An Investigation of Doppler-Shifted H_{β} Radiation
Emitted From an Electrostatically-Confined,
Spherically Symmetric, Hydrogen Plasma

by

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TABLE OF CONTENTS

	<u>Page</u>
ACKNOWLEDGMENTS	ii
LIST OF TABLES	iv
LIST OF FIGURES	v
ABSTRACT	vii
I. INTRODUCTION	1
II. EXPERIMENTAL APPARATUS AND PROCEDURE	4
Vacuum Apparatus	4
Power Supplies	7
Spectroscopic Apparatus	8
Experimental Procedure	12
III. THEORETICAL DEVELOPMENT	15
IV. THEORETICAL AND EXPERIMENTAL RESULTS	27
V. SUMMARY	42
BIBLIOGRAPHY	45
APPENDIX I: Comparison of "Exact" and Approximate Forms of $I_{PM}(\Delta\lambda)$	46
APPENDIX II: Order-of-Magnitude Estimate for the Photo- multiplier Current	49
APPENDIX III: Estimate of Fast Ion Density in the Central Region of an Electrostatically-Confined Hydrogen Plasma	51
APPENDIX IV: Abel Inversion Error Estimate	52

LIST OF TABLES

<u>Table</u>		<u>Page</u>
1	Comparison of Theoretical Values of $\Delta\lambda_{\max}$ and Values Determined After Performing Least-Mean-Square Fit of Experimental Data to Curve of the Form: $I(\Delta\lambda) = K/[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{3/2}$	 39

LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
1a	Vacuum System	.5
1b	Vacuum Apparatus Schematic	5
2	Spherical Grids	6
3	External High Voltage Filter	.8
4a	Spectroscopic Apparatus	.9
4b	Spectroscopic Viewing Arrangement	.9
5	Modified Focusing System	15
6	Plasma Model	16
7	Ion-Neutral Interaction Volume	19
8	Plasma Observation Model	20
9	Hg Wing Structure Recorded From the Central Region of an Electrostatically-Confined, Spherical Hydrogen Plasma	27
10	Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -13.5 kv)	.30
11	Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -10 kv)	31
12	Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -8 kv, Cathode Current = 10 ma)	32
13	Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -8 kv, Cathode Current = 14.5 ma)	.33
14	Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -5.2 kv)	34

<u>Figure</u>		<u>Page</u>
15	Experimental Data Demonstrating Wing Structure for Cathode Potential = -2kv	37
16	Comparison of Exact and Approximate Forms of $f(u)$	47
17	Abel Inversion Model	52
18	Intensity Distribution for a Rotationally Symmetric System	54

ABSTRACT

Spectroscopic observations through the center of an electrostatically-confined, spherical, hydrogen plasma are reported. These observations indicate the presence of a wing structure about the H_{β} line, i.e., $\lambda = 4861.3 \text{ \AA}$. The wing structure follows a $[\Delta\lambda_{\text{max}}^2 - \Delta\lambda^2]^{-3/2}$ behavior for $|\Delta\lambda| < 0.7 \Delta\lambda_{\text{max}}$, where $\Delta\lambda$ is the wavelength shift from the undisturbed H_{β} line, and $\Delta\lambda_{\text{max}}$ is the maximum Doppler shift consistent with the applied potential of the electrostatic confinement device. A theory which yields the above wavelength dependence is proposed. The theory assumes spherical symmetry, no space charge effects, and, the presence of fast, excited, hydrogen atoms resulting from charge exchange or dissociation collisions between H_2^+ ions and background neutrals. A comparison of the values of $\Delta\lambda_{\text{max}}$ determined from the experimental data and calculated using the Doppler formula indicates good agreement. The observed wing structure exhibits an unpredicted asymmetry, as yet unexplained, about the undisturbed H_{β} line.

I. INTRODUCTION

In experiments on electrostatic confinement with ion injection, Hirsch obtained a substantial neutron flux resulting from fusion reactions within the confinement device. (1) The estimated reaction rate for neutron production indicated a central ion density of 10^{13} per cc, or larger. Densities of this magnitude suggest the existence of a "potential well" in which substantial ion trapping occurs. The detailed physical mechanisms of the electrostatic confinement scheme were not well understood since the x-ray and fast neutron diagnostics employed by Hirsch did not present a complete picture of the potential profile, the ion density, or even the ion energies. The aim of the present work is to investigate basic physical processes of electrostatic confinement, and to extend the diagnostics associated with this scheme.

An essential feature of any plasma diagnostic is that the diagnostic, itself, should not seriously perturb the system being studied. In the electrostatic confinement device, because of spherical symmetry, strong applied electric fields, and possible strong internal electric fields associated with any "potential well" formation, the use of Langmuir probes as diagnostic tools is not desirable. Spectroscopy, on the other hand, is especially attractive, in that it is completely non-perturbing. The main problem associated with spectroscopy is that if the plasma is not in equilibrium, the data may be difficult to interpret.

For the electrostatic confinement device, two possible applications of spectroscopy immediately suggest themselves: (1) the Stark effect associated with any strong applied, or internally generated, fields, and (2) Doppler shifting of the lines associated with fast neutrals or ions in the system.

Since both the Stark and Doppler effects are largest in hydrogen, this naturally suggests its use as the experimental background gas. One then examines the H_β line, as well as the H_α and H_γ lines, to see if there is a broadening present. Either the Stark effect or Doppler shifting, or a combination of both, could cause such a broadening.

In this study, it was found that line broadening was present. However, the Stark effect was eliminated as a possible broadening mechanism since this would have required electric fields on the order of 50 kv per cm, and fields this strong were not anticipated. On the other hand, as will be shown in the following sections, Doppler shifting of the radiation could account for the results. Consequently, this work presents a study of Doppler-shifted H_β radiation in an electrostatically-confined, spherical, hydrogen plasma.

Section II contains a description of the experimental apparatus and the operating procedure followed in this study. In Section III, the theoretical model of an electrostatically-confined, hydrogen plasma is presented, along with the theory proposed to describe the model. Section IV contains a discussion of the theoretical and experimental results, and Section V is a summary of the essential results of the comparison of the theoretical and experimental data.

along with some recommendations for future study. Three appendices have been included covering the comparison of the "exact" and approximate forms of the intensity distribution, an order of magnitude estimate of the photomultiplier current, and an order of magnitude estimate for the fast ion density in the central region of the plasma. In addition, a fourth appendix has been included which contains some work performed during the preliminary stage of this investigation. Specifically, this fourth appendix contains a method for estimating the error involved when using the well-known Abel inversion technique to determine the radial emission coefficient in rotationally symmetric plasmas. This error estimate is applicable to both the usual case where one assumes the emission coefficient is negligible beyond some radius, and, more importantly, to the case where spectroscopic observations can only be made out to some radius at which point the emission coefficient is still significant.

II. EXPERIMENTAL EQUIPMENT AND PROCEDURE

The experimental apparatus may be conveniently divided into three general sections:

1. Vacuum Apparatus
2. Power Supplies
3. Spectroscopic Equipment

A detailed description of each section follows, and, in addition, the experimental procedure followed throughout this investigation is described.

Vacuum Apparatus

The vacuum apparatus is shown pictorially in Figure 1a, and in simplified schematic form in Figure 1b. Basically, the system consisted of an 18 inch Pyrex bell jar evacuated by a 260 liter/sec turbomolecular pump. Preliminary roughing down of the system was accomplished by the use of a 15 cfm mechanical pump, and the bell jar pressure was monitored with a cold cathode discharge gauge having a usable range from 10^{-5} to 10^{-2} torr. In addition to the basic system, provision was made for the purging of air from the inlet gas manifold by evacuating the inlet line, and then backfilling the line with the desired gas; which, in this case, was commercially available, high purity hydrogen. The presence of hydrogen dictated the observance of certain safety precautions, such as ensuring proper room ventilation and prohibiting smoking and open flames.

Two tungsten filaments, approximately 5 inches in length and 10 mils in diameter, were operated in parallel, at about 5.5 amps each, to provide primary electrons for ionization of the background gas.

Figure 1a. Vacuum System

Figure 1b. Vacuum Apparatus Schematic

A conical section of stainless steel, located at the base of the spheres between the filaments and maintained at ground potential, was employed to diffuse the electrons more uniformly throughout the lower portion of the bell jar.

The spheres, or spherical grids, shown in Figure 2,

Figure 2. Spherical Grids

were constructed of stainless wire which was spot welded at the crossover points. (Note that the equipment on both sides of the spheres was located as shown during this study; however, this equipment was used in a different experiment, and had no direct bearing on

the case discussed here.) The longitudinal wires were made from 20 mil diameter stainless, and the latitudes were made from 41 mil stainless. The spheres were mounted concentrically, the inner sphere being 3 inches in diameter, while the outer sphere was 6 inches in diameter. Since it was desired to make spectroscopic observations at several points along the equator, all but five of the inner sphere longitudinal wires were removed at the points where they crossed the equator; while all but four of the outer sphere longitudes were removed. The number of wires retained at the equators of both spheres had no particular significance other than providing good mechanical strength and allowing sufficient unobstructed viewing area, while not appreciably distorting the spherical geometry. Also, in connection with spectroscopic observations, a viewing port was cut in the aluminum guard shield which normally surrounds the Pyrex bell jar.

All electrical connections, with the exception of the cathode (inner sphere) high voltage, were made through a standard octal feedthru located in the baseplate. The cathode high voltage was supplied through a high voltage feedthru, also a standard item.

Power Supplies

High voltage for the cathode was supplied from a 30 kv-20 ma d.c. power supply. In addition to the internal filter of the supply, an RC network, shown in Figure 3, had also been connected across the supply output to reduce the ripple voltage in connection with a separate experiment. Although the addition of the RC filter resulted in a reduced amount of ripple, it was felt that the amount of ripple present was not of major importance to the results of this experiment.

Figure 3. External High Voltage Filter

A radiation check indicated a significant amount of x-rays present at voltages above 15 kv; consequently, radiation badges were worn when operating the system at the higher potentials, and, in addition, a 1/8 inch thick, three foot square sheet of steel was used as a shield between the bell jar and the spectroscopic equipment. A viewing port was cut in the shield to allow for spectroscopic observations.

Filament power was supplied by a 20 volt-30 amp regulated d.c. source, and anode (outer sphere) power was supplied from a 250 volt d.c. supply which was capable of delivering as much as 350 ma d.c. at greatly reduced voltages (about 60 volts).

Spectroscopic Equipment

Figure 4a is a photograph of the spectroscopic equipment, and Figure 4b is a simplified diagram of the experimental viewing arrangement. The arrows in Figure 4b are meant to indicate the path of a photon which originates in the plasma, is focused on the entrance slit, dispersed by the grating, and then changed into a current element upon striking the photomultiplier cathode. Essentially, then,

Figure 4a. Spectroscopic Apparatus

Figure 4b. Spectroscopic Viewing Arrangement

the spectroscopic apparatus consisted of an external focusing system, a 0.5 meter Ebert scanning spectrometer, and an electronic recording system.

The focusing arrangement, consisting of a plane mirror, and an 18 inch focal length, concave mirror, both mounted on the same table as the monochromator, is convenient for viewing spherical, or cylindrical, plasmas. Proper focusing of the equipment is accomplished by locating a thin wire at the point of interest, and then backlighting the wire to form a shadow which is directed at the monochromator entrance slit by adjusting the mirrors. The shadow is then brought to a sharp focus at the slit by adjusting the distance between the wire and the table. One of the advantages of this type of viewing arrangement is that under suitable conditions, to be described more fully in Section III, the plasma segment under observation may be approximated by a rectangular slice through the plasma; the area of the slice being approximately equal to the area of the slit. The use of the slice approximation leads to a more readily handled theoretical analysis.

The spectrometer was equipped with an 1180 groove/mm grating, a slit housing containing the input and output slits, and a housing in which the photomultiplier tube was located. The monochromator had a spectral range of approximately 1900-9100 Å, a reciprocal linear dispersion at the exit slit of 16 Å/mm, and a capability of resolving to better than 0.2 Å. Also, provisions were available for both automatic and manual grating drive, at any of several rates.

The input and output slits were continuously adjustable from 5 to 400 microns in width and from 0 to 20 millimeters in height. One control adjusted both slit widths simultaneously, and a second control adjusted both slit heights.

The photomultiplier tube used was an RCA type 1P21. The tube housing had been modified, to allow for nitrogen flushing, by attaching two sections of copper tubing to the sides of the housing. The tubing was coiled, as can be seen in Figure 4a, to prevent stray light from entering the housing and affecting the operation of the photomultiplier tube. Nitrogen, from commercially available tanks, was forced through the housing at a gentle rate of flow, providing a constant, dry atmosphere within the housing. The modification resulted in significant improvement in the low signal operation of the photomultiplier tube, which was of prime importance since the intensity of the radiation emitted by the hydrogen plasma was low (approximately 4 orders of magnitude less than that of a standard hydrogen-filled Geisler tube).

The monochromator and the focusing mirrors were mounted on a movable table which had been designed to allow for both automatic and manual scanning in the horizontal plane. Also, the table design allowed for variations in the automatic scanning rate by a simple gear change. Although little use was made of this feature for the purposes of this experiment, it was conveniently used, in the manual mode, to locate the point of maximum emission in the horizontal plane.

The electronic recording system used in conjunction with the monochrometer is a standard item available from the same manufacturer. Essentially, the system consists of a stabilized, high voltage d.c. power supply for the photomultiplier tube, a high gain stable amplifier, and a 10 inch strip chart recorder, all mounted in a movable cabinet. The photomultiplier output, usually on the order of 10^{-10} to 10^{-9} amps for this experiment, was smoothed and amplified, and then reproduced by the recorder as a plot of light intensity versus wavelength over the wavelength interval of interest. The smoothing of the signal was accomplished by selecting one of six internally-available, low-pass, RC filters to adjust the over-all response time of the electronic recording system. Experience dictated the choice of filter for this experiment in that the response time of the system was to be long in relation to the rise time of small, rapid fluctuations, and still short in comparison to the rise time of changes in the average light intensity emitted from the plasma. Provisions were also available both for "bucking out" the photomultiplier dark current, and for adjusting the photomultiplier high voltage. Tests were conducted to determine the voltage setting which resulted in the best signal-to-noise ratio for the photomultiplier tube.

Experimental Procedure

The experimental procedure divided naturally into a preliminary, or warm-up, stage, an operational stage, and a shutdown stage.

During the preliminary stage, which lasted approximately 30 minutes, proper room ventilation was established and the electronic recording system was energized, including the photomultiplier high

voltage supply. Nitrogen flushing of the photomultiplier housing was initiated, and the entire electronic recording system was allowed to stabilize for at least 15 minutes. Also, during this stage the inlet gas manifold was purged of air and backfilled with H_2 . The focusing of the monochrometer, as previously described, was accomplished, and the bell jar, which was usually maintained under vacuum overnight, was pumped down to less than 10^{-5} torr. It should be mentioned that, aside from establishing proper room ventilation, no particular significance should be attached to the order in which the preliminary steps have been described, as day-to-day variations did occur. The main aim was to perform all of these particular tasks before proceeding to the operational stage.

The operational stage began with the slow heating of the tungsten filaments. When the filaments were at normal operating conditions, the outer sphere (anode) 250 volt power supply was energized. H_2 was then admitted into the vacuum chamber until the pressure had risen and stabilized at about 1 micron, at which point a faint glow was observed throughout the bell jar. Anode current, on the order of approximately 100 ma, was also noted. The cathode high voltage was then energized and set to the desired value, resulting in an intensification of the glow at the center of the spheres. It should be noted that at the higher values of cathode potential, there was a tendency for the cathode current to exceed the high voltage supply limit unless the filament current was varied so as to reduce the cathode current. On certain occasions, the filament current had to be increased slightly, rather than decreased, in order to reduce the cathode current.

After reaching the desired operating conditions, the monochromator was located so as to view the maximum intensity region of the plasma. The strip-chart recorder and the wavelength scan rates were set as desired, usually 1 inch/minute and 5 Å/minute, respectively, and the monochromator was then allowed to scan the wavelength range of interest, recording the intensity profile as a function of wavelength. The cathode high voltage was then adjusted to some different value and another wavelength scan was recorded. This procedure was followed for several values of high voltage, up to a maximum of 15 kv negative.

After the desired data had been recorded, the shutdown stage was initiated by turning off the cathode high voltage supply. The various other power supplies, along with the hydrogen and nitrogen gas supplies were secured, and the system was allowed to pump for at least 30 minutes thereafter, during which time the bell jar pressure dropped below 10^{-5} torr, and the bell jar, itself, cooled down to room temperature. In all cases the last item to be secured was the room ventilation fan.

III. THEORETICAL DEVELOPMENT

The calculation of the intensity distribution makes use of the slice approximation referred to in Section II. The basis for the use of the approximation will now be given. Figure 5 is a sketch

Figure 5. Modified Focusing System

of the focusing system, modified by replacing the plane and concave mirrors with a 3 inch diameter converging lens of equivalent focal length. The observed plasma segment appears as shown due to the finite width w of the entrance slit. A comparable sketch would apply for the slit height h .

In this experiment the quantities L , R_p , and w , as shown in Figure 5, were approximately 36, 0.5, and 0.004 inches respectively; consequently, the ratios $2 R_p/L$ and w/R_p were on the order of 0.03 and 0.001. Since the ratio $2 R_p/L$ was small, it was assumed that the monochromator grating subtended the same small solid angle from any point on the diameter of the plasma. This assumption, in

conjunction with the small ratio w/R_p , is the same as assuming that the observed plasma segment is equivalent to a plasma slice. In similar circumstances Elder, Jerrick, and Birkeland have tacitly made use of the slice approximation in writing their initial equations. (2) More importantly, the fact that there is good agreement between the theory and experiment is probably the best justification for using the slice approximation.

A sketch of the model used in calculating the intensity distribution as a function of wavelength is shown in Figure 6. The anode is maintained at approximately ground potential, while the cathode is at

Figure 6. Plasma Model

some highly negative potential ϕ . Ions, formed between the grids, are accelerated radially inwards through the cathode potential ϕ . At some point inside the cathode the various types of ions undergo either dissociative or charge exchange collisions resulting in

excited hydrogen atoms traveling with approximately the same velocities as the original ions. The hydrogen atoms then de-excite, emitting Doppler shifted radiation, with the amount of shift depending on the line-of-sight component of the radiator's velocity.

An estimate of the ion density inside the inner sphere can be calculated in a straightforward manner. Let I_c be the measured cathode current, then $I_c = I_i + I_e$, where I_i and I_e are the ion and electron current contributions, respectively. More simply, $I_i = a I_c$, with $0 \leq a \leq 1$. If the transparency factor for the cathode is t , then $tI_i (=atI_c)$ is the ion current passing through the cathode. Note that since the typical mean-free path in this experiment was on the order of 10 cm, or less, it has been assumed that the fast ions make at most one pass through the inner sphere, which results in the transparency being restricted to $0 \leq t < 1$. Assuming steady state and spherical symmetry inside the cathode, it follows that $tI_i = atI_c = 4\pi r^2 J_i(r)$, or $J_i(r) = atI_c/4\pi r^2$, where $J_i(r)$ is the ion current density. The r^{-2} dependence of $J_i(r)$ introduces a singularity at the origin which comes about from neglecting non-radial ion motion which can arise either from collisions, or from interactions with the cathode grid wires. In addition to these small angle deflections, any departure from spherical symmetry in the inner sphere might be expected to add angular momentum to the system. An intuitively simple method of introducing this lack of focusing through non-radial motion is to replace r^2 by $r^2 + r_a^2$, where $r_a/R_p \ll 1$. Then, with the definition $R^2 \equiv r^2 + r_a^2$, there results $J_i(R) = atI_c/4\pi R^2$. Now, the assumptions of spherical symmetry and monoenergetic ions,

implying no space charge effects inside the inner sphere, lead to $J_i(R) = qN_i(R)v_r$, where $v_r = [2q\phi/m]^{1/2}$, q = ion charge, m = ion mass, ϕ = magnitude of the cathode potential. Therefore, on substituting for $J_i(R)$, it follows that the fast ion density inside the inner sphere is given by

$$N_i(R) = \Delta t I_c / 4\pi q v_r R^2 \quad (3.1a)$$

or,
$$N_i(R) = K / q v_r R^2 \quad (3.1b)$$

where
$$K = \Delta t I_c / 4\pi \quad (3.1c)$$

An estimate of the fast ion density at the center of the plasma has been calculated, in Appendix III, and the results indicate a fast ion density on the order of $10^7 - 10^8$ ions/cc at $r = 0$.

Now, suppose σ is the cross section for the production of fast hydrogen atoms resulting from interactions between ions and the background neutrals. Then, from the fast ion distribution, it is possible to calculate the fast hydrogen atom production rate per unit volume. Making use of a fairly standard technique, the situation can be considered as shown in Figure 7. (3, 4) An ion, traveling with velocity v_r can be thought of as sweeping out the volume shown, in a time Δt . Assuming a background neutral density N_n which is uniform in space, then the quantity $\sigma v_r N_n$ can be considered as representing the number of collisions per ion per second with the background neutrals. Furthermore, the quantity $P(R) = N_i(R)\sigma v_r N_n$ represents the total number of ion-neutral collisions per second per unit volume, i.e., $P(R)$ is the fast hydrogen atom production rate per

Figure 7. Ion-Neutral Interaction Volume

unit volume. Specifically, making use of Eq. 3.1b,

$$P(R) = \sigma v_r N_n N_i(R), \quad (3.2a)$$

or
$$P(R) = \sigma N_n K / q R^2 \quad (3.2b)$$

Now, if a fraction f of the fast atoms are excited and radiate, upon de-excitation, at the H_β frequency, then, because of steady state, the quantity $fP(R)$ is the rate at which H_β radiation is produced per unit volume. In other words, $fP(R)$ is the volume emission coefficient for H_β radiation. With the definition $fP(R) \equiv \epsilon_\beta(R)$, it follows that

$$\epsilon_\beta(R) = f \sigma N_n K / q R^2 \quad (3.3)$$

By making use of both the spherical nature of $\epsilon_\beta(R)$ and the slice approximation, a model of the plasma light source, including the segment under observation, may be depicted as shown in Figure 8. In this model, it is assumed that the density of H_β radiators does not vary significantly over lengths on the order of the slit width w , measured perpendicular to the plane of Figure 8, thereby reducing the

Figure 8. Plasma Observation Model

the problem to two dimensional polar coordinates, as shown. In addition, R_p is that radius beyond which the emission coefficient can be assumed negligible. It should also be recalled that the radiation to be considered is essentially H_β radiation emitted from radially-moving neutral hydrogen atoms.

As observed from the monochrometer, all of the H_β photons emitted from the volume element ΔV , located between θ and $\theta + d\theta$, as shown in Figure 8, will be Doppler-shifted to appear as photons in the range λ to $\lambda + d\lambda$, where $\lambda = \lambda_\beta [1 - (v_r/c) \cos \theta]$, λ_β is the unshifted H_β wavelength at $4861 \text{ \AA}$, and v_r is positive for velocities directed radially outward. At this point it is convenient to define the following quantities:

$$\Delta\lambda = \lambda - \lambda_\beta \quad (3.4a)$$

$$\Delta\lambda_{\max} = \lambda_\beta v_r/c \quad (3.4b)$$

Then, making use of Eqs. 3.4a and 3.4b, the following results are easily derived:

$$\Delta\lambda = -\lambda_{\max} \cos \theta \quad (3.5a)$$

$$d\Delta\lambda = \Delta\lambda_{\max} \sin \theta d\theta \quad (3.5b)$$

$$\sin \theta = [1 - (\Delta\lambda/\Delta\lambda_{\max})^2]^{1/2} \quad (3.5c)$$

If one assumes that the plasma emits isotropically, then the amount of radiation entering the monochrometer will be proportional to the ratio ρ of the area of the concave mirror to the area of a sphere having the same radius as the radius of curvature of the mirror, which in this case was 36 inches. Consequently, by defining the quantity dN to be the number of photons per second entering the monochrometer in the range $\Delta\lambda$ to $\Delta\lambda + d\Delta\lambda$, it follows that

$$dN = 4\rho \int_{\Delta\lambda} \epsilon_\beta(R) dV \quad (3.6)$$

Note that the factor of 4 in Eq. 3.6 results from the 4 shaded regions in Figure 8 which are equivalent because of the assumption spherical symmetry. From Figure 8 it is readily seen that for $\theta_{\min} < \theta < \pi - \theta_{\min}$, $r \sin \theta = h/2$, or $r = h/2 \sin \theta$. Also, since it has been assumed that the H_β radiator density is approximately constant over lengths on the

order of the slit width, the volume element is chosen in cylindrical coordinates, i.e., $dV = r d\theta dz dr$. Therefore,

$$dN = 4P \int_{\Delta V} E_p(R) dV = 4P d\theta \int_{-w/2}^{w/2} dz \int_0^{h/2 \sin \theta} E_p(R) r dr \quad (3.7a)$$

or

$$dN = \frac{4Pw d\theta f \sigma N_n K}{q} \int_0^{h/2 \sin \theta} \frac{r dr}{r^2 + r_a^2} \quad (3.7b)$$

Eq. 3.7b is easily solved to yield

$$dN = C_o d\theta \ln \left[1 + (h/2 r_a \sin \theta)^2 \right] \quad (3.8)$$

where

$$C_o = \frac{2Pw f \sigma N_n K}{q}$$

Note that Eq. 3.8 is valid only for

$$\theta_{\min} < \theta < \pi - \theta_{\min}$$

Now, making use of Eqs. 3.5b and 3.5c, it follows that

$$dN = \frac{C_o d\lambda}{[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{1/2}} \ln \left[1 + \frac{(h\Delta\lambda_{\max}/2r_a)^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \right] \quad (3.9)$$

Eq. 3.9 is valid only for

$$|\Delta\lambda| < \Delta\lambda_{\max} \cos \theta_{\min}$$

It is noteworthy that the number of photons per second per unit wavelength entering the monochrometer in the range $\Delta\lambda$ to $\Delta\lambda + d\Delta\lambda$ is given by

$$X(\Delta\lambda) \equiv \frac{dN}{d\Delta\lambda} = \frac{C_0}{[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{1/2}} \ln \left[1 + \frac{C_1^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \right] \quad (3.10)$$

where

$$C_1 = h\Delta\lambda_{\max}/2f_a$$

In order to estimate the number of photons per second entering the monochrometer slit, one should multiply $X(\Delta\lambda)$ by the slit response function, $S(\Delta\lambda)$, centered at $\Delta\lambda$, and integrate the product over the monochrometer bandwidth, i.e.,

$$\chi(\Delta\lambda) = \int_{-\infty}^{\infty} X(\Delta\lambda^*) S(\Delta\lambda - \Delta\lambda^*) d\Delta\lambda^* \quad (3.11)$$

where $\chi(\Delta\lambda)$ = number of photons at wavelength $\Delta\lambda$ entering the slit per second.

If $\Delta\lambda_{\max}$ is on the order of the monochrometer bandwidth, then one must evaluate the convolution integral, Eq. 3.11, in order to achieve any realistic results. However, in this experiment the monochrometer bandwidth was approximately $1 \text{ \AA}$, which was much smaller than the wavelength interval of interest $|2\Delta\lambda_{\max}|$; consequently, the following approximations were made:

$$\begin{aligned} 1. \quad S(\Delta\lambda - \Delta\lambda^*) &= 1 & |\Delta\lambda - \Delta\lambda^*| &\leq \frac{1}{2} \text{ \AA} \\ &= 0 & |\Delta\lambda - \Delta\lambda^*| &> \frac{1}{2} \text{ \AA} \end{aligned}$$

$$2. \quad X(\Delta\lambda^*) \simeq X(\Delta\lambda) \quad |\Delta\lambda - \Delta\lambda^*| \leq \frac{1}{2} \text{Å}$$

Using these approximations the convolution integral can be written as

$$X(\Delta\lambda) = \int_{-\infty}^{\infty} X(\Delta\lambda^*) S(\Delta\lambda - \Delta\lambda^*) d\Delta\lambda^* \simeq X(\Delta\lambda) \int_{-\frac{1}{2}\text{Å}}^{\frac{1}{2}\text{Å}} 1 \cdot d\Delta\lambda^* \quad (3.12)$$

therefore

$$X(\Delta\lambda) \simeq X(\Delta\lambda) \cdot 1 \text{Å} \quad (3.13)$$

Since $X(\Delta\lambda)$ is the number of photons per second entering the monochromator at wavelength $\Delta\lambda$, it follows that the photomultiplier current as a function of wavelength $I_{PM}(\Delta\lambda)$ is simply related to $X(\Delta\lambda)$ by:

$$I_{PM}(\Delta\lambda) = q L M E X(\Delta\lambda) \quad (3.14)$$

where q = electronic charge

M = tube amplification factor

E = cathode material quantum efficiency

$L = \frac{\text{\# of photons/sec striking cathode}}{\text{\# of photons/sec entering monochromator}} = \text{loss factor}$

Therefore, in terms of the previously defined quantities:

$$I_{PM}(\Delta\lambda) = \frac{q L M E C_0 (1\text{Å})}{[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{1/2}} \ln \left[1 + \frac{C_1^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \right] \quad (3.15)$$

for

$$|\Delta\lambda| < \Delta\lambda_{\max} \cos \Theta_{\min}$$

In the case where

$$\frac{C_1^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \ll 1$$

the log function may be expanded as follows:

$$\lim_{z \rightarrow 1} \ln[z + \epsilon] \approx \lim_{z \rightarrow 1} \left\{ \ln z + \frac{\epsilon}{z} \right\} = \epsilon$$

where

$$\epsilon = \frac{C_1^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2}.$$

Then the photomultiplier current may be approximated by:

$$I_{PM}(\Delta\lambda) \approx K_1 / [\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{3/2} \quad (3.16)$$

where

$$K_1 = q L M E C_0 C_1^2 (1\text{\AA}).$$

Note that even though the value of r_a is unknown, the approximation that

$$\frac{C_1^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} = \frac{(h\Delta\lambda_{\max}/2r_a)^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \ll 1$$

is not quite as arbitrary as it may seem. Appendix I contains a comparison of Eqs. 3.15 and 3.16, and the results of the comparison show that for a 1 mm slit height, if $r_a \geq 1.3$ mm, then the approximation results in at most 12% error for $\Delta\lambda \leq 0.8 \Delta\lambda_{\max}$. In fact, if $r_a \geq 2.5$ mm, then the error is less than 5% for $\Delta\lambda \leq 0.8 \Delta\lambda_{\max}$. Consequently, it would appear that the approximate form of $I_{PM}(\Delta\lambda)$, i.e.,

Eq. 3.16, should have a large range of applicability, providing r_a is on the order of 1 mm, or more.

An order of magnitude estimate has been performed, in Appendix II, to determine if the predicted and measured values of the photomultiplier current were sufficiently close. The results of the calculation are based on estimated values for certain quantities, such as the cross section for the production of fast, excited, hydrogen atoms, the fraction of hydrogen atoms which radiate at the H_β frequency, and the size of the quantity r_a , among others; and consequently, these results are approximate at best. However, in spite of these assumptions, the estimated and measured values of the photomultiplier current were found to be of the same order of magnitude, and this good agreement was felt to be encouraging.

IV. THEORETICAL AND EXPERIMENTAL RESULTS

The experimental results for several different cathode high voltage settings are shown in Figure 9. The principle H_β line is

Figure 9. H_β Wing Structure Recorded From the Central Region of an Electrostatically-Confined, Spherical, Hydrogen Plasma

thought to be due to background radiation from slowly-moving neutral hydrogen atoms, and is approximately one order of magnitude stronger in intensity than the wing structure. This data is presented to demonstrate the H_β wing structure and has been selected from all of

the data available because the estimated values of the wing intensity extrapolated to the center of the unshifted H_β line are approximately equal for these curves.

Based on the results of Section III, the wing structure should demonstrate a wavelength dependence of the form:

$$I(\Delta\lambda) \propto \frac{\ln \left[1 + \frac{(h/2r_a)^2}{1 - \Delta\lambda^2/\Delta\lambda_{\max}^2} \right]}{[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{1/2}} \quad (4.1)$$

or

$$I(\Delta\lambda) \propto [1 - \Delta\lambda^2/\Delta\lambda_{\max}^2] \quad (4.2)$$

if

$$\frac{(h/2r_a)^2}{1 - \Delta\lambda^2/\Delta\lambda_{\max}^2} \ll 1$$

Furthermore, if the wing structure does fit the prescribed curve, the quantity $\Delta\lambda_{\max}$ should be a good indication of how closely the theoretical and experimental results compare.

In order to relate the experimental results to the theory, the values of the intensity at 1 Å intervals between 2 and 8 Å away from the center of the unshifted H_β line were estimated from the experimental curves. Next, a computer program was written to perform a least-mean-square (LMS) fit of these experimentally determined values to a curve of the form given by Eq. 4.1. To perform the LMS fit, use was made of the subprogram NLIN2, which was readily available from the FORTRAN-IV share program library.

Initially, an attempt was made to fit the experimental data to the general form:

$$I(\Delta\lambda) = \frac{B_1^* \ln \left[1 + \frac{B_3^*}{B_2 - \Delta\lambda^2} \right]}{[B_2 - \Delta\lambda^2]^{1/2}} \quad (4.3)$$

where B_1^* = proportionality constant
 $B_2 = \Delta\lambda_{\max}^2$
 $B_3^* = [h\Delta\lambda_{\max}/2r_a]^2$

It was found, however, that B_1^* and B_3^* were related in some as yet undetermined manner which prevented the NLIN2 subprogram from converging to an LMS fit. Consequently, an attempt was then made, again using NLIN2, to fit the data to the approximate form of $I(\Delta\lambda)$, i.e.,

$$I(\Delta\lambda) \approx B_1 [B_2 - \Delta\lambda^2]^{-3/2} \quad (4.4)$$

where B_1 = proportionality constant
 $B_2 = \Delta\lambda_{\max}^2$

This attempt met with good success and the results of the LMS fit to the data for varying experimental conditions are shown in Figures 10 to 14, where the curves obtained by the LMS technique are plotted as the smooth solid lines in all cases. Because of an obvious asymmetry in the experimental data, it seemed appropriate to generate separate LMS curves for the $\Delta\lambda < 0$ and $\Delta\lambda > 0$ portions of the data. Contrary

Figure 10. Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential) = -13.5 kv)

Figure 11. Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -10 kv)

Figure 12. Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -8 kv, Cathode Current = 10.5 ma)

Figure 13. Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -8 kv, Cathode Current = 14.5 ma)

Figure 14. Comparison of Experimental Data and Negative 3/2 Power Curve Generated by Least-Mean-Square Technique (Cathode Potential = -5.2 kv)

to the standard method of presenting data, the value of the independent variable $\Delta\lambda$ increases from right to left in the figures. This unusual situation arises from the way the data was recorded, that being a scan in wavelength from lower to higher values of $\Delta\lambda$. In addition, the maximum Doppler shift consistent with each cathode potential has been indicated on the curves, for reference purposes.

Some general features of the data presented in Figures 10 to 14 are readily apparent. The LMS curves normally fit the data within a few percent for $|\Delta\lambda| < 0.5 \Delta\lambda_{\max}$; and all of the experimental curves indicate the presence of an additional line in the range $-5 \text{ \AA} < \Delta\lambda < -4 \text{ \AA}$, i.e., $4856 \text{ \AA} < \lambda < 4857 \text{ \AA}$, which is thought to be the $4856.5 \text{ \AA}$ line in the neutral H_2 spectrum. (5) It can also be seen that the LMS curve generated from the experimental data for $\Delta\lambda < 0$ is consistently higher than its counterpart for $\Delta\lambda > 0$; and the distance to the peak on the negative side of the data is less than the distance to the peak on the positive side. The difference between the negative $3/2$ power curve and the experimental data in the far-wing region, i.e., outside the peaks, has not been satisfactorily explained as yet; however, it is thought that this difference is due to a combination of several factors. In particular, it will be recalled that to evaluate the convolution integral given by Eq. 3.11 in Section III, it was assumed that both the radiated intensity and the slit function were constant over wavelength intervals on the order of the slit function half-width. Obviously, in the far-wing region where the negative $3/2$ power curve is rapidly increasing, these approximations are not valid.

It will also be recalled that the slice approximation was used and that the resulting theory is applicable only for $|\Delta\lambda| < |\Delta\lambda_{\max} \cos \theta_{\min}|$, where $\cos \theta_{\min} = [1 - h^2/4R_p^2]^{1/2}$. Since typically $h \approx 1$ mm and $R_p \geq 15$ mm, it follows that $\theta_{\min} \approx 0.07$ radians, and consequently this contribution to the far-wing deviation is felt to be quite small.

Finally, the approximate form of the intensity distribution, Eq. 4.4, rapidly deviates from the "exact" form, Eq. 4.3, in the far-wing region, and the use of the "exact" form would bring the experimental and theoretical curves closer. This would not cause the rapid drop in intensity for large $\Delta\lambda$, however.

Because of the assumptions just described, no attempt was made to fit the -2 kv data, shown in Figure 15, to a negative 3/2 power curve. The data is presented simply to indicate the presence of the wing structure at the lower values of cathode potential.

Figures 12 and 13 both represent data for -8 kv on the cathode. The experimental data shown in Figure 12 was recorded with 10.5 ma cathode current, while that of Figure 13 was for 14.5 ma cathode current. The original purpose in recording the two sets of data at the same cathode potential was to acquire some feeling about the physical nature, if any, of the quantity r_a , introduced in Section III to account for the presence of small amounts of angular momentum in the system. It was thought that changing the cathode current would change r_a , and that this change would be indicated by a degradation, or improvement, in the LMS fit to the experimental data. Figures 12 and 13 indicate that the negative 3/2 power curve fits well in both

Figure 15. Experimental Data Demonstrating Wing Structure for Cathode Potential = 2 kv Neg.

cases. In fact, the values of $\Delta\lambda_{\max}$ predicted by the two curves agree within 7% with the value calculated using the Doppler formula. Because of current limitations on the high voltage power supply, no further attempt was made to investigate r_a .

Contrary to what is predicted by the theory, the experimental data exhibits an asymmetry about the unshifted H_β line. Since it was felt that the asymmetry arose from some feature of the experimental technique, a series of tests were conducted to determine the source of the problem. For example, the monochromator itself was moved to the other side of the vacuum system, with the thought in mind that changing the line of observation by 180 degrees should reverse the asymmetry if it originates in the plasma. However, there was no change in the recorded data. In addition, observations of the H_α and H_γ lines indicated a similar asymmetry in their wing structures, which further supports the general feeling that the asymmetry arises from the experimental technique. Additional checks that were made on the experimental procedure included testing the monochromator grating drive, removing the mirror focusing arrangement and then observing the plasma, and, observing different sections of the plasma. However, these checks yielded no information and consequently, the asymmetry in the experimental data is unexplained as yet.

Table 1 is a comparison of the values of $\Delta\lambda_{\max}$ predicted by the LMS curves and those values calculated using the Doppler formula, assuming H_2^+ ions. The results indicate that there is good agreement between the theoretical and LMS values for $\Delta\lambda_{\max}$ for radiators moving away from the observer, and poor agreement for radiators moving towards

TABLE 1

Comparison of Theoretical Values of $\Delta\lambda_{\max}$ and Values Determined After Performing Least-Mean-Square Fit of Experimental Data to Curve of the Form:

$$I(\Delta\lambda) = K/[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{3/2}$$

Cathode Potential Φ (-kv)	$\Delta\lambda_{\max}$ Theory ($\Delta\lambda_{\max} = \lambda_B v/c$)	$\Delta\lambda_{\max}$ LMS ($\text{\AA}$)	
		$\Delta\lambda \geq 0$ (a)	$\Delta\lambda \leq 0$ (b)
13.2	18.4	18.4	15.5
10.2	15.8	15.5	13.9
8.0 ^(c)	14.2	14.3	12.9
8.0 ^(d)	14.2	13.3	11.7
5.2	11.4	10.2	9.3

Notes: (a) $\Delta\lambda \geq 0$ For Radiators Moving Away From Observer

(b) $\Delta\lambda \leq 0$ For Radiators Moving Towards Observer

(c) Cathode = -8 kv, 10.5 ma; Anode = 240 V, 160 ma

(d) Cathode = -8 kv, 14.5 ma; Anode = 170 V, 230 ma

the observer. Moreover, the agreement is better for higher values of cathode potential than for lower values. It is possible that at the lower values of cathode potential the approximation leading to the negative $3/2$ power law becomes increasingly inaccurate, and hence there is less of a correspondence between the theoretical and LMS values of $\Delta\lambda_{\max}$. It is interesting to note that if the theoretical values of $\Delta\lambda_{\max}$ were based on H_3^+ ions, rather than H_2^+ , then there was fair agreement between the theoretical and LMS values of $\Delta\lambda_{\max}$ for radiators moving towards the observer, and poor agreement for radiators moving away from the observer. However, Bleakney has indicated that at similar pressures, 90% or more, of the ions resulting from electron- H_2 collisions are H_2^+ ; consequently, in this experiment it has been assumed that the predominant ion species was H_2^+ . (6)

Although a detailed error analysis has not been performed, several major sources of error have been considered. To begin with, the error sources may be divided into two classes: the first class would include errors arising from the process of estimating the values of intensity from the experimental curve; and the second class would include errors involved in actually recording the experimental curve. This second class of error sources, which would include such things as drifting of the photomultiplier high voltage supply and changes in the response time of the electronic recording system, was felt to be unimportant in comparison to the first class. When the value of the intensity was estimated from the experimental curve, the major sources of error lie in the assumption of a baseline reference; the neglecting of any contribution to the intensity pattern resulting from the

unshifted H_β line; and the neglecting of the contributions resulting from the presence of other lines in the wavelength range of interest, such as the $4865.5 \text{ \AA}$ line in the H_2 spectrum. It is felt, however, that in this case the intensity estimate is good to within 2 out of 10 relative units, or 20%. Regardless of what the maximum error is, the data presented in Table 1 for radiators moving away from the observer is within 11% of the values calculated using the Doppler formula, and this is certainly within the experimental error.

Spectroscopic observations have also been made at locations away from the center of the hydrogen plasma. The monochromator was located in the equatorial plane of the plasma, and wavelength scans were made at several different radii. Typically, the wing amplitude would decrease rapidly as the distance from the plasma center was increased, with the wings dissolving into the background when the radius was between 0.5 and 1.0 inch from the center of the plasma. Note that the data taken with both -10 kv and -15 kv on the cathode indicates that as the distance from the plasma center is increased, the shape of the wings changes from the standard negative $3/2$ power curve to that of the curve for $r_a = 0.25 \text{ mm}$ shown in Figure 16 of Appendix I.

In conclusion, it is felt that the combination of accurate values of $\Delta\lambda_{\text{max}}$, as derived from the LMS technique, and the off-center data, just discussed, constitute good support for the model and theory proposed to explain the electrostatically-confined hydrogen plasma, at least for pressures in the micron range.

V. SUMMARY

A model of an experimentally observed electrostatically-confined hydrogen plasma has been proposed. The essential features of the model are spherical geometry, no space charge effects, and the presence of fast, excited hydrogen atoms resulting from charge exchange or dissociation collisions involving H_2^+ ions and background neutrals. Based on this model, a theory has been developed which predicts that the radiation emitted by an electrostatically-confined hydrogen plasma will be Doppler shifted in such a manner that the intensity distribution will be of the form:

$$I(\Delta\lambda) \propto \frac{\ln \left[1 + \frac{(h\Delta\lambda_{\max}/2r_a)^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \right]}{[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{1/2}}$$

In particular, if

$$\frac{(h\Delta\lambda_{\max}/2r_a)^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \ll 1$$

then the log term can be expanded and the expression reduces to

$$I(\Delta\lambda) \propto [\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{-3/2}$$

where $\Delta\lambda$ is the wavelength shift from the undisturbed H_β line, and $\Delta\lambda_{\max}$ is the maximum Doppler shift consistent with a given applied potential on the cathode of the electrostatic confinement device.

The quantity $\Delta\lambda_{\max}$ is determined first by using the Doppler formula, and then by fitting the experimental data, using a least-mean-square technique, to a curve of the negative 3/2 power form. A comparison of the values of $\Delta\lambda_{\max}$ as determined from the two independent methods indicates that there is good agreement between the theory and the experimental data for radiators moving away from the observer, and that there is poor agreement for radiators moving towards the observer. The poor agreement between the theory and experimental data for radiators moving towards the observer is directly related to an asymmetry in the data which is not predicted by the theory. No satisfactory explanation of this asymmetry has been uncovered as yet, although its source is thought to be in the experimental technique, rather than the plasma. However, it should be noted that even though there is an asymmetry in the data, both sides of the data can be well represented by a negative 3/2 power curve, up to about 0.6 or 0.7 $\Delta\lambda_{\max}$, which is felt to be the limit of the useful range of the approximation leading to the negative 3/2 power curve. At about 0.7 $\Delta\lambda_{\max}$ the theory deviates from the data, and to date no completely satisfactory explanation of this feature of the data has been produced. However, the good agreement between the theory and experiment for radiators moving away from the observer, plus some qualitative evidence, described at the end of Section IV, which resulted from spectroscopic observations made away from the center of the plasma, lend strong support to the theory. Consequently, there is strong support for the theoretical model describing the observed electrostatically-confined hydrogen plasma. The presence of space charge

effects in the system would be indicated by a spread in the velocity distribution of the radiating hydrogen atoms which result from H_2^+ ions colliding with the background neutrals. However, the theoretical development presented here, which assumes monoenergetic radiators, i.e., no spread in the H_2^+ ion velocity distribution, describes the experimental data accurately, and, consequently, it seems reasonable to accept the "no space charge effect" postulate. Acceptance of this postulate means that there should be no trapping of ions inside the cathode region. In other words, no "potential well" formation has been observed in this electrostatically-confined hydrogen plasma, for pressures in the micron range. Black, using an electron beam probe, has independently confirmed the absence of any potential wells in both hydrogen and helium plasmas confined in this same device. (7)

While the proposed model is believed to accurately describe the electrostatic confinement of a hydrogen plasma, there are several serious questions which are presently unanswered. Future work in this area should be aimed at providing satisfactory explanations for both the asymmetry in the data and the deviation between the data and the theory for $|\Delta\lambda| > 0.7 \Delta\lambda_{\max}$. Extension of the theoretical development to off-center locations should also be of some value. Furthermore, although the quantity r_a was arbitrarily introduced in the theory to account for the presence of small amounts of angular momentum in the system, it appears to be a meaningful quantity in the electrostatic confinement of hydrogen plasmas, and consequently r_a seems worthy of further investigation.

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APPENDIX I

COMPARISON OF "EXACT" AND APPROXIMATE FORMS OF $I_{PM}(\Delta\lambda)$

The essential result of Section III is the description of the photomultiplier current as a function of wavelength, i.e.,

$$I_{PM}(\Delta\lambda) = \frac{9LME C_p (1\text{\AA})}{[\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{1/2}} \ln \left[1 + \frac{C_1^2}{\Delta\lambda_{\max}^2 - \Delta\lambda^2} \right]$$

where all of the quantities have been defined in Section III. By defining $u = \Delta\lambda/\Delta\lambda_{\max}$, I_{PM} can be written in normalized form as:

$$\frac{I_{PM}(\Delta\lambda)}{I_{PM}(0)} = f(u) = \frac{\ln \left[1 + \frac{C_2^2}{1-u^2} \right]}{[1-u^2]^{1/2} \ln [1+C_2^2]}$$

where $C_2 = C_1/\Delta\lambda_{\max} = h/2r_a$

Now, as stated in Section III, if

$$\frac{C_2^2}{1-u^2} \ll 1$$

then both of the log terms can be expanded, resulting in

$$f(u) \simeq f^*(u) \equiv [1-u^2]^{-3/2}$$

Since the slit height h was known, a computer program was written to calculate the values of $f(u)$, for $0 \leq u \leq 0.9$, assuming different values for the parameter r_a . In addition, the values of $f^*(u)$ for $0 \leq u \leq 0.9$ were calculated. The results, shown plotted in Figure 16, indicate that for values of r_a on the order of 2.5 mm,

Figure 16. Comparison of Exact and Approximate Forms of $f(u)$

or more, there is less than 5% error between $f(u)$ and $f^*(u)$ for $u \leq 0.8$. It is interesting to note that for $r_a \geq 25$ mm, and for $u \leq 0.9$, the error in the approximation is less than 1%.

The above results seem to indicate that for r_a on the order of 1 mm, or more, the approximate form of $I_{PM}(\Delta\lambda)$ should be sufficient.

APPENDIX II

ORDER-OF-MAGNITUDE ESTIMATE FOR THE PHOTOMULTIPLIER CURRENT

In order to estimate typical values of $I_{PM}(\Delta\lambda)$, it is sufficient to consider the approximate expression, i.e.,

$$I_{PM}(\Delta\lambda) \simeq K_1 [\Delta\lambda_{\max}^2 - \Delta\lambda^2]^{-3/2}$$

where

$$K_1 = \frac{LMEPwf\sigma N_0 at I_c h^2 \Delta\lambda_{\max}^2 (1\text{\AA})}{8\pi r_g^2}$$

and all of the above quantities have been defined previously in Section III. Evaluating the above expression at $\Delta\lambda = 0$ results in

$$I_{PM}(0) \simeq K_1 / \Delta\lambda_{\max}^3$$

When operating the system with -10 kv on the cathode, the following quantities were felt to be appropriate:

$$\left. \begin{array}{l} L^+ \simeq 1 \\ M \simeq 2 \times 10^6 \\ E \simeq 0.12 \end{array} \right\} \text{Nominal values for 1P21. (8)}$$

⁺ The assumption that $L \simeq 1$ means that every photon entering the monochromator strikes the photomultiplier cathode. However, the external focusing system has an f number of 6, while the internal system of the monochromator has an effective f number of 8.6, consequently, $L \simeq (6/8.6) \simeq 0.7$. Since the main concern here is an order-of-magnitude estimate, the $L \simeq 1$ assumption has been used.

$$w \approx 0.1 \text{ mm}$$

$$\sigma \approx 10^{-15} \text{ cm}^2$$

$$N_n \approx 1.6 \times 10^{14} \text{ per cc (Perfect gas law)}$$

$$t \approx 0.9$$

$$I_c \approx 10 \text{ ma}$$

$$h \approx 1 \text{ mm}$$

$$\Delta\lambda_{\text{max}} \approx 16 \text{ \AA}$$

$$a \approx 0.5$$

$$\rho \approx 4 \times 10^{-4}$$

$$f \approx 0.1 \text{ to } 0.01$$

$$r_a \approx 1 \text{ cm}$$

Consequently, after substituting for the various factors,

$$I_{PM}(0) \approx 2 \times 10^{-9} \rightarrow 2 \times 10^{-10} \text{ amps.}$$

Since typical measured values of $I_{PM}(0)$ were on the order of $3 - 4 \times 10^{-10}$ amps neglecting background radiation, and several of the above quantities are outright guesses based on intuition, particularly f , a , and r_a , it was felt that there was reasonable agreement between the measured and predicted values of $I_{PM}(0)$.

APPENDIX III
ESTIMATE OF FAST ION DENSITY IN THE CENTRAL REGION
OF AN ELECTROSTATICALLY-CONFINED HYDROGEN PLASMA

Equation 3.1a in Section III predicts that the fast ion density will be given by

$$N_i(R) = 2tI_c / 4\pi q v_r R^2$$

Therefore, at $r = 0$

$$N_i(0) = 2tI_c / 4\pi q v_r r_a^2$$

Assuming H_2^+ ions and 10 kv operation, v_r can be calculated from $v_r = [2q\phi/m]^{1/2}$, and is found to be approximately 10^8 cm/sec. Substituting for v_r in the above equation, and using the values stated in Appendix II for the other quantities, there results

$$N_i(0) \simeq 2 \times 10^7 / r_a^2 \quad \text{cm}^{-3}$$

Consequently, the fast ion density in the central region of the plasma is on the order of 10^7 per cc, assuming r_a is approximately 1 cm.

APPENDIX IV

ABEL INVERSION ERROR ESTIMATE

The Abel inversion technique is a well-known method of obtaining the radial emission coefficient $\epsilon(r)$ for spherically or cylindrically symmetric plasmas. (9) Figure 17 shows a typical case where the Abel inversion might be applied. Intensity measurements $I(y)$ would

Figure 17. Abel Inversion Model

be made at a series of points along the y-axis, out to the radius R_P , beyond which spectral emission may be considered negligible. If it is assumed that the plasma is cylindrically symmetric, optically thin, and in a steady state, then, following Griem, the intensity $I(y)$ is given by

$$I(y) = \int_{-x_0}^{x_0} \epsilon(r) dx = 2 \int_y^{R_P} \frac{\epsilon(r) r dr}{[r^2 - y^2]^{1/2}} \quad (\text{IV.1})$$

Application of the Abel inversion results in the radial emission coefficient being given by

$$\epsilon(r) = -\frac{1}{\pi} \int_r^{R_p} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} \quad (\text{IV.2})$$

where $I'(y)$ is the derivative of $I(y)$. (9) Equation IV.2 has been reduced, by both Barr and Bockasten, to a sum of the form

$$\epsilon(r_i) \approx \frac{1}{R_p} \sum_K A_{iK} I(y_K) \quad (\text{IV.3})$$

where the A_{iK} are certain coefficients determined by those authors, and $I(y_K)$ is the value of the measured intensity at the point y_K . (10, 11)

If, for some reason, intensity measurements can only be made out to some radius R_M , which is less than R_p (see Figure 17), then the emission coefficient can be calculated from

$$\epsilon(r) = -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} - \frac{1}{\pi} \int_{R_M}^{R_p} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} \quad (\text{IV.4})$$

provided some information is available concerning $I'(y)$ for $R_M < y < R_p$.

Figure 18 is a sketch of what the intensity $I(y)$ might be expected to look like in cases of cylindrical, or spherical, symmetry. Generally speaking, the intensity would be a maximum at the center

Figure 18. Intensity Distribution for Rotationally Symmetric System

of the plasma and decrease monotonically as the distance from the center increases. Also shown in Figure 18 are the quantities R_M and R_P as previously defined.

First, consider the case where one assumes that the intensity drops immediately to zero at $y = R_M$. Using this assumption in conjunction with the derivative of a step function, the emission coefficient is calculated from

$$\epsilon_1(r) = -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} - \frac{1}{\pi} \int_{R_M}^{R_P} \frac{[-I_M \delta(\xi - R_M)] d\xi}{[\xi^2 - r^2]^{1/2}}$$

or

$$\epsilon_1(r) = -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} + \frac{I_M}{\pi [R_M^2 - r^2]^{1/2}} \quad (IV.5)$$

where I_M is the measured intensity at $y = R_M$.

Next, consider the case where one assumes that the intensity has the constant value I_M out to the radius R_p . In this case the solution to Eq. IV.4 is identical to that given by Eq. IV.5, above, with R_M replaced by R_p , i.e.,

$$\epsilon_2(r) = -\frac{1}{\pi} \int_r^{R_p} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} + \frac{I_M}{\pi [R_p^2 - r^2]^{1/2}} \quad (\text{IV.6})$$

Note, however, that $I'(y) = 0$ for $R_M < y < R_p$ (see Figure 18), and, therefore, Eq. IV.6 may be written as

$$\epsilon_2(r) = -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} + \frac{I_M}{\pi [R_p^2 - r^2]^{1/2}} \quad (\text{IV.7})$$

and, consequently,

$$\epsilon_2(r) \geq -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} \quad (\text{IV.8})$$

The actual value of the emission coefficient, given by Eq. IV.4, will be somewhere between $\epsilon_1(r)$ and $\epsilon_2(r)$. This fact can be demonstrated by considering Eq. IV.4. If the denominator in the second integral is changed to $[R_M^2 - r^2]^{1/2}$ rather than $[\xi^2 - r^2]^{1/2}$, and if the upper limit on this same integral is allowed to become very large, then, since $I(y)$ is assumed monotonically decreasing, it follows readily that

$$\epsilon(r) \leq -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} - \frac{1}{\pi [R_M^2 - r^2]^{1/2}} \int_{R_M}^{\infty} I'(\xi) d\xi$$

or

$$\epsilon(r) \leq -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} - \frac{I(\xi) \Big|_{\xi=R_M}^{\xi=\infty}}{\pi [R_M^2 - r^2]^{1/2}}$$

hence

$$\epsilon(r) \leq -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} + \frac{I_M}{\pi [R_M^2 - r^2]^{1/2}} = \epsilon_1(r). \quad (\text{IV.9})$$

Therefore, $\epsilon(r) \leq \epsilon_1(r)$. Again, in the second integral of Eq. IV.4, replace the denominator by $[R_p^2 - r^2]^{1/2}$, then it follows that

$$\epsilon(r) \geq -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} - \frac{1}{\pi [R_p^2 - r^2]^{1/2}} \int_{R_M}^{R_p} I'(\xi) d\xi$$

which reduces to

$$\epsilon(r) \geq -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} + \frac{I_M}{\pi [R_p^2 - r^2]^{1/2}} = \epsilon_2(r) \quad (\text{IV.10})$$

Consequently then,

$$\varepsilon_2(r) \leq \varepsilon(r) \leq \varepsilon_1(r)$$

Furthermore, making of Eq. IV.8,

$$-\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]^{1/2}} \leq \varepsilon(r) \leq -\frac{1}{\pi} \int_r^{R_M} \frac{I'(\xi) d\xi}{[\xi^2 - r^2]} + \frac{I_M}{\pi [R_M^2 - r^2]^{1/2}}$$

Finally, with R_p replaced by R_M in Eq. IV.3,

$$\frac{1}{R_M} \sum_k A_{ik} I(y_k) \leq \varepsilon(r_i) \leq \frac{1}{R_M} \sum_k A_{ik} I(y_k) + \frac{I_M}{\pi [R_M^2 - r^2]^{1/2}} \quad (\text{IV.11})$$

Consequently, both the minimum and maximum values of the radial emission coefficient can be determined in a straightforward manner for rotationally symmetric plasmas where the intensity may be assumed to decrease monotonically from the center of the plasma out.