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MASTER

A DIGITAL COMPUTER PROGRAM
FOR INVESTIGATING FLOW INSTABILITIES

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A computer program to investigate the stability of compressible flow in a model formulated by Paul Warner has been written. His analysis results in the boundary value problem.

$$\frac{c^2}{dx^2} \left[Fo \frac{d^2 Q_1}{dx^2} \right] + \frac{v_1^2}{Fo} Q_1 = -v_1^2 Fo(v_2 + x) - \frac{3(1-v_2^2)^3}{8 Fo^3} \quad (1)$$

$$\text{where } Q_2 = -v_1^{-1} \left[\frac{d^2 Q_1}{dx^2} - \frac{3}{2} (1 - v_2^2) \right] \quad (2)$$

$$Fo = \left[1 - (1-v_2^2)x \right]^{\frac{1}{2}}$$

with the boundary conditions

$$Q_1(0) = Q_1(1) = 0$$

$$Q_2(0) = Q_2(1) = 0$$

$$\text{or} \quad \frac{d^2 Q_1(0)}{dx^2} = \frac{d^2 Q_1(1)}{dx^2} = \frac{3}{2} (1 - v_2^2)$$

The obvious attack of this problem is to replace equation (1) by a difference equation, and generate n equations with n unknowns. This method works fine as far as equation (1) is concerned, but failed when these Q_1 were used to approximate Q_2 in equation (2). The most troublesome area (where Q_2 contained its greatest inaccuracy) is where v_1 is very small or approaching zero. If equation (1) is solved analytically, for $v_1 = 0$, the solution is

$$Q_1^{(c)} = \frac{3}{4} (1 - v_2^2) (x^2 - 1) \quad (3)$$

and its second derivative is a constant

$$\frac{d^2 Q_1^{(0)}}{dx^2} = \frac{3}{2}(1 - v_2^2) \quad (4)$$

Since the second derivative is equal to $\frac{3}{2}(1 - v_2^2)$ where $v_1 = 0$ the function

$$\left[\frac{d^2 Q_1}{dx^2} - \frac{3}{2}(1 - v_2^2) \right]$$

should go to zero as v_1 approaches zero because

$$\lim_{v_1 \rightarrow 0} \left[\frac{d^2 Q_1(v_1)}{dx^2} - \frac{3}{2}(1 - v_2^2) \right] = 0$$

With this in mind (where $v_1 \rightarrow 0$) it appeared that better results could be obtained if the solution was of the form

$$Q_1 = Q_1^{(0)} + y(v_1, x) \quad (5)$$

This would require that $y(0, x) = 0$ and that y be a continuous function of v_1 . The above substitution was made and varying values of v_1 were tried to give a test to the continuity of y . The computer program demonstrated that $y(v_1, x)$ approached zero continuously as v_1 approached zero because as v_1 was made smaller Q_1 approached $Q_1^{(0)}$.

The resulting manipulation of this substitution is shown in Appendix I.

Now, with equation (5) and the two new equations obtained by the substitution, Q_2 can be computed with little loss of accuracy by

replacing the differential equation by a set of difference equations.

This may be done in equation (1) by letting $[Fo y'']$ be approximated by the finite-difference formulae

$$[Fo y'']_i = Fo_i \frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} = (D2F)_i$$

and

$$\left[\frac{d^2}{dx^2} (Fo y'') \right]_i = \frac{(D2F)_{i-1} - 2(D2F)_i + (D2F)_{i+1}}{(\Delta x)^2}$$

Also in equation (1) let

$$C_i = \frac{v_1^2}{Fo_i} \quad \text{and} \quad B(x_i) = \frac{-v_1^2}{Fo_i} \left[\frac{3(1-v_2^2)}{4} (x_i^2 - x_1^2) + Fo_i^2 (v_3 + x_1) \right]$$

Then (1) becomes

$$Fo_{i-1} y_{i-2} - (Fo_{i-1} + Fo_i) y_{i-1} + (Fo_{i-1} - 4Fo_i + Fo_{i+1}) y_i + Fo_{i+1} y_{i+1} - 2(Fo_i + Fo_{i+1}) y_{i+2} + Fo_{i+1} y_{i+3} = (\Delta x)^4 B(x_i)$$

This difference equation represents a set of n equations in n unknowns of y_i , $i = 1, 2, 3, \dots, n-1, n$. The boundary values are $y(x_0) = 0$ and $y(x_{n+1}) = 0$.

Once the y_i are computed by this method the Q_{2i} may be approximated directly by the difference-equation representative of equation (6)

$$Q_{2i} = - \frac{1}{v_1} \left[\frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} \right]$$

The boundary conditions are

$$y(x_0) = y''(x_0) = y(x_{n+1}) = y''(x_{n+1}) = 0$$

The Q_{1i} may be computed by the original substitution equation (5)

which is

$$Q_{1i} = Q_1^{(0)} + y_i = \frac{3}{4} (1-v_2^2)(x_i^2 - x_1^2) + y_i$$

The computer program works fine using this type of substitution.

Appendix I

$$(1) \quad \frac{d^2}{dx^2} \left[F_0 \frac{d^2 Q_1}{dx^2} \right] + \frac{v_1^2}{F_0} Q_1 = -v_1^2 F_0 (v_3 + x) - \frac{3}{8} \frac{(1-v_2^2)^3}{F_0^3}$$

Expand the left hand side of equation (1)

$$F_0 \frac{d^4 Q_1}{dx^4} + 2F_0' \frac{d^3 Q_1}{dx^3} + F_0'' \frac{d^2 Q_1}{dx^2} + \frac{v_1^2}{F_0} Q_1 =$$

and differentiate equation (5) four times to get:

$$Q_1 = Q_1^{(0)} + y$$

$$\frac{dQ_1}{dx} = \frac{dQ_1^{(0)}}{dx} + \frac{dy}{dx}$$

$$\frac{d^2 Q_1}{dx^2} = \frac{d^2 Q_1^{(0)}}{dx^2} + \frac{d^2 y}{dx^2} = \frac{3}{2} (1-v_2^2) + \frac{d^2 y}{dx^2}$$

$$\frac{d^3 Q_1}{dx^3} = \frac{d^3 y}{dx^3}$$

$$\frac{d^4 Q_1}{dx^4} = \frac{d^4 y}{dx^4}$$

Make the appropriate substitution in the left hand side of

equation (1) and combine terms to get:

$$\frac{d^2}{dx^2} \left[F_0 \frac{d^2 y}{dx^2} \right] + \frac{v_1^2}{F_0} y = - \frac{v_1^2}{F_0} \left[\frac{3(1-v_2^2)}{4} (x^2 - x) + F_0^2 (v_3 + x) \right]$$

Using the same substitution in equation (2) gives

$$Q_2 = -v_1^{-1} \left[\frac{d^2 y}{dx^2} \right] \quad (6)$$

The boundary conditions using the new variable are:

$$Q_1 = Q_1^{(0)} + y = \frac{3}{4} (1-v_2^2)(x^2-x) + y$$

$$y(0) = Q_1(0) = 0$$

$$y(1) = Q_1(1) = 0$$

$$\frac{d^2 y}{dx^2}(0) = v_1 Q_2(0) = 0$$

$$\frac{d^2 y}{dx^2}(1) = v_1 Q_2(1) = 0$$