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Informal Report

PARTICLE TRAPPING DURING PASSAGE THROUGH A
HIGH-ORDER NONLINEAR RESONANCE

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ABSTRACT

A theory of particle trapping and transport during passage through a high order one-dimensional nonlinear resonance is developed. It is a dynamic theory of what is generally referred to as the resonance "lock-in" process. The main result is an expression for the trapping efficiency as a function of the resonance excitation width, the nonlinear detuning, and the speed of passage through the resonance. The trapping efficiency shows a characteristic exponential dependence on crossing speed and a dependence on phase-space area which follows from rather general arguments connected with the invariance of local particle density. The question of what phase-space region trapped particles are drawn from is discussed. It is shown how the answer to this question is related to the adiabaticity of the system and the phase-space topology. The phenomenon of particle "explosions", occurring at phase-space amplitudes where nonlinear stabilization becomes ineffective, is investigated. The theory has been verified with a computer simulation for crossing a 5th order resonance.

1. Introduction

Beam loss in the stacked coasting beams of the ISR storage rings has been associated with high order nonlinear resonances.¹ In specific instances, 5th and 8th order resonances have been correlated with beam loss.² The evidence is strikingly simple. The beam density across the width of the stack is directly related to the tune. The observation is simply that beam loss occurs preferentially at positions corresponding to these resonances.

Although it is clear that high order resonances are a significant factor in ISR beam loss, the manner in which their effect becomes manifest is not clear at all. There are, in effect, two diverging lines of thought. First, the nonlinearity can display isolated resonance characteristics in the traditional sense.³ In the second place, there is the suggestion of a multiresonance effect, in which many high order resonances combine to produce a quasirandom force inducing diffusive beam growth, so-called Arnold diffusion.^{4,5} Of course, the main impetus for the second line of thought is the belief that the traditional resonance picture cannot provide a beam loss mechanism, that is, cannot explain the transport of particles from a beam position in the center of a chamber to the aperture limit. It thus seems that the multiple overlapping resonance model is a response to the apparent inability of a single isolated resonance model to adequately describe particle transport to large amplitudes.

This being the case, we have restudied the single-resonance model and we point out that single, isolated, high order nonlinear resonances, do indeed possess a mechanism for particle loss to the aperture limit, a mechanism which, as indicated, invokes no concept other than the traditional resonance characteristics. We are referring to the well known fact that although nonlinear detuning stabilizes high order resonances in a system with static tune, in a dynamic system (i.e. one with changing tune), small amplitude particles can "lock into" a resonance and be carried to large amplitudes. The formalism for this particle trapping process has been extensively developed, for example in Ref. (3). However, the precise conditions for "lock in", or "particle trapping" were not sufficiently explored. One might wonder why, after laying the groundwork, follow-up studies were not forthcoming. The answer lies, perhaps, in the sensitivity of the trapping process to the "speed of resonance crossing". If it was believed that high

order resonances were sufficiently weak that "fast crossing" was the prevailing mechanism, then the trapping process would substantially decrease in significance, since the trapping quickly diminishes to zero with crossing speed. This, in fact, is probably a good description of the accelerator field before storage rings. With relatively fast tune variations and weak high order resonances in existing accelerators, it is not surprising that the mechanism of trapping, which is enhanced as tune variation diminishes in speed and resonance strength increased, was neglected, if not overlooked.

This is not to say that resonances, along with fast crossing, do not cause beam growth. However, this subject has received attention elsewhere⁶ and we will not pursue it here.

With the advent of storage rings, we must reexamine our preconceptions of fast crossing and weak resonances. Take, for example, the ISR. If we consider high order resonances excited by the beam-beam interaction and tune variation caused by the diffusion of momentum due to intrabeam scattering^{7,8} (i.e. electromagnetic p-p scattering), it is clear that the fast crossing weak resonance combination does not apply. In fact, it is the attempt to respond to the "slow crossing strong resonance" implication in this case that is the primary reason for this study.

We review in Section 2, the theory of isolated resonances. Specifically, we show how resonance excitation and nonlinear detuning combine to create stable, moving islands forming at the center of the particle betatron phase space, growing in area and moving outward to larger amplitudes as the resonance is crossed. These islands, having stable fixed points in their interior, are capable of holding and carrying particles with them. They form the basis for our mechanism of particle transport to aperture limits. In this section, we thus lay the groundwork for a dynamic approach to trapping by considering the process as a sequence of static steps, assuming at each step, the existence of a phase space topology made up of quasi-invariant trajectories and fixed points. In Section 3, we present particle trapping under dynamic conditions, i.e. during resonance passage. We introduce the concept of trapping efficiency, simply the fraction of particles carried outward during a single passage. Then we consider the various factors which influence the trapping efficiency.

What is the trapping efficiency as a function of the rate of tune variation (the question of adiabaticity)? What is the influence of the initial particle distribution? From what phase-space region are the trapped particles drawn? What is the functional relationship between the trapping efficiency and the two nonlinear variables, i.e. the resonance excitation width and nonlinear detuning strength? Is there a connection between the trapping efficiency and the area of the stable islands? We find that standard isolated resonance theory can indeed provide answers to these questions. In Section 4 we show numerical results for passage through a 5th order resonance, and we correlate these results with the theoretical framework developed in Sections 2 and 3.

Our analysis applies to one-dimensional resonances. Although we have not attempted the extension to two-dimensions, we point out that in the coupling resonance case, the excitation width and nonlinear detuning are still the dominant characteristics. The complication is a result of the increase in initial condition parameters from 2 to 4. We believe that by an appropriate redefinition of tune and a proper interpretation of resonance width and detuning strength, our one-dimensional analysis is actually applicable to the coupling resonance case.

In general, we find a strong correlation between the theory developed and the numerical computations for a 5th order resonance. In particular, we derive an expression for the trapping efficiency as a function of the nonlinear resonance parameters as well as the crossing speed. We find the interesting result that the trapping efficiency is proportional to (i) the area of the moving islands in the vicinity of their exit point from the beam, relative to the beam area (recall that the islands expand during their outward migration in phase space) and (ii) an exponential factor, decreasing with the crossing speed. This expression gives a remarkable fit to data obtained from a computer simulation of a 5th order resonance. We find also that 5th order resonances are capable of inducing trapping efficiencies of the order of 30%. This is an impressive figure in that, using an isolated resonance-diffusion feeding mechanism, a trapping efficiency much less than this value is sufficient to describe the 5th order resonance loss observed at the ISR.⁸

2. One-Dimensional Nonlinear Resonance Theory

General Description

The single particle motion can be described by the Hamiltonian

$$H \left(y, \frac{dy}{ds}, s \right) = H^{(0)} \left(y, \frac{dy}{ds}, s \right) + H^{(1)} \left(y, \frac{dy}{ds}, s \right), \quad (2-1)$$

where $H^{(0)} = \frac{1}{2}[(dy/ds)^2 + K(s) y^2]$ is the unperturbed Hamiltonian, $H^{(1)}$ is a small perturbation, y is the displacement from the equilibrium orbit, s is the distance along the central orbit from a given reference azimuth, and $K(s)$ is the gradient focusing function.

The unperturbed equation of motion written in terms of the unperturbed displacement y_0 is

$$\frac{d^2 y_0(s)}{ds^2} + K(s) y_0(s) = 0. \quad (2-2)$$

To solve Eq. (2-2), we first change the "time scale" by the transformation

$$\theta = \frac{1}{v} \int_0^s \frac{ds'}{\beta(s')}, \quad (2-3)$$

with $v = 1/2\pi \int_0^C ds'/\beta(s') =$ the tune of the system,

$C =$ the circumference of the central orbit,

$\beta =$ the well-known β -function.⁹

We then have

$$\left(\frac{y_0}{\beta} \right)'' + v^2 \left(\frac{y_0}{\beta} \right) = 0, \quad$$

where prime means taking the derivative with respect to θ . The solution is easily found to be

$$y_0(\theta) = \sqrt{a_0 \beta(\theta)} \cos(v\theta + \varphi_0), \quad (2-4)$$

where a_0 and φ_0 are integration constants determined by the initial conditions of the particle under consideration. Note that a set of particles can be represented by the corresponding set of constants $\{a_0, \varphi_0\}$. In particular, the constant a_0 is just the emittance. Differentiating Eq. (2-4) with respect to θ , there results:

$$y_0'(\theta) = -\sqrt{a_0 \beta(\theta)} \sin(v\theta + \varphi_0) + \frac{1}{2} \sqrt{a_0 / \beta(\theta)} \beta'(\theta) \cos(v\theta + \varphi_0). \quad (2-5)$$

Then, eliminating φ_0 from Eqs. (2-4) and (2-5), we obtain the unperturbed invariant, i.e. the emittance referred to above:

$$a_0 = \frac{y_0'^2}{v^2 \beta} - \frac{\beta'}{v^2 \beta^2} y_0 y_0' + \frac{y_0^2}{\beta} \left(1 + \frac{\beta'^2}{4v^2 \beta^2} \right) \quad (2-6)$$

Turning now to the perturbed system, we consider the Hamiltonian in terms of the "time variable" θ , i.e.

$$H(y, y', \theta) = H^{(0)}(y, y', \theta) + H^{(1)}(y, y', \theta) \quad .$$

To solve this perturbed case, we remove the constraints that a_0 and φ_0 are constants of the motion and replace them by θ -dependent functions $a(\theta)$ and $\varphi(\theta)$ respectively. This is sufficient to describe the particle motion in the presence of the perturbation $H^{(1)}$. We now write the unperturbed solution as:

$$q_0 = y_0 = F(\theta, a_0, \varphi_0)$$

$$p_0 = y_0' = G(\theta, a_0, \varphi_0) \quad .$$

These quantities satisfy the Hamiltonian equations:

$$\frac{dq_0}{d\theta} = \frac{\partial H^{(0)}}{\partial p_0}$$

$$\frac{dp_0}{d\theta} = - \frac{\partial H^{(0)}}{\partial q_0} \quad ,$$

or

$$\frac{dF(\theta, a_0, \varphi_0)}{d\theta} = \frac{\partial F(\theta, a_0, \varphi_0)}{\partial \theta} = \frac{\partial H^{(0)}}{\partial G(\theta, a_0, \varphi_0)}$$

$$\frac{dG(\theta, a_0, \varphi_0)}{d\theta} = \frac{\partial G(\theta, a_0, \varphi_0)}{\partial \theta} = - \frac{\partial H^{(0)}}{\partial F(\theta, a_0, \varphi_0)} \quad .$$

The above equations are valid for any a_0 and φ_0 . Specifically, if a and φ are interpreted as functions of θ , we have the identities:

$$\frac{\partial F[\theta, a(\theta), \varphi(\theta)]}{\partial \theta} = \frac{\partial H^{(0)}}{\partial G[\theta, a(\theta), \varphi(\theta)]}$$

$$\frac{\partial G[\theta, a(\theta), \varphi(\theta)]}{\partial \theta} = - \frac{\partial H^{(0)}}{\partial F[\theta, a(\theta), \varphi(\theta)]} \quad .$$

(2-7)

Now, $q = F[\theta, a(\theta), \varphi(\theta)]$ and $p = G[\theta, a(\theta), \varphi(\theta)]$ must also satisfy the equations of motion described by the complete Hamiltonian H , i.e. we have:

$$\frac{dF[\theta, a(\theta), \varphi(\theta)]}{d\theta} = \frac{dq}{d\theta} = \frac{\partial H}{\partial p} = \frac{\partial H^{(0)}}{\partial G[\theta, a(\theta), \varphi(\theta)]} + \frac{\partial H^{(1)}}{\partial G[\theta, a(\theta), \varphi(\theta)]} \quad (2-8)$$

$$\frac{dG[\theta, a(\theta), \varphi(\theta)]}{d\theta} = \frac{dp}{d\theta} = -\frac{\partial H}{\partial q} = -\frac{\partial H^{(0)}}{\partial F[\theta, a(\theta), \varphi(\theta)]} - \frac{\partial H^{(1)}}{\partial G[\theta, a(\theta), \varphi(\theta)]}$$

By subtracting Eqs. (2-7) and (2-8), we obtain the equations of motion for $a(\theta)$ and $\varphi(\theta)$. They are

$$\frac{da}{d\theta} = \frac{1}{J} \frac{\partial H^{(1)}(\theta, a, \varphi)}{\partial \varphi} \quad (2-9)$$

$$\frac{d\varphi}{d\theta} = -\frac{1}{J} \frac{\partial H^{(1)}(\theta, a, \varphi)}{\partial a},$$

where the Jacobian J , is given by

$$J = \det \begin{bmatrix} \frac{\partial F}{\partial a} & \frac{\partial F}{\partial \varphi} \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial \varphi} \end{bmatrix}.$$

Since F and G are explicitly given in Eqs. (2-4) and (2-5), we can easily compute J . The result is

$$J = -\frac{\nu}{2} \beta(\theta).$$

Hence we have

$$\begin{aligned} \frac{da}{d\theta} &= -\frac{2}{\nu\beta} \frac{\partial H^{(1)}}{\partial \varphi} \\ \frac{d\varphi}{d\theta} &= \frac{2}{\nu\beta} \frac{\partial H^{(1)}}{\partial a} \end{aligned} \quad (2-10)$$

Note that Eq. (2-10) are rigorously valid, independent of the magnitude of $H^{(1)}$.

The Fifth Order Resonance

We consider first a "pure" resonance of order n , which is described by a perturbed Hamiltonian of the form

$$H^{(1)} = h(\theta) y^n, \quad (2-11)$$

where $h(\theta)$ is a given function of θ and is small in magnitude. A resonant blowup of the particle amplitude (i.e. emittance) can take place if the tune ν is close to p/n for some positive integer p and if $h(\theta)$ possesses a p^{th} azimuthal Fourier harmonic. We treat the case of a "simple" resonance, i.e. when p and n are relatively prime.

To see how the emittance blowup occurs, we first rewrite $H^{(1)}$ in terms of its Fourier components. More explicitly, let

$$\begin{aligned} \frac{2}{\nu \beta(\theta)} H^{(1)} &= \frac{2h(\theta)}{\nu \beta(\theta)} \left\{ \frac{1}{\sqrt{a(\theta)\beta(\theta)}} [e^{i\nu\theta+i\varphi(\theta)} + e^{-i\nu\theta-i\varphi(\theta)}] \right\}^n \\ &= \frac{h(\theta)}{\nu} \cdot \frac{\beta(\theta)^{n/2-1} a(\theta)^{n/2}}{2^{n-1}} \sum_{\ell=0}^n \frac{n!}{\ell!(n-\ell)!} e^{-i(n-2\ell)[\nu\theta+\varphi(\theta)]} \\ &= a^{n/2} \sum_{\ell=0}^n \frac{n!}{\ell!(n-\ell)!} \sum_{m=-\infty}^{\infty} A_m e^{im\theta - i(n-2\ell)(\nu\theta+\varphi)} \end{aligned} \quad (2-12)$$

where we have made the Fourier expansion

$$\frac{h(\theta) \beta^{n/2-1}(\theta)}{2^{n-1} \nu} = \sum_{m=-\infty}^{\infty} A_m e^{im\theta}$$

with

$$A_m = A_{-m}^* = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-im\theta} \frac{h(\theta) \beta^{n/2-1}(\theta)}{2^{n-1} \nu}.$$

If $H^{(1)}$ is a small perturbation, we expect a and φ to be slowly varying functions of θ . [In particular, $H^{(1)} = 0$ implies a and φ are constants of the motion, as can be seen from Eq. (2-10).] In order to solve for $a(\theta)$ and $\varphi(\theta)$ from Eq. (2-10), we retain only the slowly varying parts of Eq. (2-12). A rigorous justification including limitations of the procedure can be found in Ref. (3).

By inspection, the slowly varying components of the right-hand side of Eq. (2-12) are those satisfying

$$m - (n - 2\ell)\nu \approx 0,$$

$$\text{or} \quad \frac{m}{n - 2\ell} = \frac{\nu}{n}.$$

Since $|n - 2\ell| \leq n$, and further since p and n are relatively prime, we find that the above can be satisfied only when

$$\begin{cases} m = p \\ \ell = 0 \end{cases} \quad \text{or} \quad \begin{cases} m = -p \\ \ell = n \end{cases}.$$

That is, only two terms on the right-hand side of Eq. (2-12) must be retained:

$$\begin{aligned} \frac{2}{v\beta(\theta)} H^{(1)} &= a^{n/2} \{A_{-p} e^{-ip\theta + in(v\theta + \varphi)} + A_p e^{ip\theta - in(v\theta + \varphi)}\} \\ &= 2a^{n/2} |A_p| \cos [p\theta - n(v\theta + \varphi) + \eta] , \end{aligned}$$

where η is the phase angle of the complex number A_p .

Substituting into Eq. (2-10), we obtain

$$\begin{aligned} \frac{da}{d\theta} &= -2na^{n/2} |A_p| \sin n\psi \\ \frac{d\varphi}{d\theta} &= na^{n/2-1} |A_p| \cos n\psi , \end{aligned} \tag{2-13}$$

where

$$\psi = \frac{p}{n} \theta - (v\theta + \varphi) + \frac{\eta}{n} .$$

We can then show by direct differentiation that

$$\left(v - \frac{p}{n} \right) a + 2|A_p| a^{n/2} \cos n\psi = C , \tag{2-14}$$

is a constant of the motion. Thus a particle whose motion is governed by the Hamiltonian (2-1) and (2-11) will, to a good approximation, remain on curves of constant C . It is clear from Eq. (2-14) that if $v = p/n$, all curves of constant C go to infinite amplitude. Under such conditions, a beam will be unstable, with particles exploding to infinite amplitudes, approaching the asymptotes:

$$\psi = \frac{1}{2n} (2m + 1) \pi , \quad m = 0, 1, \dots, (2n - 1) .$$

Since we are mostly concerned with the fifth order resonance, we take $n = 5$. Interesting phenomena occur when we introduce a "stabilizing term" $O(\theta)y^4$ to the expression (2-11). The term is stabilizing in the sense that for any given tune, there exists a finite region of stable motion around the central (equilibrium) orbit. Sufficiently large amplitude motion will still, of course, be unstable. However, even though the smaller amplitude motion is stable for static tune, we will see that the presence of islands in the phase space introduces in the case of slowly varying tune the possibility of particles being trapped by the stable fixed points in these

islands and transported to large amplitudes. The perturbation Hamiltonian together with the stabilizing term is given by

$$H^{(1)} = O(\theta)y^4 + h(\theta)y^5 \quad (2-15)$$

We then follow the same procedure as we did before and consider a tune $\nu \approx p/5$. The slowly varying part of $h(\theta)y^5$ is as before. The slowly varying part of $O(\theta)y^4$ is as follows:

$$\begin{aligned} \frac{2O(\theta)}{\nu\beta(\theta)} y^4 &= \frac{2O(\theta)}{\nu\beta(\theta)} \left\{ \frac{1}{2} \sqrt{a(\theta)\beta(\theta)} [e^{i\nu\theta+i\varphi(\theta)} + e^{-i\nu\theta-i\varphi(\theta)}] \right\}^4 \\ &= a^2(\theta) \sum_{l=0}^4 \frac{4!}{l!(4-l)!} \sum_{m=-\infty}^{\infty} B_m e^{im\theta - i(4-2l)(\nu\theta+\varphi)} \end{aligned}$$

where $\frac{O(\theta)}{8\nu} \beta(\theta) = \sum_{m=-\infty}^{\infty} B_m e^{im\theta}$ has been used.

Thus the only slowly varying component is

$$\begin{cases} l = 2 \\ m = 0 \end{cases},$$

hence
$$\frac{2O(\theta)}{\nu\beta(\theta)} y^4 = 6a^2(\theta) B_0$$

Combining this with our expression for $H^{(1)}$ for a "pure" resonance, the result for $H^{(1)}$ is:

$$\frac{2}{\nu\beta(\theta)} H^{(1)} = 6 B_0 a^2 + 2|A_p| a^{5/2} \cos 5\psi \quad (2-16)$$

where

$$e^{i\eta}|A_p| = \frac{1}{32\pi\nu} \int_0^{2\pi} d\theta e^{-ip\theta} h(\theta)\beta^{3/2}(\theta)$$

$$B_0 = \frac{1}{16\pi\nu} \int_0^{2\pi} d\theta O(\theta)\beta(\theta)$$

Equation (2-10) now becomes

$$\begin{aligned} \frac{da}{d\theta} &= -10a^{5/2}|A_p| \sin 5\psi \\ \frac{d\varphi}{d\theta} &= 5a^{3/2}|A_p| \cos 5\psi + 12B_0 a \end{aligned} \quad (2-17)$$

Again by differentiation, we find that

$$\left(\nu - \frac{p}{5} \right) a + 6B_0 a^2 + 2a^{5/2}|A_p| \cos 5\psi = C \quad (2-18)$$

is a constant of the motion. Comparing Eq. (2-18) with Eq. (2-14) it is obvious that the stabilizing octupole term in $H^{(1)}$ has introduced a new term in the constant of motion.

In order to see all the possible particle trajectories described by Eq. (2-18), we first define

a_0 = some reference amplitude, for example, the average emittance of the initial particle distribution.

$\Delta_L = -v + p/n$ = the linear tune shift

$\Delta_{NL} = -12B_0 a_0$ = the nonlinear tune shift at the reference amplitude

$\Delta_e = -5|A_p| a_0^{3/2}$ = the excitation width at the reference amplitude

$\xi = 5\Delta_L/2\Delta_e$

$K = 5\Delta_{NL}/4\Delta_e$

$\alpha = a/a_0$ = the relative squared amplitude.

Equation (2-18) then can be written

$$\xi\alpha + K\alpha^2 + \alpha^{5/2} \cos 5\psi = C \quad (2-19)$$

We have thus obtained the first integral, or invariant, for the perturbed system. A particle whose initial conditions correspond to a given constant C is constrained to lie on a trajectory described by Eq. (2-19). We will discuss the detailed characteristics of these trajectories in the next section.

Trajectories, Fixed Points

The trajectory of a particle can be conveniently described by the phase space variables $(\alpha^{1/2}, \psi)$ in a polar coordinate system. Since a trajectory must follow a constant C curve, those points $(\alpha_f^{1/2}, \psi_f)$ for which

$$\left. \frac{\partial C}{\partial \alpha} \right|_f = 0 \quad (2-20)$$

$$\left. \frac{\partial C}{\partial \psi} \right|_f = 0$$

have the significance that a particle at $(\alpha_f^{1/2}, \psi_f)$ will return to this point after a finite number of revolutions. Except for the central orbit, a particle returns after n revolutions where n is the order of the resonance. These points are called "fixed points".

From Eq. (2-20), we have

$$\sin 5\psi_f = 0$$

or

$$\cos 5\psi_f = \pm 1 \quad (2-21)$$

and

$$\xi + 2K\alpha_f \pm 5/2 \alpha_f^{3/2} = 0 \quad (2-22)$$

For a trajectory near the fixed points, a Taylor series expansion gives

$$C(\alpha_f + \delta\alpha_f, \psi_f + \delta\psi_f) = C(\alpha_f, \psi_f) + \frac{1}{2} \left[\frac{\partial^2 C}{\partial \alpha^2} \Big|_f (\delta\alpha_f)^2 + \frac{\partial^2 C}{\partial \psi^2} \Big|_f (\delta\psi_f)^2 \right] + \dots$$

where we have made use of the fact that

$$\frac{\partial^2 C}{\partial \psi \partial \alpha} \Big|_f = 0 \quad .$$

The nearby trajectory will then be a closed ellipse or an open hyperbola according as

$$\frac{\partial^2 C}{\partial \alpha^2} \Big|_f \cdot \frac{\partial^2 C}{\partial \psi^2} \Big|_f > 0$$

or

$$\frac{\partial^2 C}{\partial \alpha^2} \Big|_f \cdot \frac{\partial^2 C}{\partial \psi^2} \Big|_f < 0$$

respectively. In the former case, the fixed point will be stable. In the latter case, the fixed point will be unstable. By direct evaluation, the criterion is found to be

$$\begin{aligned} \alpha_f^{\frac{1}{2}} \pm \frac{8}{15} K < 0 & \text{ for stable fixed points} \\ \alpha_f^{\frac{1}{2}} \pm \frac{8}{15} K > 0 & \text{ for unstable fixed points.} \end{aligned} \quad (2-23)$$

In order to find the fixed points, we have to solve the cubic Eq. (2-22). The solution is as follows:

For later convenience, we define a parameter t ,

$$t = \frac{4K}{15} \quad .$$

(I) When ξ lies between 0 and $-10t^3$, $\alpha_f^{\frac{1}{2}}$ has three real solutions

$$\mp t = 2|t| \cos \left(\frac{n\pi + \varphi}{3} \right), \quad n = 0, 2, 4$$

where
$$\varphi = \cos^{-1} \left[\pm \frac{t}{|t|} \left(1 + \frac{\xi}{5t^3} \right) \right]$$

The condition $\alpha_f^{\frac{1}{2}} \geq 0$ restricts the physical solutions to only those n for which

$$\cos \left(\frac{n\pi + \varphi}{3} \right) \leq \mp \frac{1}{2} \frac{t}{|t|} \quad (2-24)$$

Out of 0, 2, 4, only one n can satisfy Eq. (2-24) if the RHS is $-\frac{1}{2}$. Two values of n can satisfy it if the RHS is $+\frac{1}{2}$.

To determine the nature of the fixed points, we apply Eq. (2-23), which becomes

$$\begin{aligned} \cos \left(\frac{n\pi + \varphi}{3} \right) &> \pm \frac{1}{2} \frac{t}{|t|} \quad \text{for stable fixed points} \\ \cos \left(\frac{n\pi + \varphi}{3} \right) &< \pm \frac{1}{2} \frac{t}{|t|} \quad \text{for unstable fixed points.} \end{aligned} \quad (2-25)$$

To more clearly see the phase-space topology, we suppose $K > 0$. This entails no loss of generality, since the trajectory types are related to the relative signs K and ξ . There are two sets of fixed points defined by $\cos 5\psi_f = \pm 1$, as given in Eq. (2-21).

(a) When $\cos 5\psi_f = +1$, we see from Eqs. (2-24) and (2-25) that physical fixed points must be unstable. They occur at

$$\alpha_f^{\frac{1}{2}} = -t \left[1 + 2 \cos \left(\frac{n\pi + \varphi}{3} \right) \right] < t \quad (2-26)$$

where
$$\varphi = \cos^{-1} \left(1 + \frac{\xi}{5t^3} \right),$$

and n is chosen from 0, 2, or 4 such that

$$\cos \left(\frac{n\pi + \varphi}{3} \right) < -\frac{1}{2}.$$

(b) When $\cos 5\psi_f = -1$, we have a set of stable fixed points and a set of unstable fixed points. The unstable fixed points occur at

$$\alpha_f^{\frac{1}{2}} = t \left[1 - 2 \cos \left(\frac{n\pi + \varphi'}{3} \right) \right] \quad (2-27)$$

with
$$\cos \left(\frac{n\pi + \varphi'}{3} \right) < -\frac{1}{2}$$

which in turn implies

$$3t > \alpha_f^{\frac{1}{2}} > 2t \quad .$$

The stable fixed points occur at

$$\alpha_f^{\frac{1}{2}} = t \left[1 - 2 \cos \left(\frac{n\pi + \varphi'}{3} \right) \right] \quad (2-28)$$

with

$$- \frac{1}{2} < \cos \left(\frac{n\pi + \varphi'}{3} \right) < \frac{1}{2} \quad ,$$

which implies

$$\alpha_f^{\frac{1}{2}} < 2t \quad .$$

The phase φ' is given by

$$\varphi' = \cos^{-1} \left(-1 - \frac{\xi}{5t^3} \right) \quad .$$

(II) When ξ is outside of the range 0 to $-10t^3$, $\alpha_f^{\frac{1}{2}}$ has only one real solution

$$\alpha_f^{\frac{1}{2}} = \mp t - x - z$$

where

$$\begin{aligned} x &= \left[\frac{1}{5} \sqrt{\xi(\xi + 10t^3)} \pm \left(\frac{\xi}{5} + t^3 \right) \right]^{1/3} \\ z &= \left[-\frac{1}{5} \sqrt{\xi(\xi + 10t^3)} \pm \left(\frac{\xi}{5} + t^3 \right) \right]^{1/3} \end{aligned} \quad (2-29)$$

In order that $\alpha_f^{\frac{1}{2}} > 0$ (i.e. for the solution to be physical), x and z must satisfy

$$x + z < \mp t \quad . \quad (2-30)$$

To find the nature of the fixed points, we apply criterion (2-23), i.e.

$$\alpha_f^{\frac{1}{2}} \pm 2t = \pm t - x - z > \pm 2t \quad ,$$

where we have used Eq. (2-30). This means that when

$$\pm K > 0 \quad , \quad (2-31)$$

the fixed points must be unstable.

What happens when $\pm K < 0$? In this case, using Eq. (2-23), the fixed points will be stable if $\alpha_f^{\frac{1}{2}} < 2|t|$ and will be unstable if $\alpha_f^{\frac{1}{2}} > 2|t|$.

Before going on, a few comments on x and z are helpful. First note that $xz = t^2 > 0$, i.e. x and z are of the same sign.

Also, since $|\pm (\xi/5 + t^3)| > 1/5 \sqrt{5(\xi + 10t^3)}$, the sign of x and z will be the same as $\pm (\xi/5 + t^3)$ by an inspection of Eq. (2-29). The inequality

$$|x + z| > 2\sqrt{xz} = 2|t| \quad (2-32)$$

implies that the value of $x + z$ must stay sufficiently away from 0.

Again, for clarity, we assume $K > 0$. Here we have four cases to consider, since there are two distinct regions of ξ to take into account, as well as two separate fixed point equations defined by $\cos 5\psi_f = \pm 1$.

(a) If $\xi > 0$, $\cos 5\psi_f = +1$, the fixed points do not exist. This follows from the fact that if $\xi/5 + t^3$ is positive then x and z are also positive and thus

$$\alpha_f^{\frac{1}{2}} = -t - x - z < 0,$$

which means there are no physically realizable fixed points.

(b) If $\xi > 0$, $\cos 5\psi_f = -1$, we have x and z negative which follows from $-(\xi/5 + t^3)$ being negative. Therefore Eq. (2-32) gives

$$\alpha_f^{\frac{1}{2}} = t - x - z > 3t.$$

Hence these fixed points are unstable because

$$\alpha_f^{\frac{1}{2}} > 2|t|.$$

(c) If $\xi < -10t^3$, $\cos 5\psi_f = +1$, x and z are negative. Equation (2-29) implies

$$\alpha_f^{\frac{1}{2}} = -t - x - z > t.$$

The fixed points are unstable because of Eq. (2-31).

(d) If $\xi < -10t^3$, $\cos 5\psi_f = -1$, x and z are positive. We have no physical fixed points because

$$\alpha_f^{\frac{1}{2}} = t - x - z < 0.$$

Thus, in the case of one real, physical fixed point, it is an unstable one defined by the conditions

$$K > 0, \xi > 0, \cos 5\psi_f = -1,$$

or

$$K > 0, \xi < -10t^3, \cos 5\psi_f = +1.$$

Let us now consider the general topology of the phase space as ξ changes across 0; i.e. as the tune passes through the resonance.

Assume $K > 0$ and start with $\xi > 0$. There will be an unstable fixed point at the amplitude

$$\alpha_f^{\frac{1}{2}} = t - x - z > 3t, \quad (2-33)$$

where x and z are given in Eq. (2-29).

Actually, in the phase space there are five fixed points, all at the same amplitude given in Eq. (2-33) and at the phases

$$\cos 5\psi_f = -1$$

or

$$\psi_f = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}.$$

As ξ decreases and approaches 0, $\alpha_f^{\frac{1}{2}}$ approaches $3t$. When ξ becomes negative, with $0 > \xi > -10t^3$, the previous unstable fixed points persist. Their amplitude decreases to $\alpha_f^{\frac{1}{2}} = 2t$ at the end of this range, i.e. at $\xi = -10t^3$. However, as ξ passes 0, new fixed points appear in the phase plane. In fact, two sets of fixed points, one stable, the other unstable develop.

As a function of ξ , the new stable fixed points (5 of them, at phases given by $\cos 5\psi_f = -1$) all have amplitudes $\alpha_f^{\frac{1}{2}} = 0$ when $\xi = 0$, which increases until $\alpha_f^{\frac{1}{2}} = 2t$ at the end of the range, i.e. when $\xi = -10t^3$. Each of the unstable fixed points (the 5 of them occur at phases given by $\cos 5\psi_f = +1$) has amplitude $\alpha_f^{\frac{1}{2}} = 0$ when $\xi = 0$, increasing until $\alpha_f^{\frac{1}{2}} = t$ when $\xi = -10t^3$. What is happening is this: As ξ approaches $-10t^3$, a set of unstable fixed points is moving inward along the phase lines $\cos 5\psi_f = -1$, while the stable set is moving outward on a collision course. The result is that at the amplitude $\alpha_f^{\frac{1}{2}} = 2t$, as ξ passes into the region $\xi < -10t^3$, they annihilate each other. Only a single set of unstable fixed points remains, moving outward along the phase lines $\cos 5\psi_f = +1$. This process we have described is pictured in Fig. 1.

3. Theory of Particle Trapping

In this section we derive an expression for the trapping efficiency as a function of the crossing speed, the resonance excitation width and the nonlinear detuning strength, in the case of a Gaussian amplitude distribution. However, before doing this, we consider the notion of an "explosion". The point is that for high order resonances, the large amplitude phase-space region is traversed by unbounded curves. Thus particles trapped in the outward moving islands will eventually be transported into this region. At such a point, rapid amplitude growth results. In effect,

this transition point represents a limit to the applicability of the trapping mechanism. We therefore include a discussion of "explosions" at this point. It might be mentioned here that although the transition point is an aperture boundary where particles will be lost, in real situations, the true aperture limit might be the physical chamber wall which could be at an amplitude far below the "explosion" amplitude.

The Criterion for an Explosion

We have seen in Section 2 that as ξ changes, stable and unstable fixed points are created and annihilated in a well-defined manner. In particular, as ξ decreases toward the value $-10t^3$ (assuming $K > 0$), the stable fixed points and half of the unstable fixed points move to the amplitude $\alpha_f^{\frac{1}{2}} = 2t$ along the phase lines $\cos 5\psi_f = -1$ and there annihilate each other. The particles dragged along by the stable fixed points will then start to move on unbounded constant-C curves according to Eq. (2-19) and eventually escape.

An "explosion" state is defined to exist whenever at least one trapped particle of an initially stable distribution of particles is found on an unbounded constant-C curve. Note that the restriction that the exploding particle be trapped reflects our exclusion of the case where the resonant strength is so large that the exterior unstable fixed points and therefore the unbounded C curves, actually extend into the initial beam distribution. This definition is especially convenient for numerical calculations, for which an explosion is simply a sudden extremely steep increase of the amplitude of any one of the particles. Theoretically, we will see that the "explosion" threshold is a somewhat nebulous region in both tune, i.e. ξ , and amplitude. However, we will indicate some broad criteria for the onset of this phenomenon.

We have concluded that when ξ approaches $-10t^3$, an explosion state must develop. In reality, explosion may occur before (but never after) ξ reaches this limit. The reason is simply that a particle trapped by the island but removed from the exact position of the stable fixed point may find itself in the unstable region before the annihilation of stable and unstable fixed points actually takes place. This can be seen by observing the sequence in Fig. 1. Consider a particle oscillating about the stable

fixed point moving outward. The unstable fixed point, on its inward collision course, will begin to distort its trajectory before annihilation. The trajectories actually become very concentrated. But as the unstable fixed point closes in on its trajectory, the particle's motion at some point ceases to be adiabatic and thus it is not constrained to lie on its initial trajectory. The particle may thus cross the "nearby" separation boundary onto an unbounded C trajectory. An explosion state has thereby occurred before the limiting fixed point criterion. Hence we have the result:

$$(\xi)_{\text{at explosion}} \approx -10 t^3,$$

or using the definitions of ξ and t :

$$|(\Delta_L)_{\text{at explosion}}| \leq \left| \frac{4}{27} \frac{(\Delta_{NL})^3}{(\Delta_e)^2} \right|. \quad (3-1)$$

Let ξ be changing at a rate of ϵ /revolution, where ϵ may or may not be a function of θ . This change ϵ will cause a change in the position of the unstable fixed points located at $\alpha_f^{\frac{1}{2}}$ [given by Eq. (2-27)] and $\cos 5\psi_f = -1$ by approximately

$$\delta \alpha_f^{\frac{1}{2}} \approx \epsilon \frac{\partial \alpha_f^{\frac{1}{2}}}{\partial \xi} = \frac{\epsilon}{\frac{15}{2} \alpha_f - 4K \alpha_f^{\frac{1}{2}}}. \quad (3-2)$$

Equations (3-1) and (3-2) are broad "explosion" criteria reflecting respectively the dependence of explosion onset on the phase-space topology and the rate of tune change, i.e. ϵ . What we really want to know is how the magnitude of ϵ influences exactly how far below Eq. (3-1) explosion actually occurs. This is evidently a complex question in that it depends not only on the topology of the phase space, but where the particles are as annihilation approaches. Thus, the total history of the trapping process must be known, thereby determining where in the islands the particles are actually situated.

All things considered, we must content ourselves with two rather qualitative statements concerning "explosions". First, the explosion state must occur before fixed point annihilation as expressed by Eq. (3-1). The mechanism for this is hinted at in Eq. (3-2). The fact that $\delta \alpha_f^{\frac{1}{2}} \rightarrow \infty$ as $\alpha_f^{\frac{1}{2}} \rightarrow 2t$ indicates an early explosion when ξ approaches the collision limit, $-10 t^3$. A second point relates to the rate of tune variation. From Eq.

(3-2) we can see that a reduction in ϵ will delay the "explosion" state since, for any given trajectory, it reduces the probability of a particle crossing the separatrix (i.e. the separation between stable and unstable regions).

Particle Trapping and the Trapping Efficiency

Consider a physical particle in the phase space. From the first of Eqs.(2-17) and the definition of Δ_e , we obtain

$$\frac{d\alpha^{\frac{1}{2}}}{d\theta} = \Delta_e \alpha^2 \sin 5\psi .$$

On the other hand, when the amplitude of the stable fixed point passes the particle amplitude ($\alpha_f = \alpha$ at this moment) as ξ changes, the amplitude of the stable fixed point is growing at the rate

$$\frac{d\alpha_f^{\frac{1}{2}}}{d\theta} = \frac{d\xi}{d\theta} \frac{1}{\frac{15}{2} \alpha_f - 4K\alpha_f^{\frac{1}{2}}} .$$

If this physical particle is to follow the fixed point, it will have to grow in amplitude together with the fixed point. This can be fulfilled only if

$$\Delta_e \alpha^2 \sin 5\psi \geq \frac{d\xi}{d\theta} \frac{1}{\frac{15}{2} \alpha - 4K\alpha^{\frac{1}{2}}} . \quad (3-4)$$

We now have to find how many particles satisfy Eq. (3-4).

These particles may be trapped by the passing stable fixed point (although they do not have to be trapped). An upper limit of the trapping efficiency can then be obtained in the following way:

$$\leq \frac{\text{the number of particles satisfying Eq. (3-4) in the initial distribution}}{\text{the total number of particles}} . \quad (3-5)$$

Before doing this, we first note that Eq. (3-4) can be satisfied only when

$$\Delta_e \alpha^2 \geq \frac{d\xi/d\theta}{\frac{15}{2} \alpha - 4K\alpha^{\frac{1}{2}}} . \quad (3-6)$$

For later convenience, we shall define α_o to be

$$\alpha_o = \left[\frac{\delta \Delta_L / N_{\text{rev}}}{4\pi \Delta_{NL} \Delta_e} \right]^{2/5} , \quad (3-7)$$

where $\delta\Delta_L/N_{rev}$ is the change of Δ_L per revolution, and is assumed to be constant in time for simplicity.

Since K is large, say of the order of 100, and by choosing our reference amplitude α_0 to be the average emittance, α is of the order of unity, and the denominator of the RHS of Eq. (3-6) is well approximated by $-4K\alpha^{\frac{1}{2}}$. After rearranging terms and recalling the definitions of K and ξ , we find the solution of Eq. (3-6) is

$$\alpha \geq \alpha_0 \quad (3-8)$$

which means that only particles with larger amplitude than α_0 stand a chance of being trapped. For those cases where $15\alpha/2$ is not completely negligible compared with $4K\alpha^{\frac{1}{2}}$, the next order of approximation may be needed.

The solution is easily found by iteration to be:

$$\alpha \geq \alpha_0 \left(1 + \frac{3}{4K} \alpha_0^{\frac{1}{2}} \right) \quad (3-9)$$

If we denote the solution of Eq. (3-6), when the inequality is replaced by an equality, by α_1 , the solution of Eq. (3-6) will then be

$$\alpha \geq \alpha_1$$

where

$$\alpha_1 \approx \alpha_0 \left(1 + \frac{3}{4K} \alpha_0^{\frac{1}{2}} + \dots \right) \quad (3-10)$$

Now let $D(\alpha^{\frac{1}{2}})d\alpha^{\frac{1}{2}}$ be the initial particle distribution density (assumed to be independent of ψ) normalized by

$$\int_0^\infty D(\alpha^{\frac{1}{2}})d\alpha^{\frac{1}{2}} = 1 \quad .$$

Equation (3-5) then assumes the following form:

$$\begin{aligned} P_T &\leq \int_{\alpha_1^{\frac{1}{2}}}^\infty d\alpha^{\frac{1}{2}} D(\alpha^{\frac{1}{2}}) \int_{\sin^{-1} \left[\left(\frac{\frac{d\xi/d\theta}{\frac{15}{2} \alpha - 4K\alpha^{\frac{1}{2}}}} \right) / \left(\Delta_e \alpha^2 \right) \right]}^{\pi/2} d(5\psi) 5\pi \\ &= \int_{\alpha_1^{\frac{1}{2}}}^\infty d\alpha^{\frac{1}{2}} D(\alpha^{\frac{1}{2}}) \cdot \left\{ \frac{1}{2} - \frac{1}{\pi} \sin^{-1} \left[\left(\frac{\alpha_1}{\alpha} \right)^{5/2} \right] \right\} \quad (3-11) \end{aligned}$$

Since α_1/α is positive, we obtain a less stringent limit on P_T , namely

$$P_T \leq \frac{1}{2} \int_{\alpha_1^{\frac{1}{2}}}^\infty d\alpha^{\frac{1}{2}} D(\alpha^{\frac{1}{2}}) \quad (3-12)$$

For example, if

$$D(\alpha^{\frac{1}{2}})d\alpha^{\frac{1}{2}} = 2\alpha^{\frac{1}{2}}d\alpha^{\frac{1}{2}}e^{-\alpha} \quad , \quad (3-13)$$

which means a Gaussian distribution in $\alpha^{\frac{1}{2}}$ with volume element $2\pi\alpha^{\frac{1}{2}}d\alpha^{\frac{1}{2}}$ and root mean square $\langle \alpha \rangle^{\frac{1}{2}} = 1$, Eq. (3-12) implies

$$P_T \leq \frac{1}{2}e^{-\alpha_1} \quad . \quad (3-14)$$

Equations (3-14) and (3-11) should be regarded as a crude limit, for the following two reasons:

- (i) A trapping does not have to occur at the exact time when the stable fixed point is passing by.
- (ii) The phase 5ψ of the particle can change rapidly, and the integration limits $\sin^{-1}1 < 5\psi < \sin^{-1}[(\alpha_1/\alpha)^{5/2}]$ may not be real. Indeed, the second equation of (2-17) gives $d(5\psi)/d\theta = 5(\Delta_L + \Delta_e \alpha^{3/2} \cos 5\psi + \Delta_{NL}\alpha)$, which means 5ψ can change by π in approximately 100 revolutions (using $\Delta_{NL} \sim 10^{-3} \gg \Delta_e$, $\alpha \sim 1$).

A much better estimation for this upper limit is then obtained by simply replacing the lower limit

$$\sin^{-1} \left[\frac{d\mathcal{E}/d\theta}{\frac{15}{2}\alpha - 4K\alpha^{\frac{1}{2}}} / \left(\Delta_e \alpha^2 \right) \right] \text{ by } \sin^{-1}(-1) \quad ,$$

$$\text{i.e.} \quad P_T \leq \int_{\alpha_1}^{\infty} \frac{1}{2} d\alpha^{\frac{1}{2}} D(\alpha^{\frac{1}{2}}) = e^{-\alpha_1} \quad , \quad (3-15)$$

which accounts for (ii) completely and (i) partially. The trapping efficiency reaches the value $e^{-\alpha_1}$ only when all particles with $\alpha \geq \alpha_1$ are trapped regardless of their phases. Equation (3-15) is not an estimate of the trapping efficiency, for it assumes that for slow enough crossing, all particles will be trapped. This is clearly untrue. The trapping must depend on the strength of the resonance relative to the nonlinear detuning. Although the process appears complex, we will see that the limitation can be formulated in a rather general way, as a consequence of the well-known Liouville theorem for particle densities.

In the phase space $(\sqrt{\alpha}, \psi)$, the local particle density according to the Liouville theorem, cannot change. This means that the number of particles trapped is limited by the phase-space area of the outward moving

islands. For example, if the density distribution were uniform, one might expect that as $\alpha_1 \rightarrow 0$, the trapping efficiency would simply be the island area as the islands exit the beam relative to the beam area. In general, since a particle is trapped only when its amplitude α is greater than α_1 , the area of the islands should be evaluated when the stable fixed point passes by a typical particle with $\alpha > \alpha_1$. Let the amplitude of the stable fixed point be α_s at this moment. We expect, for α_1 not too far away from unity

$$\sqrt{\alpha_s} \approx \begin{cases} \sqrt{\alpha_1} & \text{if } \alpha_1 > 1 \\ 1 & \text{if } \alpha_1 < 1 \end{cases} . \quad (3-17)$$

The reason for Eq. (3-17) is as follows: When $\alpha_1 > 1$, a particle being trapped will most likely have an amplitude close to the lower limit $\sqrt{\alpha_1}$, simply due to the denser distribution at smaller amplitudes. When $\alpha_1 < 1$, a typical particle satisfying $\sqrt{\alpha} > \sqrt{\alpha_1}$ will be at $\sqrt{\alpha} \approx 1$ because of the denser distribution at larger amplitudes.

Following from this general discussion on trapping, combined with Eq. (3-15), we suggest that the trapping efficiency can be expressed by

$$P_T = \frac{A}{\pi(\sqrt{\alpha_s})^2} e^{-\alpha_1} , \quad (3-18)$$

where A is the total area of the islands when the stable fixed point has an amplitude $\sqrt{\alpha_s}$. The ratio of A to $\pi(\sqrt{\alpha_s})^2$ is the fraction of the phase-space area within which particles can be trapped, and the factor $e^{-\alpha_1}$ is simply the fraction of available particles with $\sqrt{\alpha} > \sqrt{\alpha_1}$.

Now, we evaluate A. Since ξ/K is of the order of one, and K is a large number of the order of 100, $\xi/5t^3$ will be small, where $t = 4K/15$. The amplitude of the stable fixed point, according to Eq. (2-28) can be found to be approximately

$$\alpha_s^{\frac{1}{2}} \approx \left| \frac{\xi}{2K} \right|^{\frac{1}{2}} \left(1 + \frac{1}{6t} \left| \frac{\xi}{2K} \right|^{\frac{1}{2}} \right) ,$$

which is of the order of one, as it should if any physical particle is to be trapped by the passing stable fixed point. Similarly, using Eq. (2-26), we can find the amplitude of the unstable fixed point with $\cos 5\psi = +1$:

$$\alpha_u^{\frac{1}{2}} \approx \left| \frac{\xi}{2K} \right|^{\frac{1}{2}} \left(1 - \frac{1}{6t} \left| \frac{\xi}{2K} \right|^{\frac{1}{2}} \right) .$$

To the lowest order, $\sqrt{\alpha_s}$ is equal to $\sqrt{\alpha_u}$. This means that the shape of the island will be approximately an ellipse. As ξ increases, the island will take the shape more like a triangle as shown in Fig. 1(c).

We now need the equation for the separatrix. It is given by Eq. (2-19) with

$$C = \xi \alpha_u + K \alpha_u^2 + \alpha_u^{5/2}$$

determined by the unstable fixed point. Setting $\cos 5\psi = -1$ for the separatrix equation, writing $\alpha = \alpha_u + \delta$, we obtain

$$\xi \delta + K \delta (\delta + 2\alpha_u) - \alpha_u^{5/2} \left[\left(1 + \frac{\delta}{\alpha_u} \right)^{5/2} + 1 \right] = 0 \quad (3-19)$$

The meaning of δ is the difference between the (amplitude)² of the point on the separatrix with $\cos 5\psi = -1$ and the (amplitude)² of the unstable fixed point. It has two solutions δ_+ and δ_- . The difference between these two solutions is the width along one of the two principle axes of the elliptical island multiplied by $2/\alpha_u$, i.e.

$$\delta_+ - \delta_- \approx 2/\alpha_u (\sqrt{\alpha_+} - \sqrt{\alpha_-}) \quad (3-20)$$

The width across another principle axis is simply given by $2\pi\sqrt{\alpha_u}/5 \approx 2\pi\sqrt{\alpha_s}/5$. The area of the islands will be

$$A = 5 \times \pi \left(\frac{\sqrt{\alpha_+} - \sqrt{\alpha_-}}{2} \right) \left(\frac{2\pi\sqrt{\alpha_s}/5}{2} \right),$$

where the factor of 5 is because there are five islands in the $(\sqrt{\alpha}, \psi)$ space.

Expand Eq. (3-19), keeping terms up to δ^2 , we have

$$K\delta^2 + \left(\xi + 2K\alpha_u - \frac{5}{2} \alpha_u^{3/2} \right) \delta - 2\alpha_u^{5/2} \approx 0.$$

The term in parenthesis can be simplified, using Eq. (2-22), i.e.

$$\xi + 2K\alpha_u + \frac{5}{2} \alpha_u^{3/2} = 0.$$

We then obtain

$$K\delta^2 - 5\alpha_u^{3/2} \delta - 2\alpha_u^{5/2} \approx 0,$$

or simply

$$K\delta^2 - 2\alpha_u^{5/2} \approx 0.$$

The solution

$$\delta_{\pm} \approx \pm \left(\frac{2}{K} \alpha_u^{5/2} \right)^{1/2}$$

gives

$$A = \frac{\pi^2}{\sqrt{2}} K^{-1/2} \alpha_s^{5/4} \quad (3-21)$$

Combining Eqs. (3-18) and (3-21), we obtain for the trapping efficiency,

$$P_T = \frac{\pi}{\sqrt{2}} K^{-\frac{1}{2}} \alpha_s^{\frac{1}{2}} e^{-\alpha_1} ,$$

or

$$P_T = 2.0 \left(\frac{\Delta_e}{\Delta_{NL}} \right)^{\frac{1}{2}} \alpha_s^{\frac{1}{2}} e^{-\alpha_1} , \quad (3-22)$$

where $\sqrt{\alpha_s}$ is approximately given by Eq. (3-17) if α_1 does not deviate from unity too much.

4. Numerical Results

Computer Simulation

We have done some calculations for the 5th order resonance, with an octupole stabilizing term introduced into $H^{(1)}$. For simplicity, we further assume

$$h(\theta) = h_0 \delta(\theta) - h_0 \delta(\theta - \pi/2) + h_0 \delta(\theta - \pi) - h_0 \delta(\theta - 3\pi/2) \quad (4-1)$$

$$O(\theta) = O_0 \delta(\theta) + O_0 \delta(\theta - \pi/2) + O_0 \delta(\theta - \pi) + O_0 \delta(\theta - 3\pi/2) ,$$

where $h(\theta)$ and $O(\theta)$ are defined in Eq. (2-15). We also assume that at $\theta = 0, \pi/2, \pi$ and $3\pi/2$, the values of $\beta(\theta)$ are equal and $\beta'(\theta) = 0$.

We then somewhat arbitrarily choose $\beta = \langle \beta \rangle = 21.4$ m, at $\theta = 0, \pi/2, \pi$ and $3\pi/2$. The tune ν is assumed to be near the fifth order resonance value $102/5$, with $(\Delta_L)_{\text{initial}} = 0.003$.

The significance of the constants O_0 and h_0 are such that the particle receives a kick

$$\begin{cases} y(\theta^+) = y(\theta^-) \\ \Delta y'(\theta) = y'(\theta^+) - y'(\theta^-) = -4y^3(\theta) \cdot O_0 - 5y^4(\theta) \cdot h_0 \end{cases}$$

at $\theta = 0, \pi$, and a kick

$$\begin{cases} y(\theta^+) = y(\theta^-) \\ \Delta y'(\theta) = y'(\theta^+) - y'(\theta^-) = -4y^3(\theta) \cdot O_0 + 5y^4(\theta) \cdot h_0 \end{cases} \quad (4-2)$$

at $\theta = \pi/2, 3\pi/2$. Using formulae derived in Section 2, we can find the relationship between O_0, h_0 on the one hand, and Δ_{NL}, Δ_e on the other. To do this, we first compute A_{102} and B_0 . They are:

$$A_{102} = \frac{h_0 < \beta >^{3/2}}{8\pi\nu} , \quad B_0 = \frac{O_0 < \beta >}{4\pi\nu} .$$

Note that the integer p and the alternating signs as defined for $h(\theta)$ in Eq. (4-1) are chosen so that A_p does not vanish. We then find Δ_{NL} and Δ_e :

$$\begin{aligned}\Delta_{NL} &= - \frac{3a_o O_o < \beta >}{\pi v} \\ \Delta_e &= - \frac{5a_o^{3/2} < \beta >^{3/2} h_o}{8\pi v} .\end{aligned}\tag{4-3}$$

For the reference amplitude a_o , we simply use in all calculations the rms emittance of the initial particle distribution, i.e. $a_o = 1.6 \times 10^{-6}$ m, corresponding to the emittance of a high energy particle beam that might be found at the CERN ISR or the Brookhaven AGS. We then follow a set of three hundred particles, assuming Eq. (4-1) is the only perturbation, giving kicks according to Eq. (4-2). The parameters h_o and O_o are fixed by the values of Δ_e and Δ_{NL} according to Eq. (4-3). The only parameter left is $\delta\Delta_L/N_{rev}$ which together with Δ_e and Δ_{NL} are treated as variables.

The Initial Particle Distribution

The only parameter which determines the initial particle distribution is a_o . We assume a Gaussian distribution in $s = \sqrt{a}$, with volume element $2\pi s ds$, i.e. the probability of finding a particle in $(s, s + ds)$ is:

$$D(s) ds = C_1 e^{-C_2 s^2} 2\pi s ds .$$

The parameters C_1 and C_2 are determined by setting

$$\int_0^\infty D(s) ds = 1$$

and

$$\int_0^\infty D(s) \cdot s^2 ds = a_o .$$

We then have

$$D(s) ds = \frac{2}{s_o} e^{-(s/s_o)^2} s ds\tag{4-4}$$

where $s_o = \sqrt{a_o}$.

In order to apply the random variable technique, we first prove the following theorem:

Let $z = \int_0^s D(s') ds'$ be defined as a function of s , where $D(s)$ is any given density distribution, normalized by $\int_0^\infty D(s) ds = 1$. Let $z = B(s)$ have the inverse function $s = A(z)$, whose existence will be ensured if $D(s) > 0$ for all s . It is easily shown that $0 \leq z < 1$. Consider a random distribution of z over $(0,1)$. The number of events in $(z, z + dz)$ will be simply dz for all z . These events are mapped into the interval $[A(z), A(z + dz)]$ in s -space. But,

$$A(z + dz) = A(z) + A'(z) dz ,$$

which means that dz events are distributed over an interval of width $A'(z) dz$ in s -space. Hence the number of events in $(s, s + ds)$ for a given s and ds will be dz multiplied by the ratio ds to $A'(z) dz$. That is

$$dz \left(\frac{ds}{A'(z) dz} \right) = B'(s) ds = D(s) ds .$$

The procedure is then first to randomly generate a variable z in $(0,1)$. Compute s from z according to $s = A(z)$. The resultant random variable s will be defined by the density distribution $D(s) ds$. For our case, $D(s) ds$ is known. We find that

$$z = 1 - e^{(a/a_0)}$$

or

$$a = a_0 \ln [1/(1-z)] .$$

One possible initial particle distribution calculated for 300 particles with randomly generated z 's is plotted in Fig. 2(a). The dotted curve is our expectation of an ideal case.

The Trapping

A typical set of parameters are chosen for Fig. 2. Figures (a) to (d) correspond to four successive snapshots of the particle distribution when the number of revolutions $N = 0, 30\ 000, 50\ 000$ and $75\ 000$ respectively. It is clear that particles are being trapped and carried away. Figure 3 shows the initial ($N = 0$) and final ($N = 75\ 000$) distributions in the $(\sqrt{\alpha}, \psi)$ phase space for the same parameters. Three hundred particles are used. A trapping efficiency $P_T \sim 12\%$ is observed.

The Trapping Efficiency

Listed in Table I are our data of trapping efficiency for various sets of parameters. We find that a good fit is given by

$$P_T = 2.4 \left(\frac{\Delta_e}{\Delta_{NL}} \right)^{\frac{1}{2}} \alpha_s^{\frac{1}{2}} e^{-\alpha_1} \quad , \quad (4-5)$$

where

$$\alpha_s = \begin{cases} 1 & \text{if } \alpha_1 < 1 \\ \alpha_1 & \text{if } \alpha_1 > 1 \end{cases} .$$

It is extremely interesting that Eq. (4-5) deviates from Eqs. (3-22) and (3-17) only slightly in the overall numerical constant. Equation (4-5) makes sense only when

$$2.4 \left(\frac{\Delta_e}{\Delta_{NL}} \right)^{\frac{1}{2}} \leq 1 \quad , \quad (4-6)$$

which follows from Eq. (3-15). But we are quite safe to assume Eq. (4-6). It is violated when $K \leq 7$, which in turn means that soon after the tune crosses the resonance, the unstable fixed point located at $\alpha_f^{\frac{1}{2}} \approx 4K/5$ and $\cos 5\psi_f = -1$ are so close to the particle distribution that an explosion immediately occurs. In Fig. 4, we have plotted P_T vs α_0 . The parameters are typical, with $\alpha_1 \approx \alpha_0$ to within 1-2%. The dashed curve shows the upper limit of P_T , according to Eq. (3-15). The dotted curve shows Eq. (4-5). In Fig. 5 (a to c) we have plotted P_T vs $(\Delta_e/\Delta_{NL})^{\frac{1}{2}}$, for different α_0 's. A linear behavior is clear. In these figures, the dotted curves are given by Eq. (4-5). The dashed curves are $e^{-\alpha_1}$. The error bars are given by:

$$P_T \pm \Delta P_T = \frac{m}{n} \pm \frac{\sqrt{m(1-m/n)}}{n} \quad , \quad (4-7)$$

where m = the number of particles trapped

n = the total number of particles.

Equation (4-7) is derived by a simple random walk consideration.

Which Particles are Trapped?

We have shown the initial and final particle distributions for four cases in Fig. 6 (a - c), arranged in decreasing α_1 (increasing adiabaticity). We clearly see that only particles with $\alpha \geq \alpha_1$ are trapped. We also see that as (Δ_e/Δ_{NL}) increases, more particles with $\alpha \geq \alpha_1$ are trapped. Figure 6(b) shows how a large Δ_e/Δ_{NL} affects those particles with $\alpha \geq \alpha_1$.

Most of these particles are carried away by the stable fixed points. Figure 6(c) corresponds to a small α_1 , but (Δ_e/Δ_{NL}) is not as large as Fig. 6(b). It is clear that a small α_1 plus a large (Δ_e/Δ_{NL}) would sweep out most of the particles in the beam. Note that when α_1 becomes infinitesimal, the trapping efficiency does not approach 1. Instead it approaches some limit determined by the available area in order not to violate the Liouville theorem.

5. Conclusions

We have developed a theory of particle trapping and transport during passage through a high order one-dimensional nonlinear resonance - that is, a dynamic theory of what has been referred to as the resonance "lock-in" process. Our main result is an expression for the trapping efficiency as a function of the resonance excitation width, the nonlinear octupole detuning and the speed of passage through the resonance. As we have seen, the trapping efficiency follows from rather general arguments connected with the invariance of the local particle density, i.e. the Liouville theorem. In addition, for an initial Gaussian particle density distribution, we find that the trapping efficiency has an exponential fall-off with resonance crossing speed. We have verified the theory with a computer simulation for crossing a 5th order resonance. We have also studied the phenomenon of particle "explosions" (occurring at phase-space amplitudes where nonlinear stabilization becomes ineffective) as well as the influence of the initial distribution function. With regard to this latter point, we have, specifically looked into the question of what phase-space region trapped particles are drawn from. We have shown how the answer to this question is related to both the adiabaticity of the system and the phase-space topology. Again, the computer simulation is in remarkable agreement with the dynamic theory.

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Figure Captions

- Fig. 1 The Phase Space Topology for Resonance Crossing. Each curve represents a constant - C curve according to Eq. (2-19). The variable $K = 5\Delta_{NL}/4\Delta_e$ is kept constant, while the tune crosses a 5th order resonance. Blackened dots represent fixed points.
- (a) Before the tune crosses the resonance ($\xi = 5\Delta_L/2\Delta_e > 0$).
 - (b) After ξ becomes negative. A set of stable and unstable fixed points are produced at the origin, moving outward as ξ decreases.
 - (c) Before ξ reaches the fixed point collision limit, $-10t^3$.
 - (d) After the annihilation of the colliding fixed points.
- Fig. 2 The Trapping Process: Instantaneous Particle Distributions During Trapping. As a typical example, we take $\Delta_e = 2 \times 10^{-5}$, $\Delta_{NL} = 2 \times 10^{-3}$ and $\delta\Delta_L/N_{rev} = 0.25 \times 10^{-6}$.
- (a) The initial distribution.
 - (b), (c), and (d) The distributions after 30 000, 50 000, and 75 000 revolutions respectively. The amplitudes of stable fixed points are indicated by arrows. The dashed curve in (a) represents the expected particle distribution.
- Fig. 3 Particle Distribution in the $(\sqrt{\alpha}, \psi)$ Phase Space.
- (a) The initial distribution.
 - (b) The distribution after 75 000 revolutions. A typical example: $\Delta_e = 2 \times 10^{-5}$, $\Delta_{NL} = 2 \times 10^{-3}$, $\delta\Delta_L/N_{rev} = 0.25 \times 10^{-6}$.
- Fig. 4 The Trapping Efficiency, P_T , as a Function of the Crossing Speed. $\alpha_o^{5/2} = (\delta\Delta_L/N_{rev})/4\pi \Delta_{NL} \Delta_e$ is the adiabaticity parameter. Δ_{NL} and Δ_e are kept fixed. The dashed curve is the upper bound of P_T . The dotted curve is our theoretical fit.
- Fig. 5 The trapping efficiency as a function of $(\Delta_e/\Delta_{NL})^{1/2}$.
- (a), (b), and (c) correspond to $\alpha_o = 0.756, 0.998$ and 1.317 respectively. The solid line curves are simply $e^{-\alpha_o}$. The dashed curves are the upper limits $e^{-\alpha_1}$ of the trapping efficiency. The dotted curves are the theoretical fit.

Fig. 6 Influence of Initial Phase Space Position on Trapping. Solid curves are initial distributions. Dotted curves are the distributions after trapping.

- (a) α_1 large,
- (b) Δ_e / Δ_{NL} large,
- (c) α_1 small.

Note that (1) only particles with amplitude $\sqrt{\alpha} \geq \sqrt{\alpha_1}$ are trapped. (2) A large Δ_e / Δ_{NL} means a strong trapping effect for those particles with $\sqrt{\alpha} \geq \sqrt{\alpha_1}$.

Table I (a). Data of the Computer Simulation Experiment
for $\alpha_0 = 0.998$

$\frac{\delta A_L}{N_{rev}} (10^{-6})$	$\Delta_e (10^{-5})$	$\Delta_{NL} (10^{-3})$	$P_T = \frac{\text{no. of particles trapped}}{\text{total no. of particles}}$
2	4	4	26/300
1.15	3	3	27/300
0.5	2	2	29/300
0.25	2	1	41/300
1	4	2	42/300
0.57	3	1.5	38/300
1	2	4	20/300
2.3	3	6	17/300
1.56	2.5	5	54/600
0.5	4	1	62/300
1.14	6	1.5	59/300
1	8	1	79/300

Table I (b). Data of the Computer Simulation Experiment
for $\alpha_o = 0.756$

$\frac{\delta\Delta_L}{N_{rev}} (10^{-6})$	$\Delta_e (10^{-5})$	$\Delta_{NL} (10^{-3})$	$P_T = \frac{\text{no. of particles trapped}}{\text{total no. of particles}}$
0.25	2	2	35/300
1	4	4	100/900
0.56	3	3	63/600
2	8	4	46/300
0.5	4	2	117/900
0.78	5	2.5	99/600
0.28	3	1.5	43/300
0.5	2	4	23/300
2	4	8	21/300
1.14	3	6	25/300
0.25	4	1	59/300
1	8	2	62/300
0.5	8	1	88/300
0.5	10	0.8	105/300

Table I (c). Data of the Computer Simulation Experiment
for $\alpha_0 = 1.317$

$\frac{\delta\Delta_L}{N_{rev}} (10^{-6})$	$\Delta_e (10^{-5})$	$\Delta_{NL} (10^{-3})$	$P_T = \frac{\text{no. of particles trapped}}{\text{total no. of particles}}$
1	2	2	21/300
2.25	3	3	22/300
4	4	4	57/600
2	4	2	31/300
4.5	6	3	35/300
0.5	2	1	28/300
3	6	2	79/600
0.75	3	1	40/300
1	4	1	37/300
4	8	2	44/300
2.25	6	1.5	38/300
1	1	4	12/300
2.25	1.5	6	12/300
2.25	9	1	66/300
2	8	1	60/300

Table I (d). Data of the Computer Simulation Experiment
for $\Delta_e = 2 \times 10^{-5}$ and $\Delta_{NL} = 2 \times 10^{-3}$

$\frac{\delta\Delta_L}{N_{rev}} (10^{-6})$	$P_T = \frac{\text{no. of particles trapped}}{\text{total no. of particles}}$
20	6/600
10	4/300
5	10/300
3	11/300
1.5	18/300
1	21/300
0.75	39/600
0.5	29/300
0.25	35/300
0.1	47/300

LENGTH
SCALE:
26.7

$$\kappa = 100$$

$$\xi = 3 \times 10^2$$

$$\kappa = 100$$

$$\xi = -5 \times 10^4$$

(a) BEFORE CREATION

(b) AFTER CREATION

$$\kappa = 100$$

$$\xi = -1.5 \times 10^5$$

$$\kappa = 100$$

$$\xi = -2 \times 10^5$$

(c) BEFORE ANNIHILATION

(d) AFTER ANNIHILATION

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6