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IDO-16983
February 1965

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SOME METHODS OF CALCULATION OF THE PARAMETERS
IN THE
REACTOR KINETICS EQUATIONS

G. E. Putnam

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IDO-16983
AEC Research and Development Report
Mathematics and Computers
TID-4500 (37th Ed.)
Issued: February 1965

SOME METHODS OF CALCULATION OF THE PARAMETERS
IN THE
REACTOR KINETICS EQUATIONS

BY
G. E. Putnam

PHILLIPS
PETROLEUM
COMPANY

Atomic Energy Division
Contract AT(10-1)-205
Idaho Operations Office
U. S. ATOMIC ENERGY COMMISSION

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ABSTRACT

The spatially independent form of the reactor kinetics equations is written in terms of several time dependent parameters; namely, the reactivity $\rho(t)$, the delayed neutron fraction $\bar{\beta}(t)$, the prompt neutron lifetime $\Lambda(t)$, the concentration of the i^{th} species of delayed neutron precursor $c_i(t)$, and the source $Q(t)$. The formulae for the effective values of these parameters are derived in a manner which somewhat parallels that of A. F. Henry in the "The Application of Reactor Kinetics to the Analysis of Experiments," Nuclear Science and Engineering: 3, 52-70 (1958).

Ways of computing the values of the parameters according to these formulae are described in detail with the emphasis given entirely to methods which utilize practically any of the commonly available reactor computer codes without recourse to the calculation of neutron importance functions. Methods are outlined which lead to an indirect evaluation of the commonly encountered integrals of functions weighted by the importance function; that is, the methods involve no integration.

In the appendix, the arbitrariness of the definitions of some of the parameters is discussed and several different but equally valid sets of definitions are given by way of example. The criterion for the consistency of a set of definitions is given.

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SECTION 1

INTRODUCTION

The difficulty and expensiveness of solving the space, energy, and time dependent nuclear reactor kinetics equations make it highly desirable to reduce the problem, in some appropriate manner, to the solution of a space and energy independent set of equations. The reduced set of equations, which depends only on time, is so widely used in the nuclear reactor field for both the theoretical prediction of reactor behavior and the analysis of experiments that a proper definition of the time dependent parameters in the equations is of considerable importance.

However, a considerable problem remains in that, given a proper definition of the parameters, the reactor engineer must find a practical means of computing the parameters in a manner consistent with the definitions.

This report is devoted to the latter problem of finding the practical methods of calculation after the definitions of the parameters have been established.

The primary purpose is to enable the reactor engineer to compute the effective values of the reactor kinetics equation parameters to a degree of accuracy consistent with the degree of sophistication of the reactor computer codes at his disposal. Almost as importantly, the methods outlined permit the engineer to use the less sophisticated (and far cheaper) computer codes for his calculations when conditions do not require greater accuracy.

Finally there is a considerable amount of confusion which commonly but understandably arises from the complete arbitrariness of the definition of any one of the parameters $\rho(t)$, $\bar{\beta}_1(t)$, and $\Lambda(t)$ in the kinetics equations. The main problem here becomes one of defining the other parameters in a manner consistent with the definition of the first. To meet this problem, the criterion for a consistent set is defined and several examples of commonly used consistent sets are given in the appendix.

SECTION 2

DEFINITION OF THE PROBLEM

The fundamental problem here is to find and define, in considerable detail, practical methods of calculation for the effective values of the parameters which appear in the space independent form of the reactor kinetics equations.

The importance of this problem derives from the fact that it is encountered as one of a sequence of problems which arises when one chooses a particular approach to the solution of the general reactor kinetics problem. This general kinetics problem is to describe the time and space dependent behavior (usually in terms of neutron flux or fission rate) of a nuclear reactor following the introduction of a time dependent perturbation of the physical properties of the reactor, given some well-defined initial condition (at time $t = 0$) of the reactor.

One approach to the solution of this physical problem is as follows:

1) By some appropriate means, reduce the physical problem from one of finding the flux as a function of both space and time to one of finding the flux as the product of a space dependent function (possibly at a number of time points) and a time dependent function. The equations which determine this time dependent function are called the "reactor kinetics equations".

2) Find some suitable (usually approximate) means of calculating the space dependent function at the desired time points.

3) Find some suitable means of calculating the time dependent parameters which determine the coefficients of the reactor kinetics equations. (The definitions of these parameters are determined, of course, by the means chosen to reduce the equations of the general kinetics problem to the reactor kinetics equations.)

4) Find a suitable means of obtaining the solution to the reactor kinetics equations.

5) Find the value of the neutron flux in space and time by multiplying the space dependent function from 2) by the time dependent function obtained in 4).

As stated earlier, the problem in the sequence 1) through 5) with which this report is primarily concerned is the problem stated in 3). However, the exact definitions of the parameters and even of the time dependent functions in the reactor kinetics equations depend on the assumptions and derivations of problem 1). For this reason, these assumptions and derivations are reviewed in the next section.

The problem stated in 2) is then discussed and a method for finding an approximate space dependent function is found which is suitable for use in a large fraction of the commonly encountered reactor kinetics problems. The choice of the space dependent function will be shown to be highly expedient not only because it simplifies the task of computing the space dependent function, but mainly because it forms the basis for all of the methods which are employed in this report to compute the parameters of the kinetics equations (the equations for the time dependent function).

SECTION 3

DERIVATION OF THE REACTOR KINETICS EQUATIONS

3.1 The Basic Balance Equation for the Time Dependent Reactor

If a time dependent perturbation of the physical properties of a nuclear reactor is begun at some instant, say, time $t = 0$; then the directional neutron flux density $\mathcal{F}(\bar{r}, \bar{\Omega}, u, t)$ at each point in the reactor will vary with time according to the following equations.*

$$\begin{aligned}
 (1) \quad & -\bar{\Omega} \cdot \nabla \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) - \Sigma(\bar{r}, u, t) \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) \\
 & + \int_{\bar{\Omega}'} \int_{u'} du' d\Omega' \left[\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \right. \\
 & + \frac{1}{4\pi\lambda_c} \sum_j f^j(u) (1 - \beta^j) v^j(u') \Sigma_f^j(\bar{r}, u', t) \left. \right] \mathcal{F}(\bar{r}, \bar{\Omega}', u', t) \\
 & + s(\bar{r}, \bar{\Omega}, u, t) + \frac{1}{4\pi} \sum_i \lambda_i c_i(\bar{r}, t) f_i(u) = \frac{\partial}{\partial t} \left[\frac{\mathcal{F}(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \right]; \\
 (2) \quad & \frac{\partial c_i(\bar{r}, t)}{\partial t} = \int_{\bar{\Omega}'} \int_{u'} du' d\Omega' \frac{1}{\lambda_c} \sum_j \beta_1^j v^j(u') \Sigma_f^j(\bar{r}, u', t) \mathcal{F}(\bar{r}, \bar{\Omega}', u', t) \\
 & - \lambda_i c_i(\bar{r}, t),
 \end{aligned}$$

where i denotes the i^{th} delayed neutron precursor. (Usually, $1 \leq i \leq 6$.)

If the boundary conditions which exist at the outer surface of the reactor and the initial conditions at time $t = 0$ are both well-defined, then it can be said that these conditions together with Eqs. (1) and (2) comprise the complete definition of the basic physical problem. That is, $\mathcal{F}(\bar{r}, \bar{\Omega}, u, t)$ can be determined by solving Eqs. (1) and (2) provided the boundary conditions and initial conditions are definitely stated.

* The equations, terminology, and definitions of sub-section 3.1 are taken almost directly from A. F. Henry, Nuclear Science and Engineering, Vol. 3, No. 1, Jan. 1958.

In these equations $\mathcal{F}(\bar{r}, \bar{\Omega}, u, t)$ is the directional neutron flux density at point $\bar{r}$, time t , lethargy u , and direction $\bar{\Omega}$. The units are neutrons/(cm²·steradian·lethargy·sec).

$\Sigma(\bar{r}, u, t)$ is the total macroscopic collision cross section.

$\Sigma_s(\bar{r}, u', u, \bar{\Omega}', \bar{\Omega}, t)$ is the total macroscopic scattering cross section for scattering from lethargy u' and direction $\bar{\Omega}'$ to u and $\bar{\Omega}$. The term

$$(3) \int_{\bar{\Omega}'} \int_{u'} \frac{1}{\lambda_c} \sum_j f^j(u) (1 - \beta^j) v^j(u') \Sigma_f^j(\bar{r}, u', t) \mathcal{F}(\bar{r}, \bar{\Omega}', u', t) du' d\bar{\Omega}'$$

is the fission neutron source of promptly emitted neutrons [neutrons/(cm³·lethargy·sec)], and λ_c is a correction factor which is introduced for practical reasons defined in 3.2. Superscript j distinguishes the various fissionable isotopes. Thus $v^j(u')$ is the total number of neutrons emitted from a fission of the j^{th} isotope when the lethargy of the neutron causing the fission is u' . These emitted neutrons have a spectrum of $f^j(u)$ (neutrons per unit lethargy) and $\int_0^\infty f^j(u') du' = 1$. $\Sigma_f^j(\bar{r}, u', t) (\text{cm}^{-1})$ is the macroscopic fission cross section.

$s(\bar{r}, \bar{\Omega}, u, t)$ is the fixed volume source [neutrons/(cm³·sec·lethargy·steradian)].

$C_i(\bar{r}, t) (\text{emitters/cm}^3)$ and $\lambda_i (\text{sec}^{-1})$ represent the concentration and decay constant of the i^{th} group of delayed neutron emitters.

$v(u)$ is the neutron speed (cm/sec) corresponding to lethargy u .

$$(4) \int_{\bar{\Omega}'} \int_{u'} \frac{1}{\lambda_c} \sum_j v^j(u') \beta_i^j \Sigma_f^j(\bar{r}, u', t) \mathcal{F}(\bar{r}, \bar{\Omega}', u', t) du' d\bar{\Omega}'$$

is the source [emitters produced/(cm³·sec)] for the i^{th} group of delayed neutron emitters. The fraction of neutrons emitted as delayed neutrons from the i^{th} emitter group due to a fission in element j is β_i^j and $\beta^j = \sum_i \beta_i^j$; $f_i(u)$ is the spectrum of the neutrons emitted by the i^{th} emitter group and $\int_0^\infty f_i(u) du = 1$.

Equation (1) is the mathematical statement of the fact that the number of neutrons (in an infinitesimal volume about $\bar{r}$ and in the lethargy interval about u and moving within the solid angle $d\bar{\Omega}$ about the direction $\bar{\Omega}$) increases at a rate equal to the rate of loss (due to leakage, scattering, or absorption) subtracted from the rate of gain (due to

scattering, fissions, fixed volume sources, and decay of delayed neutron emitters).

3.2 The Definition of λ_c

If one were able to find the exact solution to Eqs. (1) and (2), given the exact boundary conditions, initial conditions, and the exact values of all the cross sections and other parameters; then there would be no need to introduce λ_c into the equations. However, one does not have accurate knowledge of the cross sections, parameters, initial conditions, etc., and regardless of this fact it is a practical impossibility to obtain the exact solution to Eqs. (1) and (2) for even the simplest reactors. Various approximations and simplifications are always made to reduce the general equations to a form which can be solved with a reasonable expenditure of time and effort.

It is necessary to account for all of these errors in some manner if one is to obtain predictions of reactor behavior from Eqs. (1) and (2) which bear any relevance at all to the behavior of the given reactor. It is obvious on examination that a small error in $\Sigma_f^j(\bar{r}, u, t)$ or $\Sigma(\bar{r}, u, t)$, for example, could result in an Eq. (1) which would predict an increasing flux level when, in fact, the actual reactor flux could be decreasing, and vice versa.

The customary solution to this difficulty is to find the largest eigenvalue λ_c which satisfies the steady-state equation

$$(5) \quad -\bar{\Omega} \cdot \nabla \psi_0(\bar{r}, \bar{\Omega}, u) - \Sigma_0(\bar{r}, u) \psi_0(\bar{r}, \bar{\Omega}, u) + \int_{\Omega'} \int_{u'} du' d\Omega' \left[\Sigma_{s0}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) + \frac{1}{4\pi\lambda_c} \sum_j f_t^j(u) v^j(u') \Sigma_{f0}^j(\bar{r}, u') \right] \psi_0(\bar{r}, \bar{\Omega}', u') = 0,$$

where

$$(6) \quad f_t^j(u) \equiv f^j(u)(1 - \beta^j) + \sum_i f_i(u) \beta_i^j.$$

This equation represents some critical state of the given reactor and the solution is obtained by using exactly the same approximations, methods of obtaining cross sections and other constants, and procedures that will be used to solve Eqs. (1) and (2).

Therefore, λ_c is adjusted such that whatever methods are used to predict the reactor behavior (according to Eqs. (1) and (2)) will also predict a critical or steady-state condition for the given reactor. It follows that if the reactor state described by Eqs. (1) and (2) is only slightly perturbed from the state represented by Eq. (5) and if the perturbations or small changes themselves are reasonably accurate then there is some assurance that the behavior of the fluxes will be reasonably well-predicted.

In summary, this means that before one attempts to solve Eqs. (1) and (2), one should first find λ_c , the eigenvalue at some critical state of the given reactor, by using the same set of computer codes and other methods for obtaining all cross sections and other reactor constants as is used for obtaining the solution to Eqs. (1) and (2). (As stated earlier, the method of solution of Eqs. (1) and (2) with which this report is concerned involves the definition of $\phi(\bar{r}, \bar{\Omega}, u, t)$ as the product of a space dependent function and a time dependent function. It is therefore only necessary to use whatever codes and methods to compute λ_c as are used to compute the space dependent function for use in forming $\phi(\bar{r}, \bar{\Omega}, u, t)$.)

3.3 The Choice of the Weighting Function ψ^*

In order to reduce the general set of Eqs. (1) and (2) to a set of equations in which only time remains as an independent variable, the obvious expedient is to multiply the set of equations by some appropriately chosen weighting function and then integrate with respect to each space, lethargy, and direction variable over the complete range of each variable. This procedure, followed by some further manipulation, leads to a set of equations which has the form of the usual set of reactor kinetics equations which is independent of space, lethargy, and direction.

In general, the weighting function used in this integration over space, lethargy, and direction can be a function of $\bar{r}, \bar{\Omega}, u$, and t ; hence, the weighting function is defined $\psi^*(\bar{r}, \bar{\Omega}, u, t)$.

Now if Eq. (1) is multiplied by $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ and Eqs. (2) are multiplied by $\psi^*(\bar{r}, \bar{\Omega}, u, t) f_1(u)$ and then the equations are integrated with respect to V (volume), Ω , and u , the result is

$$\begin{aligned}
(7) \quad & - \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \nabla \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) d\Omega du dv \\
& - \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) \Sigma(\bar{r}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) d\Omega du dV \\
& + \int_V \int_u \int_{\Omega} \int_{\Omega'} \int_{u'} du' d\Omega' \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi^*(\bar{r}, \bar{\Omega}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}', u', t) d\Omega du dV \\
& + \frac{1}{\lambda_c} \int_V \int_u \int_{\Omega'} \int_{u'} du' d\Omega' \sum_j f_i^j(u) (1 - \beta^j) \nu^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}', u', t) du dV \\
& + \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) s(\bar{r}, \bar{\Omega}, u, t) d\Omega du dV \\
& + \int_V \int_u \sum_i \lambda_i f_i(u) \phi^*(\bar{r}, u, t) C_i(\bar{r}, t) du dV \\
& = \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) \frac{\partial}{\partial t} \left[\frac{\mathcal{H}(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \right] d\Omega du dV.
\end{aligned}$$

$$\begin{aligned}
(8) \quad & \int_V \int_u \phi^*(\bar{r}, u, t) \frac{\partial}{\partial t} \left[f_i(u) C_i(\bar{r}, t) \right] du dV = \\
& + \frac{1}{\lambda_c} \int_V \int_u \int_{\Omega'} \int_{u'} du' d\Omega' \sum_j f_i^j(u) \beta_1^j \nu^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}', u', t) du dV \\
& - \lambda_i \int_V \int_u f_i(u) C_i(\bar{r}, t) \phi^*(\bar{r}, u, t) du dV.
\end{aligned}$$

In Eqs. (7) and (8) $\phi^*(\bar{r}, u, t)$ is, by definition,

$$(9) \quad \phi^*(\bar{r}, u, t) = \frac{1}{4\pi} \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) d\Omega.$$

A vast simplification of Eqs. (7) and (8) can now be effected by some manipulations which can be performed following a judicious choice of the weighting function $\psi^*(\bar{r}, \bar{\Omega}, u, t)$. Although the weighting function is quite arbitrary, it is common practice (and extremely expedient) to choose it as the solution to a source-free adjoint equation corresponding to a well-defined condition of the reactor. For present purposes the

equation which is chosen is that which is the source-free adjoint of the steady-state form of Eq. (1). This equation is

$$(10) \quad \bar{\Omega} \cdot \nabla \psi^*(\bar{r}, \bar{\Omega}', u', t) - \Sigma(\bar{r}, u', t) \psi^*(\bar{r}, \bar{\Omega}', u', t) \\ + \int_{\bar{\Omega}} \int_u \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi^*(\bar{r}, \bar{\Omega}, u, t) d\bar{\Omega} du \\ + \frac{1}{\lambda(t)} \int_u \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) du = 0, \text{ where } \lambda(t) \text{ is the}$$

eigenvalue which allows the stationary solution of Eq. (10) to exist.

A. F. Henry, in the NS&E article, chooses the weighting function to be the solution to the source-free adjoint equation corresponding to "some critical condition in the reactor (generally the initial or final state)". This choice is unfortunate because this critical condition is, for most reactors, a very arbitrary condition in that an infinite set of control rod, reflector, or fuel configurations usually exist for which the reactor is critical. It follows that this choice would lead to the illogical result that all of the parameters in the reactor kinetics equations would be functions of a reactor state which might be only remotely related to the state of the reactor at the particular instant for which the kinetics equation parameters are wanted. In effect, the parameters, at a given instant, would depend not only on the state of the reactor at that instant but also on one's quite arbitrary choice of a "critical condition". This dependency on an arbitrary condition is not only unreasonable but unnecessary.

[Perhaps the most important reason for rejecting this arbitrary weighting function is that its adoption does not permit the expression of kinetic equation parameters as simple functions of easily obtained eigenvalues. (See Section 5.2 and Appendix B)]

3.4 The Development of the Kinetics Equations

Eq. (10) is now multiplied by $\mathcal{H}(\bar{r}, \bar{\Omega}', u', t)$, integrated over all V , $\bar{\Omega}'$, and u' , and then subtracted from Eq. (7). The result is

$$\begin{aligned}
(11) \quad & - \int_V \int_u \int_{\Omega} d\Omega du dV \left[\psi^*(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \nabla \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) \right. \\
& + \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \nabla \psi^*(\bar{r}, \bar{\Omega}, u, t) \Big] \\
& - \int_V \int_u \int_{\Omega} d\Omega du dV \left[\psi^*(\bar{r}, \bar{\Omega}, u, t) \Sigma(\bar{r}, u, t) \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) - \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) \Sigma(\bar{r}, u, t) \psi^*(\bar{r}, \bar{\Omega}, u, t) \right] \\
& + \int_V \int_u \int_{\Omega} \int_{\Omega'} \int_{u'} du' d\Omega' \left[\Sigma_S(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi^*(\bar{r}, \bar{\Omega}, u, t) \mathcal{F}(\bar{r}, \bar{\Omega}', u', t) \right. \\
& - \Sigma_S(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \mathcal{F}(\bar{r}, \bar{\Omega}', u', t) \psi^*(\bar{r}, \bar{\Omega}, u, t) \Big] d\Omega du dV \\
& + \int_V \int_u \int_{\Omega} \int_{u'} du' d\Omega' \left\{ \frac{1}{\lambda_c} \sum_j \left[f_t^j(u) \cdot \sum_i f_1(u) \beta_1^j \right] \nu^j(u') \Sigma_F^j(\bar{r}, u', t) \psi^*(\bar{r}, u, t) \mathcal{F}(\bar{r}, \bar{\Omega}, u', t) \right. \\
& - \frac{1}{\lambda(t)} \sum_j f_t^j(u) \nu^j(u') \Sigma_F^j(\bar{r}, u', t) \mathcal{F}(\bar{r}, \bar{\Omega}, u', t) \psi^*(\bar{r}, u, t) \Big\} du dV \\
& + \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) s(\bar{r}, \bar{\Omega}, u, t) d\Omega du dV \\
& + \int_V \int_u \sum_i \lambda_i f_1(u) \psi^*(\bar{r}, u, t) C_1(\bar{r}, t) du dV \\
& = \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) \frac{\partial}{\partial t} \left[\frac{\mathcal{F}(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \right] d\Omega du dV,
\end{aligned}$$

where $f_t^j(u)(1-\beta_1^j)$ in Eq. (7) has been replaced by its equivalent

$$f_t^j(u) - \sum_i f_1(u) \beta_1^j, \text{ according to definition (6).}$$

The first integral in Eq. (11) can be eliminated in the following manner. One can make use of a well known identity to write

$$(12) \quad \bar{\Omega} \cdot \nabla \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) = \text{div} \left[\bar{\Omega} \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) \right] - \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) \text{div} \bar{\Omega}$$

One now uses the fact that $\text{div} \bar{\Omega} = 0$ and, in addition, Green's theorem is used to write

$$\begin{aligned}
(13) \int_V \psi^*(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \nabla \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) dV &= \int_V \psi^*(\bar{r}, \bar{\Omega}, u, t) \operatorname{div} [\bar{\Omega} \mathcal{H}(\bar{r}, \bar{\Omega}, u, t)] dV \\
&= - \int_V \nabla \psi^*(\bar{r}, \bar{\Omega}, u, t) \cdot \bar{\Omega} \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) dV + \int_S \psi^*(\bar{r}, \bar{\Omega}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \bar{n} dS
\end{aligned}$$

where S is the surface of the reactor and $\bar{n}$ is the outwardly directed unit vector normal to the surface element dS . If (13) is used in the first term of Eq. (11), the result is

$$\begin{aligned}
(14) & - \int_V \int_u \int_{\Omega} d\Omega du dV \left[- \nabla \psi^*(\bar{r}, \bar{\Omega}, u, t) \cdot \bar{\Omega} \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) \right. \\
& \left. + \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \nabla \psi^*(\bar{r}, \bar{\Omega}, u, t) \right] - \int_u \int_{\Omega} \int_S \psi^*(\bar{r}, \bar{\Omega}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \bar{n} dS d\Omega du \\
& = - \int_u \int_{\Omega} \int_S \psi^*(\bar{r}, \bar{\Omega}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) \bar{\Omega} \cdot \bar{n} dS d\Omega du.
\end{aligned}$$

One can now assume, as usual, that the boundary conditions on $\mathcal{H}(\bar{r}, \bar{\Omega}, u, t)$ state that

(15) $\mathcal{H}(\bar{r}, \bar{\Omega}, u, t) = 0$ for inwardly directed $\bar{\Omega}$ and that the boundary conditions on $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ state that

(16) $\psi^*(\bar{r}, \bar{\Omega}, u, t) = 0$ for outwardly directed $\bar{\Omega}$; hence $\psi^*(\bar{r}, \bar{\Omega}, u, t) \mathcal{H}(\bar{r}, \bar{\Omega}, u, t) = 0$ for all $\bar{\Omega}$ and the surface integral in (14) is therefore zero.

The second and third integrals in Eq. (11) are obviously zero so

that Eq. (11) is reduced to

$$\begin{aligned}
(17) \quad & \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) \frac{\partial}{\partial t} \left[\frac{\mathcal{F}(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \right] d\Omega du dV \\
& = \left[\frac{1}{\lambda_c} - \frac{1}{\lambda(t)} \right] \int_V \int_u \int_{\Omega} \int_{u'} du' d\Omega \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \mathcal{F}(\bar{r}, \bar{\Omega}, u', t) du dV \\
& - \frac{1}{\lambda_c} \int_V \int_u \int_{\Omega} \int_{u'} du' d\Omega \sum_j \sum_i f_i(u) \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \mathcal{F}(\bar{r}, \bar{\Omega}, u', t) du dV \\
& + \int_V \int_u \phi^*(\bar{r}, u, t) S(\bar{r}, u, t) du dV \\
& + \int_V \int_u \sum_i \lambda_i f_i(u) \phi^*(\bar{r}, u, t) C_i(\bar{r}, t) du dV,
\end{aligned}$$

where it has been assumed that $s(\bar{r}, \bar{\Omega}, u, t)$ is isotropic such that

$$(18) \quad s(\bar{r}, \bar{\Omega}, u, t) = S(\bar{r}, u, t)/4\pi \text{ and}$$

$$(19) \quad S(\bar{r}, u, t) \equiv \int_{\Omega} s(\bar{r}, \bar{\Omega}, u, t) d\Omega, \text{ by definition.}$$

The formalism of A. F. Henry is adopted and extended by defining

$$(20) \quad \mathcal{F}(\bar{r}, \bar{\Omega}, u, t) \equiv \psi'(\bar{r}, \bar{\Omega}, u, t) T(t),$$

$$(21) \quad \phi'(\bar{r}, u, t) = \int_{\Omega} \psi'(\bar{r}, \bar{\Omega}, u, t) d\Omega, \text{ and}$$

$$(22) \quad \lambda_i C_i(\bar{r}, t) f_i(u) \equiv \gamma_i(\bar{r}, u, t) T_i(t), \text{ with the added restrictions that}$$

$$(23) \quad \int_V \int_u \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u, t) \frac{\partial}{\partial t} \left[\frac{\psi'(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \right] d\Omega du dV = 0 \text{ and}$$

$$(24) \quad \int_V \int_u \phi^*(\bar{r}, u, t) \frac{\partial \gamma_i(\bar{r}, u, t)}{\partial t} du dV = 0.$$

The convention is adopted that

$$(25) \quad T(0) = T_i(0) = 1.$$

Eqs. (17) and (8) can now be written

$$\begin{aligned}
 (26) \quad & \frac{\partial T(t)}{\partial t} \int_V \int_u \int_{\Omega} \frac{\psi^*(\bar{r}, \bar{\Omega}, u, t) \psi'(\bar{r}, \bar{\Omega}, u, t)}{v(u)} d\Omega du dV = \\
 & \left[\frac{1}{\lambda_c} - \frac{1}{\lambda(t)} \right] T(t) \int_V \int_u \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \phi'(\bar{r}, u', t) du' du dV \\
 & - \frac{T(t)}{\lambda_c} \sum_i \int_V \int_u \int_{u'} \sum_j f_i(u) \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \phi'(\bar{r}, u', t) du' du dV \\
 & + \int_V \int_u \phi^*(\bar{r}, u, t) S(\bar{r}, u, t) du dV + \sum_i T_i(t) \int_V \int_u \gamma_i(\bar{r}, u, t) \phi^*(\bar{r}, u, t) du dV; \\
 (27) \quad & \frac{\partial T_i(t)}{\partial t} \int_V \int_u \frac{\gamma_i(\bar{r}, u, t)}{\lambda_i} \phi^*(\bar{r}, u, t) du dV =
 \end{aligned}$$

$$\begin{aligned}
 & \frac{T(t)}{\lambda_c} \int_V \int_u \int_{u'} \sum_j f_i(u) \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \phi'(\bar{r}, u', t) du' du dV \\
 & - T_i(t) \int_V \int_u \gamma_i(\bar{r}, u, t) \phi^*(\bar{r}, u, t) du dV.
 \end{aligned}$$

Eqs. (26) and (27) are functions only of the time variable and they are the final reactor kinetics equations. The various integrals in these equations define the kinetics equation parameters and one can put Eqs. (26) and (27) into the usual familiar form by defining the parameters as follows.

$$(28) \quad \Lambda(t) \equiv \frac{1}{U(t)} \int_V \int_u \int_{\Omega} \frac{\psi^*(\bar{r}, \bar{\Omega}, u, t) \psi'(\bar{r}, \bar{\Omega}, u, t)}{v(u)} d\Omega du dV.$$

$$(29) \quad F(t) \equiv \frac{1}{U(t)} \int_V \int_u \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \phi'(\bar{r}, u', t) du' du dV.$$

$$(30) \quad \rho(t) \equiv \left[\frac{1}{\lambda_c} - \frac{1}{\lambda(t)} \right] F(t).$$

$$(31) \quad \bar{\beta}_i(t) \equiv \frac{1}{\lambda_c U(t)} \int_V \int_u \int_{u'} \sum_j f_i(u) \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \phi'(\bar{r}, u', t) du' du dV.$$

$$(32) \bar{\rho}(t) \equiv \sum_i \bar{\rho}_i(t).$$

$$(33) S'(t) \equiv \int_V \int_u \phi^*(\bar{r}, u, t) S(\bar{r}, u, t) du dV, \text{ where } S(\bar{r}, u, t) = \int_{\Omega} s(\bar{r}, \bar{\Omega}, u, t) d\Omega.$$

$$(34) \bar{S}(t) \equiv S'(t)/U(t).$$

$$(35) Q(t) \equiv \bar{S}(t)/\Lambda(t).$$

$$(36) \bar{\gamma}_i(t) \equiv \int_V \int_u \gamma_i(\bar{r}, u, t) \phi^*(\bar{r}, u, t) du dV = \frac{\lambda_i}{T_i(t)} \int_V \int_u f_i(u) c_i(\bar{r}, t) \phi^*(\bar{r}, u, t) du dV.$$

$$(37) \Gamma_i(t) \equiv \bar{\gamma}_i(t)/U(t)\Lambda(t).$$

$$(38) c_i(t) \equiv \Gamma_i(t)T_i(t)/\lambda_i.$$

$U(t)$ is a purely arbitrary function of t which serves only as a normalizing factor. Since $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ is the solution of a homogeneous equation, the magnitude of the function contains an arbitrary constant as a factor. The primary purpose of $U(t)$ is to remove this arbitrary factor from the definitions of the kinetics equation parameters by choosing $U(t)$ to be proportional to the magnitude of $\psi^*(\bar{r}, \bar{\Omega}, u, t)$.

Eqs. (26) and (27) can now be written

$$(39) \frac{\partial T(t)}{\partial t} = \frac{\rho(t) - \bar{\rho}(t)}{\Lambda(t)} T(t) + \sum_i \lambda_i c_i(t) + Q(t), \text{ and (for } \frac{\partial \psi^*(\bar{r}, \bar{\Omega}, u, t)}{\partial t} = 0)$$

$$(40) \frac{\partial c_i(t)}{\partial t} = \frac{\bar{\beta}_i(t)T(t)}{\Lambda(t)} - \lambda_i c_i(t),$$

which is the usual form of the space independent reactor kinetics equations.

In terms of $T_i(t)$, the kinetics equations are

$$(41) \frac{\partial T(t)}{\partial t} = \frac{\rho(t) - \bar{\rho}(t)}{\Lambda(t)} T(t) + \sum_i \Gamma_i(t)T_i + Q(t), \text{ and}$$

$$(42) \quad \frac{\partial T_i(t)}{\partial t} = \frac{\bar{\beta}_i(t)\lambda_i}{\Lambda(t)\Gamma_i(t)} T(t) - \lambda_i T_i(t).$$

The normalizing function $U(t)$ is usually chosen to be

$$(43) \quad U(t) = \left[\frac{1}{\lambda_c^m \lambda^n(t)} \right] \int_V \int_u \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi^*(\bar{r}, u, t) \phi'(\bar{r}, u', t) du' du dV$$

so that

$$(44) \quad F(t) = \lambda_c^m \lambda^n(t)$$

and

$$(45) \quad \rho(t) = \frac{[\lambda(t) - \lambda_c]}{\lambda_c \lambda(t)} \lambda_c^m \lambda^n(t),$$

where m and n are integers or zero.

It is noted that Eq. (40) is less general than Eq. (42) because Eq. (40) is generally true only if $\frac{\partial \psi^*(\bar{r}, \bar{\Omega}, u, t)}{\partial t} = 0$, whereas Eq. (42) is true on the basis of restriction (24). In usual practice this loss of generality is of no consequence because the kinetics equation parameters $[\beta(t), \Lambda(t), \text{etc.}]$ are calculated and used as if $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ were assumed constant for the time period of interest.

Moreover, if $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ is not constant in time and one adopts the form of the kinetics equations (41) and (42) for the sake of the added generality, one then faces the very difficult problem of finding the time dependent $\Gamma_i(t)$. Since it is generally impractical to find the time dependent $\Gamma_i(t)$, the quantity $\Gamma_i(t)$ is usually assumed to be constant. But for this assumption to be valid in general, $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ must be again constant.

For all practical purposes, then, the choice between the form of Eq. (40) versus the form of Eq. (42) rests simply on whether or not one wishes to add the definition (22) and the restriction (24) in order to obtain the somewhat simpler form of Eq. (42) for the sake of the calculation. [Eq. (42) reduces to $\frac{\partial T_i(t)}{\partial t} = \lambda_i [T(t) - T_i(t)]$ in many problems. See subsection 6.4.]

The assumption that $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ is constant at the value $\psi^*(\bar{r}, \bar{\Omega}, u, 0)$ does not modify the definitions of any of the parameters. It should nevertheless be emphasized that in the definition of $\rho(t)$ the value of $\lambda(t)$ must be the eigenvalue of the source-free, steady-state form of Eq. (1) for time equal to t . That is, $\lambda(t)$ is not the solution of Eq. (10) at $t = 0$, but it is the solution of the equation

$$-\bar{\Omega} \cdot \nabla \psi(\bar{r}, \bar{\Omega}, u, t) - \Sigma(\bar{r}, u, t) \psi(\bar{r}, \bar{\Omega}, u, t) + \int_{\bar{\Omega}'} \int_{u'} du' d\Omega' [\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) + \frac{1}{\lambda(t)} \frac{1}{4\pi} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u', t)] \psi(\bar{r}, \bar{\Omega}', u', t) = 0.$$

3.5 Alternate Expressions for $\rho(t)$

$\rho(t)$ can be expressed in other manners. Let the equation for some critical state of the reactor be

$$(46) \quad -\bar{\Omega} \cdot \nabla \psi_o(\bar{r}, \bar{\Omega}, u) - \Sigma_o(\bar{r}, u) \psi_o(\bar{r}, \bar{\Omega}, u) + \int_{\bar{\Omega}'} \int_{u'} \Sigma_{so}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_o(\bar{r}, \bar{\Omega}', u') du' d\Omega' + \frac{1}{\lambda_c} \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_{fo}^j(\bar{r}, u') \phi_o(\bar{r}, u') du = 0, \text{ where}$$

$$(47) \quad \phi_o(\bar{r}, u) = \int_{\bar{\Omega}} \psi_o(\bar{r}, \bar{\Omega}, u) d\bar{\Omega}.$$

If Eq. (46) is multiplied by $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ and subtracted from the product of $\psi_o(\bar{r}, \bar{\Omega}', u')$ and Eq. (10), the result, after integrating over

all V , $\bar{\Omega}$, and u is

$$\begin{aligned}
(48) \quad & - \int_V \int_u \int_{\bar{\Omega}} \left[\Sigma(\bar{r}, u, t) - \Sigma_0(\bar{r}, u) \right] \psi_0(\bar{r}, \bar{\Omega}, u) \psi^*(\bar{r}, \bar{\Omega}, u, t) d\bar{\Omega} du dV \\
& + \int_V \int_u \int_{\bar{\Omega}} \int_{\bar{\Omega}'} \int_{u'} du' d\bar{\Omega}' \left[\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \right. \\
& \left. - \Sigma_{s0}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \right] \psi_0(\bar{r}, \bar{\Omega}, u) \psi^*(\bar{r}, \bar{\Omega}', u', t) d\bar{\Omega} du dV \\
& + \int_V \int_u \int_{u'} du' \sum_j f_t^j(u) \left[\frac{v^j(u') \Sigma_f^j(\bar{r}, u', t)}{\lambda(t)} - \frac{v^j(u') \Sigma_{f0}^j(\bar{r}, u')}{\lambda_c} \right] \phi_0(\bar{r}, u) \phi^*(\bar{r}, u', t) du' dV = 0,
\end{aligned}$$

where the usual boundary conditions and Green's Theorem have been applied to eliminate the divergence terms.

After some manipulation it follows from Eq. (48) that

$$\begin{aligned}
(49) \quad \rho(t) &= \frac{[\lambda(t) - \lambda_c] \lambda_c^m \lambda^n(t)}{\lambda_c \lambda(t)} = \frac{-1}{U_1(t)} \int_V \int_u \int_{\bar{\Omega}} \Delta \Sigma(\bar{r}, u, t) \psi_0(\bar{r}, \bar{\Omega}, u) \psi^*(\bar{r}, \bar{\Omega}, u, t) d\bar{\Omega} du dV \\
& + \frac{1}{U_1(t)} \int_V \int_u \int_{\bar{\Omega}} \int_{\bar{\Omega}'} \int_{u'} du' d\bar{\Omega}' \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi_0(\bar{r}, \bar{\Omega}, u) \psi^*(\bar{r}, \bar{\Omega}', u', t) d\bar{\Omega} du dV \\
& + \frac{1}{\lambda_c U_1(t)} \int_V \int_u \int_{u'} du' \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \right] \phi_0(\bar{r}, u) \phi^*(\bar{r}, u', t) du' dV, \text{ where} \\
(50) \quad U_1(t) &= \frac{1}{\lambda_c^m \lambda^n(t)} \int_V \int_u \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi_0(\bar{r}, u) \phi^*(\bar{r}, u', t) du' du dV,
\end{aligned}$$

and

$$(51) \quad \Delta \Sigma(\bar{r}, u, t) \equiv \Sigma(\bar{r}, u, t) - \Sigma_0(\bar{r}, u),$$

$$(52) \quad \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \equiv \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) - \Sigma_{s0}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}), \text{ and}$$

$$(53) \quad \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \right] \equiv \sum_j f_t^j(u) v^j(u') \left[\Sigma_f^j(\bar{r}, u', t) - \Sigma_{f0}^j(\bar{r}, u') \right]$$

Still another expression for $\rho(t)$ can be obtained as follows.

The equation adjoint to Eq. (46) for a critical state of the reactor is

$$\begin{aligned}
 (54) \quad & \bar{\Omega} \cdot \nabla \psi^*(\bar{r}, \bar{\Omega}', u') - \Sigma_O(\bar{r}, u') \psi^*(\bar{r}, \bar{\Omega}', u') \\
 & + \int_{\bar{\Omega}} \int_u \Sigma_{sO}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi^*(\bar{r}, \bar{\Omega}, u) d\bar{\Omega} du \\
 & + \frac{1}{\lambda_c} \int_u \sum_j f_t^j(u) v^j(u') \Sigma_{fO}^j(\bar{r}, u') \phi^*(\bar{r}, u) du = 0, \text{ where}
 \end{aligned}$$

$$(55) \quad \phi_O^*(\bar{r}, u) \equiv \frac{1}{4\pi} \int_{\bar{\Omega}} \psi^*(\bar{r}, \bar{\Omega}, u) d\bar{\Omega}.$$

Also, the equation for the source-free steady-state form of Eq. (1) (for which Eq. (10) is the adjoint) is

$$\begin{aligned}
 (56) \quad & -\bar{\Omega} \cdot \nabla \psi(\bar{r}, \bar{\Omega}, u, t) - \Sigma(\bar{r}, u, t) \psi(\bar{r}, \bar{\Omega}, u, t) \\
 & + \int_{\bar{\Omega}'} \int_{u'} \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi(\bar{r}, \bar{\Omega}', u', t) du' d\bar{\Omega}' \\
 & + \frac{1}{4\pi\lambda(t)} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi(\bar{r}, u', t) du' = 0, \text{ where}
 \end{aligned}$$

$$(57) \quad \phi(\bar{r}, u, t) \equiv \int_{\bar{\Omega}} \psi(\bar{r}, \bar{\Omega}, u, t) d\bar{\Omega}.$$

If Eq. (54) is multiplied by $\psi(\bar{r}, \bar{\Omega}', u', t)$ and subtracted from the product of $\psi^*(\bar{r}, \bar{\Omega}, u)$ and Eq. (56), the result is, after integration over all V , $\bar{\Omega}$, and u and after application of the usual boundary conditions and Green's Theorem,

$$\begin{aligned}
(58) \quad & - \int_V \int_u \int_{\Omega} \Delta \Sigma(\bar{r}, u, t) \psi_0^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u, t) d\Omega du dV \\
& + \int_V \int_u \int_{\Omega} \int_{\Omega'} \int_{u'} du' d\Omega' \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi_0^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}', u', t) d\Omega du dV \\
& + \int_V \int_u \int_{u'} du' \sum_j f_t^j(u) \left[\frac{v^j(u') \Sigma_f^j(\bar{r}, u', t)}{\lambda(t)} - \frac{v^j(u') \Sigma_{f0}^j(\bar{r}, u')}{\lambda_c} \right] \phi_0^*(\bar{r}, u) \phi(\bar{r}, u', t) du dV = 0
\end{aligned}$$

It follows from Eq. (58) that

$$\begin{aligned}
(59) \quad \rho(t) &= \frac{[\lambda(t) - \lambda_c] \lambda_c^m \lambda^n(t)}{\lambda_c \lambda(t)} = \frac{-1}{U_2(t)} \int_V \int_u \int_{\Omega} \Delta \Sigma(\bar{r}, u, t) \psi_0^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u, t) d\Omega du dV \\
&+ \frac{1}{U_2(t)} \int_V \int_u \int_{\Omega} \int_{\Omega'} \int_{u'} du' d\Omega' \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi_0^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}', u', t) d\Omega du dV \\
&+ \frac{1}{\lambda_c U_2(t)} \int_V \int_u \int_{u'} du' \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \right] \phi_0^*(\bar{r}, u) \phi(\bar{r}, u', t) du dV, \text{ where} \\
U_2(t) &= \frac{1}{\lambda_c^m \lambda^n(t)} \int_V \int_u \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \phi_0^*(\bar{r}, u) \phi(\bar{r}, u', t) du' du dV.
\end{aligned}$$

Hence, one has three alternative expressions for $\rho(t)$; that is, expressions (45), (49), and (59) are equally rigorous ways of expressing $\rho(t)$.

SECTION 4

THE APPROXIMATION OF $\psi'(\bar{r}, \bar{\Omega}, u, t)$

The reactor kinetics parameters $\Lambda(t)$ and $\bar{\beta}_i(t)$ according to the definitions (28), (31), and (43) are dependent upon the spatial distribution $\psi'(\bar{r}, \bar{\Omega}, u, t)$. However, the general solution for this function requires the solution of Eqs. (1) and (2), which is, of course, the original problem; and the difficulty of the solution of this problem is what one is attempting to avoid when one derives the space independent kinetic equations. Therefore, the only expedient is to compute the kinetic equation parameters with a function which approximates $\psi'(\bar{r}, \bar{\Omega}, u, t)$ but which is fairly easy to obtain.

The kinetic equation parameters are also functions of $\psi^*(\bar{r}, \bar{\Omega}, u, t)$, and this function or its diffusion theory counterpart is an example of one that is "fairly easy" to obtain. It is the solution of a steady-state problem and some reactor computer codes can solve for it in many cases.

$\psi(\bar{r}, \bar{\Omega}, u, t)$, the solution to the source-free, steady-state form of Eq. (1), is most often used to approximate $\psi'(\bar{r}, \bar{\Omega}, u, t)$; this is primarily because it (or its diffusion theory counterpart) is so easy to obtain by using most of the common reactor computer codes available. That is to say the use of $\psi(\bar{r}, \bar{\Omega}, u, t)$ in place of $\psi'(\bar{r}, \bar{\Omega}, u, t)$ is fortunately warranted in a great many practical reactor kinetics problems, but even in many other cases when the validity of the approximation is dubious or worse it is still used as the "best conveniently available" approximation of $\psi'(\bar{r}, \bar{\Omega}, u, t)$.

A. F. Henry, in the NS and E article, presents arguments which justify the use of $\psi(\bar{r}, \bar{\Omega}, u, t)$ in almost all reactor kinetic problems of interest. Briefly, the approximation is justified under the following conditions:

(a) The reactor flux is effectively independent of any fixed sources (as indicated, for example, by a large value of the fission neutron

production rate compared to the source neutron production rate in the fissioning region), and

(b) $\rho(t)$ is constant or changes only slightly in the time required for the flux shape to readjust to a change in the reactor properties. $\Lambda(t)$ is a measure of the neutron lifetime so that if N is the number of prompt neutron lifetimes (or generations) required for the flux shape to readjust then $N\Lambda(t)$ is the time required for readjustment. Hence, if 0.001 is considered a small change in $\rho(t)$, a rate of change $\frac{d\rho(t)}{dt} \leq 0.001/N\Lambda(t)$ can be tolerated. A rough estimate of N for a given reactor can be obtained by observing the number of outer iterations (or fission neutron source iterations) required by an ordinary one-dimensional diffusion theory computer code for convergence of the flux shape after starting with a crude guess for the fission neutron source distribution. In such computer codes, an outer iteration begins with a source of newly-born neutrons and ends after it computes how these neutrons are lost (by absorption or leakage). (The new fission neutron source distribution which corresponds to that fraction of the absorptions which cause fissions can then be found in order to begin another iteration.) Since a neutron generation starts with a given source of neutrons in a given reactor and ends when all these neutrons are lost, an outer iteration corresponds to one neutron generation.

In order to define $\Lambda(t)$, $F(t)$, $\rho(t)$, and $\beta_i(t)$ in terms of $\psi(\bar{r}, \bar{\Omega}, u, t)$, one simply substitutes $\psi(\bar{r}, \bar{\Omega}, u, t)$ in place of $\psi'(\bar{r}, \bar{\Omega}, u, t)$ in the definitions of the quantities (28) through (31). To modify the definitions of $\bar{S}(t)$, $Q(t)$, $\Gamma_i(t)$, and $c_i(t)$, however, a simple substitution of $\psi(\bar{r}, \bar{\Omega}, u, t)$ for $\psi'(\bar{r}, \bar{\Omega}, u, t)$ is not sufficient. This is because $\psi(\bar{r}, \bar{\Omega}, u, t)$, the solution of Eqs. (1) and (2), has a unique value. In view of definitions (20) and (25), $\psi'(\bar{r}, \bar{\Omega}, u, t)$ also has a unique value. However, the value of $\psi(\bar{r}, \bar{\Omega}, u, t)$, the solution of Eq. (56), can be determined only within an arbitrary constant multiplier, C , because Eq. (56) is an homogeneous equation. One can define

$$(60) \quad C \equiv \frac{\psi'(\bar{r}, \bar{\Omega}, u, t)}{\psi(\bar{r}, \bar{\Omega}, u, t)} = \frac{\phi'(\bar{r}, u, t)}{\phi(\bar{r}, u, t)}.$$

If $\psi(\bar{r}, \bar{\Omega}, u, t)$ replaces $\psi'(\bar{r}, \bar{\Omega}, u, t)$, then the constant C is cancelled out because it appears in both the numerator and denominator of the

expressions for $\Lambda(t)$, $F(t)$, $\rho(t)$, and $\beta_i(t)$.

However, the constant C appears as a normalizing factor in the denominators of $\bar{S}(t)$, $Q(t)$, $\Gamma_i(t)$, and $c_i(t)$ because of the fact that $\psi'(\bar{r}, \bar{\Omega}, u, t)$ appears only in the denominators of these expressions.

SECTION 5

METHODS FOR COMPUTING THE KINETIC EQUATION PARAMETERS

In this section, methods are derived for computing the effective values of the kinetic equation parameters (in terms of the approximate distribution, $\psi(\bar{r}, \bar{\Omega}, u, t)$) which enable the use of many of the presently available reactor computer codes.

A few reactor codes exist which are capable of computing the adjoint function $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ or its diffusion theory counterpart, $\phi^*(\bar{r}, u, t)$, usually in multi-group form; and some of these codes provide, as output, the integrals necessary for the computation of the kinetic equation parameters. The methods derived in this section, however, enable one to compute the parameters without the need to compute $\psi^*(\bar{r}, \bar{\Omega}, u, t)$ or perform the integrations directly. This enables the reactor engineer to have a much wider choice of computer codes and it thereby gives him more freedom in choosing both the degree of sophistication and the level of expenditure in obtaining the parameters.

5.1 The Definitions of the Parameters in Terms of $\psi(\bar{r}, \bar{\Omega}, u, t)$

In the subsequent treatment the independent time variable t is omitted from the notations for reasons of simplification. It is understood, however, that all of the parameters are actually the ones for the particular reactor condition which exists at some specific time of interest t .

For the sake of summary and simplification of notation, one can re-define all the equations and parameters of interest as follows:

It is henceforth assumed that

$$(61) \psi'(\bar{r}, \bar{\Omega}, u, t) = C\psi(\bar{r}, \bar{\Omega}, u)$$

where C is a constant of normalization which makes $C\psi(\bar{r}, \bar{\Omega}, u)$ equal to the actual flux distribution at time t ; and the equation for $\psi(\bar{r}, \bar{\Omega}, u)$ is

the source-free, steady-state version of Eq. (1).

$$\begin{aligned}
 (62) \quad & -\bar{\Omega} \cdot \nabla \psi(\bar{r}, \bar{\Omega}, u) - \Sigma(\bar{r}, u) \psi(\bar{r}, \bar{\Omega}, u) \\
 & + \int_{\Omega'} \int_u \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi(\bar{r}, \bar{\Omega}', u') du' d\Omega' \\
 & + \frac{1}{4\pi\lambda} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' = 0, \text{ where}
 \end{aligned}$$

$$(63) \quad \phi(\bar{r}, u) = \int_{\Omega} \psi(\bar{r}, \bar{\Omega}, u) d\Omega.$$

The equation for the adjoint or importance function $\psi^*(\bar{r}, \bar{\Omega}, u)$ is the adjoint to Eq. (62).

$$\begin{aligned}
 (64) \quad & \bar{\Omega} \cdot \nabla \psi^*(\bar{r}, \bar{\Omega}', u') - \Sigma(\bar{r}, u') \psi^*(\bar{r}, \bar{\Omega}', u') \\
 & + \int_{\Omega} \int_u \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi^*(\bar{r}, \bar{\Omega}, u) du d\Omega \\
 & + \frac{1}{\lambda} \int_u \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \phi^*(\bar{r}, u) du = 0, \text{ where}
 \end{aligned}$$

$$(65) \quad \phi^*(\bar{r}, u) = \frac{1}{4\pi} \int_{\Omega} \psi^*(\bar{r}, \bar{\Omega}, u) d\Omega.$$

The equation for some arbitrary but critical reactor condition which, for present purposes, serves only to define the critical eigenvalue (or correction factor) λ_c is

$$\begin{aligned}
 (66) \quad & -\bar{\Omega} \cdot \nabla \psi_o(\bar{r}, \bar{\Omega}, u) - \Sigma_o(\bar{r}, u) \psi_o(\bar{r}, \bar{\Omega}, u) \\
 & + \int_{\Omega'} \int_u \Sigma_{s_o}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_o(\bar{r}, \bar{\Omega}', u') du' d\Omega' \\
 & + \frac{1}{4\pi\lambda_c} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_{f_o}^j(\bar{r}, u') \phi_o(\bar{r}, u') du' = 0, \text{ where}
 \end{aligned}$$

$$(67) \quad \phi_o(\bar{r}, u) = \int_{\Omega} \psi_o(\bar{r}, \bar{\Omega}, u) d\Omega.$$

The reactor kinetics equations parameters are

$$(68) \quad U = \frac{1}{\lambda_c^m \lambda^n} \int_V \int_u \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV,$$

$$(69) \quad \Lambda = \frac{1}{U} \int_V \int_u \int_{\Omega} \frac{\psi^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u)}{v(u)} d\Omega du dV,$$

$$(70) \quad \rho = \frac{\lambda - \lambda_c}{\lambda_c \lambda} \quad F = \frac{(\lambda - \lambda_c) \lambda_c^m \lambda^n}{\lambda_c \lambda},$$

$$(71) \quad F = \frac{1}{U} \int_V \int_u \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV = \lambda_c^m \lambda^n$$

$$(72) \quad \bar{\beta}_i = \frac{1}{\lambda_c U} \int_V \int_u \int_{u'} \sum_j f_i(u) \beta_i^j \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV,$$

$$(73) \quad \bar{\beta} = \sum_i \bar{\beta}_i,$$

$$(74) \quad S' = \int_V \int_u \phi^*(\bar{r}, u) S(\bar{r}, u) du dV \quad (\text{note: } S(\bar{r}, u) = \int_{\Omega} s(\bar{r}, \bar{\Omega}, u, t) d\Omega),$$

$$(75) \quad \bar{S} = S' / UC,$$

$$(76) \quad Q = \bar{S} / \Lambda$$

$$(77) \quad \bar{\gamma}_i = \int_V \int_u \gamma_i(\bar{r}, u) \phi^*(\bar{r}, u) du dV \quad (\text{note: } \gamma_i(\bar{r}, u) = \frac{\lambda_i f_i(u) c_i(\bar{r}, t)}{T_i(t)}),$$

$$(78) \quad \Gamma_i = \bar{\gamma}_i / UC\lambda, \quad \text{and}$$

$$(79) \quad c_i(t) = \Gamma_i T_i(t) / \lambda_i.$$

5.2 Expression of the Parameters in Terms of Eigenvalues

It is noted that ρ , according to (70), is a function only of the easily obtained eigenvalues λ_c and λ . The other parameters can also be expressed as functions only of eigenvalues, as will now be shown.

The source-free, steady-state form of Eq. (1), that is, Eq. (62), can be regarded as the equation for a fictitious reactor which is critical by virtue of the λ in the denominator of the fission neutron source term. A small perturbation of the reactor properties is now made and a small extraneous source is added such that the new reactor constants and source are defined:

$$(80) \quad \Sigma_p(\bar{r}, u) \equiv \Sigma(\bar{r}, u) + \delta\Sigma(\bar{r}, u);$$

$$\Sigma_{sp}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \equiv \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) + \delta\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega});$$

$$\left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]_p \equiv \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') + \delta \left[\sum_j f_f^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right];$$

$$g(\bar{r}, \bar{\Omega}, u) \equiv \text{a source of neutrons (which requires one to impose the constraint* } \frac{1}{\lambda_p} \int_V \int_{u'} \sum_j v^j(u') \Sigma_f^j(\bar{r}, u') \phi_p(\bar{r}, u') du' dV = A, \text{ where } A \text{ is a constant.)}. \text{ All}$$

of these perturbations are sufficiently small to warrant the assumption that the solution to the perturbed reactor equation $\psi_p(\bar{r}, \bar{\Omega}, u)$ is only slightly different from $\psi(\bar{r}, \bar{\Omega}, u)$; hence, one can write

$$(81) \quad \psi_p(\bar{r}, \bar{\Omega}, u) = \psi(\bar{r}, \bar{\Omega}, u) + \delta\psi(\bar{r}, \bar{\Omega}, u).$$

In order to satisfy the steady state equation for the perturbed reactor the eigenvalue is perturbed slightly to λ_p , where $\lambda_p = \lambda + \delta\lambda$.

* The presence of the source $g(\bar{r}, \bar{\Omega}, u)$ in the reactor equation would, without this constraint, result in the existence of a unique solution for the flux for any value of λ_p . The imposition of the constraint, however, requires one, once again, to find the value of λ_p which will allow the flux to satisfy both the reactor equation and the constraint.

The equation for the perturbed reactor can be written

$$\begin{aligned}
(82) \quad & -\bar{\Omega} \cdot \nabla \psi_p(\bar{r}, \bar{\Omega}, u) - \Sigma_p(\bar{r}, u) \psi_p(\bar{r}, \bar{\Omega}, u) \\
& + \int_{\Omega'} \int_{u'} \Sigma_{sp}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_p(\bar{r}, \bar{\Omega}', u') du' d\Omega' + g(\bar{r}, \bar{\Omega}, u) \\
& + \frac{1}{4\pi\lambda_p} \int_{u'} \left[\sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi_p(\bar{r}, u') du' = 0, \text{ where} \\
\phi_p(\bar{r}, u') &= \int_{\Omega} \psi_p(\bar{r}, \bar{\Omega}, u) d\Omega = \phi(r, u) + \int_{\Omega} \delta\psi(\bar{r}, \bar{\Omega}, u) d\Omega.
\end{aligned}$$

The following steps are now taken: Eq. (82) is multiplied by $\psi^*(\bar{r}, \bar{\Omega}, u)$ and the product of $\psi_p(\bar{r}, \bar{\Omega}', u')$ and Eq. (64) is subtracted from the result; the result is integrated over all V , Ω , and u ; and the usual boundary conditions and Green's Theorem are used to eliminate the divergence terms. The result of this manipulation is the equation

$$\begin{aligned}
(83) \quad & - \int_V \int_{\Omega} \int_u \delta\Sigma(\bar{r}, u) \psi^*(\bar{r}, \bar{\Omega}, u) \psi_p(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
& + \int_V \int_{\Omega} \int_{u'} \int_{\Omega'} \int_{u'} \delta\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi^*(\bar{r}, \bar{\Omega}, u) \psi_p(\bar{r}, \bar{\Omega}', u') du' d\Omega' du d\Omega dV \\
& + \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) g(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
& + \frac{1}{\lambda_p} \int_V \int_u \int_{u'} \left[\sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi^*(\bar{r}, u) \phi_p(\bar{r}, u') du' du dV \\
& - \frac{1}{\lambda} \int_V \int_u \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi^*(\bar{r}, u) \phi_p(\bar{r}, u') du' du dV = 0.
\end{aligned}$$

If the last term of Eq. (83) is multiplied by $\frac{\lambda}{\lambda_p}$ and the product is subtracted from the next to the last term and is added to the last, it is apparent that Eq. (83), after division by U , can be expressed in terms of $\psi(\bar{r}, \bar{\Omega}, u)$ as follows.

$$\begin{aligned}
(84) \quad & -\frac{1}{U} \int_V \int_{\Omega} \int_u \delta \Sigma(\bar{r}, u) \psi^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
& + \frac{1}{U} \int_V \int_{\Omega} \int_u \int_{\Omega'} \int_{u'} \delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}', u') du' d\Omega' du d\Omega dV \\
& + \frac{1}{U} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) g(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
& + \frac{1}{\lambda_p U} \int_V \int_u \int_{u'} \delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \psi^*(\bar{r}, u) \psi(\bar{r}, u') du' du dV \\
& + \lambda_c^m \lambda^n \left(\frac{1}{\lambda_p} - \frac{1}{\lambda} \right) = 0.
\end{aligned}$$

All terms containing the products of perturbations (such as $\delta \Sigma(\bar{r}, u) \delta \psi(\bar{r}, \bar{\Omega}, u)$) have been neglected.

$\Sigma(\bar{r}, u)$ is the total collision cross section which is given by

$$(85) \quad \Sigma(\bar{r}, u) = \Sigma_a(\bar{r}, u) + \int_{\Omega} \int_{u'} \Sigma_s(\bar{r}, u, u', \bar{\Omega}' \cdot \bar{\Omega}) du' d\Omega,$$

where $\Sigma_a(\bar{r}, u)$ is the absorption cross section. Now, if one perturbs the reactor by adding a small amount of $1/v$ absorber everywhere, then one can specify

$$(86) \quad \delta \Sigma(\bar{r}, u) = \delta \Sigma_a(\bar{r}, u) = \frac{\alpha_\ell}{v(u)},$$

$$g(\bar{r}, \bar{\Omega}, u) = 0$$

$$\delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) = 0, \text{ and}$$

$$\delta \left[\sum_j f_t^j(u) v^j(u) \Sigma_f^j(\bar{r}, u) \right] = 0.$$

Also, if one defines $\lambda_\ell \equiv \lambda_p$ for this case, then it follows immediately from Eq. (84) that

$$(87) \frac{1}{U} \int_V \int_{\Omega} \int_u \frac{\alpha_{\ell} \psi^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u) du d\Omega dV}{v(u)} = \frac{(\lambda - \lambda_{\ell}) \lambda_c^m \lambda^n}{\lambda \lambda_{\ell}},$$

so that by the definition of Λ in (69),

$$(88) \Lambda = (\lambda - \lambda_{\ell}) \lambda_c^m \lambda^n / \alpha_{\ell} \lambda \lambda_{\ell}.$$

Hence, the effective neutron lifetime can be computed as a function of the two easily obtained eigenvalues λ (for the reactor) and λ_{ℓ} (for the reactor perturbed by adding the absorber $\delta \Sigma_a(\bar{r}, u) = \frac{\alpha_{\ell}}{v(u)}$ everywhere to the reactor).

The expression for $\bar{\beta}_i$ in terms of eigenvalues is derived in a strictly analogous manner. Let the reactor perturbation be such that

$$(89) g(\bar{r}, \bar{\Omega}, u) = 0$$

$$\delta \Sigma(\bar{r}, u) = 0,$$

$$\delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) = 0, \text{ and}$$

$$\delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] = \alpha_{\beta i} \sum_j f_i(u) \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u')$$

where $\alpha_{\beta i}$ is a constant chosen small enough to assure that $\alpha_{\beta i} \sum_j f_i(u) \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u')$ is a sufficiently small perturbation; and let

the eigenvalue λ_p for this case be called $\lambda_{\beta i}$. It then follows from Eq. (84) that

$$(90) \frac{\alpha_{\beta i}}{\lambda_{\beta i} U} \int_V \int_u \int_{u'} \sum_j f_i(u) \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u') \phi^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV \\ = \frac{(\lambda_{\beta i} - \lambda) \lambda_c^m \lambda^n}{\lambda \lambda_{\beta i}}.$$

Therefore, according to definition (72) for $\bar{\beta}_i$,

$$(91) \bar{\beta}_i = \frac{(\lambda_{\beta i} - \lambda) \lambda_c^m \lambda^n}{\alpha_{\beta i} \lambda \lambda_c}.$$

In a similar manner let

$$(92) \quad \delta \Sigma(\bar{r}, u) = 0,$$

$$\delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) = 0,$$

$$\delta \left[\sum_j f_i^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] = 0, \text{ and}$$

$$g(\bar{r}, \bar{\Omega}, u) = \alpha_s \frac{S(\bar{r}, u)}{4\pi}$$

where α_s is a constant chosen sufficiently small to assure that the source of strength $\alpha_s S(\bar{r}, u)$ causes only a slight perturbation of $\psi(\bar{r}, \bar{\Omega}, u)$ and λ for any fixed value of the constraint constant A . [See (80)]. Then

$$(93) \quad \frac{\alpha_s}{U} \int_V \int_u \phi^*(\bar{r}, u) S(\bar{r}, u) du dV = \frac{(\lambda_s - \lambda) \lambda_c^m \lambda^n}{\lambda \lambda_s}.$$

Therefore, from the definition of $\bar{S}$ in (75), one can write

$$(94) \quad \bar{S} = \frac{(\lambda_s - \lambda) \lambda_c^m \lambda^n}{c \alpha_s \lambda \lambda_s}.$$

Eqs. (88) and (76) can be immediately employed to give

$$(95) \quad Q = \bar{S}/\Lambda = \frac{\alpha_s \lambda_s (\lambda_s - \lambda)}{c \alpha_s \lambda_s (\lambda - \lambda_s)}.$$

Finally, let

$$(96) \quad \delta \Sigma(\bar{r}, u) = 0,$$

$$\delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) = 0,$$

$$\delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(u') \right] = 0, \text{ and}$$

$$g(\bar{r}, \bar{\Omega}, u) = \frac{\alpha_{\gamma_i} \gamma_i(\bar{r}, u)}{\lambda_i 4\pi} = \frac{\alpha_{\gamma_i}}{4\pi} \frac{f_i(u) c_i(\bar{r}, t)}{T_i(t)}.$$

If α_{γ_i} is chosen small, it follows from Eq. (84) that

$$(97) \quad \frac{\alpha_{\gamma_i}}{\lambda_i U} \int_V \int_u \gamma_i(\bar{r}, u) \phi^*(\bar{r}, u) du dV = \frac{(\lambda_{\gamma_i} - \lambda) \lambda_c^m \lambda^n}{\lambda \lambda_{\gamma_i}},$$

where λ_{γ_i} is the value of λ_p in Eq. (84) when $g(\bar{r}, \bar{\Omega}, u) = \frac{\alpha_{\gamma_i}}{4\pi\lambda_i} \gamma_i(\bar{r}, u)$. It follows immediately from (78), (79) and (88) that

$$(98) \quad \Gamma_i = \frac{\lambda_i (\lambda_{\gamma_i} - \lambda) \alpha_{\ell} \lambda_{\ell}}{\lambda_{\gamma_i} c \alpha_{\gamma_i} (\lambda - \lambda_{\ell})} \text{ and}$$

$$(99) \quad c_i(t) = \frac{\alpha_{\ell} \lambda_{\ell}}{c \alpha_{\gamma_i} \lambda_{\gamma_i}} \frac{(\lambda_{\gamma_i} - \lambda)}{(\lambda - \lambda_{\ell})} T_i(t).$$

5.3 Expression of the Parameters in Terms of Reactor Multiplication per Neutron Generation

A neutron generation can be described as any well-defined collection of neutrons. If such a collection of neutrons occurs or is introduced in a reactor or other bounded volume one can speak of the instant of introduction of these neutrons as the "birth" of the neutrons; and if the neutrons leak out of the reactor or volume or are absorbed at any time then that instant can be said to be the "death" of the neutrons. The time elapsing between the birth and death of any neutron is called the lifetime ℓ of the neutron. It is evident that the lifetime ℓ of a neutron depends upon where in the reactor or volume it is introduced (in terms of $\bar{r}, \bar{\Omega}$, and u , but not t) and also upon the properties of the given reactor or volume. (The "properties" of a volume can be defined in terms of the geometrical description of the volume and the cross sections $\Sigma_S(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t)$ and $\Sigma(\bar{r}, u, t)$, insofar as the lifetime of a neutron is concerned.)

If one defines

(100) $B_1(\bar{r}, \bar{\Omega}, u, t) dV d\bar{\Omega} du dt \equiv$ neutrons injected at $\bar{r}, \bar{\Omega}, u$, and t into a given reactor in the intervals $dV, d\bar{\Omega}, du$, and dt , then one can speak of the total of such neutrons injected as

$$(101) \quad B_1 \equiv \int_0^\infty \int_0^\infty \int_{\bar{\Omega}} \int_V B_1(\bar{r}, \bar{\Omega}, u, t) dV d\bar{\Omega} du dt.$$

Now, Eq. (62) (which is the source-free, steady-state form of Eq. (1)) can be viewed as representing a fictitious critical reactor in which the number of neutrons per fission is $\nu^j(u)/\lambda$ for the j^{th} nuclide when the

neutron causing the fission is at lethargy u . It is interesting to examine the result of injecting a source of neutrons $B_1(\bar{r}, \bar{\Omega}, u, t)$ into this reactor for a finite length of time.

If the B_1 neutrons are injected over a finite length of time into the fictitious reactor represented by Eq. (62) they will produce a neutron flux "pulse" as they diffuse and are slowed down, leaked out and absorbed. The equation which describes how this neutron flux pulse which follows the introduction of the neutron generation (without consideration, as yet, of any fission neutron daughters) is the inhomogeneous transient equation

$$(102) \quad -\bar{\Omega} \cdot \nabla \mathcal{F}_1(\bar{r}, \bar{\Omega}, u, t) - \Sigma(\bar{r}, u) \mathcal{F}_1(\bar{r}, \bar{\Omega}, u, t) + B_1(\bar{r}, \bar{\Omega}, u, t) + \int_{\Omega'} \int_u \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \mathcal{F}_1(\bar{r}, \bar{\Omega}', u', t) du' d\Omega' = \frac{\partial \left[\frac{\mathcal{F}_1(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \right]}{\partial t}.$$

One can now consider how many of the B_1 neutrons are eventually absorbed, how many are leaked and, finally how many fission neutron daughters are subsequently formed as the direct result of these absorptions. Therefore, one can assume the B_1 neutrons are injected over a finite period of time such that one can define

$$(103) \quad \bar{B}_1(\bar{r}, \bar{\Omega}, u, 0) = \lim_{t \rightarrow \infty} B_1(\bar{r}, \bar{\Omega}, u, t) = 0 \text{ and therefore}$$

$$(104) \quad \frac{\mathcal{F}_1(\bar{r}, \bar{\Omega}, u, 0)}{v(u)} = \lim_{t \rightarrow \infty} \frac{\mathcal{F}_1(\bar{r}, \bar{\Omega}, u, t)}{v(u)} = 0.$$

When Eq. (102) is integrated over all time one obtains

$$(105) \quad -\bar{\Omega} \cdot \nabla \psi_1(\bar{r}, \bar{\Omega}, u) - \Sigma(\bar{r}, u) \psi_1(\bar{r}, \bar{\Omega}, u) + B_1(\bar{r}, \bar{\Omega}, u) + \int_{\Omega'} \int_u \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_1(\bar{r}, \bar{\Omega}', u') du' d\Omega' = 0, \text{ where}$$

$$(106) \quad \psi_1(\bar{r}, \bar{\Omega}, u) \equiv \int_0^\infty \mathcal{F}_1(\bar{r}, \bar{\Omega}, u, t) dt \text{ and}$$

$$(107) \quad B_1(\bar{r}, \bar{\Omega}, u) = \int_0^\infty B_1(\bar{r}, \bar{\Omega}, u, t) dt.$$

The right hand side of Eq. (102) vanishes because

$$(108) \int_0^\infty \frac{\partial [\mathcal{F}_1(\bar{r}, \bar{\Omega}, u, t)/v(u)]}{\partial t} dt = \frac{\mathcal{F}_1(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \Big|_0^\infty = 0,$$

by virtue of (104).

[If Eq. (105) is integrated over all V , $\bar{\Omega}$, and u , one obtains (See the note in Appendix C, Eq. C.10)

$$(109) - \int_S \int_u \int_{\bar{\Omega}} \bar{n} \cdot \bar{\Omega} \psi_1(\bar{r}, \bar{\Omega}, u) d\bar{\Omega} du dS - \int_V \int_u \Sigma_a(\bar{r}, u) \phi_1(\bar{r}, u) du dV \\ + \int_V \int_{\bar{\Omega}} \int_u B_1(\bar{r}, \bar{\Omega}, u) du d\bar{\Omega} dV = 0, \text{ where } \Sigma_a(\bar{r}, u)$$

is the absorption cross section and $\phi_1(\bar{r}, u) = \int_{\bar{\Omega}} \psi_1(\bar{r}, \bar{\Omega}, u) d\bar{\Omega}$. Hence, it

becomes obvious that of the B_1 neutrons injected $\int_V \int_u \Sigma_a(\bar{r}, u) \phi_1(\bar{r}, u) du dV$ absorbed and $\int_S \int_u \int_{\bar{\Omega}} \bar{n} \cdot \bar{\Omega} \psi_1(\bar{r}, \bar{\Omega}, u) d\bar{\Omega} du dS$ leak out.]

If one now solves Eq. (105) for $\psi_1(\bar{r}, \bar{\Omega}, u)$, the total number of fission neutron daughters produced by the absorptions of this neutron generation can be calculated as

$$(110) B_2 = \int_V \int_{\bar{\Omega}} \int_u \frac{1}{4\pi\lambda} \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_1(\bar{r}, u') du' d\bar{\Omega} du dV,$$

where λ is from the solution of Eq. (62) for the given reactor

Define

$$(111) B_2'(\bar{r}, \bar{\Omega}, u) \equiv \frac{1}{4\pi} \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_1(\bar{r}, u') du',$$

and define

$$(112) \lambda_1 \equiv \frac{1}{B_1} \int_V \int_{\bar{\Omega}} \int_u B_2'(\bar{r}, \bar{\Omega}, u) du d\bar{\Omega} dV,$$

then one can express the new source distribution as

$$(113) B_2(\bar{r}, \bar{\Omega}, u) \equiv \frac{B_2'(\bar{r}, \bar{\Omega}, u)}{\lambda}.$$

$B_2(\bar{r}, \bar{\Omega}, u)$ can be regarded as a new generation of neutrons and, according to definitions (110), (111), and (112), the ratio of the total number of neutrons in the new generation to the total number in the initial distribution can be expressed as

$$(114) \quad \frac{B_2}{B_1} = \frac{\lambda_1}{\lambda}.$$

Now, this new source of neutrons $B_2(\bar{r}, \bar{\Omega}, u)$ can be allowed to diffuse and die in the reactor; Eq. (105) can be written in terms of a $\psi_2(\bar{r}, \bar{\Omega}, u)$ function and $B_2(\bar{r}, \bar{\Omega}, u)$, and the solution $\psi_2(\bar{r}, \bar{\Omega}, u)$ can be used to find the fission neutron source distribution for the next generation of neutrons. That is,

$$(115) \quad B_3(\bar{r}, \bar{\Omega}, u) = \frac{1}{\lambda} B'_3(\bar{r}, \bar{\Omega}, u) \text{ where}$$

$$(116) \quad B'_3(\bar{r}, \bar{\Omega}, u) \equiv \frac{1}{4\pi} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \phi_2(\bar{r}, u') du'.$$

Again, one defines a quantity

$$(117) \quad \lambda_2 \equiv \frac{1}{B_2} \int_V \int_{\Omega} \int_u B'_3(\bar{r}, \bar{\Omega}, u) du d\Omega dV$$

so that

$$(118) \quad \frac{B_3}{B_2} = \frac{1}{B_2} \int_V \int_{\Omega} \int_u B_3(\bar{r}, \bar{\Omega}, u) du d\Omega dV = \frac{\lambda_2}{\lambda}.$$

Each succeeding neutron generation can be described in a strictly analogous manner: and one can write that the ratio of the number of fission neutrons born on the $n + 1^{\text{th}}$ neutron generation to the number of fission neutrons born on the n^{th} generation is

$$(119) \quad \frac{B_{n+1}}{B_n} = \frac{\lambda_n}{\lambda}.$$

The quantity λ_n/λ , as defined, is termed the "reactor multiplication for the n^{th} neutron generation."

It follows that

$$(120) \quad B_n = B_1 \left(\frac{B_2}{B_1} \right) \left(\frac{B_3}{B_2} \right) \cdots \left(\frac{B_n}{B_{n-1}} \right) = B_1 \prod_{m=1}^{n-1} \frac{\lambda_m}{\lambda}.$$

Now, if this successive generation process converges such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$, then it follows from Eq. (62) that

$$(121) \quad \lim_{n \rightarrow \infty} \phi_n(\bar{r}, u) = A \phi(\bar{r}, u), \text{ where } A \text{ is some arbitrary constant.}$$

Therefore, if $\lim_{n \rightarrow \infty} \prod_{m=1}^n (\lambda_m / \lambda)$ exists, one can define a B_∞ as

$$(122) \quad B_\infty = B_1 \lim_{n \rightarrow \infty} \prod_{m=1}^n \frac{\lambda_m}{\lambda}.$$

This means that following the introduction of the B_1 neutrons into the fictitious critical reactor the number of neutrons born in each generation becomes constant at a definite value B_∞ , which depends only on the distribution $B_1(\bar{r}, \bar{\Omega}, u)$ and the properties of the reactor.

In addition, since this reactor is critical, the introduction of the B_1 neutrons into the reactor eventually results in the establishment of a definite steady-state reactor power which can be called $P_{B_1}^*$. This power is obviously a unique value which depends on both the distribution $B_1(\bar{r}, \bar{\Omega}, u)$ and the reactor properties. Now, the neutrons in the distribution $B_2(\bar{r}, \bar{\Omega}, u)$ are all of the daughters and the only daughters of the B_1 generation. Therefore, one can say immediately that the steady-state power P_{B_2} which would result from the introduction into the reactor of a $B_2(\bar{r}, \bar{\Omega}, u)$ distribution of neutrons instead of the $B_1(\bar{r}, \bar{\Omega}, u)$ distribution must be equal to P_{B_1} . It follows by similar reasoning that the introduction of the $B_\infty(\bar{r}, \bar{\Omega}, u)$ distribution into the reactor results in a steady-state power P_{B_∞} which is also equal to P_{B_1} .

The steady-state distribution $B_\infty(\bar{r}, \bar{\Omega}, u)$ can be expressed in the form

$$(123) \quad B_\infty(\bar{r}, \bar{\Omega}, u) = B_\infty S_\infty(\bar{r}, \bar{\Omega}, u),$$

* It is assumed that B_1 is so large that one can validly speak only of the mean behavior of the neutrons in the B_1 distribution.

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in $dV, d\Omega, du$ about the point $\bar{r}, \bar{\Omega}, u$ then

$$(126) \quad \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV = A \lim_{t \rightarrow \infty} \left[\begin{array}{l} \text{the power level in the critical} \\ \text{reactor following the injection} \\ \text{of the } B_1(\bar{r}, \bar{\Omega}, u) dV d\Omega du \text{ neutrons} \end{array} \right].$$

Obviously, from (126), the power level eventually attained following the introduction of the total $B_1 = \int_V \int_{\Omega} \int_u B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV$ neutrons into the reactor can be expressed as

$$(127) \quad P_{B_1} = \frac{1}{A} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV.$$

But P_{B_1} is equal to P_{B_∞} , so that if one makes use of (125) and (122) one can write

$$(128) \quad \frac{1}{A} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV = P_{B_\infty} = B_\infty P_{S_\infty} \\ = P_{S_\infty} B_1 \lim_{n \rightarrow \infty} \prod_{m=1}^n \left(\frac{\lambda_m}{\lambda} \right).$$

Now, consider the introduction of a number of neutrons B_0 into the reactor where the distribution of the neutrons is given by

$$(129) \quad B_0(\bar{r}, \bar{\Omega}, u) = \frac{1}{4\pi \lambda_c^m \lambda^n} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du';$$

or, in terms of S_∞ ,

$$B_0(\bar{r}, \bar{\Omega}, u) = B_0 S_\infty(\bar{r}, \bar{\Omega}, u), \text{ where}$$

$$(130) \quad B_0 = \int_V \int_{\Omega} \int_u B_0(\bar{r}, \bar{\Omega}, u) du d\Omega dV = \frac{1}{\lambda_c^m \lambda^n} \left[\int_V \int_{u'} \sum_j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV \right].$$

$\phi(\bar{r}, u)$ is again the steady-state solution of the scalar flux from Eq. (62).

Since this first generation distribution corresponds exactly to that of the steady-state neutron generation, the distribution of the next generation is identical to the first, and the reactor multiplication for the n^{th} generation, λ_n/λ is equal to unity for all n . The steady-state power level P_{B_0} eventually attained in the reactor is obviously given by

$$P_{B_0} = B_0 P_{S_\infty}.$$

This power P_{B_0} can be expressed, as was P_{B_1} , in terms of the importance or adjoint function $\psi^*(\bar{r}, \bar{\Omega}, u)$ as

$$P_{B_0} = \frac{1}{A} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_0(\bar{r}, \bar{\Omega}, u) du d\Omega dV,$$

so that one can write

$$(131) \quad \frac{1}{A} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_0(\bar{r}, \bar{\Omega}, u) du d\Omega dV = B_0 P_{S_{\infty}}.$$

One can use (128) and (131) immediately to arrive at the result

$$(132) \quad \frac{\frac{1}{A} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV}{\frac{1}{A} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_0(\bar{r}, \bar{\Omega}, u) du d\Omega dV} = \frac{B_1}{B_0} \lim_{n \rightarrow \infty} \prod_{m=1}^n \frac{\lambda_m}{\lambda}.$$

and from the definition of $B_0(\bar{r}, \bar{\Omega}, u)$ in (129) it follows that

$$(133) \quad \frac{\int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV}{\frac{1}{\lambda^m \lambda^n} \int_V \int_u \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \phi^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV} = \frac{B_1}{B_0} \lim_{n \rightarrow \infty} \prod_{m=1}^n \frac{\lambda_m}{\lambda}.$$

By the definition of U in (68), it is apparent that the denominator or the left hand side of Eq. (133) is equal to U ; hence,

$$(134) \quad \frac{1}{U} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV = \frac{B_1}{B_0} \lim_{n \rightarrow \infty} \prod_{m=1}^n \frac{\lambda_m}{\lambda}.$$

Eq. (84) in subsection 5.2 can be used to find the following result. Let $g(\bar{r}, \bar{\Omega}, u)$ equal $\alpha_b B_1(\bar{r}, \bar{\Omega}, u)$, where α_b is sufficiently small. Then if

$$(135) \quad \begin{aligned} \delta \Sigma(\bar{r}, u) &= 0, \\ \delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}', \bar{\Omega}) &= 0, \text{ and} \\ \delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] &= 0; \end{aligned}$$

it follows from Eq. (84) that

$$(136) \quad \frac{1}{U} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV = \frac{[\lambda_b - \lambda]}{\alpha_b \lambda \lambda_b} \lambda_c^m \lambda^n,$$

where λ_b denotes the value of λ_p when the perturbation is a source equal to $\alpha_b B_1(\bar{r}, \bar{\Omega}, u)$.

It is apparent that there are now two means available for computing the importance weighted average of a general function $B_1(\bar{r}, \bar{\Omega}, u)$. That is, one can use either the formula of (134) in terms of reactor multiplication constants or one can use the function of eigenvalues of (136). It is of interest to note the respective sources of each of the alternative formulae.

a). Given the definition of the function $\psi^*(\bar{r}, \bar{\Omega}, u)$ as a physical concept, one derives expression (134) for

$$\frac{1}{U} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV$$

as a ratio of powers P_{B_1}/P_{B_0} which can be computed from a series of reactor multiplication factors.

b). Given the definition of $\psi^*(\bar{r}, \bar{\Omega}, u)$ as simply the solution of Eq. (64), one derives expression (136) as a function of the eigenvalues λ and λ_b .

Also, one can consider the above process a method of proof of the interesting equality,

$$(137) \quad \frac{[\lambda_b = \lambda] \lambda_c^{\text{III}} \lambda^{\text{II}}}{\alpha_b \lambda \lambda_b} = \frac{R_1}{B_0} \lim_{\ell \rightarrow \infty} \prod_{k=1}^{\ell} \frac{\lambda_k}{\lambda},$$

where α_b is a vanishingly small quantity.

[It is perhaps reassuring to note that Eq. (137) can be proved without making use of the definition of $\psi^*(\bar{r}, \bar{\Omega}, u)$ as a physical concept. That is, one can show that (137) follows from the definition of $\psi^*(\bar{r}, \bar{\Omega}, u)$ as simply the solution of Eq. (64) and from the definition of λ_b as an eigenvalue of the perturbed equation (82). (See Appendix D)].

In conclusion, it is necessary only to emphasize that two alternative methods have now been defined; and by means of either of the methods one can compute the values of Λ , $\bar{\beta}_i$, $\bar{S}$, Q , Γ_i , and $c_i(t)$. The first method is that outlined in 5.2 wherein the fictitious critical reactor is perturbed by adding the small source $\alpha_b B_1(\bar{r}, \bar{\Omega}, u)$ in order to obtain a

perturbed value of the eigenvalue. The desired kinetics equation parameter is then computed as a function of the perturbed eigenvalue. The second method, that of this subsection, consists of computing the reactor multiplication factor λ_n/λ for each neutron generation n produced in the fictitious critical reactor following the introduction of a particular first generation $B_1(\bar{r}, \bar{\Omega}, u)$.

The detailed procedure for computing each of the kinetic equation parameters by means of this second method can be implied by the outline presented on the following table.

OUTLINE OF COMPUTATION OF THE KINETICS EQUATION PARAMETERS IN TERMS OF REACTOR MULTIPLICATION

Kinetics Equation Parameter	Λ	$\bar{\beta}_i$	$\bar{S}$	Γ_i
$B_1(\bar{r}, \bar{\Omega}, u)$	$\psi(\bar{r}, \bar{\Omega}, u)/v(u)$	$\frac{1}{4\pi\lambda_c} f_i(u) \int_{u'} \sum_j \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du'$	$\frac{1}{4\pi} S(\bar{r}, u)$	$\frac{1}{4\pi} \gamma_i(\bar{r}, u)$
B_1	$\int_V \int_u \frac{\phi(\bar{r}, u)}{v(u)} du dV$	$\frac{1}{\lambda_c} \int_V \int_{u'} \sum_i \beta_i v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV$	$\int_V \int_u S(\bar{r}, u) du dV$	$\int_V \int_u \gamma_i(\bar{r}, u) du dV$
Reactor Mult. th for m th Generation	$\lambda_{\ell m}/\lambda$	$\lambda_{\beta_i m}/\lambda$	λ_{sm}/λ	$\lambda_{\gamma_i m}/\lambda$
Formula for the Kinetics Equation Parameter	$\frac{B_1}{B_0} \prod_{m=1}^{\infty} \frac{\lambda_{\ell m}}{\lambda}$	$\rho_i \frac{\lambda_c^{m\lambda^n}}{\lambda_c} \prod_{m=1}^{\infty} \frac{\lambda_{im}}{\lambda} *$	$\frac{B_1}{B_0 C} \prod_{m=1}^{\infty} \frac{\lambda_{sm}}{\lambda}$	$\frac{B_1}{\Lambda B_0 C} \prod_{m=1}^{\infty} \frac{\lambda_{\gamma_i m}}{\lambda}$

NOTE: Q and $c_i(t)$ can be computed from the other parameters by means of formulae (76) and (79).

* By definition, $\beta_i \equiv \int_V \int_{u'} \sum_j \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV / \int_V \int_{u'} \sum_j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV$

such that $B_1/B_0 = (\beta_i/\lambda_c)(\lambda_c^{m\lambda^n})$.

SECTION 6

CALCULATION OF THE PARAMETERS BY MEANS OF
CONVENTIONAL, MULTIGROUP REACTOR COMPUTER CODES

It is assumed that one has available one of many conventional computer codes which handles the lethargy dependence of the variables by means of the usual multigroup or few group approximations. The word "conventional" is taken to mean the following:

1) All cross sections are considered to be only "regionwise" variable with respect to $\bar{r}$; ie, a cross section is constant with respect to $\bar{r}$ in each region.

2) The code can solve (in some approximate manner) the usual eigenvalue problem defined by Eq. (62) or Eq. (66) and include in the output the values for at least the eigenvalue λ , the scalar fluxes $\phi_g(\bar{r})$ for each group, the converged steady-state fission neutron source distribution $S_f(\bar{r})$, and certain average quantities such as the average scalar flux for each region ϕ_g^r . It is also assumed that the successive eigenvalues of the outer iterations are available.

3) The code can solve the inhomogeneous problem defined by Eq. (105), where $B_{1g}(\bar{r}, \bar{\Omega})$ is an isotropic source in group g and provide, as output, the spatial distribution of the first generation of fission neutrons $S_{f1}(\bar{r})$ and the first "eigenvalue" λ_1 .

The more important equations and parameters can be expressed in the more familiar and directly useable multigroup form as follows. The multigroup form of Eq. 62 is

$$(138) \quad -\bar{\Omega} \cdot \nabla \psi_g(\bar{r}, \bar{\Omega}) - \Sigma_g^r \psi_g(\bar{r}, \bar{\Omega}) + \sum_{g'} \int_{\bar{\Omega}'} \Sigma_{sg' \rightarrow g}^r (\bar{\Omega}' \cdot \bar{\Omega}) \psi_{g'}(\bar{r}, \bar{\Omega}') d\bar{\Omega}'$$

$$+ \frac{1}{4\pi\lambda} \chi_g \sum_{g'} \left[\nu \Sigma_{f, g'}^r \phi_{g'}(\bar{r}) \right] = 0,$$

where

$$\nu \Sigma_{f,g'}^r \equiv \sum_j \nu_{g'}^j \Sigma_{f,g'}^{j,r},$$

and

$$\chi_g \equiv \int_{u_{g-1}}^{u_g} f_t(u) du.$$

(It is assumed here, as is usually done, that all the j fission spectra are the same.) The subscript g and the superscript r denote values for the g^{th} group and r^{th} region.

One can define the previously mentioned sources as

$$(139) \quad S_f(\bar{r}) = \sum_g \nu \Sigma_{f,g}^r \phi_g(\bar{r}) \quad \text{and}$$

$$S_{f1}(\bar{r}) = \sum_g \nu \Sigma_{f,g}^r \phi_{1,g}(\bar{r}),$$

where the g subscripts on the fluxes denote integration over the lethargy interval (u_{g-1}, u_g) . $\phi_{1,g}(\bar{r})$ is understood to be the solution to the multigroup form of Eq. (105) for the scalar flux produced by the first neutron generation source. $\sum_g B_{1g}(\bar{r}, \bar{\Omega})$.

The adjoint equation Eq. (64) can be expressed in multigroup form as

$$(140) \quad \bar{\Omega} \cdot \nabla \psi_g^*(\bar{r}, \bar{\Omega}') - \Sigma_g^r \psi_g^*(\bar{r}, \bar{\Omega}') + \sum_g \int_{\Omega} \Sigma_{sg \rightarrow g}^r (\bar{\Omega}' \cdot \bar{\Omega}) \psi_g^*(\bar{r}, \Omega) d\Omega \\ + \frac{1}{\lambda} \sum_g \nu \Sigma_{f,g}^r \chi_g \phi_g^*(\bar{r}) = 0.$$

The multigroup forms of the expressions for the reactor kinetics equation parameters are:

$$(141) \quad U \equiv \frac{1}{\lambda_c^m \lambda^n} \int_V \sum_g \chi_g \phi_g^*(\bar{r}) \sum_{g'} \nu \Sigma_{f,g'}^r \phi_{g'}(\bar{r}) dV.$$

$$(142) \quad \Lambda \equiv \frac{1}{U} \sum_g \int_V \int_{\Omega} \left| \frac{\bar{1}}{v} \right|_g^r \psi_g^*(\bar{r}, \bar{\Omega}) \psi_g(\bar{r}, \bar{\Omega}) d\Omega dV,$$

where $\left| \frac{\bar{1}}{v} \right|_g^r$ is the average of the reciprocal neutron velocity for group g and region r .

$$(143) \quad \rho \equiv \left| \frac{\lambda - \lambda_c}{\lambda_c \lambda} \right| \left| \lambda_c^m \lambda^n \right|.$$

$$(144) \quad \bar{\beta}_i \equiv \frac{1}{\lambda_c U} \int_V \sum_g \chi_{\beta_{i,g}} \phi_g^*(\bar{r}) \sum_j \sum_{g'} \beta_i^j v_{g'}^j \Sigma_{f,g'}^{j,r} \phi_{g'}^r(\bar{r}) dV,$$

where

$$(145) \quad \chi_{\beta_{i,g}} \equiv \int_{u_{g-1}}^{u_g} f_i(u) du.$$

For later purposes one can define

$$(146) \quad \beta_i^r \equiv \sum_j \sum_{g'} \beta_i^j v_{g'}^j \Sigma_{f,g'}^{j,r} \phi_{g'}^r / \sum_{g'} v \Sigma_{f,g'}^r \phi_{g'}^r,$$

where $\phi_g^r = \frac{1}{V_r} \int_{V_r} \phi_g(\bar{r}) dV$, and one can also define a β_i given by

$$(147) \quad \beta_i \equiv \sum_r \sum_{g'} \sum_j \beta_i^j v_{g'}^j \Sigma_{f,g'}^{j,r} \phi_{g'}^r V_r / \sum_r \sum_{g'} v \Sigma_{f,g'}^r \phi_{g'}^r V_r.$$

$$(148) \quad S' \equiv \int_V \sum_g \phi_g^*(r) S_g(r) dV,$$

where

$$(149) \quad S_g(\bar{r}) \equiv \int_{u_{g-1}}^{u_g} S(\bar{r}, u) du$$

and, as before, one defines $\bar{S} \equiv \frac{S'}{UC}$ and $Q \equiv \bar{S}/\Lambda$.

$$(150) \quad \bar{\gamma}_i \equiv \int_V \sum_g \gamma_{i,g}(r) \phi_g^*(\bar{r}) dV \equiv \frac{\lambda_i}{T_i(t)} \int_V c_i(\bar{r}, t) \sum_g \chi_{\beta_{i,g}} \phi_g^*(r) dV.$$

Also, as before,

$$\Gamma_i \equiv \bar{\gamma}_i / UCA \text{ and } c_i(t) \equiv \Gamma_i T_i(t) / \lambda_i,$$

or

$$(151) \quad c_i(t) \equiv \frac{1}{\Lambda UC} \int_V c_i(\bar{r}, t) \sum_g \chi_{\beta_{i,g}} \phi_g^*(r) dV.$$

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computed by Eq. (88) which states

$$(152) \quad \Lambda = \frac{(\lambda - \lambda_\ell)}{\alpha_\ell \lambda \lambda_\ell} (\lambda_c^m \lambda^n) .$$

It is recalled that this method is based on the assumption that α_ℓ is vanishingly small; but, in practice, this criterion cannot be strictly met. The values $\alpha_\ell (1/v)_g^r$ which are added to the absorption cross sections must be large enough to perturb the cross sections when they are expressed in terms of the number of digits the computer handles. The suggested way to overcome this difficulty is to choose three or four values of α_ℓ and compute the corresponding values for Λ . If these values are plotted, one can extrapolate to the value of Λ for a zero α_ℓ . It is usually found that as α_ℓ approaches small but still significant values (with respect to the unperturbed cross sections), the curve $\Lambda(\alpha_\ell)$ becomes a straight line which, of course, one can easily extrapolate to $\Lambda(0)$. As α_ℓ approaches still smaller and insignificant values, anomalous behavior of the curve is apparent. This is due to the total disappearance of the perturbation from some of the cross section values used by the machine. This part of the curve should therefore be avoided in any extrapolation.

6.3 The Effective Delayed Neutron Precursor Fraction $\bar{\beta}_1$

There are, as shown in 5.3, two ways of expressing the value of $\frac{1}{U} \int_V \int_{\Omega} \int_u \Psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV$ for an arbitrary distribution $B_1(\bar{r}, \bar{\Omega}, u)$. According to Eqs. (134) and (136) it can be represented either by

$$\frac{B_1}{B_0} \lim_{n \rightarrow \infty} \prod_{m=1}^n \left(\frac{\lambda_m}{\lambda} \right) ,$$

where B_1 and B_0 are given by (101) and (130); or it can be represented by $\lambda_c^m \lambda^n (\lambda_b - \lambda) / \alpha_b \lambda \lambda_b$, where λ_b is the eigenvalue solution to Eq. (82) when the only perturbation is a source $g(\bar{r}, \bar{\Omega}, u) = \alpha_b B_1(\bar{r}, \bar{\Omega}, u)$ which is vanishingly small compared to B_0 and when there exists the constraint

$$(154) \quad \frac{1}{\lambda_b} \int_V \int_{\Omega} \sum_j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV = \frac{\lambda_c^m \lambda^n B_0}{\lambda_b} = A .$$

In addition, if one considers $B_1(\bar{r}, \bar{\Omega}, u)$ to be given by

$$(155) \quad B_1(\bar{r}, \bar{\Omega}, u) = \frac{1}{4\pi\lambda_c} \int_{u'} f_i(u') \sum_j \beta_i^j \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du';$$

then it follows from Eq. (90) that in this case

$$(156) \quad \frac{1}{U} \int_V \int_{\Omega} \int_u \psi^*(\bar{r}, \bar{\Omega}, u) B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV = \frac{(\lambda_{\beta_i} - \lambda) \lambda_c^m \lambda^m}{\alpha_{\beta_i} \lambda_c \lambda},$$

where λ_{β_i} is the eigenvalue of Eq. (82) when the only perturbation is

$$\delta \left[\sum_j f_i^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \right] = \alpha_{\beta_i} f_i(u) \sum_j \beta_i^j \nu^j(u') \Sigma_f^j(\bar{r}, u').$$

There are therefore three ways to express β_i . In multigroup notation, one can consider a source

$$(157) \quad B_{1,g}(\bar{r}, \bar{\Omega}) = \frac{1}{4\pi\lambda_c} \chi_{\beta_i,g} \sum_{g'} \sum_j \beta_i^j \nu_{g'}^j \Sigma_{f,g'}^j \phi_{g'}(\bar{r})$$

and express the value of $\bar{\beta}_i$ either by

$$(158) \quad \bar{\beta}_i = \frac{B_1}{B_0} \lim_{n \rightarrow \infty} \prod_{m=1}^n \frac{\lambda_m}{\lambda},$$

or by

$$(159) \quad \bar{\beta}_i = \frac{(\lambda_b - \lambda) (\lambda_c^m \lambda^n)}{\alpha_b \lambda \lambda_b},$$

where λ_b is the eigenvalue of Eq. (82) when the only perturbation is the addition of the vanishingly small source $\alpha_b B_{1,g}(\bar{r}, \bar{\Omega})$ to each group g while the flux is subject to the constraint (154). As the third method one can perturb Eq. (138) by adding the quantity

$$(160) \quad \alpha_{\beta_i} \chi_{\beta_i,g} \sum_j \beta_i^j \nu_{g'}^j \Sigma_{f,g'}^j \text{ to } \chi_g \nu \Sigma_{f,g}^r,$$

in the last term of the equation. The perturbed eigenvalue λ_{β_i} can then be used to express $\bar{\beta}_i$ as

$$(161) \quad \bar{\beta}_i = \frac{(\lambda_{\beta_i} - \lambda) \lambda_c^m \lambda^n}{\alpha_{\beta_i} \lambda \lambda_c}.$$

Of these three expressions for $\bar{\beta}_1$, only (158) and (161) will now be considered. This is because of the fact that the conventional reactor computer codes do not offer the option to compute an eigenvalue (such as λ_0) of Eq. (82) when there is present the fixed volume source $g(\bar{r}, \bar{\Omega}, u)$ and the necessary constraint. (In passing, it should be stated that many of the codes could easily add this option because the usual "outer iteration" or fission source iteration scheme automatically imposes the necessary constraint (154).)

6.3.1 The Method of Perturbed Eigenvalue

$\bar{\beta}_1$ is often calculated by means of the expression (161), but several factors limit the accuracy of the result. First, α_{β_1} must, in practice, be large enough to be significant rather than "vanishingly small" as theory requires. Hence, the best one can do is calculate several $\bar{\beta}_1$ values at different α_{β_1} points and extrapolate to zero α_{β_1} . There are no serious errors involved here, but the extra trouble involved in making several calculations in order to assure elimination of the error due to finite α_{β_1} values is of importance. Second, an approximation must always be made in perturbing $\chi_g v \Sigma_{f,g}^r$ by adding the quantity (160) to it. Conventional codes do not allow for the direct addition of the quantity, and one must therefore find an approximate method of effecting the addition. The addition is usually performed, in effect, by adding to χ_g , a region dependent quantity $\Delta \chi_g^r$ where

$$(162) \quad \Delta \chi_g^r = \alpha_{\beta_1} \chi_{\beta_1, g} \beta_1^r = \frac{\alpha_{\beta_1} \chi_{\beta_1, g} \sum_j \sum_{g'} \beta_1^{j, j} \Sigma_{f, g'}^{j, r} \phi_{g'}^r}{\sum_{g'} v \Sigma_{f, g'}^r \phi_{g'}^r}$$

such that

$$(\chi_g + \Delta \chi_g^r) \sum_{g'} v \Sigma_{f, g'}^r \phi_{g'}^r(r)$$

is approximately equal to

$$\chi_g \sum_{g'} v \Sigma_{f, g'}^r \phi_{g'}^r(r) + \alpha_{\beta_1} \chi_{\beta_1, g} \sum_j \sum_{g'} \beta_1^{j, j} \Sigma_{f, g'}^{j, r} \phi_{g'}^r(\bar{r}).$$

The approximation is valid and rigorous if, in any region, each group flux $\phi_g(\bar{r})$ has exactly the same distribution in $\bar{r}$; ie, if $\phi_g(\bar{r}) = C_g f(\bar{r})$ where C_g is a constant. If the regions can be made small enough, the error due to this approximation can be reduced to small proportions. It should be noted that the χ values used by the code are now region dependent and this unfortunately is not an allowable option in many codes one might otherwise choose to employ.

This method of computing β_i can be summarized as follows. It is assumed that the unperturbed eigenvalue has been computed by solving Eq. (138) and that the region averages of the group fluxes, $\phi_g^r = \frac{1}{V_r} \int_{V_r} \phi_g(\bar{r}) dV$, are available. β_i^r is computed from formula (146), and (162) is used to compute the $\Delta\chi_g^r$ value for each group and region. These $\Delta\chi_g^r$ values are then added to the unperturbed χ_g^r values, and the eigenvalue problem is re-run to obtain λ_{β_i} . The value of $\bar{\beta}_i$ is then calculated from (161), given the λ_c value from a previous calculation and given the desired choice for m and n.

Another source of error with this method should be noted. When the multigroup form of Eq. (82) is solved for λ_{β_i} , the set of group constants $\Sigma_{sg}^r(\bar{n} \cdot \bar{n})$, Σ_g^r , and $v\Sigma_{fg}^r$ (and D_g^r if diffusion theory is used), is usually obtained by an auxiliary computer code which suitably averages with respect to a neutron lethargy distribution which results from a fission neutron source spectrum given by $f_t(u)$. Now, $f_t(u)$ is nearly equal to (or usually assumed equal to) the prompt fission neutron spectrum, so that the group constants (especially if the groups are broad, high energy ones) can be radically different from those that would have been obtained if the fission neutron spectrum had been assumed to be $f_i(u)$. However, the spectrum of the neutrons which one, in effect, is adding to the fission neutron source when one adds $\Delta\chi_g^r$ to χ_g is $f_i(u)$ and not $f_t(u)$; hence these $\Delta\chi_g^r$ extra neutrons should really be subject to or exposed to average cross sections derived from an $f_i(u)$ spectrum. The fact that these $\Delta\chi_g^r$ extra neutrons are exposed, instead, to cross sections derived from the $f_t(u)$ spectrum accounts for an amount of error that can be significant (a few percent in $\bar{\beta}_i$) if the groups are broad.

In summation, this method of computation of $\bar{\beta}_i$ can be viewed as a good method provided:

- a) the regions are small enough to warrant the assumption that $\phi_g(r) = C_g f(r)$ within a region, where C_g is a group dependent constant,
- b) the code allows χ_g to be region dependent, and
- c) the groups are small enough in width in the high energy ranges (say above 10 kev) to warrant the assumption that the average cross sections derived for a delayed neutron spectrum $f_i(u)$ would not be significantly different from those derived for a prompt neutron spectrum $f_t(u)$.

6.3.2 The Method of Generation Multiplication

The use of formula (158) in obtaining $\bar{\beta}_i$ avoids most of the difficulties and errors encountered with the previous method which used formula (161).

It is assumed, as before, that an eigenvalue calculation has been run from which λ for the unperturbed reactor was obtained and from which $\phi_g(\bar{r})$ and $\phi_g^r = \frac{1}{V_r} \int_{V_r} \phi_g(r) dV$ are available as output.

It is also assumed that λ_c is available.

B_0 can be calculated from the multigroup forms of (129) and (130) by means of the following formula.

$$(163) \quad B_0 = \frac{1}{\lambda_c \lambda_n} \sum_r \sum_{g'} v \Sigma_{f,g'}^r \phi_{g'}^r V_r.$$

One now defines and calculates a $v \Sigma_{f\beta_i,g'}^r$ from the formula,

$$(164) \quad v \Sigma_{f\beta_i,g'}^r = \sum_j \beta_i^j v \Sigma_{f,g'}^j \phi_{g'}^r,$$

and β_i is then computed from (147) with the use of (164)

$$(165) \quad \beta_i = \frac{\sum_r \sum_{g'} v \Sigma_{f\beta_i,g'}^r \phi_{g'}^r V_r}{\sum_r \sum_{g'} v \Sigma_{f,g'}^r \phi_{g'}^r V_r}.$$

An inhomogeneous problem (or an equivalent "one iteration" problem) is run next with an isotropic source in each group given by

$$(166) \quad S_{\beta_i,g}(\bar{r}) = \chi_{\beta_i,g} \frac{1}{\lambda_c} \sum_{g'} v \Sigma_{f\beta_i,g'}^r \phi_{g'}^r(r)$$

and with all cross sections and multigroup constants derived from an assumed fission neutron spectrum equal to $f_i(u)$.

The source is the desired B_1 source since, by (157) and (164)
 $B_{1g}(\bar{r}, \bar{\Omega}) = \frac{1}{4\pi} S_{\beta_{i,g}}(r)$. Moreover, the total number of neutrons in this source is

$$(167) \quad B_1 = \sum_g \int_V S_{\beta_{i,g}}(r) dV = \frac{1}{\lambda_c} \sum_r \sum_{g'} V_r \nu \Sigma_{f\beta_{i,g'}}^r \phi_{g'}^r = \lambda_c^m \lambda^n B_0 \beta_i / \lambda_c.$$

Upon solution of this inhomogeneous problem, the fluxes $\phi_{1,g}(r)$ and flux integrals $\phi_{1,g}^r$ are available, and one can compute

$$(168) \quad \int_V S_{f1}(\bar{r}) dV = \sum_r \sum_{g'} (\nu \Sigma_{f,g'}^r)^{f_i} \phi_{1,g'}^r V_r,$$

and then

$$(169) \quad \lambda_1 = \int_V S_{f1}(\bar{r}) dV / B_1.$$

The $(\nu \Sigma_{f,g'}^r)^{f_i}$ are the values derived from the $f_i(u)$ spectrum averaging procedure.

At this point one can compute

$$(170) \quad \bar{\beta}_{i,1} = \frac{B_1}{B_0} \frac{\lambda_1}{\lambda} = \frac{\lambda_c^m \lambda^n \beta_i}{\lambda_c} \left(\frac{\lambda_1}{\lambda} \right),$$

where $\bar{\beta}_{i,1}$ can be considered the first estimate of β_i or the "one iteration" value of β_i . This value is usually of sufficient accuracy to warrant the neglect of the higher product factor, $\left[\left(\frac{\lambda_2}{\lambda} \right) \left(\frac{\lambda_3}{\lambda} \right) \dots \left(\frac{\lambda_n}{\lambda} \right) \dots \left(\frac{\lambda_\infty}{\lambda} \right) \right]$. The reason that this first estimate of β_i is usually quite accurate is that $S_{\beta_{i,g}}(r)$ is not much different in shape from the converged source shape $S_f(r)$, and the source $S_{f1}(r)$ is almost identical to $S_f(r)$. Moreover, the lethargy spectrum of the $S_{f1}(r)$ neutrons is $f_t(u)$, as it is for $S_f(r)$, so that λ_2 is very nearly equal to λ .

* In the "one iteration" cases, λ_1 , $S_{f1}(r)/\lambda_1$, and $\int_V S_{f1}(r) dV$ are usually available as output.

If one desires to compute the higher order product, one simply runs a homogeneous problem in which all cross sections and group constants are derived from an assumed fission spectrum equal to $f_t(u)$, the χ 's used are the χ_g 's, and the fission neutron source guess is $S_{f1}(r) = \sum_{g'} (v \Sigma_{f,g'}^r)^{f1} \phi_{1,g'}(r)$. The only output from this problem which is required is, of course, the values of the successive eigenvalues obtained. The first eigenvalue is λ_2 , the second is λ_3 , etc.

$\bar{\beta}_1$ is then calculated immediately from Eq. (158).

It should be emphasized that no fission neutron source extrapolation should be performed on the neutron sources from one outer iteration to the next.

Also, the reactor computer code or codes one uses to perform these calculations often carry out a re-normalization of fluxes or sources at various stages of the outer iterations. It is obviously necessary that one know exactly what normalizations are performed, if any, in order to make sure that the proper values are obtained for use in the formulae given here.

Finally, it is assumed that during the outer iterations of the solution of the homogeneous problem, the fluxes $\phi_g^n(\bar{r})$ obtained in each group are correct in the sense that they would represent the true solution to an inhomogeneous problem in which the fission neutron source is the source guess for the n^{th} iteration. This means that in two-dimensional codes and some one-dimensional transport theory codes, the "inner iterations" used to converge the fluxes for a given fission neutron source guess (outer iteration) must be continued until the group fluxes are quite well-converged.

If it is not possible to obtain correct or well-converged fluxes on each outer iteration, then a series of inhomogeneous problems must be run to obtain the successive values $\lambda_2, \lambda_3, \lambda_4$, etc.

6.4 The Effective Precursor Parameter $\Gamma_i(t)$

If the form of the kinetics equations used is that of Eqs. (39) and (40), then one needs only the initial value of $c_i(t)$; that is, the value at $t = 0$.

Since $T_i(0) = 1$, by convention (25),

$$(171) \quad c_i(0) = \frac{\Gamma_i(0)}{\lambda_i}.$$

Therefore the remainder of this subsection will be concerned only with the calculation of $\Gamma_i(t)$, the parameter which is needed if the form of the kinetics equations used is (41) and (42).

In many instances the initial distribution $C_i(\bar{r}, t)$ corresponds to the distribution for some steady-state reactor condition. In these cases $\Gamma_i(0)$ is, of course, very easy to find because it is simply equal to $\bar{\beta}_i(0)/\Lambda(0)$ by virtue of the fact that $\frac{\partial T_i(0)}{\partial t} = 0$ and $T_i(0) = T(0) = 1$ in Eq. (42).

The calculation of $\Gamma_i(t)$ for any known $C_i(\bar{r}, t)$ distribution is most expeditiously performed by means of the method of generation multiplication.

One first obtains a set of multigroup constants (Dg's, Σ g's, etc.) for the reactor which are derived from a fission spectrum equal to $f_i(u)$ - just as was done in subsection 6.3.2 for the $\bar{\beta}_i$ calculation. Then with these multigroup constants, an inhomogeneous problem (or an equivalent "one iteration" problem) is run with an isotropic source in each group equal to

$$(172) \quad S_{ci,g}(\bar{r}) = \chi_{\beta_i,g} C_i(\bar{r}, t),$$

where

$$\chi_{\beta_i,g} = \int_{u_{g-1}}^{u_g} f_i(u) du.$$

The $B_{1,g}(\bar{r}, \bar{\Omega})$ corresponding to this source is

$$(173) \quad B_{1,g}(\bar{r}, \bar{\Omega}) = \frac{1}{4\pi} \chi_{\beta_i,g} C_i(\bar{r}, t)$$

and the total number of neutrons per second in the B_1 source is

$$(174) \quad B_1 = \int_V C_i(\bar{r}, t) dV.$$

After solution of this inhomogeneous problem, the fluxes $\phi_{1,g}(\bar{r})$ and $\phi_{1,g}^r$ are available, and one computes

$$(175) \quad \int_V S_{f1}(\bar{r}) dV = \sum_r \sum_{g'} (\nu \Sigma_{f,g'}^r)^{fi} \phi_{1,g'}^r V_r$$

and

$$(176) \quad \lambda_1 = \int_V S_{f1}(\bar{r}) dV / B_1,$$

where $(\nu \Sigma_{f,g'}^r)^{fi}$ again denotes the values obtained when the $f_i(u)$ fission neutron spectrum is used to obtain multigroup reactor constants. A first estimate of $\Gamma_i(t)$ can be computed from

$$(177) \quad \Gamma_i(t) = \frac{\lambda_i}{\Lambda(t) T_i(t) C} \frac{B_1}{B_0} \frac{\lambda_1}{\lambda}$$

where Λ is the effective lifetime obtained as in subsection 6.2, λ is the eigenvalue of the unperturbed reactor,

$$B_0 = \frac{1}{\lambda_m \lambda_n} \sum_r \sum_{g'} \nu \Sigma_{f,g'}^r \phi_{g'}^r V_r, \text{ and } \phi_g^r = \frac{1}{V_r} \int_{V_r} \phi_g(\bar{r}) dV, \text{ as before. } \phi_g(r) \text{ is}$$

again the flux for group g in the unperturbed reactor. The constant C is defined as the ratio $\Psi'(\bar{r}, \bar{\Omega}, u, t) / \Psi(\bar{r}, \bar{\Omega}, u)$. It would usually be estimated by use of the formulae

$$(178) \quad C = \frac{1}{T(t)} \left[\sum_r \sum_{g'} \nu \Sigma_{f,g'}^r \int_{V_r} \int_{\Omega} \mathcal{F}_g(\bar{r}, \bar{\Omega}, t) d\Omega dV \right] / \sum_r \sum_{g'} \nu \Sigma_{f,g'}^r \phi_{g'}^r V_r,$$

where $\mathcal{F}_g(\bar{r}, \bar{\Omega}, t)$ is the multigroup form of $\mathcal{F}(\bar{r}, \bar{\Omega}, u, t)$, and it can be assumed that the quantity in brackets is known. Since $c_i(t) = \frac{\Gamma_i(t) T_i(t)}{\lambda_i}$, a first estimate value of $c_i(0)$ can be obtained as

$$(179) \quad c_i(0) = \frac{1}{\Lambda(0)} \frac{B_1}{B_0} \frac{\lambda_1}{\lambda}.$$

If the higher order product $\left(\frac{\lambda_2}{\lambda} \right) \left(\frac{\lambda_3}{\lambda} \right) \dots \left(\frac{\lambda_n}{\lambda} \right)$ is desired, a homogeneous problem is run with the fission neutron source guess for the first iteration given by

$$(180) \quad S_{f1}(\bar{r}) = \sum_{g'} (\nu \Sigma_{f,g'}^r)^{f1} \phi_{1,g'}(r)$$

and with all group constants (D_g 's, χ_g 's, Σ 's, etc.) derived by using a fission spectrum equal to $f_t(u)$, the normal spectrum. The eigenvalues obtained on each successive iteration are of course, $\lambda_2, \lambda_3, \lambda_4$, etc. It is assumed (as stated at the end of subsection 6.3.2) that the code used does not extrapolate from one outer iteration source guess to the next and the fluxes $\phi_g(\bar{r})$ computed for any given source guess are correct (or sufficiently well-converged). Otherwise, a series of inhomogeneous calculations must be performed to obtain the values of $\lambda_2, \lambda_3, \lambda_4$, etc..

Because of the difficulty involved in knowing $C_i(\bar{r}, t)$ at any other time than $t = 0$, it will usually be impractical to find $\Gamma_i(t)$ at any other time than $t = 0$. Consequently, one usually will assume that $\Gamma_i(t) = \Gamma_i(0)$, $\Gamma_i(t)\Lambda(t) = \Gamma_i(0)\Lambda(0)$, $\Gamma_i(t)/\bar{\beta}_i(t) = \Gamma_i(0)/\bar{\beta}_i(0)$, or $\Gamma_i(t)\Lambda(t)/\bar{\beta}_i(t) = \Gamma_i(0)\Lambda(0)/\bar{\beta}_i(0)$ - whichever assumption best applies to the given kinetics problem.

It can be noted that when it is assumed that $\Gamma_i(t)\Lambda(t)/\bar{\beta}_i(t) = \Gamma_i(0)\Lambda(0)/\bar{\beta}_i(0)$ and the distribution $C_i(\bar{r}, 0)$ is an equilibrium distribution then Eq. (42) takes on the particularly simple form

$$(181) \quad \frac{\partial T_i(t)}{\partial t} = \lambda_i [T(t) - T_i(t)].$$

6.5 The Effective Source Q

The effective source is most easily found for a general isotropic fixed volume source distribution $S(\bar{r}, u)$ neutrons/cm³ sec by means of the method of neutron generation multiplication.

Neutron group constants for each region r are obtained from an assumed fission spectrum equal to $S^r(u)$, where it is assumed that in the region r $S(\bar{r}, u)$ is a separable function of $\bar{r}$ and u such that $S(\bar{r}, u)$ can be represented as $f_s^r(u)S^r(\bar{r})$. (The regions can usually be chosen small enough to make this assumption valid.) Hence, if $\chi_{s,g}^r \equiv \int_{u_{g-1}}^u f_s^r(u) du$, this isotropic source in each group can be represented as

$$(182) \quad \chi_{s,g}^r S^r(\bar{r})$$

(With many codes, it is possible to enter the source in each group directly as $S_g(\bar{r}) \equiv \int_{u_{g-1}}^g S(\bar{r}, u) du$. In such cases it is unnecessary to make the assumption of separability, and the source should be entered directly in the form $S_g(\bar{r})$.)

This source is used as input to an inhomogeneous (or equivalent one-iteration) problem to obtain the number

$$(183) \quad \lambda_1 = \sum_r \sum_g (\nu \Sigma_{f,g}^r)_s^r \phi_{1,g}^r V_r / \sum_r \int_{V_r} S^r(\bar{r}) dV,$$

where $(\nu \Sigma_{f,g}^r)_s^r$ is the averaged value of $\nu \Sigma_{f,g}^r$ obtained for region r and group g when the assumed fission neutron spectrum is $f_s(u)$ and where

$\phi_{1,g}^r = \frac{1}{V_r} \int_{V_r} \phi_{1,g}(\bar{r}) dV$ and $\phi_{1,g}(\bar{r})$ is, as before, the solution for the flux in group g from the inhomogeneous problem.

The second generation source distribution

$$S_{f1}(\bar{r}) = \sum_g \left[\nu \Sigma_{f,g}^r \right]_s^r \phi_{1,g}(\bar{r})$$

is then used as the first source guess in a standard homogeneous eigenvalue calculation (with all group constants equal to those corresponding to the usual fission neutron spectrum, $f_t(u)$). The values of $\lambda_2, \lambda_3, \lambda_4$, etc. are obtained from this calculation, provided that there is no fission source extrapolation, etc., as explained at the end of subsection 6.3.2.

Q is then calculated from

$$(184) \quad Q = \frac{\bar{S}}{\Lambda} = \frac{B_1}{\Lambda C B_0} \prod_{m=1}^{m'} \left(\frac{\lambda_m}{\lambda} \right), \text{ where } \frac{\lambda_{m'}}{\lambda} = 1.$$

B_1 is given either by

$$\sum_r \int_{V_r} S^r(\bar{r}) dV; \text{ or by } \sum_r \int_{V_r} \sum_g S_g(\bar{r}) dV; B_0 = \frac{1}{\lambda_c^m \lambda^n} \sum_r \sum_g \nu \Sigma_{f,g}^r \phi_g^r V_r;$$

and ϕ_g^r and λ are the solutions for the unperturbed reactor Eq. (138).

λ_c is the eigenvalue obtained from the calculation of the critical reactor condition. The constant C is usually obtained by means of Eq. (178).

As an approximation, the value of C is usually assumed equal to its value at time $t = 0$. Λ is found as described in subsection 6.2.

APPENDIX A
THE CONSISTENCY OF THE DEFINITIONS OF A SET
OF KINETIC EQUATION PARAMETERS

It can be seen from the definitions of U , ρ , Λ , $\bar{\beta}_i$, and Γ_i that because of the presence of the arbitrary quantity $\lambda_c^m \lambda^n$ in U the values of ρ , Λ , and $\bar{\beta}_i$ are completely arbitrary. It is noted that Q and $c_i(t)$ are not arbitrary because ΔU is well-defined and not arbitrary, as can be seen from Eq. (69). Therefore, further consideration will be given only to ρ , Λ , and $\bar{\beta}_i$.

Although these reactor kinetics equation parameters are completely arbitrary in value, this has no effect whatever on the solution of the kinetics equations because of the fact that the parameters occur only as ratios in the equations; namely, ρ/Λ , and $\bar{\beta}_i/\Lambda$. (The factor $\lambda_c^m \lambda^n$ is divided out in each of these ratios such that these ratios are not arbitrary and the equations are unaffected by the arbitrary choice of m and n .)

The expression for ρ , Λ , and $\bar{\beta}_i$ which one obtains for each of four of the most common choices of m and n are shown in Table A. It is obvious that the criterion for a consistent set is simply that the same m and n be used to compute each parameter; or, more generally, the criterion is that the ratios ρ/Λ , and $\bar{\beta}_i/\Lambda$ be the correct and well-defined values. For the sake of illustration, the values of the parameters in Table A are assumed those obtained in each case as follows.

$$(A.1) \quad \rho = \frac{(\lambda - \lambda_c)}{\lambda \lambda_c} (\lambda_c^m \lambda^n)$$

$$(A.2) \quad \Lambda = \frac{(\lambda - \lambda_\ell)}{\alpha_\ell \lambda \lambda_\ell} (\lambda_c^m \lambda^n),$$

where λ_ℓ is the perturbed eigenvalue of Eq. (82) which corresponds to the perturbation of the reactor by the addition of an absorption cross section equal to $\alpha_\ell/v(u)$. (α_ℓ is vanishingly small in the ideal case.)

$$(A.3) \quad \bar{\beta}_i = \frac{(\lambda_c^m \lambda^n)}{\lambda_c} \beta_i \prod_{m=1}^{m'} \left(\frac{\lambda_m^{\beta_i}}{\lambda} \right),$$

where $\bar{\beta}_i$ is computed by the method of neutron generation multiplication and

$$(A.4) \quad \beta_i = \frac{\int_V \int_{u'} \sum_j \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV}{\int_V \int_{u'} \sum_j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV}.$$

The $\lambda_m^{\beta_i}/\lambda$ value is the m^{th} neutron generation multiplication factor resulting from the insertion (into the fictitious critical reactor represented by Eq. (62)) of a first generation equal to

$$(A.5) \quad B_1^{\beta_i}(\bar{r}, \bar{\Omega}, u) = \frac{1}{4\pi\lambda_c} f_i(u) \int_{u'} \sum_j \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du'.$$

It is assumed that m' , the index of the last multiplication factor in Eq. (A.3) is sufficiently high that $\prod_{m=1}^{\infty} \left(\frac{\lambda_m}{\lambda} \right) = 1$, as must be assumed in practice.

The non-arbitrary definitions of the parameter ratios can be listed:

$$(A.6) \quad \frac{\rho}{\Lambda} = \frac{\alpha_\ell \lambda_\ell (\lambda - \lambda_c)}{\lambda_c (\lambda - \lambda_\ell)}, \text{ and}$$

$$(A.7) \quad \frac{\bar{\beta}_i}{\Lambda} = \frac{\alpha_\ell \lambda_\ell \beta_i \prod_{m=1}^{m'} (\lambda_m^{\beta_i}/\lambda)}{\lambda_c (\lambda - \lambda_\ell)}.$$

The relationship which exists between each of the consistent sets and certain physical quantities β_i , ℓ_g , and ℓ is also shown on Table A. The quantities are defined as follows.

β_i is the unweighted or simple average of the fraction of the total number of neutrons produced in the reactor which are produced by the i^{th} delayed neutron precursor.

$$(A.8) \quad \beta_i = \frac{\frac{1}{\lambda_c} \int_V \int_{u'} \sum_j \beta_i^j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV}{\frac{1}{\lambda_c} \int_V \int_{u'} \sum_j v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV}.$$

The average neutron lifetime ℓ is derived and defined in Appendix C. It is the unweighted average of the time a neutron spends in the reactor between birth (by prompt emission at fission or by decay of a delayed neutron precursor) and death (by leakage or absorption). The formula for ℓ derived in Appendix C is

$$(A.9) \quad \ell = \frac{(\lambda/\lambda_c) \int_V \int_{u'} \frac{\phi(\bar{r}, u')}{v(u')} du' dV}{1/\lambda_c \int_V \int_{u'} \sum_j \nu^j(u') \Sigma_F^j(\bar{r}, u') \phi(\bar{r}, u') du' dV}.$$

This lifetime ℓ depends on the fission neutron source distribution and the properties of the given reactor. There is another mean "generation" time ℓ_g which one can define and which depends on the neutron distribution $\psi(\bar{r}, \bar{\Omega}, u)/v(u)$ and the properties of the given reactor. ℓ_g can be regarded as the mean time required for a neutron in the reactor (distributed as $\psi(\bar{r}, \bar{\Omega}, u)/v(u)$) to produce either another neutron or a delayed neutron precursor. Hence, if there are a total of N neutrons in the reactor and each one causes another neutron or precursor to be produced in ℓ_g secs then the production rate is N/ℓ_g neutrons/sec. Equivalently, if the neutron flux were maintained at $\psi(\bar{r}, \bar{\Omega}, u)$ the neutron production rate would eventually attain the rate R , where R is given by

$$(A.10) \quad R(\text{neutrons/sec}) = \frac{1}{\ell_g} \int_V \int_{u'} \frac{\phi(\bar{r}, u') du' dV}{v(u')}.$$

Since the production rate R is equal to

$$\frac{1}{\lambda_c} \int_V \int_{u'} \sum_j \nu^j(u') \Sigma_F^j(\bar{r}, u') \phi(\bar{r}, u') du' dV$$

if the neutron flux is $\psi(\bar{r}, \bar{\Omega}, u)$, it follows that ℓ_g can be found from the formula

$$(A.11) \quad \ell_g = \frac{\int_V \int_{u'} \frac{\phi(\bar{r}, u') du' dV}{v(u')}}{\frac{1}{\lambda_c} \int_V \int_{u'} \sum_j \nu^j(u') \Sigma_F^j(\bar{r}, u') \phi(\bar{r}, u') du' dV}.$$

From this formula and (A.9) it is apparent that

$$(A.12) \quad \ell_g = \frac{\lambda_c}{\lambda} \ell.$$

If $\psi^*(\bar{r}, \bar{\Omega}, u)$ were equal to unity, the formulae for $\bar{\beta}_i$ and Λ become, respectively,

$$(A.13) \quad \bar{\beta}'_i = (\lambda_c^m \lambda^n) \beta_i / \lambda_c ,$$

and

$$(A.14) \quad \Lambda' = (\lambda_c^m \lambda^n) \ell / \lambda$$

or

$$(A.15) \quad \Lambda' = (\lambda_c^m \lambda^n) \ell_g / \lambda_c ,$$

where the primes denote that the quantities are defined in terms of unweighted (by $\psi^*(\bar{r}, \bar{\Omega}, u)$) averages.

It might be noted that when $m = 0$ and $n = 1$, Λ' is indeed equal to the mean lifetime ℓ , so that it is appropriate to speak of Λ as the effective lifetime. But when $m = 1$ and $n = 0$, Λ might more appropriately be called an effective "generation" time rather than a lifetime because of the direct correspondence of the unweighted value Λ' to ℓ_g in this case.

Values of m and n		Reactivity ρ	Difference in Reactivity $\rho_2 - \rho_1$	Effective Neutron Lifetime Λ	Effective ith Delayed Neutron Fraction $\bar{\beta}_i$	Unweighted Effective ith Delayed Neutron Fraction $\bar{\beta}_i$ (For $\psi^* = 1$)	Unweighted Neutron Lifetime Λ' (For $\psi^* = 1$)
m	n						
0	0	$\frac{\lambda - \lambda_c}{\lambda \lambda_c}$	$\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}$	$\frac{\lambda - \lambda_l}{\alpha_l \lambda \lambda_l}$	$\frac{\beta_i}{\lambda_c} \prod_{m=1}^{m'} \left(\frac{\lambda \beta_i}{\lambda} \right)$	$\frac{1}{\lambda_c} \beta_i$	$\frac{1}{\lambda} l$ or $\frac{1}{\lambda_c} l_g$
0	1	$\frac{\lambda - \lambda_c}{\lambda_c}$	$\frac{\lambda_2 - \lambda_1}{\lambda_c}$	$\frac{\lambda - \lambda_l}{\alpha_l \lambda_l}$	$\frac{\lambda}{\lambda_c} \beta_i \prod_{m=1}^m \left(\frac{\lambda \beta_i}{\lambda} \right)$	$\frac{\lambda}{\lambda_c} \beta_i$	l or $\frac{\lambda}{\lambda_c} l_g$
1	0	$\frac{\lambda - \lambda_c}{\lambda}$	$\frac{\lambda_c (\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}$	$\frac{\lambda_c (\lambda - \lambda_l)}{\alpha_l \lambda \lambda_l}$	$\beta_i \prod_{m=1}^{m'} \left(\frac{\lambda \beta_i}{\lambda} \right)$	β_i	$\frac{\lambda_c}{\lambda} l$ or l_g
1	1	$\lambda - \lambda_c$	$\lambda_2 - \lambda_1$	$\frac{\lambda_c (\lambda - \lambda_l)}{\alpha_l \lambda_l}$	$\lambda \beta_i \prod_{m=1}^{m'} \left(\frac{\lambda \beta_i}{\lambda} \right)$	$\lambda \beta_i$	$\lambda_c l$ or λl_g

TABLE A

SOME COMMON CONSISTENT SETS OF PARAMETER DEFINITIONS

APPENDIX B

THE EXPEDIENCY OF THE CHOICE OF THE WEIGHTING FUNCTION $\psi^*(\bar{r}, \bar{\Omega}, u)$

Let $\psi_a^*(\bar{r}, \bar{\Omega}, u)$ be the solution of the equation which is adjoint to the source free, steady-state equation for some reactor "a". The choice of this reactor "a" is completely arbitrary and the adjoint equation is written

$$\begin{aligned}
 (B.1) \quad & \bar{\Omega}' \cdot \nabla \psi_a^*(\bar{r}, \bar{\Omega}', u') - \Sigma_a^a(\bar{r}, u') \psi_a^*(\bar{r}, \bar{\Omega}', u') \\
 & + \int_{\Omega} \int_u \Sigma_s^a(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_a^*(\bar{r}, \bar{\Omega}, u) du d\Omega \\
 & + \frac{1}{\lambda_a} \int_u \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]^a \phi_a^*(\bar{r}, \bar{\Omega}, u) du = 0, \text{ where} \\
 & \phi_a^*(\bar{r}, \bar{\Omega}, u) = \frac{1}{4\pi} \int_{\Omega} \psi_a^*(\bar{r}, \bar{\Omega}, u) d\Omega.
 \end{aligned}$$

Now, consider the perturbed, source-free, steady-state form of the reactor of interest, that is, the reactor to which Eq. (1) applies. The equation for this perturbed, fictitiously critical reactor is Eq. (82), which can be written in the form

$$\begin{aligned}
 (B.2) \quad & -\bar{\Omega} \cdot \nabla \psi_p(\bar{r}, \bar{\Omega}, u) - \left[\Sigma_p^a(\bar{r}, u) + \Delta \Sigma(\bar{r}, u) + \delta \Sigma(\bar{r}, u) \right] \psi_p(\bar{r}, \bar{\Omega}, u) \\
 & + \int_{\Omega'} \int_{u'} \left[\Sigma_s^a(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) + \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) + \delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \right] \psi_p(\bar{r}, \bar{\Omega}', u') du' d\Omega' \\
 & + g(\bar{r}, \bar{\Omega}, u) \\
 & + \frac{1}{\lambda_p} \int_{u'} \left\{ \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]^a + \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \right. \\
 & \left. + \delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \right\} \phi_p(\bar{r}, u') du' = 0, \text{ where} \\
 & \phi_p(\bar{r}, u) \equiv \int_{\Omega} \psi_p(\bar{r}, \bar{\Omega}, u) d\Omega \text{ and } \Sigma_p(\bar{r}, u), \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \text{ and} \\
 & \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]_p \text{ have been expressed as}
 \end{aligned}$$

$$(B.3) \quad \Sigma_p(\bar{r}, u) \equiv \Sigma^a(\bar{r}, u) + \Delta\Sigma(\bar{r}, u) + \delta\Sigma(\bar{r}, u),$$

$$\Sigma_{sp}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \equiv \Sigma_s^a(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) + \Delta\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) + \delta\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}), \text{ and}$$

$$\left[\sum_j f_t(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]_p \equiv \left[\sum_j f_t(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]^a$$

$$+ \Delta \left[\sum_j f_t(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] + \delta \left[\sum_j f_t(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right].$$

These Δ quantities represent the differences between the arbitrary reactor "a" and the reactor to which Eq. (1) applies. These quantities must be regarded as appreciably large because of the arbitrariness of reactor "a".

The δ quantities represent, as before, vanishingly small perturbations of the quantities for the unperturbed reactor of Eq. (62), the source-free steady-state form of Eq. (1). $g(\bar{r}, \bar{\Omega}, u)$ is a source, the presence of which (in order to restore the problem of solving Eq. (B.2) to that of solving a homogeneous eigenvalue problem) requires one to impose such a restriction as

$$(B.4) \quad \frac{1}{\lambda_p} \int_V \int_{u'} \left[\sum_j v^j(u') \Sigma_f^j(\bar{r}, u') \right]_p \phi_p(\bar{r}, u') du' dV = A,$$

where A is some constant. As before $g(\bar{r}, \bar{\Omega}, u)$ is vanishingly small with respect to A .

Under these conditions

$$(B.5) \quad \psi_p(\bar{r}, \bar{\Omega}, u) = \psi(\bar{r}, \bar{\Omega}, u) + \delta\psi(\bar{r}, \bar{\Omega}, u)$$

and

$$\lambda_p = \lambda + \delta\lambda$$

where the δ quantities are vanishingly small.

Eq. (B.1) is now multiplied by $\psi_p(\bar{r}, \bar{\Omega}', u')$ and subtracted from the product of Eq. (B.2) and $\psi_a^*(\bar{r}, \bar{\Omega}, u)$. The difference is then integrated over all V , Ω , and u . After applying the identity (12) and Green's Theorem,

the divergence terms can be written in terms of a surface integral. This surface integral can be made to vanish because a surface can always be chosen which is common to both reactors and on which the desired boundary conditions apply; namely, that $\psi_a^*(\bar{r}, \bar{\Omega}, u)$ be zero for all outward $\bar{\Omega}$ and that $\psi_a(\bar{r}, \bar{\Omega}, u)$ be zero for all inward $\bar{\Omega}$.

The result of subtracting the integrals of the two products is, after some manipulation,

$$\begin{aligned}
 (B.6) \quad & - \int_V \int_{\Omega} \int_u \Delta \Sigma(\bar{r}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi_p(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
 & + \int_V \int_{\Omega} \int_u \int_{\Omega'} \int_{u'} \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi_p(\bar{r}, \bar{\Omega}', u') du' d\Omega' du d\Omega dV \\
 & + \frac{1}{\lambda_p} \int_V \int_u \int_{u'} \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi_a^*(\bar{r}, u) \phi_p(\bar{r}, u') du' du dV \\
 & + \left(\frac{1}{\lambda_p} - \frac{1}{\lambda_a} \right) \int_V \int_u \int_{u'} \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]^a \phi_a^*(\bar{r}, u) \phi_p(\bar{r}, u) du' du dV \\
 & + \int_V \int_{\Omega} \int_u g(\bar{r}, \bar{\Omega}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
 & - \int_V \int_{\Omega} \int_u \delta \Sigma(\bar{r}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi_p(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
 & + \int_V \int_{\Omega} \int_u \int_{\Omega'} \int_{u'} \delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi_p(\bar{r}, \bar{\Omega}', u') du' d\Omega' du d\Omega dV \\
 & + \frac{1}{\lambda_p} \int_V \int_u \int_{u'} \delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi_a^*(\bar{r}, u) \phi_p(\bar{r}, u') du' du dV = 0.
 \end{aligned}$$

One can define

$$(B.7) \quad U_a \equiv \frac{1}{\lambda_c^m \lambda^n} \int_V \int_u \int_{u'} \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right]^a \phi_a^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV.$$

Next, Eqs. (B.5) can be used to express Eq. (B.6) in terms of $\psi(\bar{r}, \bar{\Omega}, u)$, $\delta\psi(\bar{r}, \bar{\Omega}, u)$, λ , and $\delta\lambda$. After second order terms are neglected (terms which contain the product of two or more δ quantities), the equation can be divided by U_a to obtain the result

$$\begin{aligned}
(B.8) \quad & \frac{1}{U_a} \int_V \int_{\Omega} \int_u g(\bar{r}, \bar{\Omega}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) du d\Omega dV - \frac{1}{U_a} \int_V \int_{\Omega} \int_u \delta \Sigma(\bar{r}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
& + \frac{1}{U_a} \int_V \int_{\Omega} \int_u \int_{\Omega'} \int_{u'} \delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}', u') du' d\Omega' du d\Omega dV \\
& + \frac{1}{\lambda_p U_a} \int_V \int_u \int_{u'} \delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi_a^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV = \frac{(\lambda_p - \lambda) \lambda_c^m \lambda^n}{\lambda \lambda_p} \\
& - \frac{\delta \lambda (\lambda_c^m \lambda^n) (\lambda - \lambda_a)}{\lambda_a \lambda^2} + \frac{1}{U_a} \int_V \int_{\Omega} \int_u \Delta \Sigma(\bar{r}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) \delta \psi(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\
& - \frac{1}{U_a} \int_V \int_{\Omega} \int_u \int_{\Omega'} \int_{u'} \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_a^*(\bar{r}, \bar{\Omega}, u) \delta \psi(\bar{r}, \bar{\Omega}', u') du' d\Omega' du d\Omega dV \\
& - \frac{1}{\lambda U_a} \int_V \int_u \int_{u'} \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi_a^*(\bar{r}, u) \delta \phi(\bar{r}, u') du' du dV \\
& + \frac{\delta \lambda}{\lambda^2 U_a} \int_V \int_u \int_{u'} \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi_a^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV \\
& + \left\{ \frac{\lambda_c^m \lambda^n (\lambda - \lambda_a)}{\lambda_a \lambda} + \frac{1}{U_a} \int_V \int_{\Omega} \int_u \Delta \Sigma(\bar{r}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u) du d\Omega dV \right. \\
& - \frac{1}{U_a} \int_V \int_{\Omega} \int_u \int_{\Omega'} \int_{u'} \Delta \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}', u') du' d\Omega' du d\Omega dV \\
& \left. - \frac{1}{\lambda U_a} \int_V \int_u \int_{u'} \Delta \left[\sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \right] \phi_a^*(\bar{r}, u) \phi(\bar{r}, u') du' du dV \right\}.
\end{aligned}$$

The part of Eq. (B.8) enclosed by the braces is zero, as can be easily shown by subtracting the product of Eq. (B.1) and $\psi(\bar{r}, \bar{\Omega}', u')$ from the product of Eq. (62) and $\psi_a^*(\bar{r}, \bar{\Omega}, u)$ and then integrating over all V , Ω , and u .

It is apparent from Eq. (B.8) that the left hand side is not equal to $(\lambda_p - \lambda)(\lambda_c^m \lambda^n)/\lambda \lambda_p$ unless $\lambda = \lambda_a$ and all the Δ quantities are zero. However, if and only if all the Δ quantities are zero, then $\lambda = \lambda_a$, $\psi^*(\bar{r}, \bar{\Omega}, u) = \psi_a^*(\bar{r}, \bar{\Omega}, u)$, $U_a = U$, and Eq. (B.8) becomes identical to Eq. (84). Hence, it is only when $\psi_a^*(\bar{r}, \bar{\Omega}, u)$ is equal to $\psi^*(\bar{r}, \bar{\Omega}, u)$ that the left hand side of Eq. (B.8) can be determined solely as a function of the eigenvalues λ_c , λ_p , and λ ; and the expediency of choosing the weighting function to be $\psi^*(\bar{r}, \bar{\Omega}, u)$ is amply demonstrated.

If the weighting function remains arbitrary, then to evaluate a quantity such as

$$(B.9) \quad \frac{1}{U_a} \int_V \int_{\Omega} \int_u \delta\Sigma(\bar{r}, u) \psi_a^*(\bar{r}, \bar{\Omega}, u) \psi(\bar{r}, \bar{\Omega}, u) du d\Omega dV$$

(where $\delta\Sigma(\bar{r}, u) = \alpha_g/v(u)$, as is required for the evaluation of the effective prompt neutron lifetime) one is required to solve for both $\psi_a^*(\bar{r}, \bar{\Omega}, u)$ and $\psi(\bar{r}, \bar{\Omega}, u)$ and perform the laborious integrations to evaluate both the numerator and U_a .

APPENDIX C

THE UNWEIGHTED NEUTRON LIFETIME ℓ

If one defines $B(\bar{r}, \bar{\Omega}, u) du d\Omega dV \equiv$ the rate of injection of neutrons into a given reactor at $\bar{r}$, $\bar{\Omega}$, and u in the intervals du , $d\Omega$, and dV , then the total of such neutrons injected or "born" per unit time is

$$(C.1) \quad B \equiv \int_V \int_{\Omega} \int_u B(\bar{r}, \bar{\Omega}, u) du d\Omega dV.$$

Moreover, for a neutron selected at random from the collection B , one can define a probability or distribution function such that $p(t)dt \equiv$ the probability that a randomly selected neutron from B will "die" in the interval $(t, t + dt)$, where t is the time elapsed since the birth of the neutron selected. (Equivalently, if all the neutrons in B were born at time $t = 0$, then $Bp(t)$ is the expected death rate of the neutrons at time t .)

The probability $p(t)dt$ for the entire collection of neutrons B injected per unit time can be defined in terms of the probability

$$(C.2) \quad p(\bar{r}', \bar{\Omega}', u', \bar{r}, \bar{\Omega}, u, \tau) du d\Omega dV d\tau \equiv \text{the probability that a neutron injected at } \bar{r}', \bar{\Omega}', \text{ and } u' \text{ will leak out or be absorbed in the intervals } du d\Omega dV \text{ about the points } u, \bar{\Omega}, \text{ and } \bar{r} \text{ and in the interval } d\tau \text{ at a time } \tau \text{ after the injection.}$$

It follows directly that one can define

$$(C.3) \quad p(\bar{r}', \bar{\Omega}', u', \tau) d\tau \equiv \int_V \int_{\Omega} \int_u p(\bar{r}', \bar{\Omega}', u', \bar{r}, \bar{\Omega}, u, \tau) du d\Omega dV d\tau$$

$\equiv$ the probability that a neutron injected at $\bar{r}', \bar{\Omega}', u'$ will leak out or be absorbed from the reactor in $d\tau$ at time τ after the injection; and finally,

$$(C.4) \quad p(\tau) d\tau \equiv \frac{1}{B} \int_V \int_{\Omega} \int_u p(\bar{r}', \bar{\Omega}', u', \tau) B(\bar{r}', \bar{\Omega}', u') du' d\Omega' dV' d\tau.$$

The mean lifetime of the B neutrons for the neutron generation is, by the usual definition,

$$(C.5) \quad \ell \equiv \int_0^{\infty} t p(t) dt \text{ since } \int_0^{\infty} p(t) dt = 1.0.$$

For the reactor to which Eq. (1) applies, if one lets $B(\bar{r}, \bar{\Omega}, u) dV d\Omega du$ be the birth rate of neutrons, then the net rate of loss or the death rate is

$$(C.6) \quad D(\bar{r}, \bar{\Omega}, u, t) dV d\Omega du \equiv \left[\bar{\Omega} \cdot \nabla \psi_s(\bar{r}, \bar{\Omega}, u, t) + \Sigma(\bar{r}, u, t) \psi_s(\bar{r}, \bar{\Omega}, u, t) \right. \\ \left. - \int_{\Omega'} \int_{u'} du' d\Omega' \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi_s(\bar{r}, \bar{\Omega}', u', t) \right] dV d\Omega du,$$

where $\psi_s(\bar{r}, \bar{\Omega}, u, t)$ is the solution of the inhomogeneous equation

$$(C.7) \quad \frac{\partial \left[\frac{\psi_s(\bar{r}, \bar{\Omega}, u, t)}{v(u)} \right]}{\partial t} = - \bar{\Omega} \cdot \nabla \psi_s(\bar{r}, \bar{\Omega}, u, t) - \Sigma(\bar{r}, u, t) \psi_s(\bar{r}, \bar{\Omega}, u, t) \\ + \int_{\Omega'} \int_{u'} du' d\Omega' \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi_s(\bar{r}, \bar{\Omega}', u', t) + B(\bar{r}, \bar{\Omega}, u).$$

Define the total death and birth rates:

$$(C.8) \quad D(t) = \int_V \int_{\Omega} \int_u D(\bar{r}, \bar{\Omega}, u, t) du d\Omega dV$$

and

$$(C.9) \quad B = \int_V \int_{\Omega} \int_u B(\bar{r}, \bar{\Omega}, u) du d\Omega dV.$$

[NOTE: It should be pointed out that $D(\bar{r}, \bar{\Omega}, u, t) dV d\Omega du$ is really the rate of net loss rather than total loss from the element of phase space $dV d\Omega du$; but this quantity is what is desired for a definition of the detailed death rate. This is because of the fact that the total death rate for the reactor is, according to (C.6) and (C.8)

$$(C.10) \quad D(t) = \int_V \int_{\Omega} \int_u \left[\bar{\Omega} \cdot \nabla \psi_s(\bar{r}, \bar{\Omega}, u, t) \right] dV d\Omega du + \int_V \int_{\Omega} \int_u \Sigma(\bar{r}, u, t) \psi_a(\bar{r}, \bar{\Omega}, u, t) dV d\Omega du \\ - \int_V \int_{\Omega} \int_u \int_{\Omega'} \int_{u'} \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t) \psi_s(\bar{r}, \bar{\Omega}', u', t) du' d\Omega' du d\Omega dV.$$

The first term, by Green's Theorem, is equal to

$$\int_S \int_{\Omega} \int_u \bar{n} \cdot \bar{\Omega} \psi_s(\bar{r}, \bar{\Omega}, u, t) du d\Omega ds,$$

which is obviously the rate of leakage from the surface of the reactor.

The second term is

$$(C.11) \quad \int_V \int_u \Sigma(\bar{r}, u, t) \phi_s(\bar{r}, u, t) du dV \\ = \int_V \int_u \left[\Sigma_a(\bar{r}, u, t) + \int_{u'} \Sigma_s^0(\bar{r}, u, u', t) du' \right] \phi_s(\bar{r}, u, t) du dV,$$

where $\Sigma_s^0(\bar{r}, u', u, t)$ is the zeroth moment in a normalized spherical harmonic expansion of the scattering cross section. The third term can be shown to be equal to

$$\int_V \int_{u'} \int_u \Sigma_s^0(\bar{r}, u', u, t) du \phi_s(\bar{r}, u', t) du' dV.$$

This follows easily if $\psi_s(\bar{r}, \bar{\Omega}, u, t)$ and $\Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}, t)$ are expanded in spherical harmonics. The orthogonality properties of the spherical harmonics cause all terms to be equal to zero when integrated over all Ω and Ω' except the term containing the product of the zeroth moments of the flux and cross section. The zeroth moment in the expansion for $\psi_s(\bar{r}, \bar{\Omega}, u, t)$ is $\phi_s(\bar{r}, u, t)$ so that the scattering cross section terms cancel and one can immediately write

$$(C.12) \quad D(t) = \int_S \int_{\Omega} \int_u \bar{n} \cdot \bar{\Omega} \psi_s(\bar{r}, \bar{\Omega}, u, t) du d\Omega ds + \int_V \int_u \Sigma_a(\bar{r}, u, t) \phi_s(\bar{r}, u, t) du dV \Big]$$

The total number of neutrons in the reactor at time t is

$$(C.13) \quad N(t) \equiv \int_V \int_{\Omega} \int_u \frac{\psi_s(\bar{r}, \bar{\Omega}, u, t)}{v(u)} du d\Omega dV,$$

hence, one can integrate Eq. (C.7) over all u , Ω , and V to obtain

$$(C.14) \quad \frac{dN(t)}{dt} = B - D(t).$$

But, by the definition of $p(t)dt$, it follows that if $B = 0$ for $t < 0$ then the death rate $D(t)$ is

$$(C.15) \quad D(t) = \int_0^t Bp(t-\tau)d\tau,$$

or with a change of variable such that $x = t-\tau$, $D(t)$ can be written

$$(C.16) \quad D(t) = \int_0^t Bp(x)dx.$$

Eq. (C.14) is now integrated with respect to t over the range $(0, t)$, and the result in terms of (C.16) is

$$(C.17) \quad N(t) - N(0) = \int_0^t Bdt' - \int_0^t \int_0^{t'} Bp(x)dxdt'.$$

It is now assumed (consistent with the fact that $B = 0$ for $t < 0$) that $N(0) = 0$ and Eq. (C.17) becomes

$$(C.18) \quad N(t) = Bt - B \int_0^t \int_0^{t'} p(x)dxdt'.$$

If one integrates by parts and lets $v(t') = \int_0^{t'} p(x)dx$, $\frac{dv(t')}{dt'} dt' = p(t')$ and $u(t') = t'$ and

$$(C.19) \quad \int_0^t \int_0^{t'} p(x)dxdt' = t' \int_0^{t'} p(x)dx \Big|_0^t - \int_0^t t' p(t') dt'.$$

(C.18) can now be rewritten as

$$(C.20) \quad N(t) = Bt[1-P(t)] + B \int_0^t t' p(t') dt',$$

where, by definition

$$(C.21) \quad P(t) \equiv \int_0^t p(x)dx.$$

The limit as $t \rightarrow \infty$ of Eq. (C.20) is now evaluated with the result that

$$(C.22) \quad \lim_{t \rightarrow \infty} N(t) = B \lim_{t \rightarrow \infty} t[1-P(t)] + B \lim_{t \rightarrow \infty} \int_0^t t' p(t') dt'.$$

But, by L'Hospital's rule,

$$(C.23) \quad \lim_{t \rightarrow \infty} t[1-P(t)] = \lim_{t \rightarrow \infty} \frac{1-P(t)}{1/t} = \lim_{t \rightarrow \infty} t^2 p(t).$$

Now, in all practical reactors, $p(t) \leq A_1 e^{-A_2 t}$ where A_1 and A_2 are constants*. Hence, it follows that $\lim_{t \rightarrow \infty} t^2 p(t) = 0$ and that, consequently, the limit as $t \rightarrow \infty$ of $\int_0^t t' p(t') dt'$ exists. This latter limit is ℓ , by definition, so that if $N_\infty \equiv \lim_{t \rightarrow \infty} N(t)$, one can write

$$(C.24) \quad N_\infty = B\ell \text{ or}$$

$$(C.25) \quad \ell = N_\infty / B.$$

L. A. Schmittroth (private communication) pointed out that $\lim_{t \rightarrow \infty} t[1-P(t)] = 0$ follows from the assumption that the limit

$$\ell = \lim_{T \rightarrow \infty} \int_0^T x p(x) dx \text{ exists.}$$

The proof is as follows: For $t \leq x$, $tp(x) \leq xp(x)$; therefore

$$(C.26) \quad t \int_t^\infty p(x) dx \leq \int_t^\infty xp(x) dx = \ell - \int_0^t xp(x) dx.$$

And since

$$\int_t^\infty p(x) dx = \int_0^\infty p(x) dx - \int_0^t p(x) dx = 1 - P(t),$$

it follows that

$$t[1-P(t)] \leq \ell - \int_0^t xp(x) dx.$$

* Consider, for example, A_2 to be the smallest value of the product $\sum_a (\bar{r}, u) v(u)$ which exists in the reactor.

Finally,

$$\lim_{t \rightarrow \infty} t [1 - P(t)] \leq \ell - \lim_{t \rightarrow \infty} \int_0^t x p(x) dx = \ell - \ell = 0.$$

But, since $t [1 - P(t)] \geq 0$ for all t , $0 \leq \lim_{t \rightarrow \infty} t [1 - P(t)] \leq 0$;

and therefore $\lim_{t \rightarrow \infty} t [1 - P(t)] = 0$.

It has been demonstrated that if a constant source equal to $B(\bar{r}, \bar{\Omega}, u)$ exists in the reactor starting at time $t = 0$ a steady-state condition will finally attain. In this steady-state condition the number of neutrons in the reactor is N_∞ and the lifetime can therefore be found from Eq. (C.25) if N_∞ can be evaluated (since, certainly, $B = \int_V \int_\Omega \int_u B(\bar{r}, \bar{\Omega}, u) du d\Omega dV$ is known).

N_∞ can be evaluated for the reactor of interest by simply solving the steady-state form of Eq. (C.7). If the steady-state flux solution is denoted $\psi_{s\infty}(\bar{r}, \bar{\Omega}, u)$, then

$$(C.27) \quad N_\infty = \int_V \int_\Omega \int_u \frac{\psi_{s\infty}(\bar{r}, \bar{\Omega}, u)}{v(u)} du d\Omega dV$$

and one can immediately write

$$(C.28) \quad \ell = \frac{\int_V \int_\Omega \int_u \frac{\psi_{s\infty}(\bar{r}, \bar{\Omega}, u)}{v(u)} du d\Omega dV}{\int_V \int_\Omega \int_u B(\bar{r}, \bar{\Omega}, u) du d\Omega dV}.$$

In the reactor of interest (that reactor to which Eq. (1) applies), when a fission occurs some neutrons are emitted as "prompt neutrons", and the remainder (the fraction β) are emitted at later times. If one concerns oneself with the lifetimes of all these neutrons which are caused by fissions at time t , then the birth rate of interest is obviously given by a $B(\bar{r}, \bar{\Omega}, u)$ equal to

$$(C.29) \quad \frac{1}{4\pi\lambda} \int_{\Omega'} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u', t) \mathcal{H}(\bar{r}, \bar{\Omega}', u', t) du' d\Omega'.$$

Again, the assumption is made that $\mathcal{H}(\bar{r}, \bar{\Omega}, u, t) = C \psi(\bar{r}, \bar{\Omega}, u) T(t)$, where $\psi(\bar{r}, \bar{\Omega}, u)$ is the solution of Eq. (62), the steady-state, source-free form of Eq. (1) at time t . Then the mean lifetime of the neutrons from the source (C.29) can be written

$$(C.30) \quad \ell = \frac{\int_V \int_u \frac{\phi_{s\infty}(\bar{r}, u)}{v(u)} du dV}{\frac{CT(t)}{\lambda_c} \int_V \int_u \sum_j v^j(u) \Sigma_f^j(\bar{r}, u) \phi(\bar{r}, u) du dV},$$

where

$$(C.31) \quad \phi_{s\infty}(\bar{r}, u) = \int_{\bar{\Omega}} \psi_{s\infty}(\bar{r}, \bar{\Omega}, u) d\bar{\Omega}$$

and $\psi_{s\infty}(\bar{r}, \bar{\Omega}, u)$ is now the solution of the steady-state inhomogeneous equation

$$(C.32) \quad -\bar{\Omega} \cdot \nabla \psi_{s\infty}(\bar{r}, \bar{\Omega}, u) - \Sigma(\bar{r}, u) \psi_{s\infty}(\bar{r}, \bar{\Omega}, u) + \int_{\bar{\Omega}'} \int_{u'} \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_{s\infty}(\bar{r}, \bar{\Omega}', u') du' d\bar{\Omega}' + \frac{CT}{4\pi\lambda_c} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV = 0.$$

But if Eq. (62) is multiplied by $\frac{CT\lambda}{\lambda_c}$ one obtains

$$(C.33) \quad \frac{CT\lambda}{\lambda_c} \left[-\bar{\Omega} \cdot \nabla \psi(\bar{r}, \bar{\Omega}, u) - \Sigma(\bar{r}, u) \psi(\bar{r}, \bar{\Omega}, u) + \int_{\bar{\Omega}'} \int_{u'} \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi(\bar{r}, \bar{\Omega}', u') du' d\bar{\Omega}' \right] + \frac{CT}{4\pi\lambda_c} \int_{u'} \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV = 0,$$

so that it is now obvious on comparison of Eq. (C.32) to Eq. (C.33) that $\psi_{s\infty}(\bar{r}, \bar{\Omega}, u) = \frac{CT\lambda}{\lambda_c} \psi(\bar{r}, \bar{\Omega}, u)$, and that

$$(C.34) \quad \ell = \frac{\frac{\lambda}{\lambda_c} \int_V \int_u \frac{\phi(\bar{r}, u)}{v(u)} du dV}{\frac{1}{\lambda_c} \int_V \int_u \sum_j v^j(u) \Sigma_f^j(\bar{r}, u) \phi(\bar{r}, u) du dV}.$$

By simple inspection, it is apparent that if $\psi^*(\bar{r}, \bar{\Omega}, u) = 1$ in the definition of the effective lifetime Λ (see Eq. (69), one arrives at

$$(C.35) \quad \Lambda' = \frac{\frac{\lambda_c^m \lambda^n}{\lambda_c} \int_V \int_u \frac{\phi(\bar{r}, u)}{v(u)} du dV}{\frac{1}{\lambda_c} \int_V \int_u \sum_j v^j(u) \Sigma_f^j(\bar{r}, u) \phi(\bar{r}, u) du dV} .$$

Hence, if one chooses $m = 0$ and $n = 1$, the unweighted neutron lifetime Λ' is given by

$$(C.36) \quad \Lambda' = \ell.$$

APPENDIX D

AN ALTERNATIVE DERIVATION OF THE

METHOD OF NEUTRON GENERATION MULTIPLICATION

The solution of Eq. (82) for the eigenvalue λ_p is now considered, given an arbitrary source $g(\bar{r}, \bar{\Omega}, u)$ and the constraint

$$(D.1) \quad \frac{1}{\lambda_p} \int_V \int_{u'} \sum_j \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_p(\bar{r}, u') du' dV = A.$$

It is assumed that

$$(D.2) \quad \Sigma_p(\bar{r}, u) = \Sigma(\bar{r}, u),$$

$$\Sigma_{sp}(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) = \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}), \text{ and}$$

$$\left[\sum_j f_t^j(u) \nu^j(u') \Sigma_f(\bar{r}, u') \right]_p = \left[\sum_j f_t^j(u) \nu^j(u') \Sigma_f(\bar{r}, u') \right],$$

so that except for the presence of the small source $g(\bar{r}, \bar{\Omega}, u)$, the solution to Eq. (82) would be identical to the solution to Eq. (62). That is, $\psi_p(\bar{r}, \bar{\Omega}, u)$ would be equal to $\psi(\bar{r}, \bar{\Omega}, u)$ and λ_p would be equal to λ . The equation to be solved is therefore

$$(D.3) \quad -\bar{\Omega} \cdot \nabla \psi_p(\bar{r}, \bar{\Omega}, u) - \Sigma(\bar{r}, u) \psi_p(\bar{r}, \bar{\Omega}, u) \\ + \int_{\bar{\Omega}'} \int_{u'} \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_p(\bar{r}, \bar{\Omega}', u') du' d\bar{\Omega}' + g(\bar{r}, \bar{\Omega}, u) \\ + \frac{1}{4\pi\lambda_p} \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_p(\bar{r}, u') du' = 0, \text{ where} \\ \phi_p \equiv \int_{\bar{\Omega}} \psi_p(\bar{r}, \bar{\Omega}, u) \equiv \phi(\bar{r}, u) + \delta\phi(\bar{r}, u),$$

and $\psi_p(\bar{r}, \bar{\Omega}, u)$ is subject to the constraint (D.1).

It is to be shown (as stated by Eq. (137) that

$$(D.4) \quad \lim_{\alpha_p \rightarrow 0} \frac{(\lambda_p - \lambda_c) \lambda_c^m \lambda_p^n}{\alpha_p \lambda_p \lambda_p} = \frac{B_1}{B_0} \quad \lim_{\ell \rightarrow \infty} \prod_{k=1}^{\ell} \left(\frac{\lambda_k}{\lambda} \right)$$

where λ_p is the solution of (D.3) when

$$(D.5) \quad B_1(\bar{r}, \bar{\Omega}, u) \equiv g(\bar{r}, \bar{\Omega}, u) / \alpha_p$$

$$(D.6) \quad B_1 = \int_V \int_{\Omega} \int_u B_1(\bar{r}, \bar{\Omega}, u) du d\Omega dV,$$

$$(D.7) \quad B_0 = \frac{1}{\lambda_c^m \lambda_p^n} \int_V \int_{u'} \sum_j \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi(\bar{r}, u') du' dV,$$

and the λ_k are defined as follows.

$$(D.8) \quad \lambda_1 \equiv \frac{\int_V \int_{u'} \sum_j \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_1(\bar{r}, u') du' dV}{\int_V \int_{\Omega} \int_u g(\bar{r}, \bar{\Omega}, u) du d\Omega dV},$$

where $\phi_1(\bar{r}, u')$ is the solution of the inhomogeneous equation

$$(D.9) \quad -\bar{\Omega} \cdot \nabla \psi_1(\bar{r}, \bar{\Omega}, u) - \Sigma(\bar{r}, u) \psi_1(\bar{r}, \bar{\Omega}, u) + \int_{\Omega'} \int_{u'} \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_1(\bar{r}, \bar{\Omega}', u') du' d\Omega' + g(\bar{r}, \bar{\Omega}, u) = 0.$$

$$(D.10) \quad \lambda_k \equiv \frac{\int_V \int_{u'} \sum_j \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_k(\bar{r}, u') du' dV}{\int_V \int_{u'} \sum_j \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_{k-1}(\bar{r}, u') du' dV},$$

where $\phi_k(\bar{r}, u')$ is the solution of the inhomogeneous equation

$$(D.11) \quad -\bar{\Omega} \cdot \nabla \psi_k(\bar{r}, \bar{\Omega}, u) - \Sigma(\bar{r}, u) \psi_k(\bar{r}, \bar{\Omega}, u) + \int_{\Omega'} \int_{u'} \Sigma_s(\bar{r}, u', u, \bar{\Omega}' \cdot \bar{\Omega}) \psi_k(\bar{r}, \bar{\Omega}', u') du' d\Omega' + \frac{1}{4\pi} \int_{u'} \sum_j f_t^j(u) \nu^j(u') \Sigma_f^j(\bar{r}, u') \phi_{k-1}(\bar{r}, u') du' = 0.$$

These λ_k values are obviously identical to the λ_k values defined in sub-section 5.3 when $g(\bar{r}, \bar{\Omega}, u)$ is equal to α_p times $B_1(\bar{r}, \bar{\Omega}, u)$.

Eq. (D.3) is now solved by the well-known neutron source iteration method which many one-dimensional (in $\bar{r}$) reactor computer codes presently use.

The method proceeds from a guess for the fission neutron distribution and computes the neutron fluxes which would exist in the reactor for that source guess. These fluxes are then used to compute a new fission neutron source guess which is immediately renormalized such that the number of neutrons per sec produced in the new source distribution is equal to that of the original source guess. [This renormalization automatically imposes the constraint (D.1).]

The new source is next used as the source guess for the second iteration, etc., until a persisting ("converged") source or flux distribution is attained after a sufficient number of iterations.

Since the first fission neutron source guess is arbitrary, one can consider the source guess

$$(D.12) \quad S_0(\bar{r}, \bar{\Omega}, u) = \frac{1}{4\pi\lambda_p} \int_{u'} H(\bar{r}, u, u') \phi(\bar{r}, u') du',$$

where

$$(D.13) \quad H(\bar{r}, u, u') \equiv \sum_j f_t^j(u) v^j(u') \Sigma_f^j(\bar{r}, u');$$

and one can define

$$(D.14) \quad S_0 \equiv \int_V \int_{\Omega} \int_u S_0(\bar{r}, \bar{\Omega}, u) du d\Omega dV = \frac{1}{\lambda_p} \int_V \int_u \int_{u'} H(\bar{r}, u, u') \phi(\bar{r}, u') du' du dV.$$

If this source distribution $S_0(\bar{r}, \bar{\Omega}, u)$ is entered in Eq. (D.3) in place of the converged fission source, one obtains

$$(D.15) \quad -\bar{\Omega} \cdot \nabla \psi_p^1 - \Sigma \psi_p^1 + \int_{\Omega'} \int_{u'} \Sigma_s \psi_p^1 du' d\Omega' \\ + g + \frac{1}{4\pi\lambda_p} \int_{u'} H \phi du' = 0,$$

where the independent variables have been dropped for simplicity. But Eq. (62) states that

$$(D.16) \quad -\bar{\Omega} \cdot \nabla \psi - \Sigma \psi + \int_{\Omega'} \int_{u'} \Sigma_s \psi du' d\Omega' + \frac{1}{4\pi\lambda} \int_{u'} H\phi du' = 0$$

and Eq. (D.9) states that,

$$(D.17) \quad -\bar{\Omega} \cdot \nabla \psi_1 - \Sigma \psi_1 + \int_{\Omega'} \int_{u'} \Sigma_s \psi_1 du' d\Omega' + g = 0;$$

so that it follows from the linearity of Eq. (D.15) that

$$(D.18) \quad \psi_p^1 = \frac{\lambda}{\lambda_p} \psi + \psi_1.$$

The new unnormalized fission neutron source is

$$(D.19) \quad S^1(\bar{r}, \bar{\Omega}, u) = \frac{1}{4\pi} \int_{u'} H\phi_p^1 du' \text{ where } \phi_p^1 = \int_{\Omega} \psi_p^1 d\Omega.$$

The first estimate of λ_p is λ_p^1 defined as

$$(D.20) \quad \lambda_p^1 = \frac{1}{S_o} \int_V \int_{\Omega} \int_u S^1(\bar{r}, \bar{\Omega}, u) du d\Omega dV.$$

Hence, from (D.18) and (D.19)

$$(D.21) \quad \lambda_p^1 = \frac{1}{S_o} \left[\int_V \int_{\Omega} \int_u \frac{1}{4\pi} \int_{u'} \frac{H\lambda\phi}{\lambda_p} du' du d\Omega dV + \int_V \int_{\Omega} \int_u \frac{1}{4\pi} \int_{u'} H\phi_1 du' du d\Omega dV \right] \\ = \frac{\lambda/\lambda_p}{S_o} \int_V \int_{\Omega} \int_u H\phi du' du dV + \frac{1}{S_o} \int_V \int_{\Omega} \int_u H\phi_1 du' du dV.$$

But one can write λ_1 from (D.8)

$$(D.22) \quad \lambda_1 = \int_V \int_{\Omega} \int_u H\phi_1 du' du dV / \int_V \int_{\Omega} \int_u g du d\Omega dV;$$

and one can use the definition of S_o from (D.14) to express (D.21) in the form

$$(D.23) \quad \lambda_p^1 = \lambda + (G/S_o)\lambda_1,$$

where G is defined

$$(D.24) \quad G \equiv \int_V \int_{\Omega} \int_u g(\bar{r}, \bar{\Omega}, u) du d\Omega dV.$$

Now, the source $S^1(\bar{r}, \bar{\Omega}, u)$ is renormalized (which imposes the constraint (D.1), where A has been chosen equal to S_0) by dividing it by λ_p^1 so that

$$(D.25) \quad \frac{1}{\lambda_p^1} \int_V \int_{\Omega} \int_u S^1(\bar{r}, \bar{\Omega}, u) du d\Omega dV = S_0.$$

An iteration is now complete, and the equation which is used to obtain the fluxes for the next iteration can be written

$$(D.26) \quad -\bar{\Omega} \cdot \bar{V} \psi_p^2 - \lambda_p^2 + \int_{\Omega'} \int_{u'} \Sigma_s \psi_p^2 du' d\Omega' + g(\bar{r}, \bar{\Omega}, u) \\ + \frac{\lambda}{4\pi\lambda_p \lambda_p^1} \int_{u'} H\phi du' + \frac{1}{4\pi\lambda_p^1} \int_{u'} H\phi_1 du' = 0,$$

where the last two terms are the new fission source guess $(1/\lambda_p^1)S^1(\bar{r}, \bar{\Omega}, u)$ expressed in terms of ϕ and ϕ_1 according to (D.18) and (D.19).

It now follows from the linearity of (D.26) and Eq. (D.9) for ψ_1 , Eq. (D.11) for ψ_2 , and Eq. (62) for ψ that

$$(D.27) \quad \psi_p^2 = \frac{\lambda^2}{\lambda_p \lambda_p^1} \psi + \frac{1}{\lambda_p^1} \psi_2 + \psi_1.$$

The new unnormalized source is

$$(D.28) \quad S^2(\bar{r}, \bar{\Omega}, u) = \frac{1}{4\pi} \int_{u'} H\psi_p^2 du'$$

and the new estimate for λ_p is λ_p^2 which is given by

$$(D.29) \quad \lambda_p^2 = \frac{1}{S_0} \int_V \int_{\Omega} \int_u S^2(\bar{r}, \bar{\Omega}, u) du d\Omega dV \\ = \frac{1}{S_0} \int_V \int_u \int_{u'} H \left[\frac{\lambda^2}{\lambda_p \lambda_p^1} \phi + \frac{1}{\lambda_p^1} \phi_2 + \phi_1 \right] du' du dV.$$

But it follows from (D.14) that

$$(D.30) \quad \frac{1}{S_o} \int_V \int_{u'} \int_{u''} \frac{H \lambda_p^2 \phi}{\lambda_p \lambda_p^1} du' du'' dV = \frac{\lambda_p^2}{\lambda_p^1}.$$

Also, from (D.10) for $k = 2$ and from (D.22) one derives

$$(D.31) \quad \frac{1}{S_o} \int_V \int_{u'} \int_{u''} \frac{H \phi_2}{\lambda_p^1} du' du'' dV = \frac{\lambda_1 \lambda_2}{S_o \lambda_p^1} \int_V \int_{\Omega} \int_u g du d\Omega dV = \left(\frac{G}{S_o} \right) \frac{\lambda_1 \lambda_2}{\lambda_p^1}.$$

Finally, from (D.8) one can obtain

$$(D.32) \quad \frac{1}{S_o} \int_V \int_{u'} \int_{u''} H \phi_1 du' du'' dV = \frac{G}{S_o} \lambda_1.$$

Hence

$$(D.33) \quad \lambda_p^2 = \frac{\lambda_p^2}{\lambda_p^1} + \frac{G}{S_o} \frac{\lambda_1 \lambda_2}{\lambda_p^1} + \frac{G}{S_o} \lambda_1.$$

The new normalized source $S^2(\bar{r}, \bar{\Omega}, u)/\lambda_p^2$ for the third iteration can be written

$$(D.34) \quad \frac{S^2(\bar{r}, \bar{\Omega}, u)}{\lambda_p^2} = \frac{1}{4\pi} \int_{u'} H \left[\frac{\lambda_p^2}{\lambda_p \lambda_p^1 \lambda_p^2} \phi + \frac{1}{\lambda_p^1 \lambda_p^2} \phi_2 + \frac{1}{\lambda_p^2} \phi_1 \right] du'.$$

In exactly the same manner as used previously, one obtains

$$(D.35) \quad \psi_p^3 = \frac{\lambda_p^3}{\lambda_p \lambda_p^1 \lambda_p^2} \psi + \frac{\psi_3}{\lambda_p^1 \lambda_p^2} + \frac{\psi_2}{\lambda_p^2} + \psi_1 \text{ and}$$

$$(D.36) \quad \lambda_p^3 = \frac{\lambda_p^3}{\lambda_p^1 \lambda_p^2} + \frac{G}{S_o} \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_p^1 \lambda_p^2} + \frac{G}{S_o} \frac{\lambda_1 \lambda_2}{\lambda_p^2} + \frac{G}{S_o} \lambda_1.$$

One finds that, in general,

$$(D.37) \quad \lambda_p^n = \frac{1}{\prod_{k=1}^{n-1} \lambda_p^k} \left\{ \lambda_p^n + \frac{G}{S_o} \left[\prod_{k=1}^n \lambda_k + \lambda_p^1 \prod_{k=1}^{n-1} \lambda_k + \lambda_p^1 \lambda_p^2 \prod_{k=1}^{n-2} \lambda_k + \dots \right. \right. \\ \left. \left. + \lambda_1 \lambda_2 \prod_{k=1}^{n-2} \lambda_p^k + \lambda_1 \prod_{k=1}^{n-1} \lambda_p^k \right] \right\},$$

and after multiplying this equation by $\prod_{k=1}^{n-1} \lambda_p^k$ one obtains

$$(D.38) \quad \prod_{k=1}^n \lambda_p^k = \lambda^n + \frac{G}{S_o} \left[\prod_{k=1}^n \lambda_k + \lambda_p^1 \prod_{k=1}^{n-1} \lambda_k + \lambda_p^1 \lambda_p^2 \prod_{k=1}^{n-2} \lambda_k + \dots \right. \\ \left. + \lambda_1 \lambda_2 \prod_{k=1}^{n-2} \lambda_p^k + \lambda_1 \prod_{k=1}^{n-1} \lambda_p^k \right].$$

Now λ_p^n can be written

$$(D.39) \quad \lambda_p^n = \frac{\lambda^n + \frac{G}{S_o} \left[\prod_{k=1}^n \lambda_k + \lambda_p^1 \prod_{k=1}^{n-1} \lambda_k + \dots + \lambda_1 \prod_{k=1}^{n-1} \lambda_p^k \right]}{\lambda^{n-1} + \frac{G}{S_o} \left[\prod_{k=1}^{n-1} \lambda_k + \lambda_p^1 \prod_{k=1}^{n-2} \lambda_k + \dots + \lambda_1 \prod_{k=1}^{n-2} \lambda_p^k \right]}.$$

Therefore,

$$(D.40) \quad \frac{\lambda_p^{n-\lambda}}{\lambda} = \left\{ \lambda^n + \frac{G}{S_o} \prod_{k=1}^n \lambda_k + \frac{G}{S_o} \left[\lambda_p^1 \prod_{k=1}^{n-1} \lambda_k + \lambda_p^1 \lambda_p^2 \prod_{k=1}^{n-2} \lambda_k + \dots \right. \right. \\ \left. \left. + \lambda_1 \lambda_2 \prod_{k=1}^{n-2} \lambda_p^k + \lambda_1 \prod_{k=1}^{n-1} \lambda_p^k \right] - \lambda^{n-\frac{G}{S_o}} \lambda \left[\prod_{k=1}^{n-1} \lambda_k + \lambda_p^1 \prod_{k=1}^{n-2} \lambda_k + \dots \right. \right. \\ \left. \left. + \lambda_1 \lambda_2 \prod_{k=1}^{n-3} \lambda_p^k + \lambda_1 \prod_{k=1}^{n-2} \lambda_p^k \right] \right\} / \left\{ \lambda^n + \frac{G}{S_o} \lambda \left[\prod_{k=1}^{n-1} \lambda_k + \lambda_p^1 \prod_{k=1}^{n-2} \lambda_k + \dots \right. \right. \\ \left. \left. + \lambda_1 \prod_{k=1}^{n-2} \lambda_p^k \right] \right\}, \text{ or}$$

$$(D.41) \quad \frac{\lambda_p^{n-\lambda}}{\lambda} = \frac{(G/S_o) \prod_{k=1}^n \lambda_k + (G/S_o) [A_n - B_n] \lambda^n}{\lambda^n \left[1 + \frac{G}{S_o} B_n \right]}, \text{ where}$$

$$(D.42) \quad A_n \equiv \frac{1}{\lambda^n} \left[\lambda_p^1 \prod_{k=1}^{n-1} \lambda_k + \lambda_p^1 \lambda_p^2 \prod_{k=1}^{n-2} \lambda_k + \dots + \lambda_1 \lambda_2 \prod_{k=1}^{n-2} \lambda_p^k + \lambda_1 \prod_{k=1}^{n-1} \lambda_p^k \right]$$

and

$$(D.43) \quad B_n \equiv \frac{1}{\lambda^n} \left[\lambda \prod_{k=1}^{n-1} \lambda_k + \lambda \lambda_p^1 \prod_{k=1}^{n-2} \lambda_k + \dots + \lambda \lambda_1 \lambda_2 \prod_{k=1}^{n-3} \lambda_p^k + \lambda \lambda_1 \prod_{k=1}^{n-2} \lambda_p^k \right].$$

It is obvious from (D.38) that

$$(D.44) \quad \lim_{(G/S_0) \rightarrow 0} \prod_{k=1}^n \lambda_p^k = \lambda^n;$$

therefore

$$(D.45) \quad \lim_{(G/S_0) \rightarrow 0} A_n = \frac{1}{\lambda^n} \left[\lambda \prod_{k=1}^{n-1} \lambda_k + \lambda^2 \prod_{k=1}^{n-2} \lambda_k + \dots + \lambda_1 \lambda_2 \lambda^{n-2} + \lambda_1 \lambda^{n-1} \right]$$

and

$$(D.46) \quad \lim_{(G/S_0) \rightarrow 0} B_n = \frac{1}{\lambda^n} \left[\lambda \prod_{k=1}^{n-1} \lambda_k + \lambda^2 \prod_{k=1}^{n-2} \lambda_k + \dots + \lambda_1 \lambda_2 \lambda^{n-2} + \lambda_1 \lambda^{n-1} \right].$$

Therefore, since $\lim_{(G/S_0) \rightarrow 0} A_n = \lim_{(G/S_0) \rightarrow 0} B_n$, it follows from (D.4) that

$$(D.47) \quad \lim_{(G/S_0) \rightarrow 0} \frac{\lambda_p^{n-\lambda}}{(G/S_0)^\lambda} = \frac{1}{\lambda^n} \prod_{k=1}^n \lambda_k = \prod_{k=1}^n \left(\frac{\lambda_k}{\lambda} \right). *$$

But $G = \alpha \frac{B_1}{p}$ and it follows from (D.12) and (D.7) that

$$(D.48) \quad S_0 = \frac{\lambda_c^m \lambda^n}{\lambda_p} B_0.$$

Therefore,

$$(D.49) \quad \lim_{\alpha \rightarrow 0} \frac{\lambda_p^{\ell-\lambda}}{\left(\frac{\alpha B_1 \lambda}{p_1 p} \right) \left(\frac{\lambda_c^m \lambda^n}{\lambda_p} \right)} \lambda = \prod_{k=1}^{\ell} \left(\frac{\lambda_k}{\lambda} \right), \text{ or}$$

$$(D.50) \quad \lim_{\alpha \rightarrow 0} \frac{(\lambda_p^{\ell-\lambda})(\lambda_c^m \lambda^n)}{\alpha \lambda_p \lambda} = \frac{B_1}{B_0} \prod_{k=1}^{\ell} \left(\frac{\lambda_k}{\lambda} \right).$$

Since (D.40) is true for any n , one can write

$$(D.51) \quad \lim_{\ell \rightarrow \infty} \lim_{\alpha \rightarrow 0} \frac{(\lambda_p^{\ell-\lambda})}{\alpha \lambda_p \lambda} (\lambda_c^m \lambda^n) = \frac{B_1}{B_0} \lim_{\ell \rightarrow \infty} \prod_{k=1}^{\ell} \left(\frac{\lambda_k}{\lambda} \right).$$

* L. A. Schmittroth, by private communication, was first to prove that Eq. (D.47) follows from definition (D.37).

Now, if, as can be reasonably assumed, λ_p^ℓ (which, of course, is a function of α_p) is uniformly convergent as $\alpha_p \rightarrow 0$ and $\ell \rightarrow \infty$, (D.51) can be expressed as

$$(D.52) \quad \lim_{\alpha_p \rightarrow 0} \lim_{\ell \rightarrow \infty} \frac{(\lambda_p^\ell - \lambda)}{\alpha_p \lambda_p \lambda} (\lambda_c^m \lambda^n) = \frac{B_1}{B_0} \lim_{\ell \rightarrow \infty} \prod_{k=1}^{\ell} \left(\frac{\lambda_k}{\lambda} \right);$$

and since $\lim_{\ell \rightarrow \infty} \lambda_p^\ell = \lambda_p$, by definition (provided only that $\lim_{\ell \rightarrow \infty} \lambda_p^\ell = \lambda_p$ exists), it follows that

$$(D.53) \quad \lim_{\alpha_p \rightarrow 0} \frac{(\lambda_p - \lambda)}{\alpha_p \lambda_p \lambda} (\lambda_c^m \lambda^n) = \frac{B_1}{B_0} \lim_{\ell \rightarrow \infty} \prod_{k=1}^{\ell} \left(\frac{\lambda_k}{\lambda} \right).$$

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