THE METHOD OF CHARACTERISTICS FOR PRESSURE TRANSIENTS
IN CYLINDRICAL GEOMETRY WITH AXIAL SYMMETRY - THE FORMULATION

by

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BASE TECHNOLOGY

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MASTER

NOMENCLATURE

a	sonic speed
h	enthalpy
p	pressure
r	radial coordinate
Δr	internodal distance in radial coordubate
s	entropy
t	time
Δt	time step
u	radial velocity component
$\vec{v}$	velocity vector
w	axial velocity component
z	axial coordinate
Δz	internodal distance in axial coordinate

Greek Letters

$\alpha_1, \alpha_2, \alpha_3$	constants
$\vec{\lambda}$	normal vector to the characteristic surface
$\lambda_1, \lambda_2, \lambda_3$	components of $\vec{\lambda}$ in t, r, z coordinates, respectively
θ	parametric angle measured counterclockwise from t-axis to the projection of the bicharacteristic onto the r-z plane
ρ	density

Subscripts

i	index representing a discret node in r-coordinate
j	index representing a discrete node in z-coordinate
Q	point of intercept of a local bicharacteristic with initial surface
R	point of intercept of a local streamline with initial surface

ABSTRACT

The system of governing equations for an unsteady inviscid compressible flow is formulated by the method of characteristics for numerical integration. An axially symmetric cylindrical geometry is considered which simulates a simplified geometry of steam generators. The resulting reduced equations, the bicharacteristic and compatibility equations involve one less independent variable than the original system of equations. The system of governing partial differential equations is reduced to a system of integral equations written along bicharacteristics. Numerical integration can be performed either using the compatibility relations by writing in finite differences or using the system of integral equations by various approximation techniques.

1. INTRODUCTION

As a part of the LMFBF steam generator development program, an advanced computer code is being developed to be used as designer's guides to predict the pressure transients resulting from sodium-water reaction under accident conditions. The advanced code should be capable of tracing pressure waves propagating throughout the sodium network that consists mainly of the steam generators and various connecting pipes. In such a systems analysis method, the wave phenomenon in long pipes is adequately treated by a one-dimensional approximation. However, for steam generators especially with small L/D ratios, the one-dimensional representation may not be a realistic model and a multidimensional analysis is an essential necessity.

This report presents a complete formulation of a two-dimensional problem in the method of characteristics ready to be programmed for an axisymmetric cylindrical application. The complete governing equations were considered except for the neglected viscous terms. As an inherent property of the three-variable method of characteristics, the bicharacteristics that are equivalent to the characteristics of the two-variable method, are infinitely many. In this report, an effort was made to relate the mathematical formulation as much as possible with the physical wave phenomenon throughout the development. The idea of parametric representation of the bicharacteristics similar to that of Butler¹ is adopted. In the formulation of numerical calculation, a fixed mesh network with uniform spacing was used. The vertex of the characteristic cone was placed at the new point and the interpolation was made in the initial surface. A somewhat different formulation was observed in Sauerwein^{2,3} in which a set of new independent variables was introduced and the cone vertex was placed in the initial surface.

2. BASIC DIFFERENTIAL EQUATIONS

The basic differential equations are first obtained in a vector form for a general three-dimensional flow and they are written out for the special case of axially symmetric cylindrical geometry. This study is primarily concerned with liquid substances as working fluids. Although liquids are usually considered incompressible, their small compressibility is an important feature in problems involving a large pressure variation or a rapidly changing unsteady motion. The conservation equations of mass, momentum, and energy for an inviscid compressible fluid with neglected body force are:

Continuity

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 ; \quad (1)$$

Momentum

$$\rho \frac{D\vec{v}}{Dt} + \vec{\nabla} p = 0 ; \quad (2)$$

Energy

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = 0 ; \quad (3)$$

where the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$$

An additional equation to be used is the state equation:

$$h = h(p, \rho) \quad (4)$$

The above set of equations involve six unknowns* and may be solved as they stand. For convenience, however, enthalpy h is eliminated by

*Six unknowns for the special case of an axial symmetry.

combining equations (3) and (4):

$$a^2 \frac{D\rho}{Dt} - \frac{Dp}{Dt} = 0 \quad (5)$$

where

$$a = \left[- \frac{\rho \left(\frac{\partial h}{\partial \rho} \right)_p}{\rho \left(\frac{\partial h}{\partial p} \right)_\rho - 1} \right]^{1/2}$$

is a thermodynamic state-variable that is identified to be the sonic speed upon utilizing the following relationships;

$$a^2 = \left(\frac{\partial p}{\partial \rho} \right)_s \quad (6)$$

$$\frac{Ds}{Dt} = 0 \quad (7)$$

Equation (1) is rearranged ;

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$$

and it can be combined with equation (5) to eliminate terms involving derivatives of density:

$$\rho a^2 \vec{\nabla} \cdot \vec{v} + \frac{Dp}{Dt} = 0 \quad (8)$$

Equations (2) and (8) are used in place of the original equations (1) through (4). The density ρ , in the new set of equations, does not appear in any derivatives and it is assumed constant since the relative change* in density of liquids are small. Equations (2) and (8) are written out for a cylindrical problem with axial symmetry:

* $d\rho/\rho$.

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial r} = 0 \quad (9)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = 0 \quad (10)$$

$$\rho a^2 \left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + w \frac{\partial p}{\partial z} + \rho a^2 \frac{u}{r} = 0 \quad (11)$$

These are first-order partial differential equations of the hyperbolic type involving three independent variables in which the unknowns are p , u , and w when ρ and a are given.

3. THE METHOD OF CHARACTERISTICS

The system of governing equations (9), (10), and (11) rewritten below are hyperbolic equations in three independent variables for which real characteristics exist:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial r} = 0 \quad (9)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = 0 \quad (10)$$

$$\rho a^2 \left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + w \frac{\partial p}{\partial z} + \rho a^2 \frac{u}{r} = 0 \quad (11)$$

When the equations are written along characteristic surfaces they can be written simply in two independent variables. Unfortunately, a simple generalization of the two-variable method to problems involving three or more independent variables cannot be made without losing some of the attractive features of the two-variable method. One obvious reason for this is that the reduced equations of the latter type problems are still partial differential equations whereas those of the two-variable problems

are ordinary equations. Nevertheless, a reduction of independent variables from three to two is still a simplification and, more importantly, the reduced equations involve gradient components only in directions in which solutions can be extended.

Following the theory of the method of characteristics^{4,5} for hyperbolic partial differential equations involving three independent variables, there exist real surfaces called the characteristic surfaces normal to which the solution cannot be continued. This implies that a linear combination of equations (9) through (11) can be found which does not involve any gradient components of u , w or p in the direction $\vec{\lambda}$, normal to the characteristic surface. A linear combination of equations (9), (10), and (11):

$$\begin{aligned} & \alpha_1 \left[\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial r} \right] + \alpha_2 \left[\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} \right] \\ & + \alpha_3 \left[\rho a^2 \left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + w \frac{\partial p}{\partial z} + \rho a^2 \frac{u}{r} \right] = 0 \end{aligned}$$

which after rearranging results in:

$$\begin{aligned} & \alpha_1 \rho \frac{\partial u}{\partial t} + (\alpha_1 \rho u + \alpha_3 \rho a^2) \frac{\partial u}{\partial r} + \alpha_1 \rho w \frac{\partial u}{\partial z} + \alpha_2 \rho \frac{\partial w}{\partial t} + \\ & \alpha_2 \rho u \frac{\partial w}{\partial r} + (\alpha_2 \rho w + \alpha_3 \rho a^2) \frac{\partial w}{\partial z} + \alpha_3 \frac{\partial p}{\partial t} + (\alpha_1 + \alpha_3 u) \frac{\partial p}{\partial r} + \\ & (\alpha_2 + \alpha_3 w) \frac{\partial p}{\partial z} + \alpha_3 \rho a^2 \frac{u}{r} = 0 \end{aligned} \quad (12)$$

where α_1 , α_2 and α_3 are constants. Equation (12) will be free of gradient components in the direction $\vec{\lambda}$ if:

$$\begin{aligned} & \alpha_1 \rho \lambda_1 + (\alpha_1 \rho u + \alpha_3 \rho a^2) \lambda_2 + \alpha_1 \rho w \lambda_3 = 0 \\ & \alpha_2 \rho \lambda_1 + \alpha_2 \rho u \lambda_2 + (\alpha_2 \rho w + \alpha_3 \rho a^2) \lambda_3 = 0 \\ & \alpha_3 \lambda_1 + (\alpha_1 + \alpha_3 u) \lambda_2 + (\alpha_2 + \alpha_3 w) \lambda_3 = 0 \end{aligned}$$

or, rearranging:

$$\begin{aligned}
 \rho(\lambda_1 + u\lambda_2 + w\lambda_3)\alpha_1 + \rho a^2 \lambda_2 \alpha_3 &= 0 \\
 \rho(\lambda_1 + u\lambda_2 + w\lambda_3)\alpha_2 + \rho a^2 \lambda_3 \alpha_3 &= 0 \\
 \lambda_2 \alpha_1 + \lambda_3 \alpha_2 + (\lambda_1 + u\lambda_2 + w\lambda_3)\alpha_3 &= 0
 \end{aligned} \tag{13}$$

where $\lambda_1, \lambda_2, \lambda_3$ are components of $\vec{\lambda}$ along t, r, z -coordinate, respectively. Equations (13) are homogeneous algebraic equations in α 's, of which coefficient determinant called the characteristic determinant must vanish since a nontrivial solution of α 's exists:

$$\begin{vmatrix}
 \rho(\lambda_1 + u\lambda_2 + w\lambda_3) & 0 & \rho a^2 \lambda_2 \\
 0 & \rho(\lambda_1 + u\lambda_2 + w\lambda_3) & \rho a^2 \lambda_3 \\
 \lambda_2 & \lambda_3 & \lambda_1 + u\lambda_2 + w\lambda_3
 \end{vmatrix} = 0 \tag{14}$$

The characteristic determinant when expanded and rearranged yields the characteristic equation:

$$(\lambda_1 + u\lambda_2 + w\lambda_3) [(\lambda_1 + u\lambda_2 + w\lambda_3)^2 - a^2(\lambda_2^2 + \lambda_3^2)] = 0 \tag{15}$$

From equations (13), α_2 and α_3 are determined choosing α_1 to be arbitrary:

$$\begin{aligned}
 \alpha_2 &= \frac{\lambda_3}{\lambda_2} \alpha_1 \\
 \alpha_3 &= -\frac{1}{a^2 \lambda_2} (\lambda_1 + u\lambda_2 + w\lambda_3) \alpha_1
 \end{aligned}$$

Finally, substituting these into equation (12) the compatibility equation is obtained:

$$\begin{aligned} & [\rho (\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z}) + \frac{\partial p}{\partial r}] + \frac{\lambda_3}{\lambda_2} [\rho (\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z}) + \frac{\partial p}{\partial z}] \\ & - \frac{1}{a^2 \lambda_2} (\lambda_1 + u \lambda_2 + w \lambda_3) [\rho a^2 (\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}) + \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + w \frac{\partial p}{\partial z} + \rho a^2 \frac{u}{r}] = 0 \end{aligned} \quad (16)$$

There are two types of characteristics corresponding to the two factors in equation (15):

$$\lambda_1 + u \lambda_2 + w \lambda_3 = 0 \quad (17)$$

$$(\lambda_1 + u \lambda_2 + w \lambda_3)^2 - a^2 (\lambda_2^2 + \lambda_3^2) = 0 \quad (18)$$

Equation (17) describes an infinity of the first type characteristic surfaces that are directed along the streamline where the streamline itself lies on all characteristic surfaces. The second type is defined by equation (18) which is rearranged to read:

$$\frac{\lambda_1 + u \lambda_2 + w \lambda_3}{\sqrt{\lambda_2^2 + \lambda_3^2}} = \pm a \quad (19)$$

In the case of the steady supersonic flow, equation (19) represents surfaces that are tangent to the Mach cone. For an unsteady flow, the characteristic relation (19) represents an aggregate of normal $\vec{\lambda}$ and hence an infinite number of characteristic surfaces at a point. With reference to Figure 1, the totality of the characteristic surfaces envelop locally a pair of cones called the characteristic cones at point P where the $\pm$ signs in equation (19) correspond to the lower and upper cones, respectively. The characteristic cone, in general, is an elliptic cone of which intercept with the initial plane is a circle of radius adt . There are four distinct points on this circle for which the

Now the task is to write compatibility equations along the bicharacteristic direction. Remembering that $\overline{OQ}$ is in parallel with the projection of the normal vector $\vec{\lambda}$ onto the r - z plane,

$$\begin{aligned}\lambda_2 &= \cos \theta \\ \lambda_3 &= \sin \theta\end{aligned}\tag{21a}$$

and the characteristic equation (19) can be written for the lower cone:

$$\lambda_1 = a - u \cos \theta - w \sin \theta\tag{21b}$$

Substituting equation (21) into equation (16) and rearranging yields:

$$\begin{aligned}& \cos \theta \left[\frac{\partial u}{\partial t} + (u - a \cos \theta) \frac{\partial u}{\partial r} + (w - a \sin \theta) \frac{\partial u}{\partial z} \right] + \sin \theta \left[\frac{\partial w}{\partial t} + (u - a \cos \theta) \frac{\partial w}{\partial r} \right. \\& \left. + (w - a \sin \theta) \frac{\partial w}{\partial z} \right] - \frac{1}{\rho a} \left[\frac{\partial p}{\partial t} + (u - a \cos \theta) \frac{\partial p}{\partial r} + (w - a \sin \theta) \frac{\partial p}{\partial z} \right] \\& - a \left[\sin^2 \theta \frac{\partial u}{\partial r} - \sin \theta \cos \theta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) + \cos^2 \theta \frac{\partial w}{\partial z} + \frac{u}{r} \right] = 0\end{aligned}\tag{22}$$

Recognizing the differential operator,

$$\frac{\partial}{\partial t} + (u - a \cos \theta) \frac{\partial}{\partial r} + (w - a \sin \theta) \frac{\partial}{\partial z}$$

to be the differential operator along a bicharacteristic, the compatibility equation is finally written in a characteristic form:

$$\cos \theta \frac{du}{dt} + \sin \theta \frac{dw}{dt} - \frac{1}{\rho a} \frac{dp}{dt} - a \left[\sin^2 \theta \frac{\partial u}{\partial r} - \sin \theta \cos \theta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) + \cos^2 \theta \frac{\partial w}{\partial z} + \frac{u}{r} \right] = 0\tag{23}$$

Here, the operator $\frac{d}{dt}$ represents the derivative along a bicharacteristic associated with an orientation angle θ . The compatibility equation (23) and the bicharacteristic equation (20) can be used to extend the solution from given initial surface at t to a neighboring surface at $t + \Delta t$. Equation (23) represents an infinite number of equations since θ can be assigned any value depending on which bicharacteristic is desired. In general, at least three compatibility equations are required for three different values of θ since there are three

unknowns, u , w , and p^* . The characteristic equation (17) can also be used in which case the compatibility equation upon choosing a particular λ^* such that

$$\frac{\lambda_3}{\lambda_2} = \frac{w}{u} \text{ yields:}$$

$$\rho \frac{Du}{Dt} + \frac{w}{u} \left(\rho \frac{Dw}{Dt} + \frac{\partial p}{\partial z} \right) = 0 \quad (24)$$

where $\frac{D}{Dt}$ represents the differential along the streamline. Nevertheless, equation (11) is already written in a characteristic form and may be used in place of Equation (24):

$$\frac{Dp}{Dt} + \rho a^2 \left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} \right) = 0 \quad (11)$$

The corresponding characteristic direction is:

$$dr = udt \quad (25)^*$$

$$dz = wdt$$

Notice that both compatibility equations, equations (23) and (11), contain partial derivatives, $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial z}$, $\frac{\partial w}{\partial r}$, and $\frac{\partial w}{\partial z}$. These derivatives may be evaluated in the initial plane and treated as constant for small enough time step in numerical calculation and hence integrations along bicharacteristics and streamline can be performed in much the same way as in one-dimensional unsteady problems. In this approximation procedure, the parametric angle θ can also be assumed constant. However, if these partial derivatives appearing in the compatibility equations are to be considered unknowns, more than three equations have to be used. By a proper choice of θ , the unknowns to be calculated are reduced to five and hence four compatibility equations plus the streamline equation suffice to simultaneously solve for the five unknowns.

Consider $\theta = 0, \pi/2, 3\pi/2$, and π , then equations (23) and (11) contain partial derivatives $\frac{\partial u}{\partial r}$ and $\frac{\partial w}{\partial z}$ only since $\sin\theta\cos\theta = 0$. The five equations are used to solve for the unknowns $u, w, p, \frac{\partial u}{\partial r}, \frac{\partial w}{\partial z}$. The remaining task in this approach is to

These are also equations describing the direction of a streamline.

find how the parametric angle θ varies along the bicharacteristic. Richardson¹¹ has derived an approximate analytical expression for θ accurate to a first order of time step Δt assuming that the unknowns along the bicharacteristic vary as:

$$u(\phi, \tau) = u_0 + u(\phi) \cdot \tau$$

$$w(\phi, \tau) = w_0 + w(\phi) \cdot \tau$$

$$a(\phi, \tau) = a_0 + a(\phi) \cdot \tau$$

$$\theta(\phi, \tau) = \phi + \theta(\phi) \cdot \tau$$

where u_0 , w_0 , a_0 , ϕ are values of u , w , a and θ at the apex of the characteristic cone (point P in Figure 1) and τ is the parameter representing the distance along the bicharacteristic in the negative time direction, i.e., $\tau = -t$. The assumption was modified here such that the values of u , w , a and θ are linear functions of τ :

$$\begin{aligned} u(\phi, \tau) &= u_0 + [u(\phi) - u_0] \tau \\ w(\phi, \tau) &= w_0 + [w(\phi) - w_0] \tau \\ a(\phi, \tau) &= a_0 + [a(\phi) - a_0] \tau \\ \theta(\phi, \tau) &= \phi + [\theta(\phi) - \phi] \tau \end{aligned} \quad (26)$$

and the approximate expression for the bicharacteristic curve was rederived.

It was found that the result was identical to the one given by Richardson to the first order of the parameter τ . The approximate bicharacteristic curve equation, obtained upon setting $a = \text{constant}$, is:

$$\theta = \phi + \left(-\sin\phi \cos\phi \frac{\partial u}{\partial r} + \cos^2\phi \frac{\partial u}{\partial z} - \sin^2\phi \frac{\partial w}{\partial r} + \sin\phi \cos\phi \frac{\partial w}{\partial z} \right) dt + o(dt)^2 \quad (27)$$

where θ is the parametric angle in the plane at time t of the bicharacteristic having angle ϕ in the plane at $t + dt$ and the partial derivatives are to be evaluated in the plane at time t . The derivation of the approximate bicharacteristic curve is given in Appendix A.

4. NUMERICAL INTEGRATION

Consider an initial surface on which solution is given at time t and it is required to find solution on a neighboring surface at time $t + \Delta t$. A typical integration procedure consists of finding solution at point P (see Figure 2) located in the neighborhood of a plane on which solution is known. A repetition of the procedure for the entire grid points in the new plane yields the complete solution in the new plane. The procedure can be repeated as long as desired provided that the approximate finite difference scheme is numerically stable.

Figure 2
Finite Difference Network
and the Approximate Local
Bicharacteristic at Point P

Coordinates:

$A(i, j, t)$, $B(i+1, j-1, t)$, $C(i+1, j+1, t)$,
 $D(i-1, j+1, t)$, $E(i-1, j-1, t)$, $F(i, j+1, t)$,
 $G(i-1, j, t)$, $H(i, j-1, t)$, $I(i+1, j, t)$,
 $P(i, j, t+\Delta t)$

In reference to Figure 2, uniform internodal spacings are considered for r - and z -coordinate. The spacial domain is divided into m discrete nodes in r - direction and n nodes in z -direction. The time step Δt is chosen such that the intercept of the bicharacteristic, i.e., point Q , with the initial plane falls inside the rectangle BCDE. In this manner, the domain of dependence of the differential system is inside the domain of dependence of the finite difference system and hence the Courant-Friedrichs-Lewy⁷ criterion for convergence and stability of finite difference scheme is satisfied. The time step satisfying the criterion is:

$$\Delta t \leq \frac{\min(\Delta r, \Delta z)}{\sqrt{(|u|_{\max} + a)^2 + (|w|_{\max} + a)^2}} \quad (28)$$

A. Integrate System of Finite Difference Equations

First, the partial derivatives in Equations (23) and (11) are assumed locally constant along the bicharacteristic and the streamline over the time step Δt . Also the bicharacteristic direction is assumed to remain unchanged, i.e., $\theta = \text{constant}$. Under these approximations, Equation (23) is written in finite differences for a general point P located at $i\Delta r$, $j\Delta z$, and $t + \Delta t$, along the bicharacteristic $\overline{QP}$ which is locally approximated by a straight line:

$$\begin{aligned} & \cos\theta(u_{1,j}^{t+\Delta t} - u_Q^t) + \sin\theta(w_{1,j}^{t+\Delta t} - w_Q^t) - \frac{1}{\rho a} (p_{1,j}^{t+\Delta t} - p_Q^t) \\ & - a[\sin^2\theta \frac{\partial u}{\partial r} - \sin\theta \cos\theta (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}) + \cos^2\theta \frac{\partial w}{\partial z} + \frac{u}{r}] \frac{\Delta t}{Q} \end{aligned} \quad (29)$$

Equation (29) involves values of u_Q^t , w_Q^t , and p_Q^t which are to be evaluated at an off-mesh point Q . A double linear interpolation in r - z plane using the bicharacteristic equation (20) yields:

$$u_Q^t = u_{i,j}^t - \left(\frac{\partial u}{\partial r}\right)_{i,j}^t \Delta r - \left(\frac{\partial u}{\partial z}\right)_{i,j}^t \Delta z$$

$$w_Q^t = w_{i,j}^t - \left(\frac{\partial w}{\partial r}\right)_{i,j}^t \Delta r - \left(\frac{\partial w}{\partial z}\right)_{i,j}^t \Delta z \quad (30a)$$

$$p_Q^t = p_{i,j}^t - \left(\frac{\partial p}{\partial r}\right)_{i,j}^t \Delta r - \left(\frac{\partial p}{\partial z}\right)_{i,j}^t \Delta z$$

where

$$\Delta r = \frac{(u_{i,j}^t - a \cos \theta) [1 + \left(\frac{\partial w}{\partial z}\right)_{i,j}^t \Delta t] \Delta t - \left(\frac{\partial u}{\partial z}\right)_{i,j}^t (w_{i,j}^t - a \sin \theta) \Delta t^2}{[1 + \left(\frac{\partial u}{\partial r}\right)_{i,j}^t \Delta t] [1 + \left(\frac{\partial w}{\partial z}\right)_{i,j}^t \Delta t] - \left(\frac{\partial u}{\partial z}\right)_{i,j}^t \left(\frac{\partial w}{\partial r}\right)_{i,j}^t \Delta t^2} \quad (30b)$$

$$\Delta z = \frac{(u_{i,j}^t - a \cos \theta) \left(\frac{\partial w}{\partial r}\right)_{i,j}^t \Delta t^2 - (w_{i,j}^t - a \sin \theta) [1 + \left(\frac{\partial u}{\partial r}\right)_{i,j}^t \Delta t] \Delta t}{\left(\frac{\partial w}{\partial r}\right)_{i,j}^t \left(\frac{\partial u}{\partial z}\right)_{i,j}^t \Delta t^2 - [1 + \left(\frac{\partial u}{\partial r}\right)_{i,j}^t \Delta t] [1 + \left(\frac{\partial w}{\partial z}\right)_{i,j}^t \Delta t]}$$

and

$$\begin{aligned} \left(\frac{\partial u}{\partial r}\right)_{i,j}^t &= \frac{u_{i+1,j}^t - u_{i-1,j}^t}{2\Delta r} \\ \left(\frac{\partial u}{\partial z}\right)_{i,j}^t &= \frac{u_{i,j+1}^t - u_{i,j-1}^t}{2\Delta z} \\ \left(\frac{\partial w}{\partial r}\right)_{i,j}^t &= \frac{w_{i+1,j}^t - w_{i-1,j}^t}{2\Delta r} \\ \left(\frac{\partial w}{\partial z}\right)_{i,j}^t &= \frac{w_{i,j+1}^t - w_{i,j-1}^t}{2\Delta z} \end{aligned} \quad (30c)$$

Equation (29) contains three unknowns, $u_{1,j}^{t+\Delta t}$, $w_{1,j}^{t+\Delta t}$, and $p_{1,j}^{t+\Delta t}$, at the new point P and hence three equations are required corresponding to three different values of the parametric angle θ to determine the three unknowns.

The compatibility equation corresponding to the characteristic along the streamline can also be used as one of the three required equations. With reference to Figure 2, the compatibility equation for point P in finite differences is:

$$p_{1,j}^{t+\Delta t} - p_R^t + \rho a^2 \left(\frac{\partial u}{\partial r} + \frac{dw}{dz} + \frac{u}{r} \right)_R \Delta t = 0 \quad (31)$$

A linear interpolation in r - z plane using the streamline equation (25) yields:

$$\begin{aligned} u_R^t &= u_{1,j}^t - \left(\frac{\partial u}{\partial r} \right)_{1,j}^t \Delta r - \left(\frac{\partial u}{\partial z} \right)_{1,j}^t \Delta z \\ w_R^t &= w_{1,j}^t - \left(\frac{\partial w}{\partial r} \right)_{1,j}^t \Delta r - \left(\frac{\partial w}{\partial z} \right)_{1,j}^t \Delta z \\ p_R^t &= p_{1,j}^t - \left(\frac{\partial p}{\partial r} \right)_{1,j}^t \Delta r - \left(\frac{\partial p}{\partial z} \right)_{1,j}^t \Delta z \end{aligned} \quad (32a)$$

where

$$\begin{aligned} \Delta r &= \frac{\left\{ u_{1,j}^t \left[1 + \left(\frac{\partial w}{\partial z} \right)_{1,j}^t \Delta t \right] - w_{1,j}^t \left(\frac{\partial u}{\partial z} \right)_{1,j}^t \Delta t \right\} \Delta t}{\left[1 + \left(\frac{\partial u}{\partial r} \right)_{1,j}^t \Delta t \right] \left[1 + \left(\frac{\partial w}{\partial z} \right)_{1,j}^t \Delta t \right] - \left(\frac{\partial u}{\partial z} \right)_{1,j}^t \left(\frac{\partial w}{\partial r} \right)_{1,j}^t \Delta t^2} \\ \Delta z &= \frac{\left\{ u_{1,j}^t \left(\frac{\partial w}{\partial r} \right)_{1,j}^t \Delta t - w_{1,j}^t \left[1 + \left(\frac{\partial u}{\partial r} \right)_{1,j}^t \Delta t \right] \right\} \Delta t}{\left(\frac{\partial u}{\partial z} \right)_{1,j}^t \left(\frac{\partial w}{\partial r} \right)_{1,j}^t \Delta t^2 - \left[1 + \left(\frac{\partial u}{\partial r} \right)_{1,j}^t \Delta t \right] \left[1 + \left(\frac{\partial w}{\partial z} \right)_{1,j}^t \Delta t \right]} \end{aligned} \quad (32b)$$

Here, the partial derivatives may be approximated as in equation (30c). Notice the manner the partial derivatives are taken in equation (30c). The accuracy in the approximation is within the error or order Δr^2 or Δz^2 in the limit as u or w tends small in comparison to sonic speed. As the velocity increases the error also tends to increase. When velocity is relatively high, the error can be kept small in the same order by employing an iterative numerical scheme. For example,

equations (30c) are used as initial estimates and evaluations of Δr , Δz , u_Q^t , w_Q^t , p_Q^t , u_R^t , w_R^t , and p_R^t in equations (30a), (30b), (32a), and (32b) may be repeated using improved values of the partial derivatives.

If the partial derivatives in the compatibility equations are to be treated as varying along the characteristics, an iterative scheme can be used. The above step calculation is performed for all nodal points in the new plane located at time $t+\Delta t$. Once solution in the entire new plane is obtained the partial derivatives can be evaluated in the new plane and the average values may be used to improve the solution at the advanced time. This procedure can be repeated over the same time increment until some predetermined convergence criterion is satisfied. In this iteration procedure, the parametric angle θ may be taken either as constant, or else, equation (27) may be used to calculate an average value of θ to use in the compatibility equation. The iterative procedure, however, requires an excessive computation time especially when the convergence is slow.

In the following, a non-iterative computational strategy proposed by Richardson¹¹ is described. Upon changing $\theta = 0, \pi/2, \pi$, and $3\pi/2$, the compatibility equation (23) when written in finite differences involve five unknowns, u , w , p , $\frac{\partial u}{\partial r}$ and $\frac{\partial w}{\partial z}$. Hence the four equations along the four selected bicharacteristics together with the streamline equation (11) suffice to solve for the five unknowns. The four compatibility equations are:

$$U_n \left(u_{i,j}^{t+\Delta t} - u_{Qn}^t \right) + W_n \left(w_{i,j}^{t+\Delta t} - w_{Qn}^t \right) - \left(p_{i,j}^{t+\Delta t} - p_{Qn}^t \right) = \frac{1}{2} \rho a^2 \Delta t \left\{ \right. \\ \left. \sin^2 \phi_n \left(\frac{\partial u}{\partial r} \right)_{i,j}^{t+\Delta t} + \cos^2 \phi_n \left(\frac{\partial w}{\partial z} \right)_{i,j}^{t+\Delta t} + \left(\frac{u}{r} \right)_{i,j}^{t+\Delta t} + \left(\frac{u}{r} \right)_{Qn}^t + G_{Qn}^t \right\}; \quad (33a)$$

$$n = 1, 2, 3, 4$$

where

$$\begin{aligned}
 U_n &= \frac{1}{2} \rho a (\cos \phi_n + \cos \theta_n) \\
 W_n &= \frac{1}{2} \rho a (\sin \phi_n + \sin \theta_n) \\
 G_{Qn}^t &= \sin^2 \theta_n \left(\frac{\partial u}{\partial r} \right)_{Qn}^t - \sin \theta_n \cos \theta_n \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)_{Qn}^t + \cos^2 \theta_n \left(\frac{\partial w}{\partial z} \right)_{Qn}^t
 \end{aligned} \quad (33b)$$

Here, ϕ_n denotes the orientation angle at the cone vertex P of the bicharacteristic n and θ_n represents the angle of the same bicharacteristic at point Q (refer to Figure 1). Also,

$$\begin{aligned}
 \cos \theta_n &= \cos \phi_n - \Delta \theta_n \sin \phi_n \\
 \sin \theta_n &= \sin \phi_n + \Delta \theta_n \cos \phi_n
 \end{aligned} \quad (34)$$

where

$$\begin{aligned}
 \Delta \theta_n = \theta_n - \phi_n &= \left[-\sin \phi_n \cos \phi_n \left(\frac{\partial u}{\partial r} \right)_Q^t + \cos^2 \phi_n \left(\frac{\partial u}{\partial z} \right)_Q^t - \sin^2 \phi_n \left(\frac{\partial w}{\partial r} \right)_Q^t \right. \\
 &\quad \left. + \sin \phi_n \cos \phi_n \left(\frac{\partial w}{\partial z} \right)_Q^t \right] \Delta t + O(\Delta t)^2
 \end{aligned} \quad (35)$$

The displacements of the bicharacteristics are:

$$\begin{aligned}
 r_{1,j}^{t+\Delta t} - r_{Qn}^t &= \frac{1}{2} (u_{1,j}^{t+\Delta t} - a \cos \phi_n + u_{Qn}^t - a \cos \theta_n) \Delta t \\
 z_{1,j}^{t+\Delta t} - z_{Qn}^t &= \frac{1}{2} (w_{1,j}^{t+\Delta t} - a \sin \phi_n + w_{Qn}^t - a \sin \theta_n) \Delta t
 \end{aligned} \quad (36)$$

The streamline equation is:

$$p_{1,j}^{t+\Delta t} - p_R^t + \frac{1}{2} \rho a^2 \left[\left(\frac{\partial u}{\partial r} \right)_{1,j}^{t+\Delta t} + \left(\frac{\partial w}{\partial z} \right)_{1,j}^{t+\Delta t} + \left(\frac{u}{r} \right)_{1,j}^{t+\Delta t} + \left(\frac{u}{r} \right)_R^t + H_R^t \right] \Delta t = 0 \quad (37a)$$

where

$$H_Q^t = \left(\frac{\partial u}{\partial r} \right)_R^t + \left(\frac{\partial w}{\partial z} \right)_R^t \quad (37b)$$

The displacement of the streamline is:

$$\begin{aligned} r_{i,j}^{t+\Delta t} - r_R^t &= \frac{1}{2} (u_{i,j}^{t+\Delta t} + u_R^t) \Delta t \\ z_{i,j}^{t+\Delta t} - z_R^t &= \frac{1}{2} (w_{i,j}^{t+\Delta t} + w_R^t) \Delta t \end{aligned} \quad (38)$$

Now the system of five algebraic equations, equations (33) and (37), are used to solve for the five unknowns, $u_{i,j}^{t+\Delta t}$, $w_{i,j}^{t+\Delta t}$, $p_{i,j}^{t+\Delta t}$, $(\frac{\partial u}{\partial r})_{i,j}^{t+\Delta t}$, and $(\frac{\partial w}{\partial z})_{i,j}^{t+\Delta t}$.

A closed form solution is not possible because of the nonlinearities existing in the bicharacteristics and streamline equations. But the simultaneous solution can be obtained numerically by an iterative technique. The iteration, in this case, is between the point P and the initial plane located at time t.

B. Integrate System of Integral Equations

The compatibility equation (23) can be written in a system of integral equations along selected bicharacteristics:

$$\int_t^{t+\Delta t} \cos \theta_n du + \int_t^{t+\Delta t} \sin \theta_n dw - \frac{1}{\rho a} \int_t^{t+\Delta t} dp = a \int_t^{t+\Delta t} \left[\sin^2 \theta_n \frac{\partial u}{\partial r} - \sin \theta_n \cos \theta_n \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) + \cos^2 \theta_n \frac{\partial w}{\partial z} + \frac{u}{r} \right] dt \quad (39)$$

corresponding to the bicharacteristic,

$$\begin{aligned} \int_t^{t+\Delta t} dr &= \int_t^{t+\Delta t} (u - a \cos \theta_n) dt \\ \int_t^{t+\Delta t} dz &= \int_t^{t+\Delta t} (w - a \sin \theta_n) dt \end{aligned} \quad (40)$$

The integrals can be approximated in various ways such as by the trapezoidal formula, Simpson's rule, etc., and the resulting equations can be solved simultaneously by an iterative technique. For the same network construction given in Figure 2, the solution at the cone vertex P located at a mesh point in the new plane at an advanced time requires spatial interpolations in the initial plane. A double linear interpolation such as given equation (30) can be used for this purpose although higher order interpolations such as the second-order polynomial fit² may also be used. Pal⁸ has approximated the partial derivatives in the integral equations by central differences and the integrals by the

trapezoidal formula. For an appropriately selected discrete network such as the one satisfying the condition (26), Pal proved that the resulting algebraic system was a contraction mapping, thus establishing the uniqueness and convergence.

5. BOUNDARY CONDITIONS

Since the objective of this study is to provide means for investigating multidimensional pressure wave propagation in system component such as the steam generator, the boundary conditions that are pertinent to such problems only are discussed.

A. On Solid Boundary

Consistent with the inviscidness of the analysis, the condition of zero velocity component normal to the solid surface is used. The remaining equations are the compatibility relations along selected bicharacteristics. Here, the bicharacteristics must be chosen such that they intercept the interior of the initial surface.

B. On Open Boundary Connected to One-Dimensional Components

Two cases may be considered. When the diameter ratio is large, only one or two boundary nodes are encountered and the condition of one-dimensional flow, i.e., $u = 0$, seems reasonable. Additional equations are compatibility equation from the two-dimensional domain and one from the one-dimensional domain. The above equations will determine the boundary values for the case of large diameter ratio.

When the diameter ratio is not large, the boundary nodes may be more than two. In this case, a reasonable boundary matching of one- and two-dimensional

domains may be the one allowing two dimensionality, i.e., $u \neq 0$, at the boundary. Compatibility equations from the two-dimensional domain and one such equation from the one-dimensional domain will determine the boundary value. The axial velocity component at the boundary may be averaged over the cross-section for use in the one-dimensional domain calculation for the next time step.

C. On the Axis, $r = 0$

On the axis, $r = 0$, the condition to be satisfied is $u = 0$ since the radial velocity u is antisymmetric about the axis and also $\frac{\partial w}{\partial r} = 0$ since the axial velocity w is symmetric. The remaining equations to use are the compatibility equations along bicharacteristics and the streamline.

6. DISCUSSIONS AND REMARKS

The hydrodynamics equations of mass, momentum, and energy conservations are hyperbolic only when the viscous effects are neglected. For steady state problems, the equations become hyperbolic, parabolic or elliptic depending on local flow conditions, i.e., supersonic, transonic, or subsonic. For unsteady problems, however, the equations remain hyperbolic everywhere in the region irrespective of whether the local flow is supersonic, transonic, or subsonic. In problems where the viscous effects are important, the viscous terms must be included in which case the system becomes parabolic for which the method of characteristics cannot be used. In one-dimensional problems, the viscous effects can be included while maintaining the hyperbolicity of the problem. This is done by taking account of the after-flow effects by writing the viscous term as wall friction.⁹ Nevertheless, for pressure transients resulting from sodium-water reaction the viscous effects are small. The friction effect becomes important only as the unsteadiness dies

out as in the case of blowdown or when the material velocity is excessively high.

There are other methods available for use for numerical analysis of nonlinear hyperbolic systems. The artificial viscosity method such as the one proposed by Landshoff¹⁰ is especially advantageous for cases in which shocks are present. The shock discontinuities are handled as a continuous but rapid transition without introducing additional equations such as Rankine-Hugoniot conditions. Other methods use the boundary-fitting technique in handling the shock discontinuities. These methods are comparatively very simple in machine coding. Unfortunately, however, these methods also have inherent shortcomings. For example, some solutions obtained by the artificial viscosity method are completely inadequate⁶. The free boundaries associated with these methods are not well defined and the inclusion of the artificial viscosity term causes various undesirable side effects.

Probably the most attractive property of the method of characteristics is that the troubles of the above type do not exist. The method, however, suffers from a disadvantage of requiring an extensive computer programming effort. The coding effort becomes especially excessive in problems involving irregular domain with irregular boundaries. As a result, the per time-step calculation computer time is comparatively large but this is well offset by the possible higher accuracy. Further desirable features of the method of characteristics are that it closely relates to the physical phenomenon being studied. Basically, it keeps track of propagating waves and calculates their strength. This is especially true for two-variable problems. It is also true for higher dimensional problems but only in a lesser degree. Nevertheless, the method of characteristics provides for the most accurate and informative

means of analysis for hyperbolic systems.

Unlike the generalization of the two variable method of characteristics to the three variable methods, the generalization of the three variable formulation, discussed in this report, can easily be made to the most general problem of four independent variables, i.e., the unsteady three-dimensional problem in cylindrical geometry. In this case, the characteristic cones, bicharacteristics and the normals, $\vec{\lambda}$, can no longer be drawn other than their three-variable cuts. Finally, for problems in which shock waves are present, the Rankine-Hugoniot relations are also needed to relate flow properties across the discontinuities. For pressure transient analysis in liquid sodium arising from sodium-water reaction the pressure pulse is relatively low and the sonic velocity is high and hence it is anticipated that the provision for shock analysis is unnecessary.

APPENDIX A

Approximate Solution of the Bicharacteristic Curve

Let the equation describing the characteristic cone surface be:

$$r = r(\phi, \tau), \quad z = z(\phi, \tau), \quad t = t(\phi, \tau)$$

where τ is chosen to be directed in the negative time direction, i.e.,

$\tau = -t$. The bicharacteristic equations then are

$$\begin{aligned} \frac{\partial r}{\partial \tau}(\phi, \tau) &= -u + a \cos \theta \\ \frac{\partial z}{\partial \tau}(\phi, \tau) &= -w + a \sin \theta \end{aligned} \tag{A.1}$$

Assuming locally u , w , and θ are constants, the following relationship can be obtained:

$$\cos \theta \frac{\partial r}{\partial \phi}(\phi, \tau) = -\sin \theta \frac{\partial z}{\partial \phi}(\phi, \tau) \tag{A.2}$$

This is an approximate relationship and becomes exact as $\theta \rightarrow \phi$ or $\tau \rightarrow 0$.

Now, consider a bicharacteristic, $\theta = \phi$, at a point, i.e., point P in Figure 1

$$t = t_0, \quad r = r_0, \quad z = z_0, \quad \tau = 0$$

and let the parameter τ increase. An approximate expression

is sought here of the parametric angle θ at time $t = t_0 - dt$ or $\tau = d\tau$ of a bicharacteristic identified by $\theta = \phi$ at $t = t_0$ or $\tau = 0$.

Assume that the unknowns on the bicharacteristic are linear functions of τ where the coefficients of τ are functions of ϕ only:

$$\begin{aligned}
 u &= u_0 + \left[u(\phi) - u_0 \right] \tau \\
 w &= w_0 + \left[w(\phi) - w_0 \right] \tau \\
 a &= a_0 + \left[a(\phi) - a_0 \right] \tau \\
 \theta &= \phi + \left[\theta(\phi) - \phi \right] \tau
 \end{aligned}
 \tag{A.3}$$

where u_0 , w_0 , a_0 are values of u , w and a at the apex of the characteristic cone. The analysis here is carried out in a general manner so that it includes broader class problems including fully compressible flows where the sonic speed a also changes.

Substituting (A.3) into (A.1) and integrating results in:

$$\begin{aligned}
 r(\phi, \tau) &= r_0 + (-u_0 + a_0 \cos \phi) \tau + \frac{1}{2} \left\{ - \left[u(\phi) - u_0 \right] + \cos \phi \left[a(\phi) - a_0 \right] \right. \\
 &\quad \left. - a_0 \sin \phi \left[\theta(\phi) - \phi \right] \right\} \tau^2 + O(\tau^3) ; \\
 z(\phi, \tau) &= z_0 + (-w_0 + a_0 \sin \phi) \tau + \frac{1}{2} \left\{ - \left[w(\phi) - w_0 \right] + \sin \phi \left[a(\phi) - a_0 \right] \right. \\
 &\quad \left. + a_0 \cos \phi \left[\theta(\phi) - \phi \right] \right\} \tau^2 + O(\tau^3)
 \end{aligned}
 \tag{A.4}$$

where r_0 and z_0 are the r - and z -coordinates of the apex of the characteristic cone.

Hence,

$$\begin{aligned}
 \frac{\partial r}{\partial \phi}(\phi, \tau) &= -a_0 \sin \phi \cdot \tau + \frac{1}{2} \left[-u'(\phi) + a'(\phi) \cos \phi - a(\phi) \sin \phi + 2a_0 \sin \phi \right. \\
 &\quad \left. - a_0 \theta(\phi) \cos \phi + a_0 \phi \cos \phi - a_0 \theta'(\phi) \sin \phi \right] \tau^2 + O(\tau^3) \\
 \frac{\partial z}{\partial \phi}(\phi, \tau) &= a_0 \cos \phi \cdot \tau + \frac{1}{2} \left[-w'(\phi) + a(\phi) \cos \phi - 2a_0 \cos \phi + a'(\phi) \sin \phi \right. \\
 &\quad \left. + a_0 \theta'(\phi) \cos \phi - a_0 \theta(\phi) \sin \phi + a_0 \phi \sin \phi \right] \tau^2 + O(\tau^3)
 \end{aligned}
 \tag{A.5}$$

Substituting equations (A.5) into (A.2) and equating the coefficient of the leading term on both sides, i.e., the coefficients of τ^2 , and using the chain rule:

$$\begin{aligned} \theta - \phi = & \left[\cos^2 \phi \frac{\partial u}{\partial z} - \sin^2 \phi \frac{\partial w}{\partial r} + \sin \phi \cos \phi \left(-\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) + \sin \phi \frac{\partial a}{\partial r} \right. \\ & \left. - \cos \phi \frac{\partial a}{\partial z} \right] \tau + o(\tau^2) \end{aligned} \quad (\text{A.6})$$

where the partial derivatives are to be evaluated at $\tau = \tau$. Equation (A.6) is written for $\tau = \Delta t$ in which the angle θ at $t = t_0 - \Delta t$ is expressed in terms of derivatives in the plane at $t = t_0 - \Delta t$ and the angle $\theta = \phi$ at the cone apex:

$$\begin{aligned} \theta - \phi = & \left[\cos^2 \phi \frac{\partial u}{\partial z} - \sin^2 \phi \frac{\partial w}{\partial r} + \sin \phi \cos \phi \left(-\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) \right. \\ & \left. + \sin \phi \frac{\partial a}{\partial r} - \cos \phi \frac{\partial a}{\partial z} \right] \Delta t + o(\Delta t)^2 \end{aligned} \quad (\text{A.7})$$

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