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THE ABSORPTION OF NEUTRONS IN DOPPLER BROADENED RESONANCES

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ABSTRACT

A method is developed for the calculation of the effect of Doppler broadening on the absorption of neutrons by a resonance absorber. Numerical values are given for the correction factors required in the interpretation of transmission experiments and self-indication experiments and for the self-shielding factors for slabs, spheres, cylinders, and homogeneous mixtures.

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G. M. Roe

INTRODUCTION

The probability that a neutron will be captured by a nucleus depends on the relative velocity of the neutron and the nucleus, hence when the nuclei are in thermal motion the effective cross section must be calculated by averaging over the velocity distribution of the nuclei. Bethe and Placzek*,** have shown that in the neighborhood of an isolated Breit-Wigner resonance the effective capture cross section is $\sigma_o \psi(\theta,x)$ where

$$\psi(\theta,x) = \frac{1}{2\sqrt{\pi\theta}} \int_{-\infty}^{\infty} \frac{dy}{1+y^2} e^{-(x-y)^2/4\theta} \qquad (1)$$

$$x = \frac{2}{\Gamma}(E - E_o)$$
$$\theta = \frac{4E_o kT}{A\Gamma^2} \qquad (2)$$

where

E = energy of neutron

E_o = energy of neutron at the center of the resonance

σ_o = peak cross section at the center of the resonance

Γ = full width of the resonance at half maximum

A = ratio of nuclear mass to neutron mass

k = Boltzmann's constant

T = absolute temperature

Equation (1) is based on a Maxwellian distribution of velocities for the nuclei. Lamb*** has shown that for a crystal, T in Equation (2) should be replaced by an effective temperature T'. T' is very nearly equal to T if T is larger than the Debye temperature of the crystal. Implicit in the derivation of Equation (1) is the assumption $E_o >> \Gamma$.

*Bethe, H. and Placzek, G., Phys. Rev. 51, 464 (1937).
**Bethe, H., Rev. Mod. Phys. 9, 140 (1937).
***Lamb, W.E., Phys. Rev. 55, 190 (1939).

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If a beam of neutrons whose spectrum can be treated as flat in the neighborhood of a resonance passes through a thickness ℓ of material with peak cross section σ_o (in macroscopic units), the probability of capture by the resonance is proportional to

$$\frac{1}{2}\Gamma\int_{-\infty}^{\infty} dx \left\{1 - e^{-\ell\sigma_o\psi(\theta,x)}\right\} \tag{3}$$

Numerical values of the quantity (3), obtained by a combination of numerical quadratures and series expansions, have been computed for selected values of $\ell\sigma_o$ and θ by Melkonian* and by Dardel and Persson.** These results can be used to compute the Doppler correction in the experimental measurement of the resonance parameters σ_o and Γ by transmission data on thick and thin samples. They cannot be used to compute the self-shielding factors defined below, which require the integration of (3) over a range of values of ℓ. Computation by quadratures is even less suitable for computing the temperature derivative of the self-shielding factor.

In order to estimate the temperature coefficient of reactivity of a reactor, a semianalytic method for calculating the self-shielding factor and its temperature derivative in a slab geometry was developed in 1948.*** The functions required in the interpretation of transmission experiments and self-indication experiments occur as a by-product of this analysis. These results are described below, together with some recent extensions of the method to the calculation of self-shielding factors for homogeneous mixtures, spheres, and cylinders.

PROPERTIES OF $\psi(\theta,x)$

The definition (1) is equivalent to

$$\psi(\theta,x) = \int_0^{\infty} e^{-v-\theta v^2} \cos xv \, dv \tag{4}$$

Clearly ψ satisfies

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\psi}{\partial\theta} \tag{5}$$

*Melkonian, Havens, and Rainwater, Phys. Rev. 92, 702 (1953).

**Dardel, G. V. and R. Personn, Nature 170, 1117 (1952).

***Internal memorandum by G. M. Roe and Reactor Handbook, Vol. I, 1953, p. 668.

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From Equation (4) one can obtain the following integrals:

$$\int_{-\infty}^{\infty} \psi dx = \pi \tag{6}$$

$$\int_{-\infty}^{\infty} \psi^2 dx = \frac{\pi}{2} M(\theta) \tag{7}$$

$$\int_{-\infty}^{\infty} \left\{(1 + x^2)\psi - 1\right\} dx = 2\pi\theta \tag{8}$$

$$\int_{-\infty}^{\infty} \psi^2 x^2 dx = \frac{\pi}{2} \left\{1 + \theta M(\theta)\right\} \tag{9}$$

$$\int_{-\infty}^{\infty} \frac{\psi}{1 + x^2} dx = \frac{\pi}{2} M\left(\frac{\theta}{2}\right) \tag{10}$$

where

$$\begin{aligned} M(\theta) &= \int_{0}^{\infty} e^{-v-(\theta/2)v^2} dv \\ &= \sqrt{\frac{\pi}{2\theta}}\, e^{1/2\theta} \left\{1 - \operatorname{erf}\left(\frac{1}{\sqrt{2\theta}}\right)\right\} \\ &= \frac{1}{2} \sum_{n=0} (-1)^n \frac{\left(\frac{n-1}{2}\right)!}{n!} \left(\frac{2}{\theta}\right)^{\frac{n+1}{2}} \\ &\cong 1 - \theta + 3\theta^2 - 15\theta^3 + 105\theta^4 - \dots \end{aligned} \tag{11}$$

For $\theta \ll 1 + x^2$,

$$\psi = \frac{1}{(1 + x^2)} + \theta \frac{(6x^2 - 2)}{(1 + x^2)^3} + \theta^2 \frac{(60x^4 - 120x^2 + 12)}{(1 + x^2)^5} + \dots \tag{12}$$

For x small, //

$$\psi = \alpha + \frac{1}{2} x^2 \alpha'(\theta) + \dots \qquad (13)$$

where

$$\alpha \equiv \alpha(\theta) = M(2\theta) \qquad (14)$$

For θ large and $x << 2\theta^{3/4}$

$$\psi \cong \sqrt{\frac{\pi}{4\theta}}\, e^{-x^2/4\theta}$$

or, somewhat more accurately,

$$\psi \cong \alpha e^{-x^2\alpha^2/\pi} \qquad (15)$$

The series (12) may be inverted to give

$$x^2 = \left(\frac{1}{\psi} - 1\right) + \theta\,[6 - 8\psi] + \theta^2\,[24\psi - 96\psi^2 + 64\psi^3] + \dots \qquad (16)$$

AREA UNDER AN ABSORPTION CURVE

The area under an absorption curve (fractional absorption versus energy) is given by expression (3). In the limit $\ell \to 0$ the area $\to \frac{\pi}{2}\sigma_o \Gamma \ell$. Hence, for any ℓ the area under the resonance may be written

$$\frac{\pi}{2}\sigma_o \Gamma \ell\, G(\theta, \sigma_o \ell)$$

$$G(\theta,\eta) = \frac{1}{\pi\eta}\int_{-\infty}^{\infty} dx \left\{1 - e^{-\eta\psi(\theta,x)}\right\} \qquad (17)$$

Thus G is the factor which corrects for the fact that the sample is not really thin and for the broadened shape of the resonance. In seeking an approximation to G, it will be simpler to take ψ instead of x as the independent variable, and to treat x as a function of ψ and θ. The maximum value of ψ is α, hence define

$$P = \frac{\psi(\theta,x)}{\alpha(\theta)} \qquad (18)$$

$$x \equiv x(\theta, P)$$

and integrate (17) by parts.

$$G(\theta,\eta) = \frac{2\alpha}{\pi} \int_0^1 x e^{-\eta\alpha P} dP \tag{19}$$

x is defined implicitly by Equations (1) and (18). We now seek an analytic expression which gives a good approximation to the true x over the whole range of P and yet permits the integration to be carried out in terms of known functions. Note first, from Equations (12) and (13), that

$$x \to \frac{1}{\sqrt{\alpha P}} \text{ for } P \to 0 \tag{20}$$

$$x \to \sqrt{\frac{1 - P}{\gamma}} \text{ for } P \to 1 \tag{21}$$

$$\gamma = \frac{-\alpha'(\theta)}{2\alpha(\theta)} = \frac{1}{8\theta^2\alpha} \left\{ (1 + 2\theta)\alpha - 1 \right\} \tag{22}$$

Now for integral n,

$$\int_0^1 dP \sqrt{\frac{1 - P}{P}} P^n e^{-zP}$$

is a tabulated confluent hypergeometric function. Hence if x is replaced by $\sqrt{\frac{1 - P}{P}} \left\{ \text{polynomial in P} \right\}$ the integral in Equation (19) can be reduced to a combination of tabulated functions. Equations (6) and (7) may be written

$$\int_0^1 x dP = \frac{\pi}{2\alpha} \tag{23}$$

$$\int_0^1 x P dP = \frac{\pi M(\theta)}{8\alpha^2} \tag{24}$$

The polynomial in P is chosen so that conditions (20), (21), (23), and (24) are satisfied exactly. The resulting approximation for x is

$$x \cong x_1 \tag{25}$$

$$x_1\alpha(\theta) = \sqrt{\frac{1 - P}{P}} \left\{ 1 - (1 - 4P)S_1(\theta) + 4(P - 2P^2)S_2(\theta) + 4(3P - 16P^2 + 16P^3)S_3(\theta) \right\}$$

with

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$$S_1(\theta) = 1 - \sqrt{\alpha} \tag{26}$$

$$S_2(\theta) = 4 - 2\sqrt{\alpha} - 2\,\frac{M(\theta)}{\alpha} \tag{27}$$

$$S_3(\theta) = 1 + \frac{1}{12}\left\{\frac{\alpha}{\sqrt{\gamma}} - 5\sqrt{\alpha} - 8\,\frac{M(\theta)}{\alpha}\right\} \tag{28}$$

Values of the S functions are tabulated in Table 1. The expansions for small θ and large θ are

$$\begin{aligned}
S_1(\theta) &\cong \theta - \frac{11}{2}\theta^2 + \frac{109}{2}\theta^3 - \dots \\
S_2(\theta) &\cong 3\theta^2 - 49\theta^3 + \dots \\
S_3(\theta) &\cong -\frac{1}{2}\theta^2 + \frac{67}{3}\theta^3 - \dots \\
S_1(\theta) &= 1 - \left(\frac{\pi}{4\theta}\right)^{1/4} + \dots \\
S_2(\theta) &= (4 - 2\sqrt{2}) - 2\left(\frac{\pi}{4\theta}\right)^{1/4} + \dots \\
S_3(\theta) &= \left(1 - \frac{2}{3}\sqrt{2} + \sqrt{\frac{\pi}{12}}\right) - \frac{5}{12}\left(\frac{\pi}{4\theta}\right)^{1/4} + \dots
\end{aligned} \tag{29}$$

When Equation (25) is expanded in powers of θ, the result is in exact agreement with the expansion (16) up to and including terms in the square of θ. Hence any of the approximations below which are based on the use of Equation (25) will also be correct to the same order in θ.

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Table 1

θ	M	α	$S_1(\theta)$	$S_2(\theta)$	$S_3(\theta)$
0	1	1	0	0	0
0.01	0.9903	0.9811	0.0095	0.0003	-0.0000
0.015	0.9856	0.9724	0.0139	0.0005	-0.0000
0.02	0.9811	0.9640	0.0181	0.0009	-0.0001
0.03	0.9724	0.9485	0.0261	0.0018	-0.0002
0.04	0.9640	0.9341	0.0335	0.0029	-0.0002
0.06	0.9485	0.9083	0.0469	0.0055	-0.0003
0.08	0.9341	0.8857	0.0589	0.0084	-0.0003
0.1	0.9208	0.8654	0.0697	0.0114	-0.0002
0.15	0.8911	0.8225	0.0931	0.0194	+0.0001
0.2	0.8654	0.7874	0.1126	0.0272	+0.0005
0.3	0.8225	0.7323	0.1442	0.0422	+0.0016
0.4	0.7874	0.6900	0.1693	0.0564	0.0030
0.6	0.7323	0.6271	0.2081	0.0806	0.0055
0.8	0.6900	0.5813	0.2376	0.1012	0.0079
1.0	0.6558	0.5456	0.2614	0.1187	0.0099
1.5	0.5916	0.4819	0.3058	0.1563	0.0151
2	0.5456	0.4382	0.3380	0.1859	0.0194
3	0.4819	0.3799	0.3836	0.2303	0.0261
4	0.4382	0.3414	0.4157	0.2643	0.0315
6	0.3799	0.2915	0.4601	0.3137	0.0394
8	0.3414	0.2595	0.4906	0.3500	0.0457
10	0.3132	0.2365	0.5137	0.3788	0.0506
15	0.2664	0.1990	0.5539	0.4304	0.0598
20	0.2365	0.1754	0.5812	0.4657	0.0658
30	0.1990	0.1464	0.6174	0.5162	0.0750
40	0.1754	0.1284	0.6417	0.5513	0.0815
60	0.1464	0.1065	0.6736	0.5987	0.0904
80	0.1284	0.0931	0.6948	0.6325	0.0970
100	0.1159	0.0838	0.7104	0.6561	0.1015
150	0.0960	0.0691	0.7371	0.6974	0.1093
200	0.0838	0.0602	0.7546	0.7256	0.1147
300	0.0691	0.0495	0.7774	0.7636	0.1222
400	0.0602	0.0431	0.7924	0.7888	0.1272
600	0.0495	0.0354	0.8120	0.8219	0.1337
800	0.0431	0.0307	0.8247	0.8441	0.1381
1000	0.0386	0.0275	0.8341	0.8603	0.1413
∞	0	0	1	1.1716	0.2049

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The accuracy of the approximation (25) may be checked with the numerical values of the function ψ tabulated by Rose, Miranker, Leak, and Rabinowitz.* For a given x and θ, ψ may be obtained from their tables, P computed from Equation (18) and x_1 from Equation (25). With $\theta = 4$, for example:

x	P	x_1
0	0.9991 (1.0000)	0.14 (0)
1	0.95320	0.99
2	0.8286	1.95
3	0.6590	2.97
4	0.4833	4.07
5	0.3309	5.12
6	0.2158	6.03
7	0.1378	6.87
8	0.0891	7.76
9	0.0600	8.86
10	0.0428 (0.0423)	9.83 (9.87)

The tabulated values for ψ were based on the numerical integration of Equation (5) and contain some errors slightly larger than the claimed accuracy of 1/2%. The values of P given in parenthesis were obtained independently and show that a part of the difference between x_1 and x arises from small errors in the tables for ψ. Even by ignoring this correction, the agreement between x_1 and x is quite satisfactory, especially since our final results depend only on certain average values of x and conditions (23) and (24) ensure that the error in the average will be small.

The approximation to the function G is obtained by inserting Equation (25) into Equation (19).

$$G(\theta, \eta) \cong G_1(\theta, \eta)$$

$$G_1(\theta,\eta) = W_2(\eta\alpha) - S_1(\theta)\left\{W_2(\eta\alpha) - W_3(\eta\alpha)\right\} + S_2(\theta)\left\{W_3(\eta\alpha) - W_4(\eta\alpha)\right\} \quad (30)$$
$$+ S_3(\theta)\left\{3W_3(\eta\alpha) - 8W_4(\eta\alpha) + 5W_5(\eta\alpha)\right\}$$

where the W_i are confluent hypergeometric functions

$$W_i(y) = M(i - 3/2,\ i,\ -y) \quad (31)$$

*BNL-257, "A Table of the Integral $\Psi(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \frac{\exp - (x-y)2/4t}{1 + y^2}\, dy$ M. E. Rose, *et al.*, September 1953.

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These functions have been tabulated,* or may be computed from tables of Bessel functions of imaginary argument.

$$W_2(y) = e^{-\frac{1}{2}y}\left\{I_o\left(\frac{y}{2}\right) + I_1\left(\frac{y}{2}\right)\right\}$$

$$W_3(y) = \frac{4}{y}\, e^{-\frac{y}{2}}\, I_1\left(\frac{y}{2}\right)$$

$$W_4(y) = \left\{\left(\frac{16}{y^2} + \frac{4}{y}\right) I_1\left(\frac{y}{2}\right) - \frac{4}{y} A_o\left(\frac{y}{2}\right)\right\} e^{-\frac{y}{2}}$$

$$W_5(y) = \left\{\left(\frac{12}{y^3} + \frac{4}{y^2} + \frac{1}{y}\right) I_1\left(\frac{y}{2}\right) - \left(\frac{3}{y^2} + \frac{1}{y}\right) I_o\left(\frac{y}{2}\right)\right\}\frac{32}{5}\, e^{-\frac{y}{2}}$$

$G_1(\theta,\eta)$ and $1 - G_1$ are plotted as functions of θ and $\eta\alpha(\theta)$ in Figures 1 and 2, and as functions of θ and η in Figures 3 and 4. Interpolation is easiest when $\eta\alpha$ is used as one of the variables, even though reference must be made to Table 1 in order to find α.

It is now possible to make some estimates of the accuracy of $G_1(\theta,\eta)$. The approximation used for x could be improved by writing

$$x = x_1 + \frac{1}{\alpha}\sqrt{\frac{1-P}{P}}\sum_{\sigma=1} R_\sigma(\theta)P^\sigma\left\{(4\sigma^2 + 8\sigma + 3) - (12\sigma^2 + 52\sigma + 51)P \right.$$
$$\left. + (12\sigma^2 + 80\sigma + 128)P^2 - (4\sigma^2 + 36\sigma + 80)P^3\right\} \tag{32}$$

where the terms have been adjusted so that the four conditions (20), (21), (23), and (24) are kept intact.

If only one term $R_1(\theta)$ is kept in the improved approximation for x, the new approximation for G becomes

$$G(\theta,\eta) = G_1(\theta,\eta) + R_1(\theta)\,\frac{5}{4}\left\{3W_3 - \frac{23}{2}W_4 + \frac{55}{4}W_5 - \frac{21}{4}W_6\right\} \tag{33}$$

There are a number of ways of choosing $R_1(\theta)$. One possibility is to adjust $R_1(\theta)$ so that for $P \to 1$

$$x \to \sqrt{\frac{1-P}{\gamma}}\left\{1 + \frac{\alpha''(\theta)(1-P)}{48\alpha\gamma^2} + \dots\right\} \tag{34}$$

*British Association Report 1926.

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This improves the shape of the resonance near the maximum. Another choice is to adjust $R_1(\theta)$ so that the second term in the expansion of G for large η is correct

$$G(\theta,\eta) \rightarrow \frac{1}{\sqrt{\pi\eta}} \left\{ 1 + \frac{6\theta - 1}{4\eta} + \ldots \right\} \tag{35}$$

θ	$R_1(\theta)$ based on (34)	$R_1(\theta)$ based on (35)
0.1	-0.003	-0.001
0.3	-0.001	-0.006
1	0.002	-0.018
3	0.012	-0.031
10	0.028	-0.036
30	0.045	-0.015
100	0.064	0.049
300	0.080	0.160
1000		0.354
∞	0.115	∞

The function which multiplies $R_1(\theta)$ in Equation (33) is small.

$\eta\alpha(\theta)$	$\frac{5}{4}\left\{3W_3 - \frac{23}{2}W_4 + \frac{55}{4}W_5 - \frac{21}{4}W_6\right\}$
1	0.012
2	0.034
4	0.067
6	0.080
7.5	0.082
10	0.078
20	0.051
40	0.024
100	0.007

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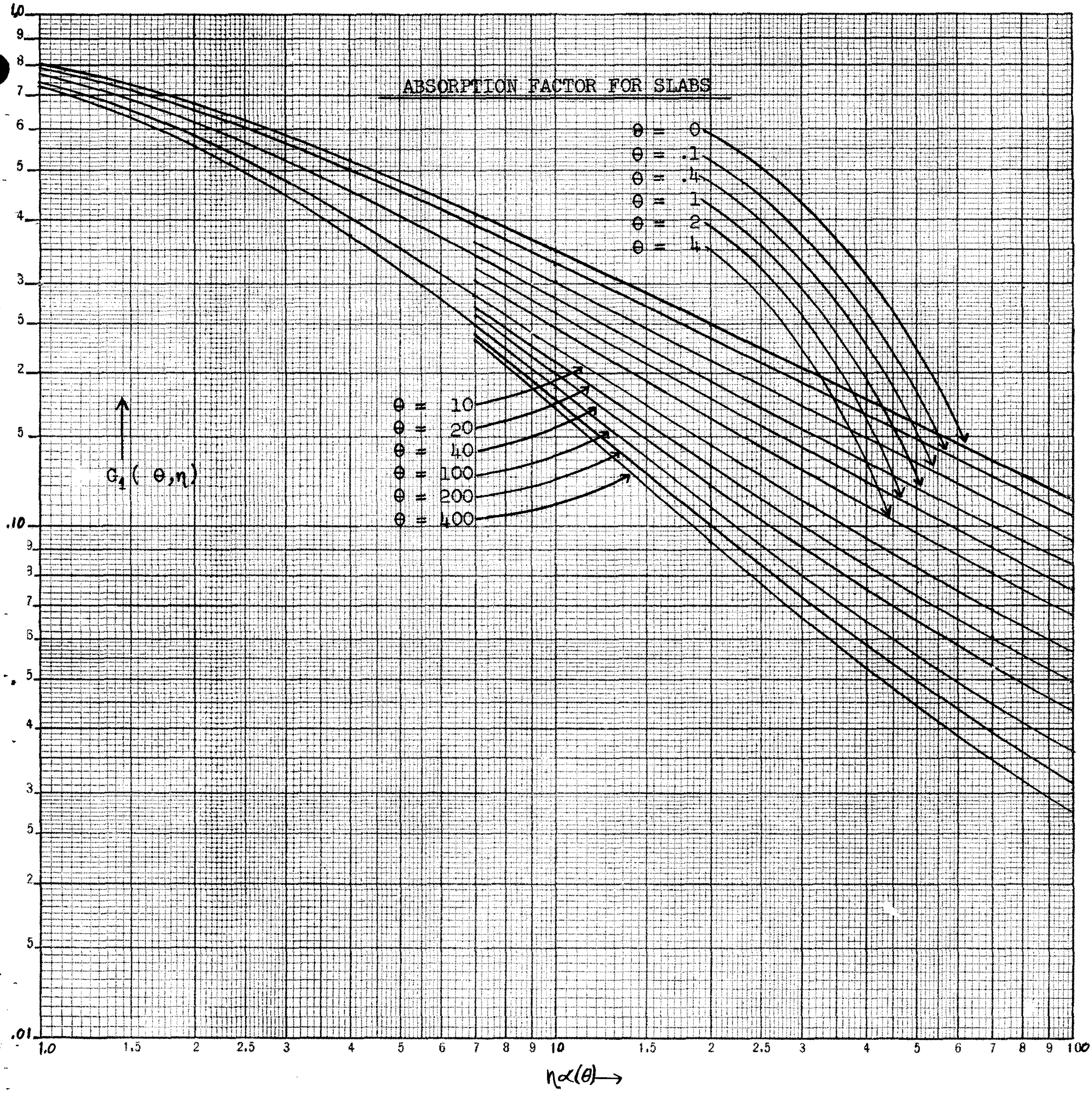

FIGURE 1

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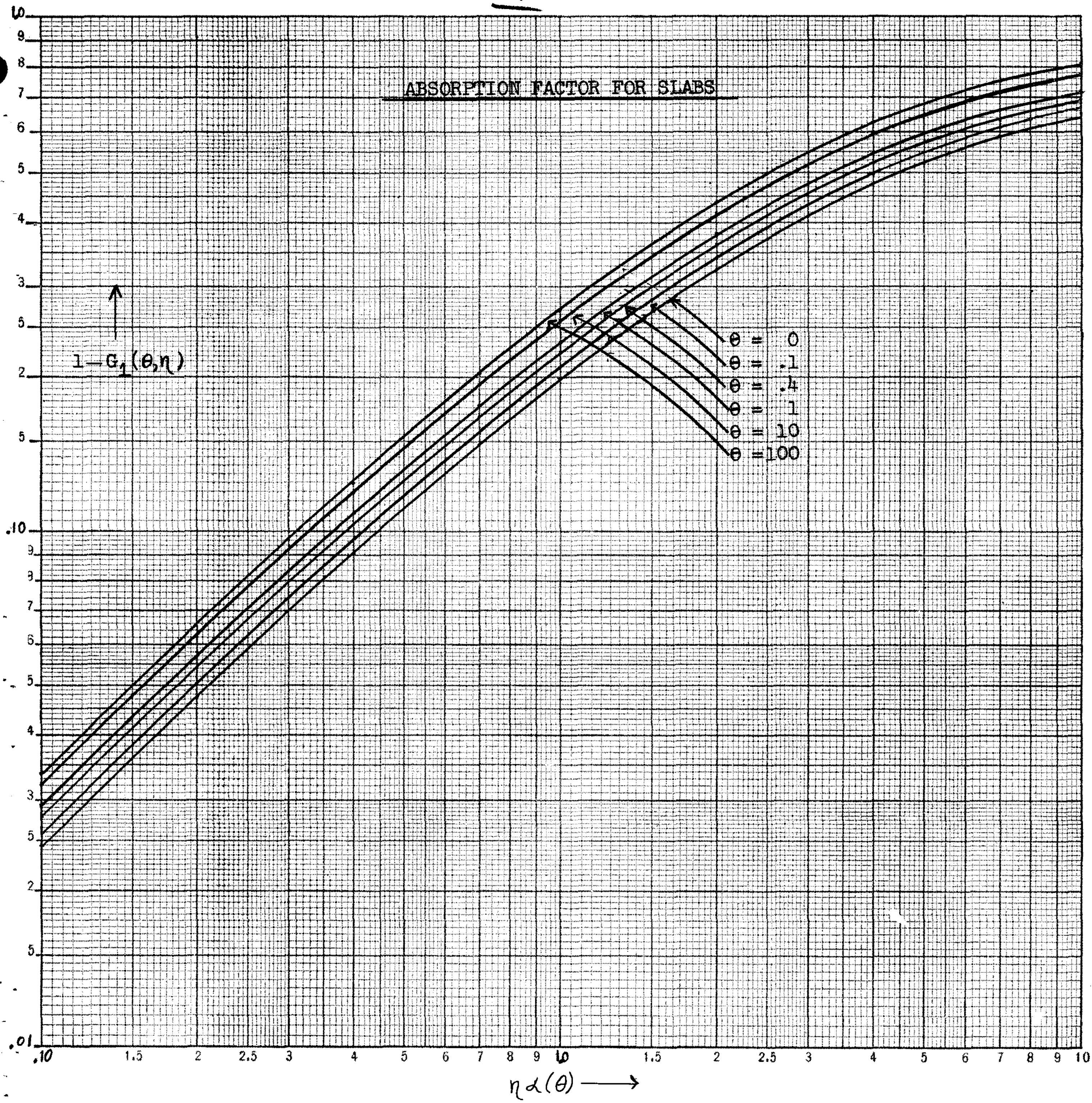

FIGURE 2

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ABSORPTION FACTOR FOR SLABS

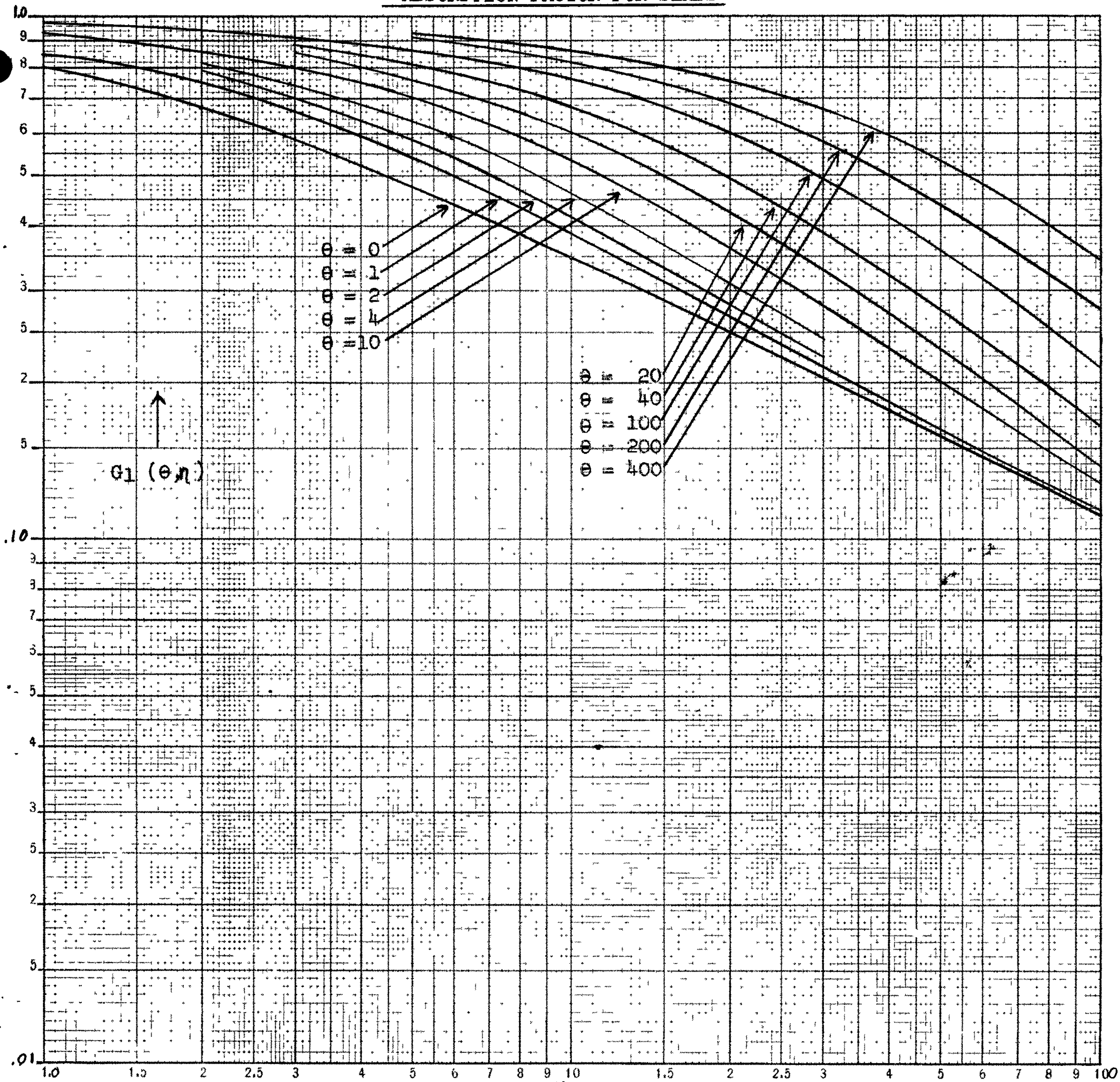

FIGURE 3

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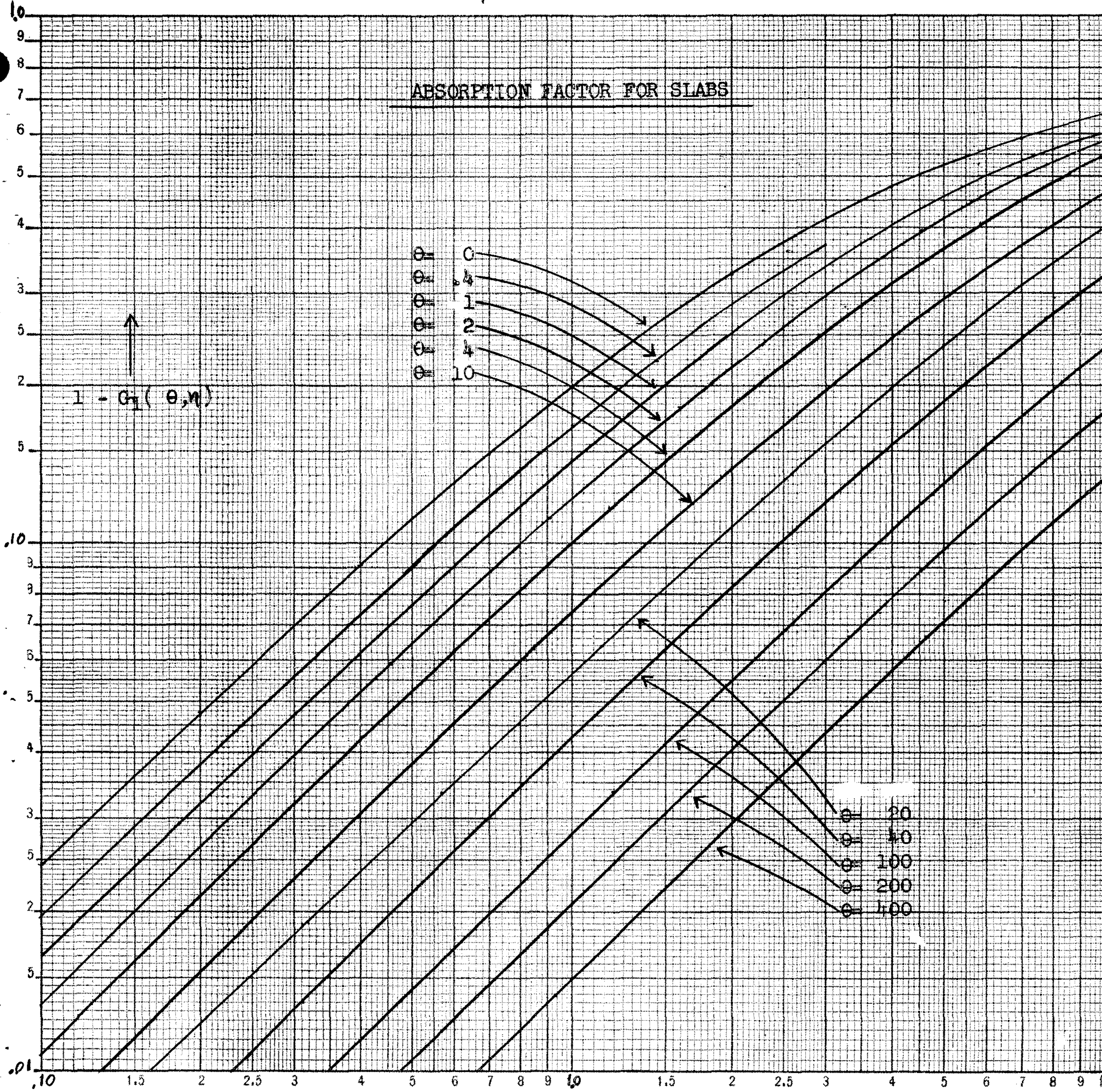

FIGURE 4

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Thus, with either choice for $R_1(\theta)$ the next correction term to be added to G_1 is always less than 0.005 provided θ is not greater than 100. The approximation $G_1(\theta,\eta)$ is least accurate when θ is large, as is to be expected from the fact that for large θ the broadened resonance is nearly Gaussian in shape and the approximation for x should then include a logarithmic term. The error in $G_1(\theta,\eta)$ when θ is large can be investigated in another way. From Equations (15) and (17) we find

$$G(\theta,\eta) \underset{\theta\to\infty}{\longrightarrow} Q(\eta\alpha) \tag{36}$$

where

$$Q(z) = \frac{1}{z\sqrt{\pi}} \int_{-\infty}^{\infty} dy \left(1 - e^{-ze^{-y^2}}\right) \tag{37}$$

$$= \sum_{n=0}^{\infty} \frac{(-z)^n}{(n+1)!\sqrt{n+1}}$$

This may be compared with the limiting value of G_1 for large θ

$\eta\alpha(\theta)$	$Q(\eta\sigma)$	$G_1(\theta\to\infty)$
0	1.0000	1.0000
0.1	0.9656	0.9655
0.2	0.9330	0.9327
0.4	0.8727	
0.6	0.8184	
0.8	0.7694	
1.0	0.7251	0.7203
1.5	0.6312	
2	0.5565	0.5429
4	0.3723	0.3422
6	0.2788	
10	0.1870	0.1380
20	0.1050	0.0583
40	0.0576	0.0227
70	0.0351	
100	0.0255	0.0060

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The agreement is better than 1% for $\eta\alpha$ not greater than unity. In actual physical problems, θ and $\eta\alpha$ will not both be large. For $\theta = 400$, $\alpha = 0.043$, so one may assume the curves in Figures 1 to 4 to be accurate to better than 1% for η not greater than 25.

SELF-INDICATION EXPERIMENTS

For an absorber of thickness t_1 and a detector (of the same material) of thickness t_2, the self-indication cross section is

$$\sigma_{si} = \frac{\int_{-\infty}^{\infty} \left\{1 - e^{-\eta_1\psi}\right\} \left\{1 - e^{-\eta_2\psi}\right\} dx}{\int_{-\infty}^{\infty} \left\{1 - e^{-\eta_2\psi}\right\} dx} \tag{38}$$

$$\eta_1 = N_1\sigma_o t_1, \quad \eta_2 = N_2\sigma_o t_2 \tag{39}$$

This may be evaluated directly in terms of the function G.

$$\sigma_{si} \cong 1 - \frac{(\eta_1 + \eta_2)G_1(\theta,\eta_1 + \eta_2) - \eta_1 G(\theta,\eta_1)}{\eta_2 G_1(\theta,\eta_2)} \tag{40}$$

In the above, both absorber and detector are assumed to be at the same temperature. If the temperatures differ, $\theta = \theta_1$ for the absorber, $\theta = \theta_2$ for the detector, then

$$\sigma_{si} \cong 1 - \frac{(\eta_1 + \eta_2)G_1(\theta_e,\eta_1 + \eta_2) - \eta_1 G(\theta_1,\eta_1)}{\eta_2 G_1(\theta_2,\eta_2)} \tag{41}$$

where, <u>approximately</u> (see page 42),

$$\theta_e \cong \bar{\theta} \left\{1 - \left[\frac{\overline{\theta^2} - \bar{\theta}^2}{\bar{\theta}^2}\right] \frac{\frac{3}{2}\bar{\theta}}{1 + 4\bar{\theta}}\right\} \tag{42}$$

$$\bar{\theta} = \frac{\eta_1\theta_1 + \eta_2\theta_2}{\eta_1 + \eta_2}$$

$$\overline{\theta^2} = \frac{\eta_1\theta_1^2 + \eta_2\theta_2^2}{\eta_1 + \eta_2}$$

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SELF-SHIELDING IN HOMOGENEOUS MIXTURES

For a resonance absorber mixed with scattering material, the probability of absorption before scattering is*

$$\frac{N_a\sigma_a}{N_s\sigma_s + N_a\sigma_a}$$

near a resonance, $\sigma_a = \sigma_a\psi(\theta,x)$, and the probability integrated over the resonance may be written

$$\frac{N_a\pi\sigma_o}{N_s\sigma_s} \; f_H(\theta,\zeta)$$

where

$$f_H(\theta,\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\psi dx}{1 + \zeta\psi} \tag{43}$$

$$\zeta = \frac{N_a\sigma_o}{N_s\sigma_s} \tag{44}$$

After integrating by parts, f_H may be written

$$f_H(\theta,\zeta) = \frac{2}{\pi} \int_o^1 \frac{axdP}{(1 + \alpha\zeta P)^2} \tag{45}$$

with P and α defined by Equations (18) and (14). The approximation (25) may now be inserted for x and the integral evaluated.

$$f_H(\theta,\zeta) \cong V_o(\alpha\zeta) - S_1(\theta)V_1(\alpha\zeta) + S_2(\theta)V_2(\alpha\zeta) + S_3(\theta)V_3(\alpha\zeta) \tag{46}$$

$$V_o(y) = (1 + y)^{-1/2}$$

$$V_1(y) = \frac{1}{y^2} \left\{ 8 - (8 + 4y - y^2)V_o \right\}$$

*See appendix for a discussion of the relevance of this probability to reactor calculations.

$$V_2(y) = \frac{4}{y^3} \left\{ -8 - 4y + (8 + 8y + y^2)V_o \right\}$$

$$V_3(y) = \frac{12}{y^4} \left\{ -32 - 32y - 6y^2 + (32 + 48y + 18y^2 + y^3)V_o \right\}$$

The functions $1 - f_H$ and f_H computed from Equation (46) are plotted in Figures 5 and 6.

Bethe and Bell have observed* that in the limiting case of large θ, f_H can be expressed as a Fermi function. When $\theta \to \infty$, the approximation (15) can be used for ψ in Equation (43). This leads to

$$f_H(\theta,\zeta) \xrightarrow[\theta \to \infty]{} \frac{1}{\sqrt{\pi}} \int_o^\infty \frac{dz}{\sqrt{z}(e^z + \zeta\alpha)} = F_{-1/2}(\ell n \zeta\alpha) \tag{47}$$

This limiting form is plotted as a dashed curve in Figure 6. For $\alpha\zeta$ less than 4, the dashed curve lies on top of the curve for $\theta = 100$.

The simple form of the integral (43) makes it possible to find a simple alternate approximation for f_H. Suppose we approximate $\psi(\theta,x)$ by

$$\psi(\theta,x) \cong \frac{\alpha(\theta) + x^2/A^2B^2}{(1 + x^2/A^2)(1 + x^2/B^2)} \tag{48}$$

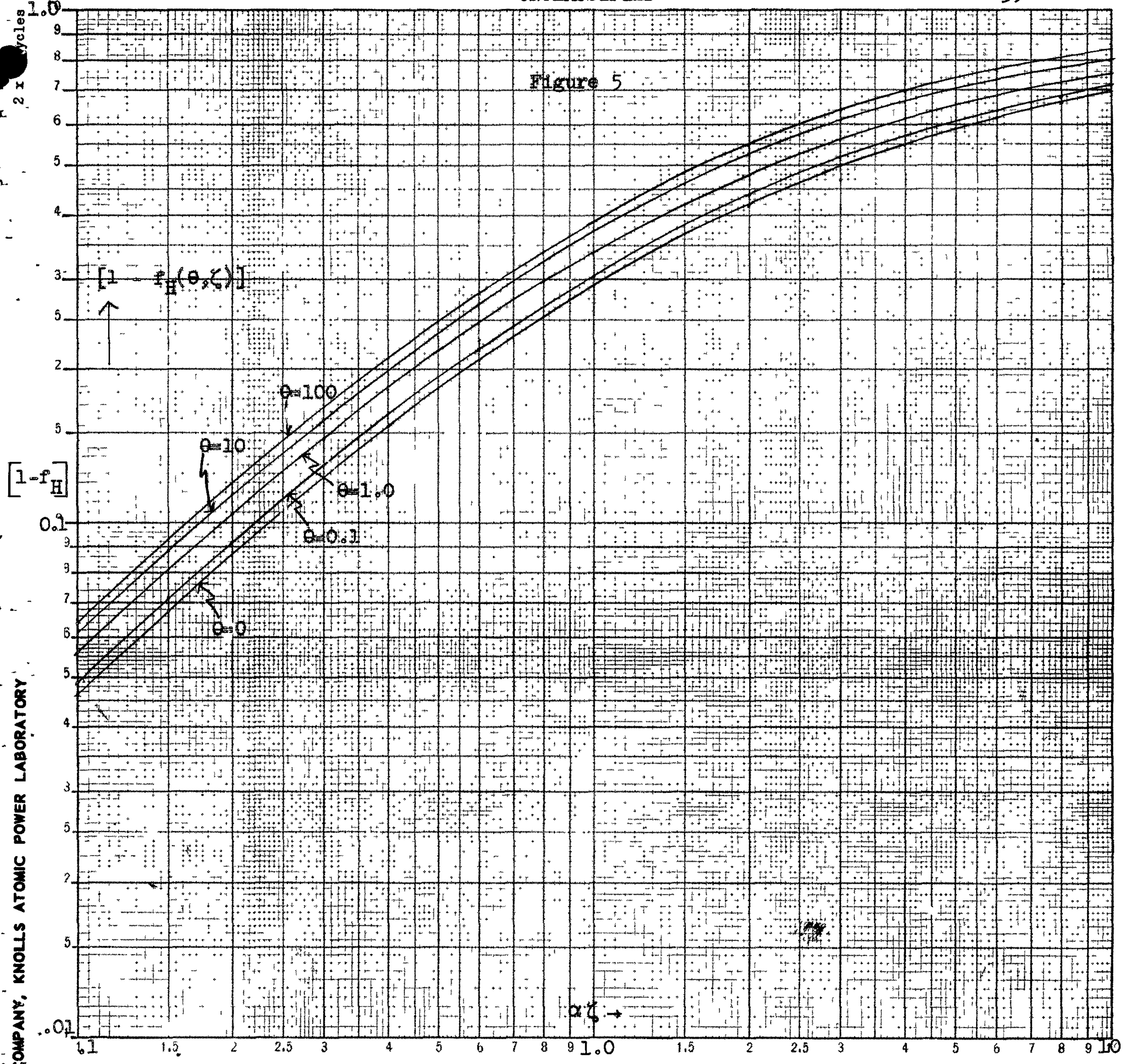
This approximation has the correct limiting values for $x \to 0$ and $x \to \infty$, and if we choose

$$A + B = 1 + \beta(\theta)$$

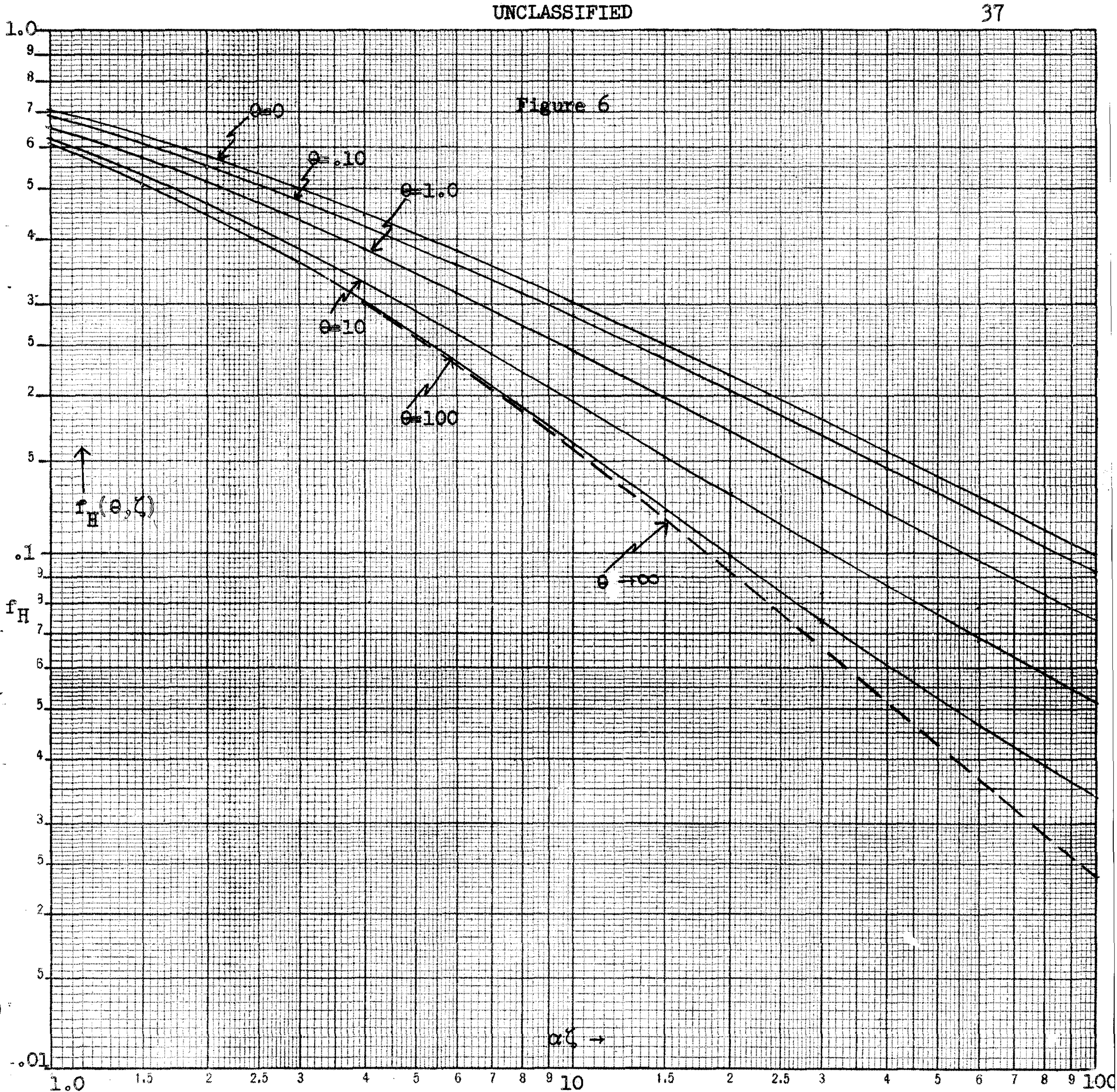
$$AB = \frac{\beta}{\alpha}$$

$$\beta(\theta) = \frac{1 - M(\theta)}{M(\theta) - \alpha(\theta)}$$

*Private communication from H. Bethe.



SELF-SHIELDING FACTORS FOR HOMOGENEOUS MIXTURES



SELF-SHIELDING FACTORS FOR HOMOGENEOUS MIXTURES

then the integrals $\int_{-\infty}^{\infty} \psi\, dx$ and $\int_{-\infty}^{\infty} \psi^2 dx$ will also have the correct values. Equations (48) and (43) then give

$$f_H(\theta,\zeta) = \frac{\beta + \sqrt{1+\zeta\alpha}}{\sqrt{1+\zeta\alpha}\left[\zeta + (1+\beta)^2 + 2\frac{\beta}{\alpha}\left(\sqrt{1+\zeta\alpha} - 1\right)\right]^{1/2}} \tag{49}$$

This approximation is more accurate than (46) in the limit $\theta \to \infty$, but is less accurate than (46) for θ not large.

SELF-SHIELDING FACTOR FOR SLABS

Consider a slab of thickness t exposed to an isotropic flux of neutrons with a spectrum which is flat in the neighborhood of a resonance. The probability that a neutron which strikes the slab will be absorbed in the slab is

$$\int_0^1 \mu\, d\mu \int_{-\infty}^{\infty} dx \left\{ 1 - e^{-\frac{1}{\mu} tN_a\sigma_o\psi(\theta,x)} \right\}$$

where μ is the cosine of the angle of incidence. In the limit $t \to 0$ this probability approaches $tN_a\sigma_o\pi$, so it will be convenient to write the probability of absorption as

$$tN_a\pi\sigma_o f(\theta,\tau)$$

where

$$\tau = N_a\sigma_o t$$

$$f(\theta,\tau) = \frac{\tau}{\pi}\int_\tau^\infty \frac{d\eta}{\eta^3}\int_{-\infty}^{\infty} dx \left\{ 1 - e^{-\eta\psi} \right\} \tag{50}$$

or, using Equation (17)

$$f(\theta,\tau) = \tau \int_\tau^\infty \frac{d\eta}{\eta^2} G(\theta,\eta) \tag{51}$$

We have already found an approximation to G so it is only necessary to integrate Equation (30).

$$f(\theta,\tau) \cong f_1(\theta,\tau) \tag{52}$$

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$$f_1(\theta,\tau) = X_2(\alpha\tau) - S_1(\theta) \left\{ X_2(\alpha\tau) - X_3(\alpha\tau) \right\} \tag{53}$$

$$+ S_2(\theta) \left\{ X_3(\alpha\tau) - X_4(\alpha\tau) \right\}$$

$$+ S_3(\theta) \left\{ 3X_3(\alpha\tau) - 8X_4(\alpha\tau) + 5X_5(\alpha\tau) \right\}$$

$$X_r(x) = x \int_x^\infty \frac{dy}{y^2} M(r - 3/2,\ r,\ -y)$$

$$= M(r - 3/2,\ r,\ -x) - \frac{x}{r}(r - 3/2) \int_x^\infty \frac{dy}{y} M(r - 1/2,\ r + 1,\ -y)$$

By making use of the recurrence formulas for the confluent hypergeometric function, the X_r functions may be written

$$X_2(x) = -T_1(x) + \frac{1}{2} m_1 + \frac{1}{2} m_2$$

$$X_3(x) = -2T_1(x) + m_1 + m_2 - m_3$$

$$X_4(x) = -\frac{5}{2} T_1(x) + \frac{5}{4} m_1 + \frac{5}{4} m_2 - \frac{11}{4} m_3 + \frac{5}{4} m_4 \tag{55}$$

$$X_5(x) = -\frac{14}{5} T_1(x) + \frac{7}{5} m_1 + \frac{7}{5} m_2 - 5m_3 + \frac{23}{5} m_4 - \frac{7}{5} m_5$$

where

$$m_i = M\left(\frac{1}{2},\ i,\ -x\right) \tag{56}$$

$$T_1(x) = \frac{x}{4} \int_x^\infty \frac{dy}{y} M\left(\frac{3}{2}, 2,\ -y\right) \tag{57}$$

The m_i have already been tabulated.* The function $T_1(x)$ has the following expansions:

$$T_1(x) = \frac{x}{4} \left\{ \ln \frac{4}{\gamma x} - 1 + \sum_{n=1} (-1)^{n+1} \frac{(n + 1/2)!\,x^n}{n(1/2)!\,n!\,(n + 1)!} \right\} \tag{58}$$

$$T_1(x) \cong \frac{1}{4} \sqrt{\frac{\pi}{x}} \sum_{n=0} \frac{(n - 1/2)!\,(n + 1/2)!}{(-1/2)!\,(1/2)!\,n!} \frac{1}{(n + 3/2)x^n} \tag{59}$$

*British Association Report 1926.

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The rate of convergence of the series in (58) may be improved by subtracting out some confluent hypergeometric functions.

$$T_1(x) = \frac{x}{4}\left\{\ln\frac{4}{\gamma x} + \frac{11}{6} - 2m_2 - \frac{1}{2}m_3 - \frac{1}{3}m_4 \right. \tag{60}$$

$$\left. + \sum_{n=1} \frac{(-1)^{n+1}(n-1/2)!\,3!}{(-1/2)!\,(n+3)!}\cdot\frac{1}{n}\cdot\frac{x^n}{n!}\right\}$$

The computed values of $[1 - f_1(\theta,\tau)]$ and $f_1(\theta,\tau)$ are plotted as functions of θ and $\tau\alpha(\theta)$ in Figures 7 and 8, and as functions of θ and τ in Figures 9 and 10. For $\theta \to \infty$ the correct limiting form for f is

$$f(\theta,\tau) \xrightarrow[\theta\to\infty]{} P(\tau\alpha(\theta)) \tag{61}$$

where

$$P(x) = x\int_x^\infty \frac{dw}{w^2}Q(w) \tag{62}$$

$$= 1 - \frac{7}{8\sqrt{2}}x - \frac{x}{2\sqrt{2}}\ln\frac{1}{\gamma x} + \sum_{n=2}\frac{(-1)^{n+1}x^n}{(n-1)\,(n+1)!\sqrt{n+1}}$$

Here, γ is Euler's constant. The limiting form P is plotted as a dashed curve in Figures 7 and 8, and provides a check on the accuracy of $f_1(\theta,\tau)$ for large θ.

For estimating the temperature coefficient of reactivity due to the Doppler effect, one needs to know

$$T\frac{\partial f}{\partial T} = \theta\frac{\partial f}{\partial\theta}$$

The derivative may be obtained simply by operating on (53) with $\frac{\partial}{\partial\theta}$, remembering that α is a function of θ. The numerical results are given in Figures KS-1285, -1286, and -1287. For convenience in interpolation the function plotted is

$$\frac{\theta\frac{\partial}{\partial\theta}f_1(\theta,\tau)}{1-\alpha(\theta)}$$

Here also one can check the accuracy by an independent calculation of the limiting form of the derivative for $\theta \to \infty$. The correct limit is

$$\theta \frac{\partial f(\theta,\tau)}{\partial\theta} \xrightarrow[\theta\to\infty]{} -\frac{1}{2}\tau\alpha P'(\tau\alpha) \qquad (63)$$

The curve marked $\theta\to\infty$ in the figures was computed from (63).

SLAB OF NONUNIFORM TEMPERATURE

For the slab treated in the previous section, suppose the temperature is proportional to $\theta(r)$, where r is the fractional distance from one edge of the slab. Then the ψ which appears in Equation (50) must be replaced by

$$\hat{\psi} = \int_0^1 \psi(\theta(r),x)dr \qquad (64)$$

It would be convenient to replace $\hat{\psi}$ by $\psi(\theta_e,x)$ with θ_e a constant; then the self-shielding factor would be approximated by $f_1(\theta_e,\tau)$. Now in getting the approximations G_1 and f_1 we effectively replaced $\psi(\theta,x)$ by a substitute function, although the operation was carried out by treating x as the dependent variable. In regard to accuracy in G_1 and f_1, the three most important conditions were that the substitute ψ have the right tail and that $\int_{-\infty}^{\infty} \psi\, dx$ and $\int_{-\infty}^{\infty} \psi^2 dx$ have the correct values. Therefore, we will adjust θ_e so that

$$\int_{-\infty}^{\infty} \psi(\theta_e,x)dx = \int_{-\infty}^{\infty} \hat{\psi}dx \qquad (65)$$

and

$$\int_{-\infty}^{\infty} \psi^2(\theta_e,x)dx = \int_{-\infty}^{\infty} \hat{\psi}^2 dx \qquad (66)$$

The condition (65) is automatically satisfied. If we write

$$\theta \equiv \theta(r)$$

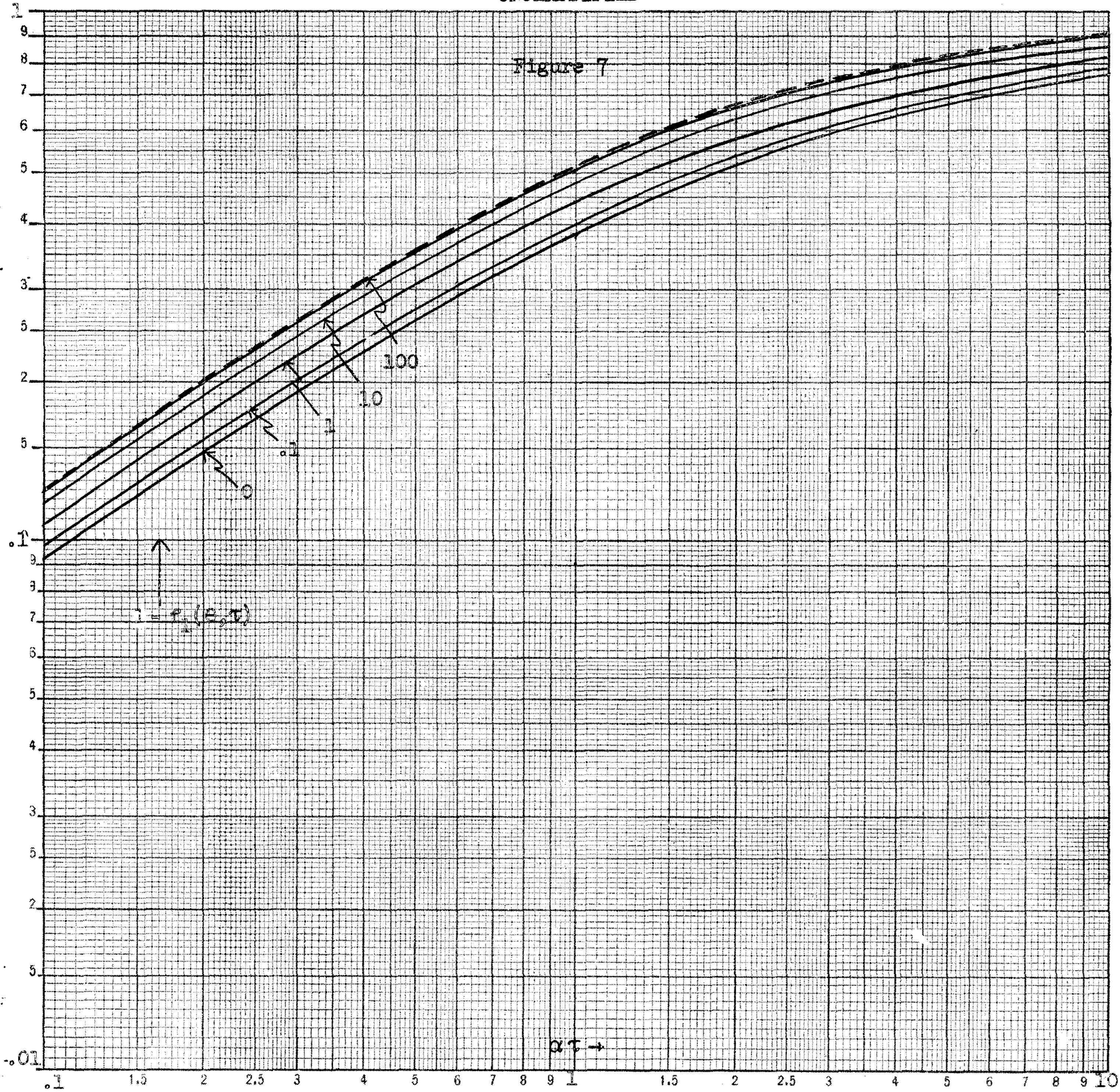
$$\theta' \equiv \theta(r')$$

condition (66) becomes

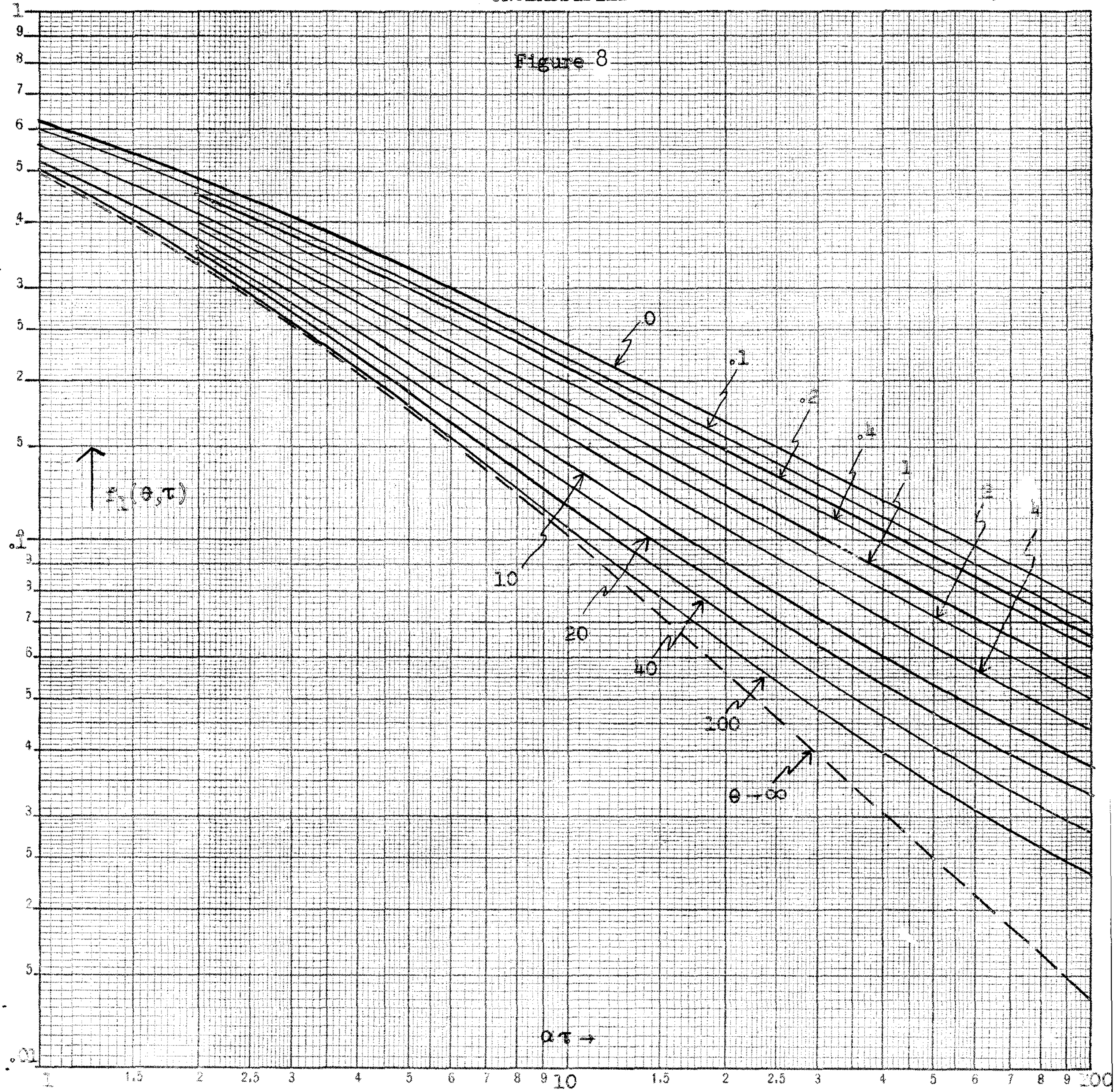
$$\frac{\pi}{2} M(\theta_e) = \int_{-\infty}^{\infty} dx \int_0^1 dr \int_0^1 dr'\, \psi(\theta,x)\, \psi(\theta',x) \qquad (67)$$

or

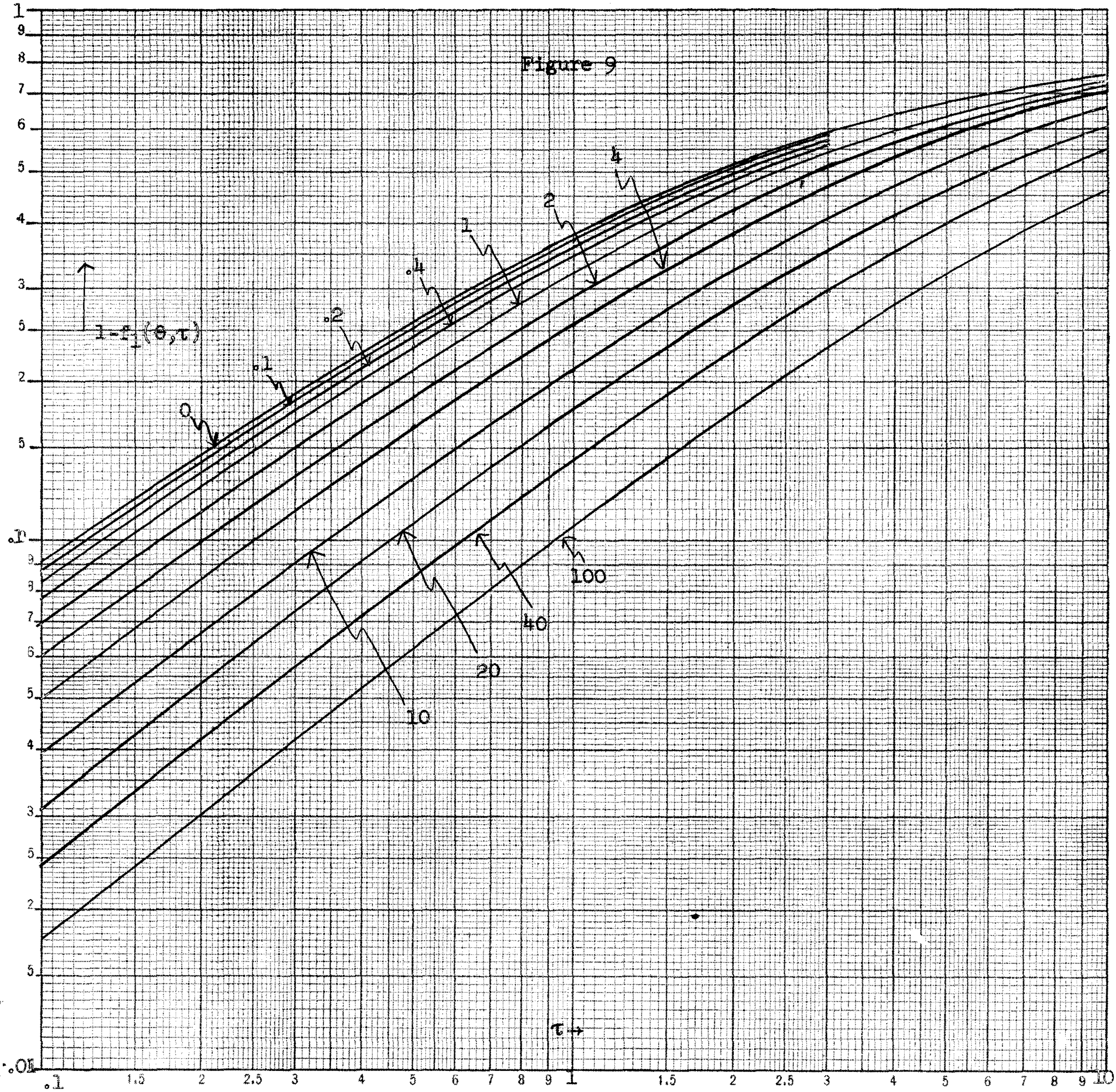
$$M(\theta_e) = \int_0^1 dr \int_0^1 dr' M\left(\frac{\theta + \theta'}{2}\right) \qquad (68)$$



SELF-SHIELDING FACTORS FOR SLABS



SELF-SHIELDING FACTORS FOR SLABS



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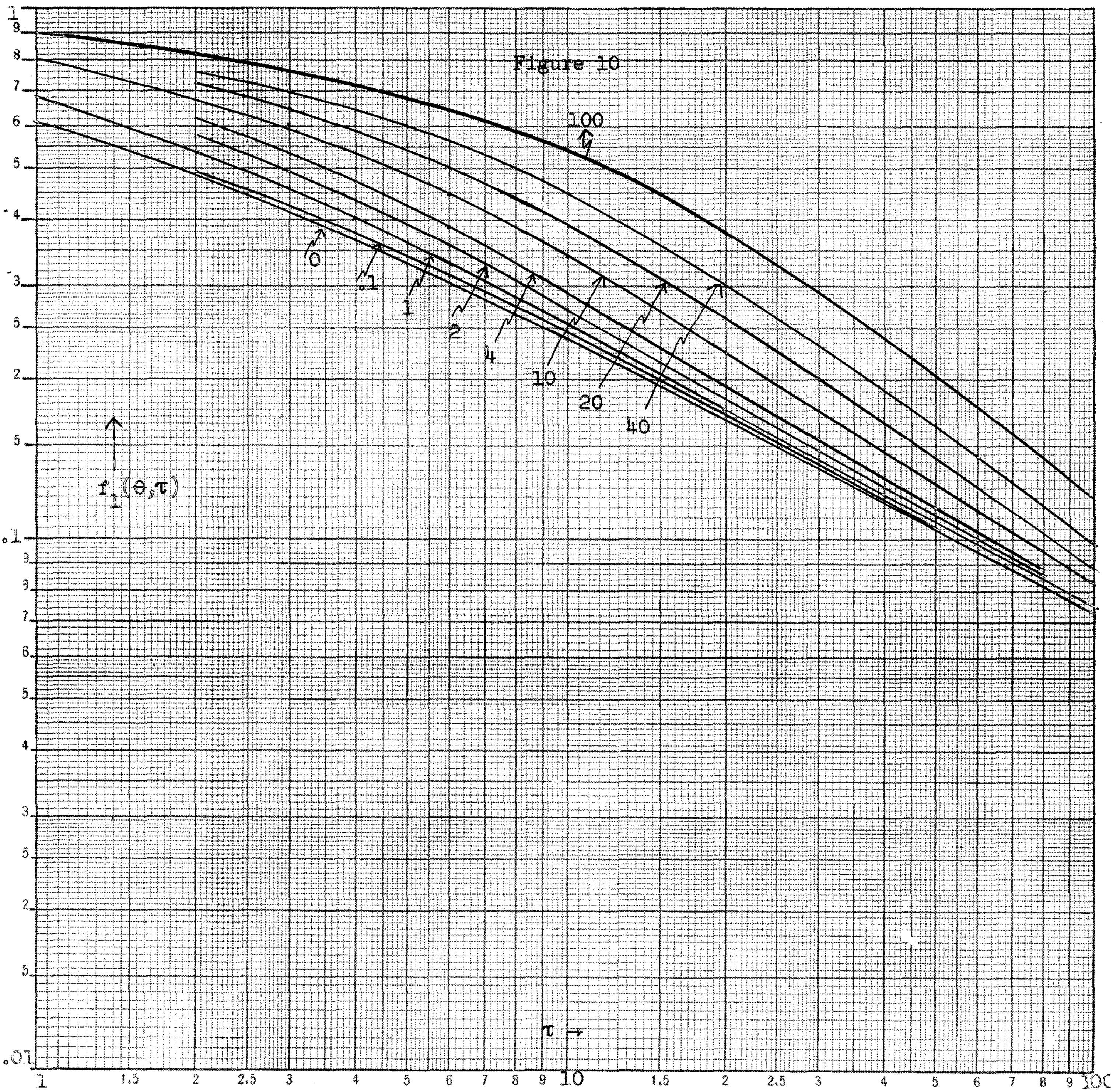

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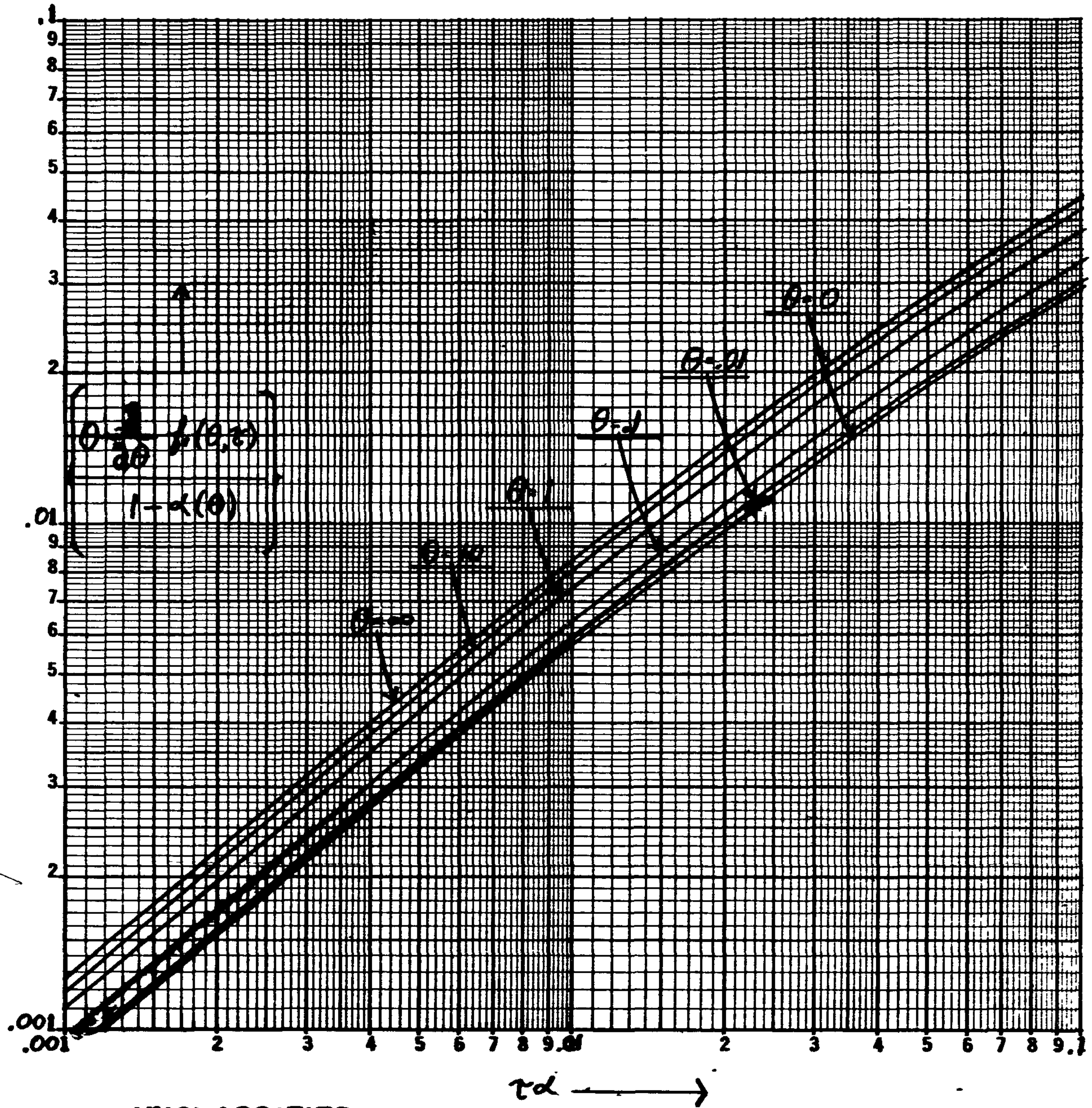

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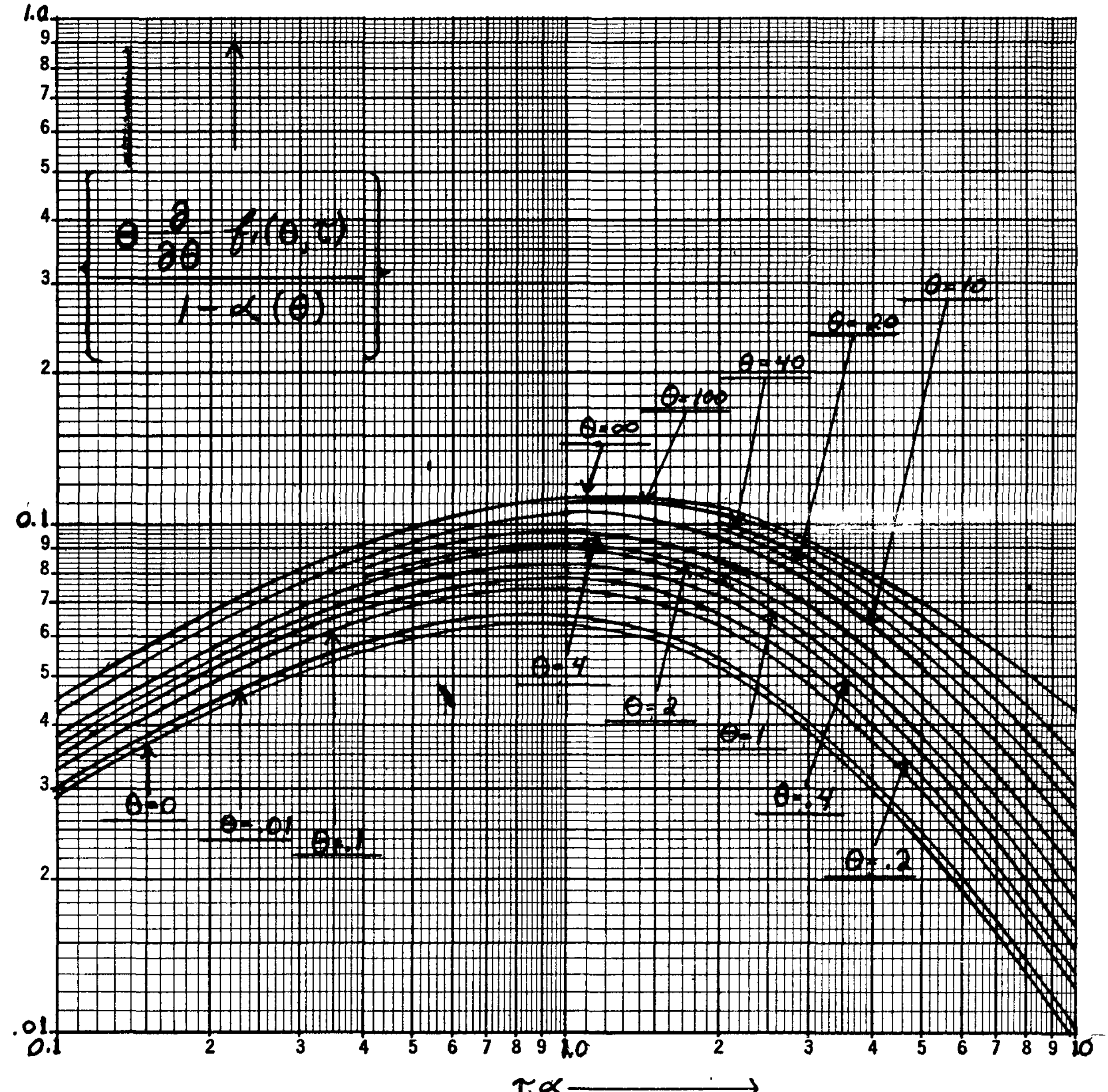

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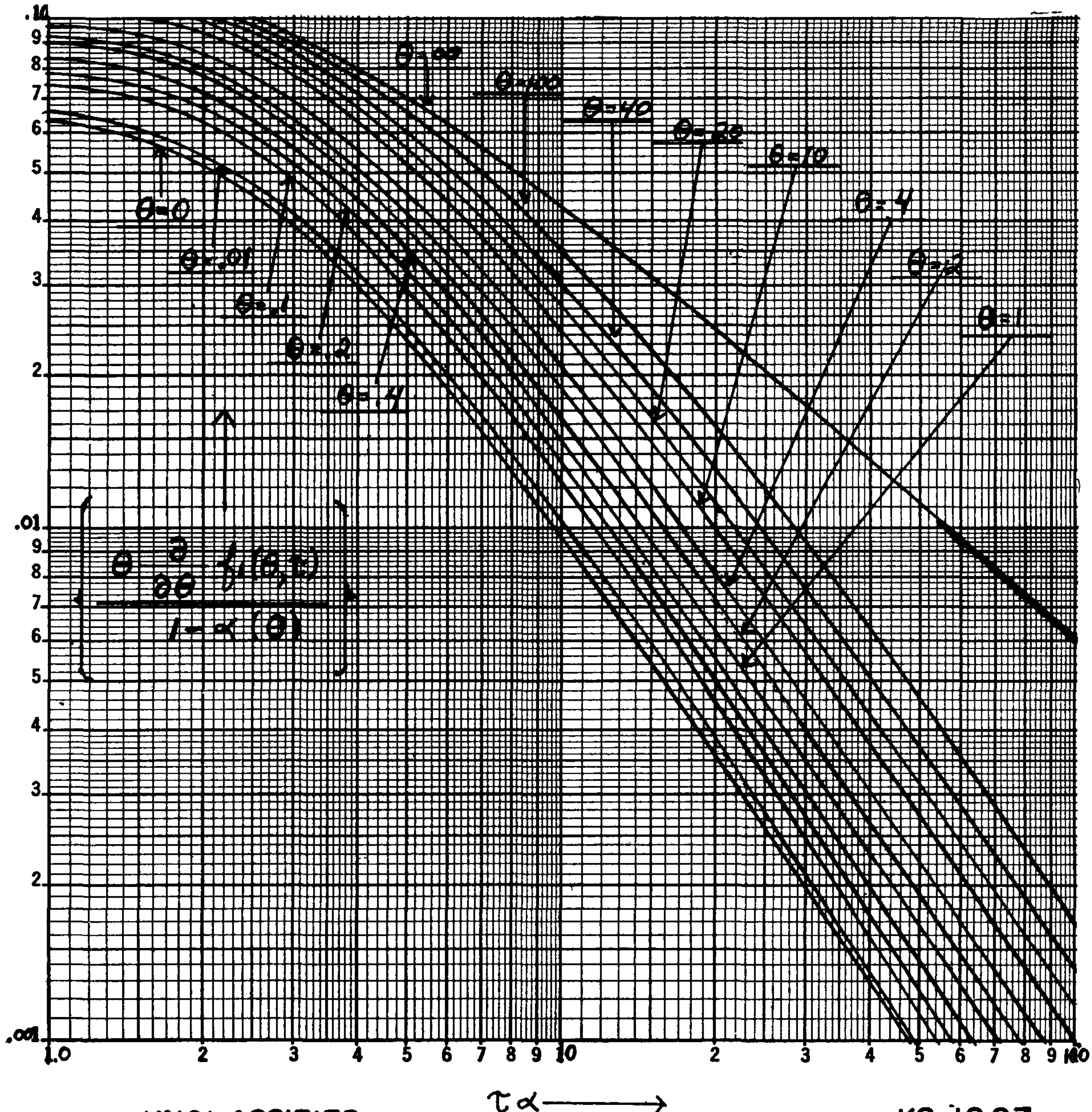

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KS-1287

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Let $\bar{\theta}$ be the average "temperature" of the slab.

$$\bar{\theta} = \int_0^1 \theta(r)dr \tag{69}$$

$$\overline{\theta^2} = \int_0^1 \theta^2(r)dr \tag{70}$$

Write

$$\theta_e = \bar{\theta} + (\theta_e - \bar{\theta})$$

$$\theta = \bar{\theta} + (\theta - \bar{\theta})$$

and expand both M's in (68) in Taylor's series about $\bar{\theta}$.

$$M(\bar{\theta}) + (\theta_e - \bar{\theta})M'(\bar{\theta}) + \frac{1}{2}(\theta_e - \bar{\theta})^2 M''(\bar{\theta}) + \ldots$$

$$= M(\bar{\theta}) + \frac{1}{4}(\overline{\theta^2} - \bar{\theta}^2)M''(\bar{\theta}) + \ldots$$

or

$$\theta_e = \theta - \frac{(\overline{\theta^2} - \bar{\theta}^2)}{\bar{\theta}} \left\{ \frac{\bar{\theta}M''(\bar{\theta})}{-4M'(\bar{\theta})} \right\} + \ldots \tag{71}$$

The quantity in brackets approaches $3/2\ \bar{\theta}$ when $\bar{\theta}$ is small, and $3/8$ when $\bar{\theta}$ is large. Since the second term is small anyway, we can replace (71) by

$$\theta_e \cong \bar{\theta} - \frac{3}{2}\ \frac{(\overline{\theta^2} - \bar{\theta}^2)}{1 + 4\bar{\theta}} \tag{72}$$

If the density of the slab is also nonuniform the integrals in (64), (69), and (70) should be modified by inserting normalized weight factors proportional to the density.

SELF-SHIELDING FACTOR FOR SPHERES

Suppose that a neutron strikes a sphere of radius r_o at an angle $\cos^{-1}\mu$ with the normal. The path length inside the sphere is $2r_o\mu$. For an isotropic flux of neutrons the angular distribution is $2\mu d\mu$. As before, assume that the

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spectrum of incident neutrons is flat in the neighborhood of a resonance. Then the average probability that a neutron striking the sphere will be absorbed in the resonance is proportional to

$$\int_{-\infty}^{\infty} dx \int_{0}^{1} 2\mu\, d\mu \left(1 - e^{-2r_o\mu N_a\sigma_o\psi(\theta,x)}\right)$$

Dividing by the limiting probability for $r_o \to 0$ gives the self-shielding factor S for a sphere

$$S(\theta,\xi) = \frac{3}{2\pi\xi} \int_{-\infty}^{\infty} dx \int_{0}^{1} \mu\, d\mu \left(1 - e^{-2\mu\xi\psi}\right) \tag{73}$$

$$\xi = N_a\sigma_o r_o \tag{74}$$

We have already found an approximation for the integration over x, and (73) may be written

$$S(\theta,\xi) \cong S_1(\theta,\xi) \tag{75}$$

$$S_1(\theta,\xi) = 3 \int_{0}^{1} \mu^2 d\mu G_1(\theta,2\xi\mu) \tag{76}$$

The function G_1 is given by Equation (30) and the integration of the confluent hypergeometric functions is simple.

$$S_1(\theta,\xi) = B_2(2\alpha\xi) - S_1(\theta)\left\{B_2(2\alpha\xi) - B_3(2\alpha\xi)\right\} \tag{77}$$
$$+ S_2(\theta)\left\{B_3(2\alpha\xi) - B_4(2\alpha\xi)\right\}$$
$$+ S_3(\theta)\left\{3B_3(2\alpha\xi) - 8B_4(2\alpha\xi) + 5B_5(2\alpha\xi)\right\}$$

$$B_r(z) = \frac{-3(r-1)}{(r-5/2)z} M(r-5/2,\ r-1,\ -z) \tag{78}$$
$$- \frac{6(r-1)(r-2)}{z^2(r-5/2)(r-7/2)} M(r-7/2,\ r-2,\ -z)$$
$$+ \frac{6(r-1)(r-2)(r-3)}{z^3(r-5/2)(r-7/2)(r-9/2)}\left\{1 - M(r-9/2,\ r-3,\ -z)\right\}$$

All of the required M functions have been tabulated.* Numerical values for S_1 are plotted in Figures 11, 12, 13 and 14. The dashed lines in Figures 11 and 12 give the correct values of S in the limiting case $\theta \to \infty$, obtained by combining (73), (15) and (37).

$$S(\theta,\xi) \xrightarrow[\theta\to\infty]{} U(2\xi\alpha) \tag{79}$$

$$U(x) = 3 \int_0^1 y^2 Q(xy)\,dy \tag{80}$$

$$U(x) = \sum_{n=0} \frac{3}{n+3} \frac{(-x)^n}{(n+1)!\sqrt{n+1}} \tag{81}$$

The accuracy of (77), which will be poorest for large θ, can now be tested by comparing the limiting value of $S_1(\theta,\xi)$ for $\theta\to\infty$ with the correct limiting value U.

$2\xi\alpha(\theta)$	$U(2\xi\alpha)$	$[S_1(\theta,\xi)]_{\theta\to\infty}$
0	1.0000	1.0000
0.1	0.9740	
0.2	0.9493	0.9491
0.4	0.9025	0.9020
0.6	0.8596	0.8586
0.8	0.8201	0.8186
1	0.7836	0.7805
1.5	0.7037	
2	0.6373	0.6281
4	0.4533	0.4368
6	0.3576	0.3253
10	0.2491	0.2057
20	0.1444	0.0965
40	0.0807	0.0407
70	0.0496	
100	0.0362	0.0115

*British Association Report 1926.

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The agreement is excellent so long as $\xi\alpha$ is not too large. The error in S_1 for $\xi\alpha$ large is of no concern since in practical cases the conditions $\xi\alpha >> 1$ and $\theta \to \infty$ are inconsistent.

Suppose the temperature of the sphere is nonuniform. Then the function ψ in Equation (73) should be replaced by

$$\overset{*}{\psi} = \frac{1}{\mu} \int_0^{\mu} d\ell\, \psi(\theta(r), x) \tag{82}$$

where

$$r = \sqrt{\ell^2 + 1 - \mu^2}$$

and $\theta(r)$ is the "temperature" at a fractional radius r. We wish to approximate ψ by $\psi(\theta_e, x)$, with θ_e a constant. Following the same reasoning as in the slab case discussed above, we chose to fix θ_e by the condition

$$\int_{-\infty}^{\infty} dx \int_0^1 \mu^3 \psi^2(\theta_e, x) d\mu = \int_{-\infty}^{\infty} dx \int_0^1 \mu^3 \overset{*}{\psi}{}^2 d\mu. \tag{83}$$

This leads to

$$\theta_e \cong \bar{\theta}$$

with the "average temperature" defined by

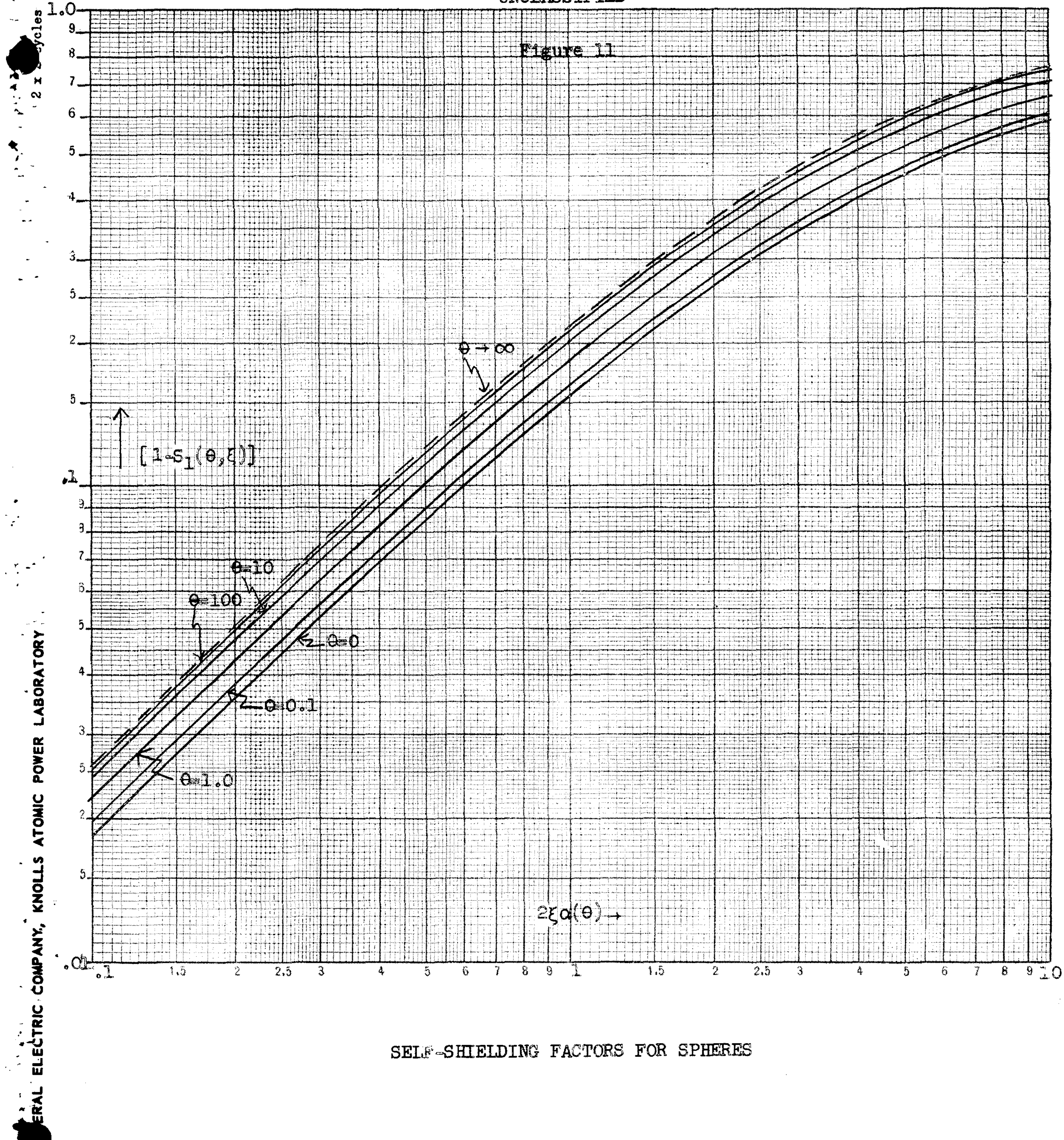
$$\bar{\theta} = 2 \int_0^1 \theta(r) r \left\{ r + (1 - r^2) \tanh^{-1} r \right\} dr \tag{84}$$

The next correction term for θ_e is not so simple as in Equation (71), but will be correspondingly small.

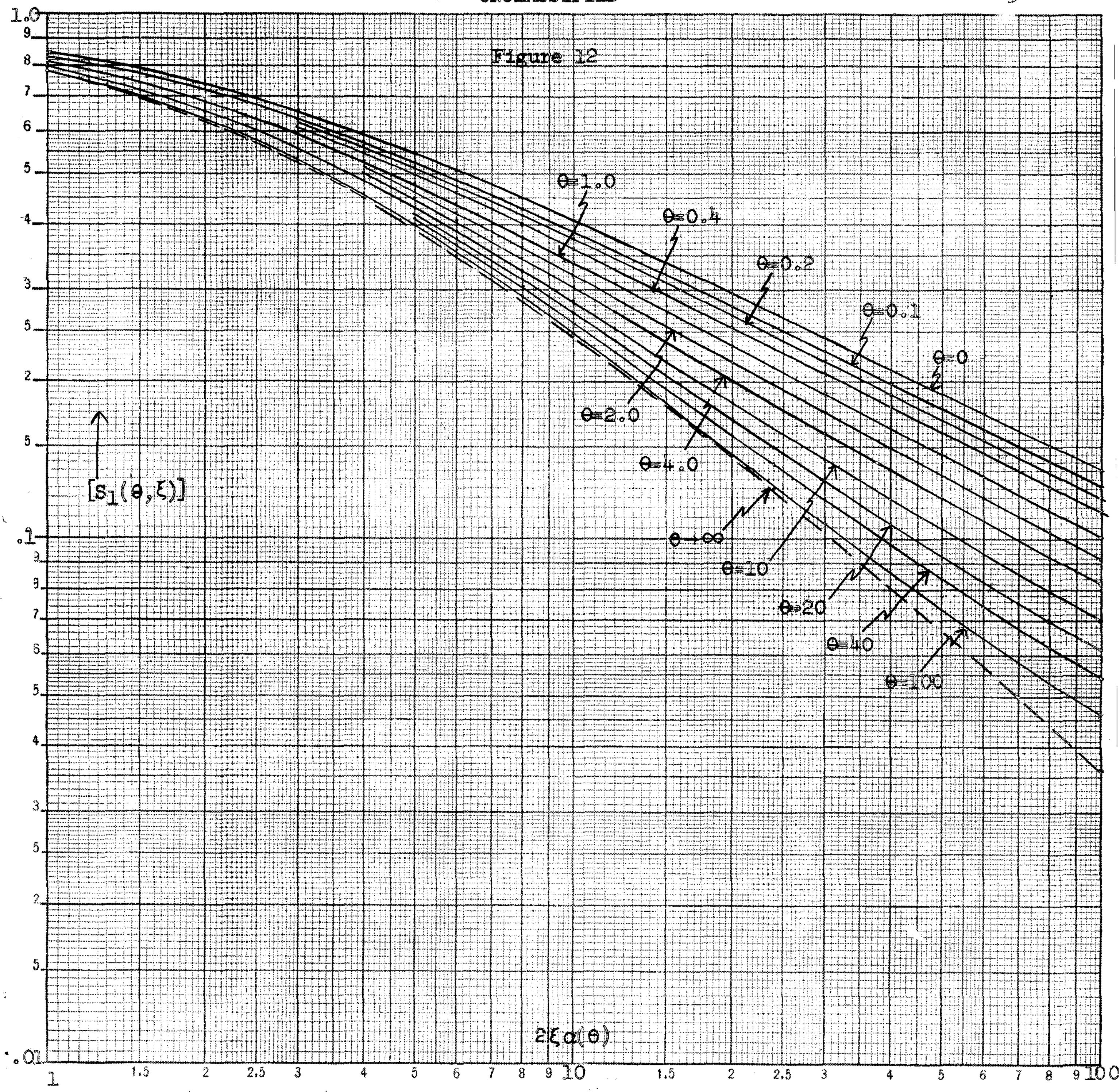
SELF-SHIELDING FACTOR FOR CYLINDERS

Consider a cylinder of radius a exposed to an isotropic flux of neutrons. Suppose a neutron enters the cylinder at P and emerges at Q. The path length inside the cylinder is

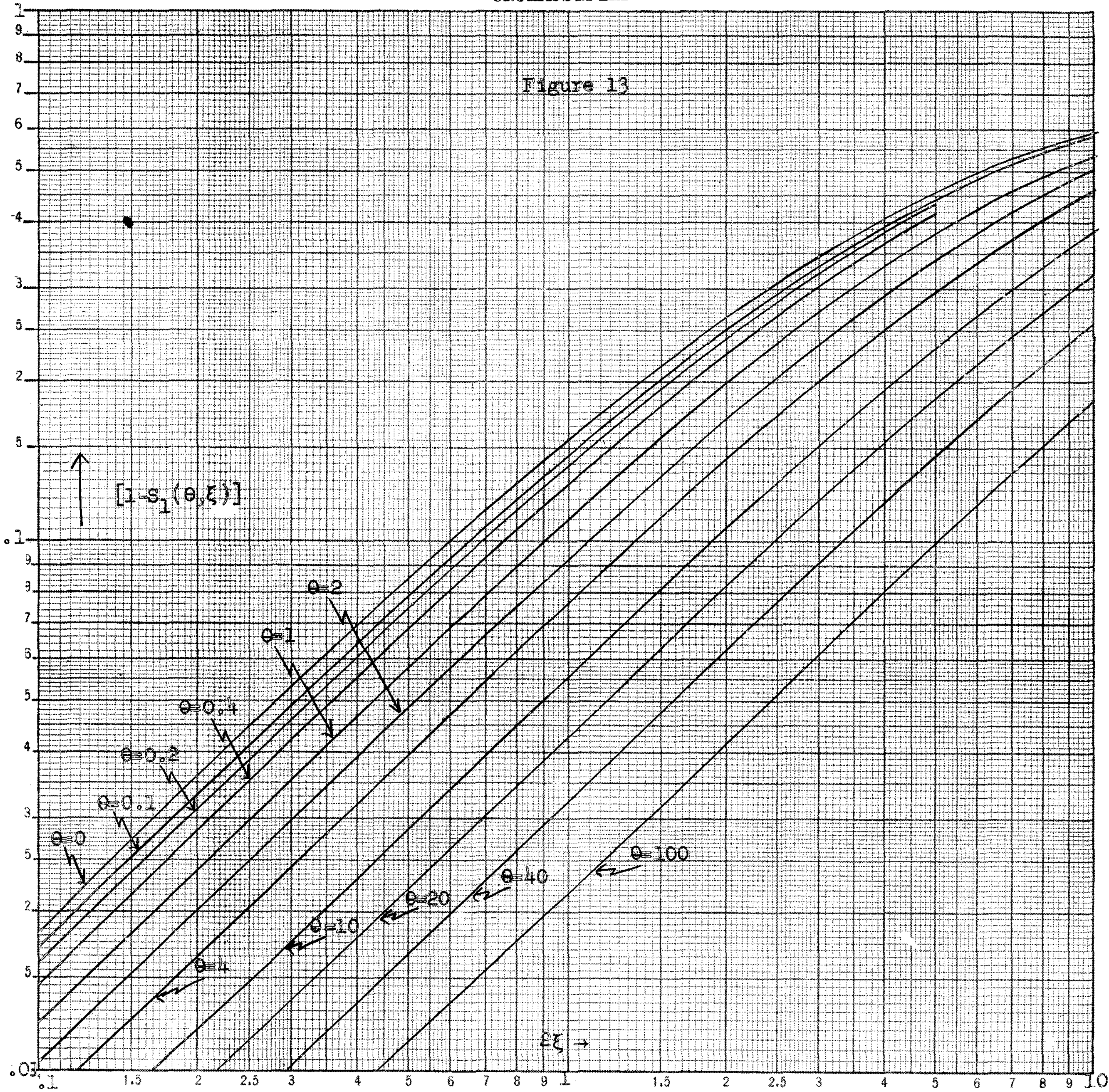
$$PQ = \frac{2a\cos\theta_1}{\cos^2\theta_1 + \sin^2\theta_1 \sin^2\phi_1}$$



SELF-SHIELDING FACTORS FOR SPHERES

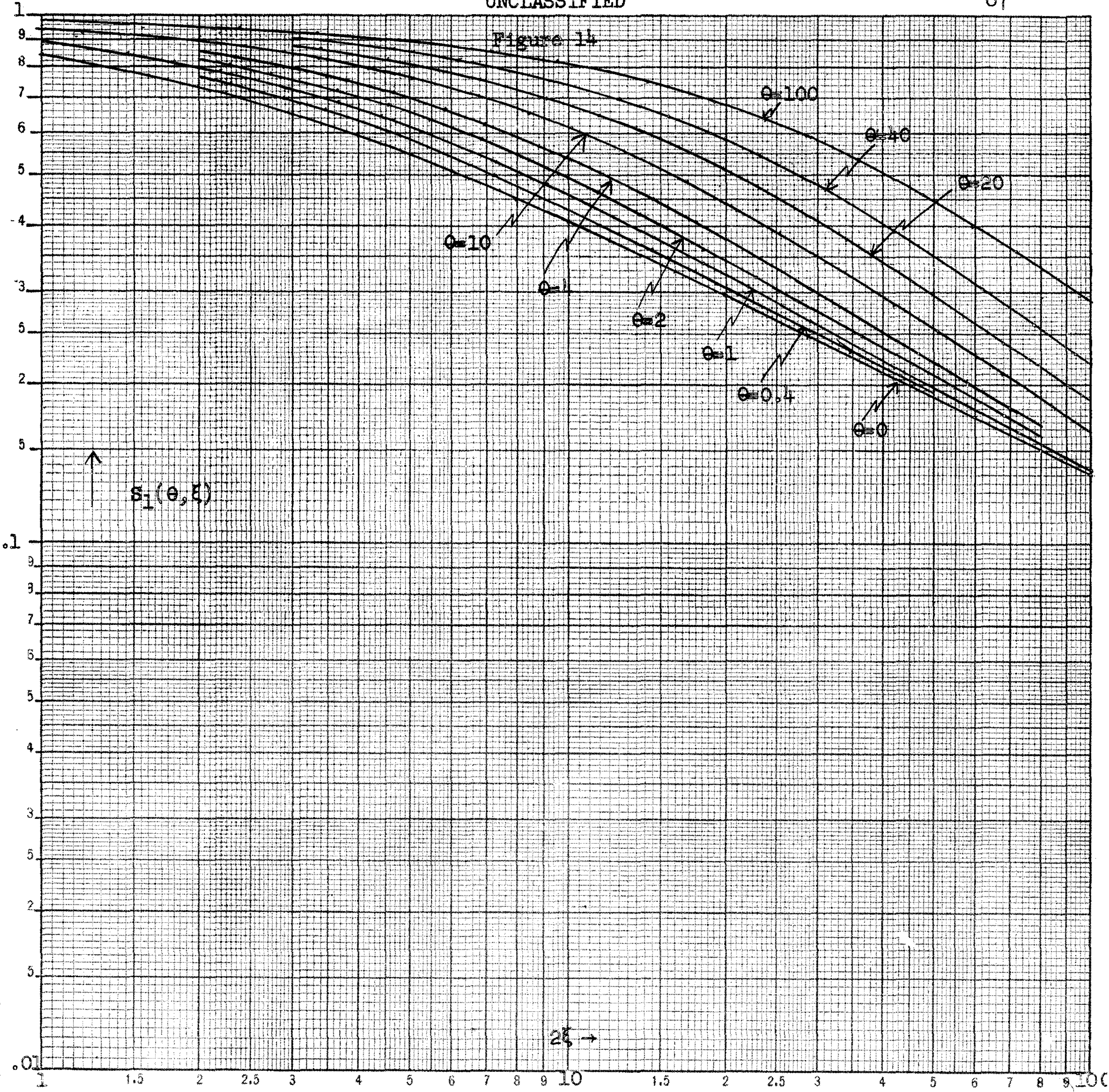


SELF-SHIELDING FACTORS FOR SPHERES



SELF-SHIELDING FACTORS FOR SPHERES

Figure 14



SELF-SHIELDING FACTORS FOR SPHERES

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where θ_1 is the angle between PQ and the normal to the surface and ϕ_1 is the angle between the projections of PQ and the cylinder axis on a plane tangent to the cylinder at P. The angular distribution for the neutrons is $d\phi_1 \cos\theta_1 d(\cos\theta_1)$, so the probability that a neutron striking the surface will be absorbed is proportional to

$$\int_0^{\pi/2} d\phi_1 \int_0^1 \cos\theta_1 d(\cos\theta_1) \left\{ 1 - e^{-(PQ)N_a\sigma_o\psi} \right\}$$

If we change to new angles A and B defined by

$$\cos B = \sqrt{\cos^2\theta_1 + \sin^2\theta_1 \sin^2\phi_1}$$

$$\cos A = \frac{\cos\theta_1}{\cos B}$$

the above probability becomes

$$\int_0^{\pi/2} dA \int_0^{\pi/2} dB \cos A \cos^2 B \left\{ 1 - e^{-2aN_a\sigma_o\psi \frac{\cos A}{\cos B}} \right\}$$

As before, we assume that the spectrum of incident neutrons is flat in the neighborhood of the resonance, average the probability over the resonance and normalize to unity for the limiting case $\sigma_o \to 0$. If

$$\rho = N_a a\sigma_o \tag{85}$$

the self-shielding factor $C(\theta,\rho)$ for cylinders is

$$C(\theta,\rho) = \frac{2}{\pi^2\rho} \int_{-\infty}^{\infty} dx \int_0^{\pi/2} dA \int_0^{\pi/2} dB \cos A \cos^2 B \left\{ 1 - e^{-2\rho\psi(\theta,x) \frac{\cos A}{\cos B}} \right\} \tag{86}$$

or

$$C(\theta,\rho) = \frac{4}{\pi} \int_0^{\pi/2} dA \int_0^{\pi/2} dB \cos^2 A \cos B \, G\left(\theta, \frac{2\rho\cos A}{\cos B}\right) \tag{87}$$

-4 Y

By changing variables to

$$W = \frac{\cos A}{\cos B} \quad V = \cos^2 A + \cos^2 B$$

the integration over V can be carried out.

$$C(\theta,\rho) = \frac{8}{3\pi} \int_0^\infty G(\theta, 2\rho w)\, dwY(w) \tag{88}$$

$$Y(w) = \frac{1}{w^2} \left\{ (1 + \frac{1}{2} w^2)K(w^2) - (1 + w^2)E(w^2) \right\} \quad \text{if } w \leqq 1 \tag{89}$$

$$Y(w) = w \left\{ \left(1 + \frac{1}{2w^2}\right) K\left(\frac{1}{w^2}\right) - \left(1 + \frac{1}{w^2}\right) E\left(\frac{1}{w^2}\right) \right\} \quad \text{if } w \geqq 1 \tag{90}$$

K and E are the complete elliptic integrals.

$$K(w^2) = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}; 1; w^2) \tag{91}$$

$$E(w^2) = \frac{\pi}{2} F(-\frac{1}{2}, \frac{1}{2}; 1; w^2) \tag{92}$$

With considerable labor $C(\theta, \rho)$ could be computed by inserting the approximation $G \cong G_1$ in (88) and integrating numerically. This would require four numerical integrations for each choice of ρ. These integrations can be avoided by using a short-cut which turns out to be extremely accurate.

For any cylinder there exists a sphere which has the same self-shielding factor as the cylinder. We define the radius of such a sphere by the relation

$$C(\theta,\rho) = S(\theta,\xi_c) \tag{93}$$

$$\xi_c \equiv \xi_c(\theta,\rho)$$

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Our object is to construct curves for the function ξ_c. We can then use the curves already obtained for S to compute the values of $C(\theta,\rho)$. Consider just the limiting forms of C and S. For ξ and ρ large

$$S(\theta,\xi) \rightarrow \frac{6}{5}\sqrt{\frac{2}{\pi\xi}}$$

$$C(\theta,\rho) \rightarrow \left[\frac{(1/4)!}{(3/4)!}\right]^2 \sqrt{\frac{2}{\pi\rho}}$$

while for ξ and ρ small

$$S(\theta,\xi) \rightarrow 1 - \frac{3}{8}\xi M(\theta)$$

$$C(\theta,\rho) \rightarrow 1 - \frac{2}{3}\rho M(\theta)$$

Hence, for ρ large

$$\xi_c(\theta,\rho) \rightarrow \frac{36}{25}\left[\frac{\frac{3}{4}!}{\frac{1}{4}!}\right]^4 \rho = 1.5221\rho \qquad (94)$$

and for ρ small

$$\xi_c(\theta,\rho) \rightarrow \frac{16}{9}\rho = 1.7778\rho \qquad (95)$$

Note that both of these limits are independent of θ, so long as θ is finite. Equation (86) may be transformed to

$$C(\theta,\rho) = \frac{2}{\pi}\int_{-\infty}^{\infty} dx\, \rho\psi^2 \int_{\rho\psi}^{\infty} \frac{I_1(y)K_1(y)}{y^2}\, dy, \qquad (96)$$

where I_1 and K_1 are Bessel functions of complex argument. For θ large we can use Equation (15) for ψ to obtain

$$C(\theta,\rho) \rightarrow \frac{2}{\sqrt{\pi}}\rho\alpha \int_{-\infty}^{\infty} dz e^{-2z^2} \int_{\rho\alpha e^{-z^2}}^{\infty} \frac{I_1(y)K_1(y)}{y^2}\, dy \quad (97)$$

$$= 1 - \frac{2}{3}\sqrt{2}\rho\alpha + \sum_{n=1}^{\infty} \frac{(n-3/2)!(\rho\alpha)^{2n}}{(-1/2)!(n-1)!n!(n+1)!\sqrt{2n+1}} \left\{ \ln\frac{1}{\rho\alpha} + \frac{1}{2(2n+1)} + \frac{1}{2}\left[\Psi(n-1) + \Psi(n) + \Psi(n+1) - \Psi(n-3/2)\right]\right\}$$

In this equation $\Psi(n)$ is the logarithmic derivative of the factorial function. The series in (97) has been computed and the results compared to the limiting form of S given by Equation (79) in order to find $\xi_c(\theta,\rho)$ as a function of $\rho\alpha(\theta)$ in the limiting case $\theta \to \infty$. This limiting ratio ξ_c/ρ is plotted as a dashed curve in Figure 15. For $\theta = 0$, Equation (96) can be reduced to

$$C(\theta,\rho) = \frac{2}{\pi} \int_0^1 \frac{dr}{r^2} \left\{ \sin^{-1}\sqrt{r} - \sqrt{r(1-r)} \right\} I_1(r\rho)K_1(r\rho)$$

$$+ \rho \int_\rho^\infty \frac{I_1(y)K_1(n)}{y^2}\, dy \qquad (98)$$

The integrals were evaluated numerically and the result compared with $S(0,\xi)$ from (77) to obtain the ratio ξ_c/ρ plotted as the solid curve in Figure 15. The curves lie so close together that the error in reading ξ_c from Figure 15 will be less than 1% for any value of θ.

The recipe for finding $C(\theta,\rho)$ is therefore: From Table 1 find $\alpha(\theta)$ and compute $\rho\alpha$, read ξ_c/ρ from Figure 15, read $S_1(\theta,\xi_c)$ from Figures 11 to 14, and set $C(\theta,\rho) \cong S_1(\theta,\xi_c)$.

SELF-SHIELDING FACTOR FOR AN ARRAY OF SLABS

Consider an infinite array of parallel slabs of alternate thicknesses t and t_m. The slabs of thickness t_m contain only moderator of cross section σ_m and number density N_m. In each of the slabs of thickness t, a fractional area ε is filled with absorber of peak resonance cross section σ_o and number density N_a. The remaining area of these slabs contains moderator. To simplify the analysis we now move all of the moderator into the t_m slabs, leaving voids in a fractional area $(1 - \varepsilon)$ of the absorber slabs. The effective thickness of the moderator slabs must be increased in the ratio

$$1 + (1 - \varepsilon)\frac{t}{t_m}$$

Let*

$$\tau = N_a\sigma_o t \qquad (99)$$

$$\nu = N_m\sigma_m(t_m + (1 - \varepsilon)t) \qquad (100)$$

*Note that τ/ν is the ratio of the spatial average of the peak macroscopic absorption cross section to the spatial average of the moderator macroscopic cross section.

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3 cyc. X 170 1mm DIVISIONS

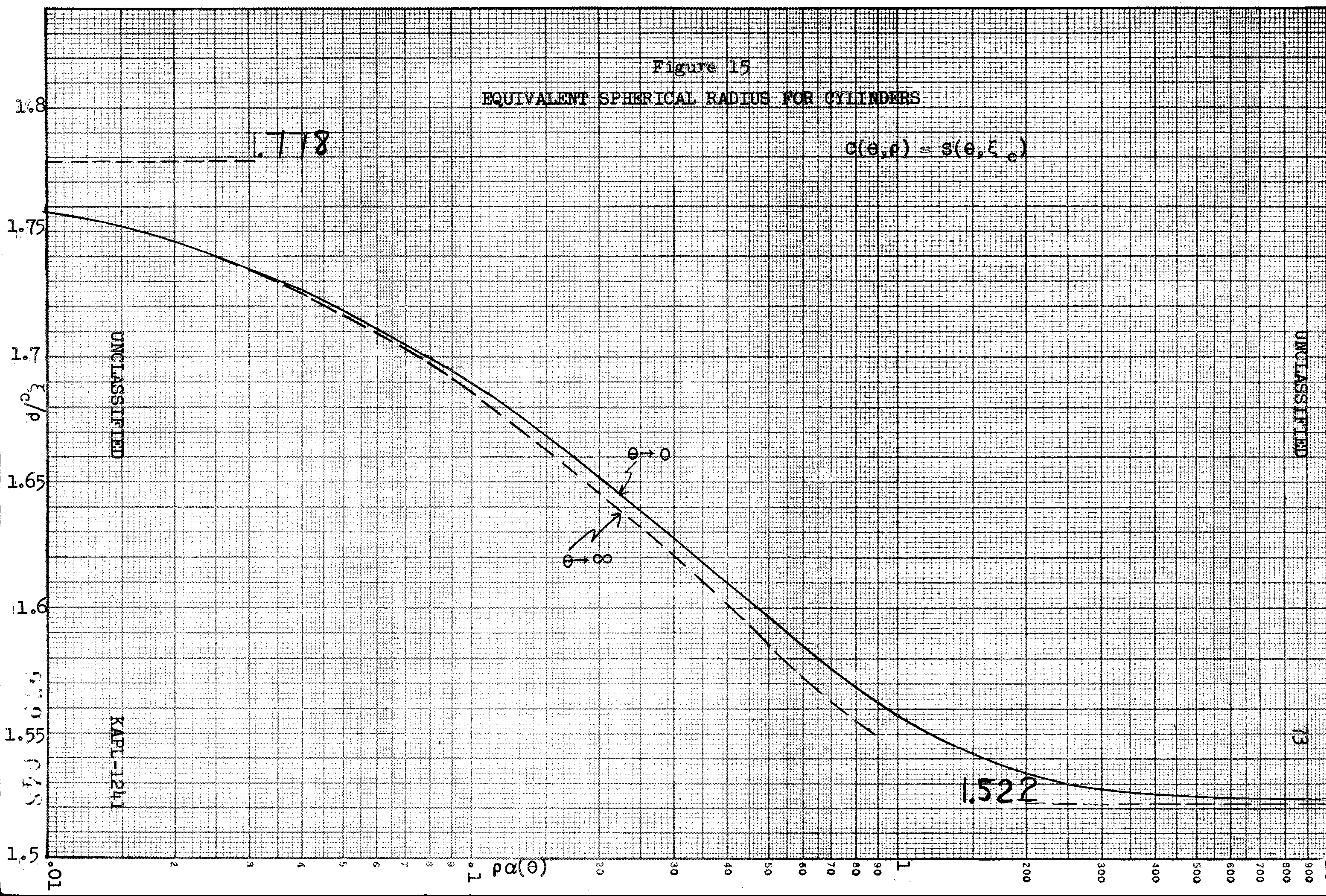

Figure 15

EQUIVALENT SPHERICAL RADIUS FOR CYLINDERS

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The probability that a neutron which has just been scattered by the moderator will be absorbed by the resonance absorber before it is again scattered (or absorbed) by the moderator may be written

$$P = \frac{\varepsilon}{\nu} \int_1^{\infty} \frac{dy}{y^3} \frac{(1 - e^{-y\tau\psi})(1 - e^{-y\nu})}{1 - \varepsilon e^{-y(\tau\psi+\nu)} - (1 - \varepsilon)e^{-y\nu}} \qquad (101)$$

where ψ is the shape factor for the resonance [Equation (1)] and $1/y$ is the cosine of the angle between the path of the neutron and the normal to the slabs. In deriving this probability the following assumptions were made:

1. For any given absorber slab the path of the neutron passes either entirely through absorber or entirely through void. That is, the slab is thin enough so that edge effects can be ignored.

2. The probability that the neutron path passes through absorber in a given slab is ε, independent of whether the path passed through void or through absorber in the preceding slab.

3. The initial points for the path of the neutron are assumed to be distributed uniformly in the moderator.

The probability P has the following limiting values:

For τ small

$$P \to \frac{\varepsilon\tau\psi}{\varepsilon\tau\psi + \nu} - \frac{\varepsilon\tau^2\psi^2}{2\nu} \ln\left(\frac{1}{\gamma\tau\psi}\right) + \ldots \qquad (102)$$

For ν large

$$P \to \frac{\varepsilon}{\nu} \int_1^{\infty} \frac{dy}{y^3} (1 - e^{-y\tau\psi}) \qquad (103)$$

For τ large

$$P \to P_{\infty} = \frac{\varepsilon}{\nu} \int_1^{\infty} \frac{dy}{y^3} \frac{1 - e^{-y\nu}}{1 - (1 - \varepsilon)e^{-y\nu}} \qquad (104)$$

$$P_{\infty} = \frac{\varepsilon}{2\nu} - \frac{\varepsilon^2}{2\nu} \frac{e^{-\nu}}{[1 - (1 - \varepsilon)e^{-\nu}]} + \frac{\varepsilon^2}{2} \frac{e^{-\nu}}{[1 - (1 - \varepsilon)e^{-\nu}]^2} \qquad (105)$$

$$- \frac{\varepsilon^2\nu}{2} \sum_{n=1} n^2(1 - \varepsilon)^{n-1} \int_{n\nu}^{\infty} \frac{e^{-x}dx}{x}$$

In order to find the self-shielding factor the probability (101) must be integrated over the resonance. This integration can be expressed in terms of the functions f and f_H which we have already obtained if we approximate the probability by

$$P \cong P_o$$

$$P_o = q\, A_H \frac{\varepsilon\tau\psi}{\varepsilon\tau\psi + q\nu} + \frac{\varepsilon A_1}{\mu\nu} \int_1^\infty \frac{dy}{y^3}\left(1 - e^{-y\mu\tau\psi}\right) \qquad (106)$$

The parameters A_1, A_H, q, and μ can be chosen so that the three limiting properties, (102), (103), and (104), are satisfied. The desired values are

$$A_1 = \frac{1}{\mu} \qquad A_H = 1 - \frac{1}{\mu}$$
$$\mu = \frac{\varepsilon}{2\nu P_\infty} \qquad q = P_\infty \qquad (107)$$

The self-shielding factor for an array of slabs is defined as (See Equation A-1)

$$f_{Array} = \frac{\int_{-\infty}^{\infty} dxP}{\int_{-\infty}^{\infty} dxP(\tau \to 0)} \qquad (108)$$

The approximation $P \cong P_o$ and the definitions (43) and (50) now lead to*

$$f_{Array} = \frac{1}{\mu} f(\theta,\mu\tau) + \left(1 - \frac{1}{\mu}\right) f_H(\theta,2\mu\tau) \qquad (109)$$

This approximation provides a smooth transition between the limiting cases of isolated slabs and homogeneous mixtures. The factor μ is given in Figure 16 as a function of ε and ν/ε. Although derived for the case of parallel slabs, Equation (109) can be modified to estimate the effect of resonance shielding in other geometries. For example, for a two dimensional array of parallel cylinders

$$f_{Array} \cong \frac{1}{\mu_1} C(\theta, \mu_1 \rho) + \left(1 - \frac{1}{\mu_1}\right) f_H(\theta, 2\mu_1\rho)$$

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*This is an improvement over the older approximation given in Reactor Handbook, p. 668. The geometrical model employed in this section and the idea of approximating f_{Array} as a linear combination of f and f_H, rather than using the function f only, were suggested by Dr. Henry Hurwitz, Jr.

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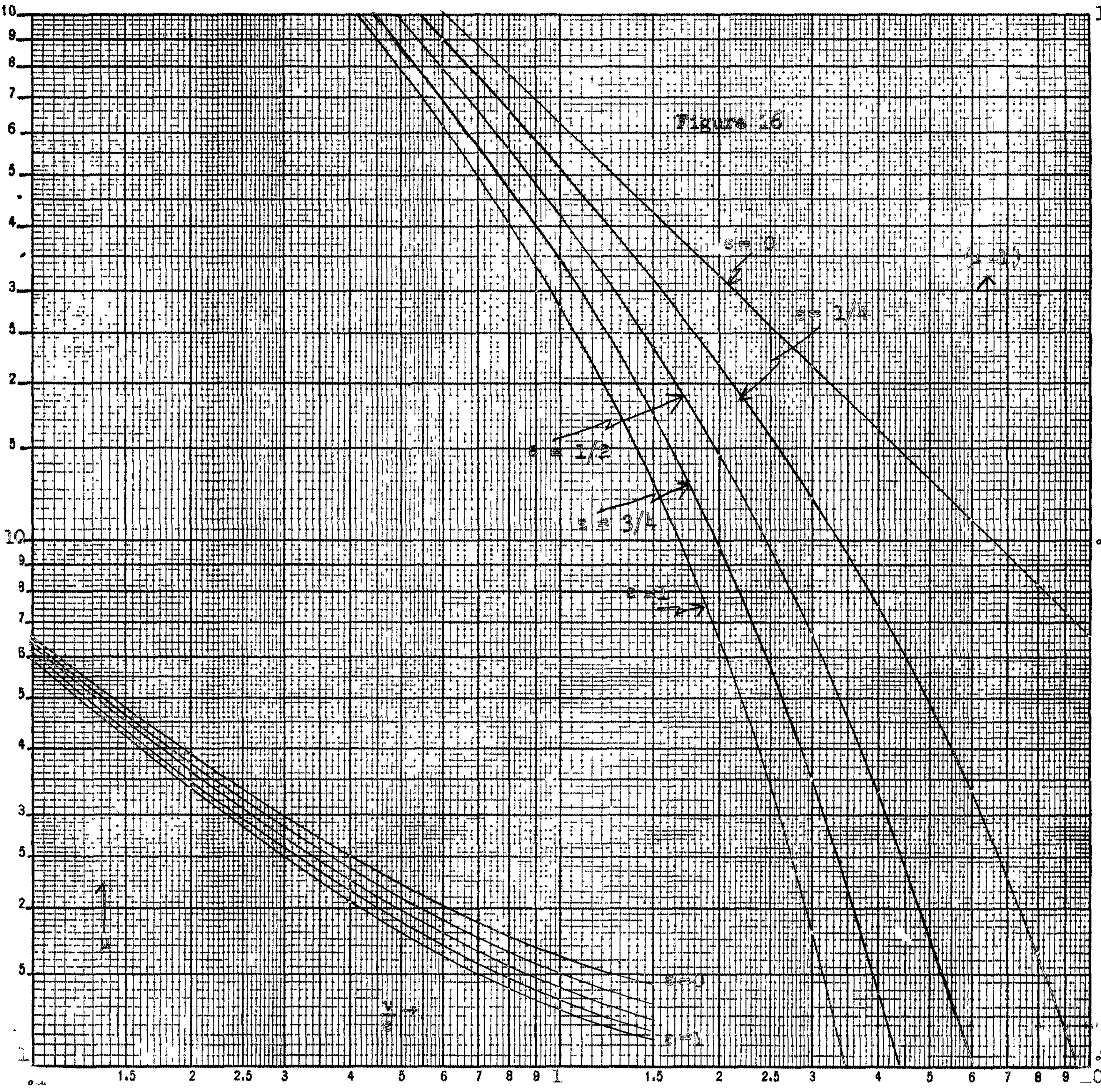
$$\mu_1 = \frac{1}{2KP_B}$$

K is adjusted so that ρ/K gives the correct ratio of the macroscopic absorption and scattering cross sections in the homogeneous limit. P_B is the probability that a neutron will be absorbed before being scattered in the limiting case of perfectly black cylinders, and must be calculated for the particular geometry involved.



EQUIVALENT SINGLE FOIL FACTOR FOR AN ARRAY OF SLABS

APPENDIX*

SELF-SHIELDING OF RESONANCES IN REACTOR CALCULATIONS

A situation frequently encountered in reactor calculations is one in which nuclear resonances have an energy width which is small compared to the average energy loss of the neutron in an elastic scattering collision. Typical capture resonances have widths of the order of magnitude of 0.05 ev, and the average energy loss in a scattering is approximately 2 E/M where E is the initial neutron energy and M is the ratio of scatterer mass to neutron mass. Thus for carbon with M = 12, the energy loss is large compared to capture resonance widths for neutron energies as low as a few ev. Even for the heaviest nuclei, the energy loss in scattering becomes large compared to resonance widths at energies above 10 ev. This situation simplifies the treatment of resonance effects since, if the resonance width is small compared to the average energy loss it can be assumed with almost complete generality that the number of neutrons scattered into the energy range E to E+dE from higher energies per unit time is independent of E over the resonance. (Note that this assumption can be used regardless of whether the resonance spacing is large or small compared to the average energy loss.)**

We now define a function $P[\sigma_a(E)]$ which gives the probability that the neutron will be absorbed*** by the material having the cross-section resonance before it is scattered out of the resonance region by a collision with one of the other materials. (We assume that scattering of a neutron in the resonance region either by the material having the resonance or by other materials will reduce the neutron energy below the resonance region.) In writing $P(\sigma_a)$ as a function only of the total resonance cross section σ_a, we are assuming that the cross sections of the other materials are independent of energy over the region of the resonance. It is clear that under the assumption of uniform spectrum of neutrons scattered into the region of the resonance the number of neutrons undergoing resonance processes is proportional to the integral of $P(\sigma_a(E))$ over the energy of the resonance. If we assume that the proportion of neutron scattering, capture, and fissions in the resonance is independent of energy in the resonance, the number of processes of each kind can be obtained directly from the total number of resonance processes occurring.

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*Prepared by H. Hurwitz, Jr.

**The existence of a monoenergetic neutron source at an energy slightly above the energy under consideration could lead to a violation of this assumption, but narrow absorption or scattering resonances above the energy under consideration would not vitiate the assumption.

***In the sense of this section, absorption may be taken to include all processes taking place in the material having the resonance, that is, scattering as well as radioactive capture and fission.

If ε is a constant over the resonance, the integral of $P[\varepsilon\sigma_r(E)]$ over the resonance will be proportional to ε for small ε, but will increase less rapidly than ε as ε increases to unity. We define the self-shielding factor f of a resonance by the relation

$$f = \frac{\text{res} \int P(\sigma_a(E)dE}{\lim\limits_{\varepsilon \to 0} \frac{1}{\varepsilon} \int P(\varepsilon\sigma_a(E))dE} \tag{A1}$$

The self-shielding factor thus defined will lie in the range between zero and unity. Note that the self-shielding factor will be less than one even for a homogeneous medium, in which case

$$P(\sigma\ (E)) = \frac{\sigma_a(E)}{\sigma_{other} + \sigma_a(E)} \tag{A2}$$

Hence in this case

$$f = \frac{\int \frac{\sigma_a(E)}{\sigma_{other} + \sigma_a(E)} dE}{\int \frac{\sigma_a(E)}{\sigma_{other}} dE} \tag{A3}$$

To apply the self-shielding factor to reactor problems, it is simply necessary to phrase the theory employed in the calculation in a manner which makes direct use of the ratio of the number of neutrons undergoing a resonance process to the number of neutrons scattered into the region of the resonance per unit energy range. This ratio is then written as the self-shielding factor times the elementary linear expression for the ratio which is valid in the limit of small cross sections [that is, the denominator of Equation (A3)]. Since the integral over the resonance of the absorption cross section is independent of the temperature, the entire temperature effect is included in the self-shielding factor. Temperature coefficients of reactivity may thus be calculated by the standard methods of adjoint perturbation theory in terms of the variation of f with temperature as obtained by methods described in the text.

As an illustration we shall discuss the application of the above procedure to the case where there is no spatial dependence in the macroscopic sense. If the scattering material has mass M relative to the neutron, the integral equation governing the slowing-down process has the form

$$\psi(u) = \int_{u-\ln(1/\alpha)}^{u} [1 - P(u')]\psi(u')\, \frac{e^{-(u-u')}}{1 - \alpha}\, du' \tag{A4}$$

$$\alpha = \left(\frac{M - 1}{M + 1}\right)^2$$

Here u is the lethargy variable which is equal to $\ell n(10^7 \text{ ev}/E)$. The quantity $\psi(u)du$ represents the number of neutrons scattered into the lethargy range du, and $(1 - P(u))$ is the probability that a neutron introduced at lethargy u will be scattered before it is absorbed. The number of neutrons, q(u), being slowed down past the lethargy u is given by

$$q(u) = \int_{u-\ell n(1/\alpha)}^{u} \psi(u')[1 - P(u')] \frac{e^{-(u-u')} - \alpha}{1 - \alpha} du' \tag{A5}$$

In accordance with the fundamental approximation of this appendix, the absorption probability P(u) can be replaced by a smoothed function $\overline{P}(u)$ which has the same average value as P(u) over lethargy intervals large compared to the resonance widths but small compared to $\ell n\ 1/\alpha$. Hence for each interval Δu, we must have

$$\overline{P}(u)\Delta u = \sum_{\substack{\text{resonance} \\ \text{in } \Delta u}} f_i \frac{\int_i \sigma_a du}{\sigma_s} \tag{A6}$$

where the index i goes over the resonances in the interval Δu, f_i is the self-shielding factor for the ith resonance as defined by Equation (A1) and $\int_i \sigma_a du$ is the integral of the absorpiton cross section over the region of the ith resonance. (The use of the lethargy variable rather than the energy variable in the resonance integrals does not alter the self-shielding factor since the relation between u and E is essentially linear in the small lethargy range which is considered.) The cross sections σ_a and σ_s are the macroscopic cross sections averaged over the structural inhomogeneities; that is, the average number of atoms per unit volume of each type times their atomic cross sections.

If $\overline{P}(u)$ is either small compared to unity or a slowly varying function of u, Equations (A4) and (A5) can be solved by the BWK approximation.* The solution has the form

$$q(u) = q_0 g(\overline{P}(u)) \exp\left(- \int_{\infty}^{u} \lambda(u')du'\right) \tag{A7}$$

where $\lambda(u)$ is related to $\overline{P}(u)$ by the implicit relationship

$$\frac{1}{1 - \overline{P}} = \int_{0}^{\ell n \frac{1}{\alpha}} \frac{[\exp(\lambda - 1)u]du}{1 - \alpha} = \frac{1 - \alpha^{(1 - \lambda)}}{(1 - \alpha)(1 - \lambda)} = F(\lambda) \tag{A8}$$

and

$$g(\overline{P}) = \frac{1}{\sqrt{\xi}} \frac{F(\lambda) - 1}{\lambda\sqrt{\left[\frac{dF}{d\lambda}\right]_\lambda}}$$

*KAPL-706, "Reactor Physics," Progress Report for November, December 1951, January 1952, p. 15.

with

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$$\xi = \left.\frac{dF}{d\lambda}\right]_0 = 1 - \frac{\alpha}{1 - \alpha} \ln 1/\alpha \qquad \text{(A10)}$$

The factor q_o is the slowing-down density at an energy just above the point at which absorption becomes important, that is, when $\lambda(u)$ becomes different from zero.

In the age diffusion approximation, Equation (A7) is further approximated by setting $\lambda(u)$ equal to $\overline{P}(u)/\xi$ and $g(\overline{P})$ equal to unity, and also, in the simplest form of the theory, setting $\overline{P}(u)$ equal to $\overline{\sigma_a}/\sigma_s$. An improvement on age theory due to Greuling and Goertzel is equivalent to approximating Equation (A8) by the relationship

$$\lambda = \frac{\overline{P}}{\xi - (\xi - \gamma)\overline{P}} \qquad \text{(A11)}$$

where

$$\gamma = \frac{1}{\xi(1 - \alpha)} \left[1 - \alpha(1 + \ln 1/\alpha + \ln^2 1/\alpha) \right] \qquad \text{(A12)}$$

Note that in using Equation (A11), or the more accurate Equation (A8), P(u) must first be averaged over a small energy range before being inserted in the equation for λ. Thus, in using Equation (A11) for a homogeneous mixture .

$$\lambda = \frac{\left\langle \dfrac{\sigma_a}{\sigma_s + \sigma_a} \right\rangle_{AV}}{\xi - (\xi - \gamma) \left\langle \dfrac{\sigma_a}{\sigma_s + \sigma_a} \right\rangle_{AV}} \qquad \text{(A13)}$$

If $\dfrac{\sigma_a}{\sigma_s + \sigma_a}$ is constant, Equation (A13) reduces to

$$\lambda = \frac{\sigma_a}{\xi\sigma_s + \gamma\sigma_a} \qquad \text{(A14)}$$

It should be noted that it is possible to use age-diffusion theory in the simple form in which $\overline{P}$ is in effect replaced by $\overline{\sigma}_a/\sigma_s$ and still obtain the full accuracy of Equation (A8). This is accomplished by multiplying $\overline{\sigma}_a$ as obtained from the cross sections by the average self-shielding factor $\overline{f}$ defined so that

$$\overline{P} = \frac{\overline{f}\ \overline{\sigma}_a}{\sigma_s} \tag{A15}$$

and then multiplying by $\beta(\overline{P})$ which is defined by the requirement

$$\lambda(\overline{P}) = \beta(\overline{P}) \cdot \frac{\overline{P}}{\xi} \tag{A16}$$

In the approximation of Equation (A11),

$$\beta = \frac{1}{1 - \left[1 - \frac{\gamma}{\xi}\right]\overline{P}} \tag{A17}$$

The final recipe therefore is to use the following effective absorption cross section in the age-diffusion equation:

$$\sigma_{a_{eff}} = \beta(\overline{P})\overline{f}\ \overline{\sigma}_a \tag{A18}$$

The factor f corrects for spatial and energy self-shielding effects and the factor β corrects for the error in the age-diffusion approximation when the average absorption is large.