²)	10.14	11.5	15.8	15.79
BT	0.270	0.286	0.401	0.405
AF (cm ²)	6.40	5.77	4.05	4.01
BF	0.0	0.0	0.0	2.61×10^{-6}

+ Determined from a TURBO Code (see (32)).

* Determined from GAM and TEMPEST Codes (see (32)).

Note: For a definition of constant terms see Chapter VI. (B^2 is the material buckling.)

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