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REARRANGEMENT INEQUALITY
ON A CIRCULAR DOMAIN

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ABSTRACT

A rearrangement inequality for multiple integrals on a circular domain is stated and proved. Its relation to a previous theorem for functions on the real line is discussed.

In this paper we shall prove an integral rearrangement inequality (Theorem 1) for functions on a circular domain - or, what is the same thing, for periodic functions. By passing to the limit of long period in two different ways, we obtain two rearrangement inequalities on the real line. The first (Corollary 1) is a special case of a more difficult theorem¹ recently proved. The second (Corollary 2) is a new theorem for which we know no other proof. It leads, in physics, to Greens' - function inequalities on open (i. e., infinite) domains².

In the book Inequalities³, by Hardy, Littlewood and Pólya, the symmetric decreasing rearrangement (SDR) of a non-negative function F on the reals is defined as that function F^* which is even in its argument, monotonic decreasing for positive argument, and equimeasurable with F in that the sets $\{x \mid F(x) > y\}$ and $\{x \mid F^*(x) > y\}$ have equal measure, for each $y > 0$.

For our present purposes, we shall say that f^* is the periodic symmetric decreasing rearrangement (PSDR) of f if f^* and f are both periodic with period 2π , and f_π^* is the SDR of f_π , where

$$\begin{aligned} f_\pi(\vartheta) &= f(\vartheta), \quad -\pi \leq \vartheta \leq \pi \\ &= 0, \quad \text{otherwise} \\ f_\pi^*(\vartheta) &= f^*(\vartheta), \quad -\pi \leq \vartheta \leq \pi \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{1}$$

Our main theorem is as follows. It resembles Theorem 371 of [3], but the sums are replaced by integrals, (x, y) by n variables, and the real line by a circle.

Theorem 1. Let $f_1, f_2, \dots, f_n$ be non-negative periodic functions on the real line, with period 2π . Let $f_1^*, \dots, f_n^*$ be their PSDR. Let $h_1, h_2, \dots, h_n$ be

non-negative functions satisfying

$$h_i(x) \geq h_i(y) \text{ if } 1 \geq x \geq y \geq -1 \quad (2)$$

Let

$$I \equiv \int_{-\pi}^{\pi} d\theta_1 \dots d\theta_n f_1(\theta_1) h_1(\cos(\theta_1 - \theta_2)) \dots f_n(\theta_n) h_n(\cos(\theta_n - \theta_1)) \quad (3)$$

$$I^* \equiv \int_{-\pi}^{\pi} d\theta_1 \dots d\theta_n f_1^*(\theta_1) h_1(\cos(\theta_1 - \theta_2)) \dots f_n^*(\theta_n) h_n(\cos(\theta_n - \theta_1)) \quad (4)$$

Then

$$I \leq I^* \quad (5)$$

Proof: We use the idea of Sec. 10.8 of [3]. This simplifies considerably when applied to integrals instead of sums; the two kinds of pairing become one, and the nine partial sums become four. As against this, it is no longer obvious that we can pass from I to I^* in a finite number of transformations, since we are no longer dealing with a finite universe of rearrangements. Also, the crucial inequality will have to be generalized to the case of n variables; Hardy et al. treat the case $n = 2$.

We shall prove Theorem I under the restriction that each f_i takes only the values 0 and 1, and has only finitely many discontinuities between $-\pi$ and π .

The function f_i^* is then 1 or 0, according as $\cos \theta_i < \text{or } > \cos \frac{1}{2} m_i$ where

$$m_i = \int_{-\pi}^{\pi} f_i(\theta) d\theta \quad (6)$$

(For our purposes it is unnecessary to specify the value of a function at a point of discontinuity.) The restriction to 0 and 1 values may afterwards be removed as shown in Sec. 10.14 of [3].

By analogy with Ω_p in ref [3], we define a transformation Ω_θ , called "refraction about θ " where θ is a fixed parameter, which may be set equal to any real number. Each periodic function f is transformed into a function $\Omega_\theta f$, given on the interval $[\theta - \pi, \theta + \pi]$ by

$$\begin{aligned}\Omega_\theta f(\theta + \chi) &= \max [f(\theta + \chi), f(\theta - \chi)] \\ \Omega_\theta f(\theta - \chi) &= \min [f(\theta + \chi), f(\theta - \chi)]\end{aligned}\quad (7)$$

for $0 \leq \chi \leq \pi$. The values of $\Omega_\theta f$ outside this interval are then determined by requiring it to be periodic with period 2π .

We define the refracted value of I by

$$I_\theta \equiv \int_{-\pi}^{\pi} d\theta_1 \dots d\theta_n \Omega_{\theta_1} f_1(\theta_1) h_1(\cos(\theta_1 - \theta_2)) \dots \Omega_{\theta_n} f_n(\theta_n) h_n(\cos(\theta_n - \theta_1)) \quad (8)$$

(Note that the f_i are all refracted about the same θ , while the h_i are left unchanged.) We shall prove that

$$I \leq I_\theta \quad (9)$$

for arbitrary θ . We shall then show that a finite sequence $(\theta_1, \theta_2, \dots, \theta_M)$ exists such that

$$\Omega_{\theta_M} \Omega_{\theta_{M-1}} \dots \Omega_{\theta_2} \Omega_{\theta_1} f_i = f_i^* \quad (10)$$

for all $i = 1, \dots, n$. From these two assertions the theorem follows immediately in the form

$$I \leq I_{\theta_1} \leq I_{\theta_1 \theta_2} \leq \dots \leq I_{\theta_1 \theta_2 \dots \theta_M} = I^* \quad (11)$$

To prove (9), we use the periodicity of the f_i and $\Omega_\theta f_i$ to shift the region of integration in (3) and (8) so that it is centered at θ . Using the symbol μ to represent a sequence $\mu_1 \dots \mu_n$ all of whose members are ± 1 , we have

$$\begin{aligned}
 I &= \sum_{\mu} \int_0^{\pi} d\chi_1 \dots d\chi_n f_1(\theta + \mu_1 \chi_1) h_1(\cos(\mu_1 \chi_1 - \mu_2 \chi_2)) \dots f_n(\theta + \mu_n \chi_n) \\
 &\quad h_n(\cos(\mu_n \chi_n - \mu_1 \chi_1)) \\
 &= \int_0^{\pi} d\chi_1 \dots d\chi_n Q(\chi_1 \dots \chi_n)
 \end{aligned} \tag{12}$$

where, writing $f_i^{\pm}$ for $f_i(\theta \pm \chi_i)$ and $h_i^{\pm}$ for $h_i(\cos(\chi_i \mp \chi_{i+1}))$,

we can express $Q(\chi_1 \dots \chi_n)$ as

$$Q = \sum_{\mu} f_1^{\mu_1} h_1^{\mu_1 \mu_2} \dots f_n^{\mu_n} h_n^{\mu_n \mu_1} \tag{13}$$

(Two superscripts standing together are to be multiplied. By χ_{n+1} we mean χ_1 .)

Likewise,

$$I_{\theta} = \int_0^{\pi} d\chi_1 \dots d\chi_n \bar{Q}(\chi_1 \dots \chi_n) \tag{14}$$

where $\bar{Q}(\chi_1 \dots \chi_n)$ is given as

$$\bar{Q} = \sum_{\mu} \bar{f}_1^{\mu_1} h_1^{\mu_1 \mu_2} \dots \bar{f}_n^{\mu_n} h_n^{\mu_n \mu_1} \tag{15}$$

in which

$$\bar{f}_i^{+} = \max(f_i^{+}, f_i^{-}) \tag{16}$$

$$\bar{f}_i^{-} = \min(f_i^{+}, f_i^{-})$$

Both (13) and (15) contain a dependence on $x_1 \dots x_n$ which our notation has suppressed. We shall prove that for any fixed $x_1 \dots x_n$,

$$Q \leq \bar{Q} \quad (17)$$

This will suffice to prove (9).

We recall that all the $f_i^{\pm}$ are 0 or 1. If, for some i , $f_i^+ = 0$, then $f_i^+ = f_i^- = 0$, and $Q = \bar{Q} = 0$. Therefore, in proving (17), we may assume that all f_i^+ are 1.

Let s be the number of values of i for which $f_i^- = 0$. If $s = 0$ or 1, then $Q = \bar{Q} = 2^{-s} \left[\prod_{i=1}^n (h_i^+ + h_i^-) + \prod_{i=1}^n (h_i^+ - h_i^-) \right]$, as is easily seen. Therefore, assume that $s \geq 2$.

Let $i_1 < \dots < i_s$ be the values of i for which $f_i^- = 0$.

Let v_j stand for μ_{i_j} . Then

$$Q = C_{i_1, i_2}^{v_1 v_2} \dots C_{i_s, i_1}^{v_s v_1} \quad (18)$$

$$\bar{Q} = C_{i_1, i_2}^{++} \dots C_{i_s, i_1}^{++} \quad (19)$$

where

$$C_{i, i'}^{w'} = \sum_{\mu_{i+1}} \dots \sum_{\mu_{i'-1}} h_i^{v_{i+1} \mu_{i+1}} h_{i+1}^{\mu_{i+1} \mu_{i+2}} \dots h_{i'-1}^{\mu_{i'-1} v'} \quad (20)$$

if $i < i'$. For the case $i > i'$, (20) has an obvious modification by cyclic permutation of the subscripts.

Clearly, $C_{i, i'}^{++} = C_{i, i'}^{--} = C_{i, i'}^+$, $C_{i, i'}^{+-} = C_{i, i'}^{-+} = C_{i, i'}^-$, where

$$C_{i, i'}^+ - C_{i, i'}^- = (h_i^+ - h_i^-) (h_{i+1}^+ - h_{i+1}^-) \dots (h_{i'-1}^+ - h_{i'-1}^-) \quad (21)$$

again with a cyclic modification if $i > i'$.

For any i , we have

$$\cos(\chi_i - \chi_{i+1}) - \cos(\chi_i + \chi_{i+1}) = 2 \sin \chi_i \sin \chi_{i+1} \geq 0 \quad (22)$$

since the χ 's are all between 0 and π . It follows from (2) that $h_i^+ \geq h_i^-$ for all i , and therefore

$$C_{i, i'}^+ \geq C_{i, i'}^- \quad (23)$$

by (21). Applying (23) to (18) and (19), and recalling that the h 's and hence the C 's are non-negative, we obtain (17). This completes the proof of (9).

We now take up the problem of passing from the f_i to the f_i^* by a finite number of refractions, as indicated by (10). This will be done in two stages. The first stage will convert all the f_i into functions f_i' defined by

$$\begin{aligned} f_i'(\theta) &= 1 \text{ if } \cos(\theta - \alpha_i) > \cos \frac{1}{2} m_i \\ &= 0 \text{ if } \cos(\theta - \alpha_i) < \cos \frac{1}{2} m_i \end{aligned} \quad (24)$$

where m_i are given by (6) and α_i are some set of real numbers. The second stage will alter the f_i' so as to bring all the α_i to multiples of 2π , while preserving the form of (24). The final functions, then, will be identical to the f_i^* . Each stage will be accomplished in a finite number of refractions.

For the first stage, we examine the discontinuities of f_i . We call such a discontinuity positive or negative according as the increment of f_i is positive or negative. Obviously, the discontinuities alternate in sign. We call two discontinuities equivalent if they are separated by a multiple of 2π , otherwise inequivalent.

Let d_i be the number of inequivalent discontinuities of each sign in f_i and let $\Omega_\theta d_i$ be the corresponding number for $\Omega_\theta f_i$. We shall show that

$$a) \Omega_\theta d_i \leq d_i$$

and

$$b) \text{ if } d_i > 1, \theta \text{ can be chosen so that } \Omega_\theta d_i < d_i.$$

To prove these statements, we fix θ and χ_i , and note the following consequences of (7):

- i) if f_i is continuous at both points $\theta \pm \chi_i$, then so is $\Omega_\theta f_i$;
- ii) if f_i has a discontinuity at one of the points $\theta \pm \chi_i$ but not both, the same is true of $\Omega_\theta f_i$;
- iii) if f_i has discontinuities of opposite sign at $\theta \pm \chi_i$, so does $\Omega_\theta f_i$;
- iv) if f_i has discontinuities of the same sign at $\theta \pm \chi_i$, then $\Omega_\theta f_i$ is continuous at both points.

Statement a) follows from i) and ii). To prove b), we observe that if $d_i > 1$ we may choose θ and χ_i so that positive discontinuities appear at both points $\theta \pm \chi_i$, and apply iv).

By applying a) and b) in a series of successive refractions, it is clear that we can transform the original f_i into functions $f_i^!$ each having $d_i^! = 1$. The number of refractions required, in terms of the original d_i , is just $\sum_1^n (d_i - 1)$.

Since the $f_i^!$ have each only one discontinuity of each sign, they must be of the form (24). We must, however, prove that the m_i appearing in (24) are the same as those defined by (6). We do this by writing the right side of (6) as

$$\int_0^\pi [f_i(\theta + \chi) + f_i(\theta - \chi)] d\chi \quad (25)$$

which is clearly unchanged when f is replaced by $\Omega_\theta f$. Hence the right side of (6) is invariant under refractions and has the same value for f_i^1 as for f_i . But if f_i^1 is substituted from (24), the integral becomes just the m_i of (24). Hence this m_i is the same as that given by f_i in (6).

We have now completed the first stage and pass to the second. It is easily seen that if the f_i^1 of (24) are refracted about θ , the new functions $\Omega_\theta f_i^1$ will still be of the form (24) with the same values of m_i but with the α_i replaced by $\Omega_\theta \alpha_i$ where

$$\begin{aligned}\Omega_\theta \alpha_i &= \alpha_i & \text{if } \sin(\alpha_i - \theta) \geq 0 \\ &= 2\theta - \alpha_i & \text{if } \sin(\alpha_i - \theta) < 0\end{aligned}\tag{26}$$

The crucial feature of (26) is that the quantities m_i do not enter, so that all the α_i obey the same transformation law. This permits us to draw two immediate inferences:

$$a') \text{ if } \alpha_i \equiv \alpha_j \pmod{2\pi} \text{ then } \Omega_\theta \alpha_i = \Omega_\theta \alpha_j \pmod{2\pi}$$

and

$$b') \text{ if } \alpha_i \not\equiv \alpha_j \pmod{2\pi} \text{ then } \theta \text{ can be chosen (e.g. } \theta = \frac{1}{2}(\alpha_i + \alpha_j) \text{) so that } \Omega_\theta \alpha_i \equiv \Omega_\theta \alpha_j \pmod{2\pi}.$$

From a') and b') it is clear that all the α 's can be made congruent $\pmod{2\pi}$ by a series of $n-1$ refractions. An n th refraction serves to bring the α 's into congruence with zero. This completes the second stage.

We have shown, then, that $\theta_1, \dots, \theta_M$ can be chosen so as to satisfy (10) for all the i , with

$$\begin{aligned}
 M &= \sum_{i=1}^n (d_i - 1) + (n - 1) + 1 \\
 &= \sum_{i=1}^n d_i
 \end{aligned}
 \tag{27}$$

This completes the proof of Theorem 1.

We now return to aperiodic functions on the real line, and derive our two corollaries.

Corollary 1. Let $F_1, \dots, F_n$ be non-negative functions of a real variable,

and $F_1^*, \dots, F_n^*$ be their SDR,

Let $H_1, \dots, H_n$ be non-negative functions satisfying

$$H_i(x) \geq H_i(x') \quad \text{if} \quad 0 < x < x' \tag{28}$$

$$\text{Let} \quad J \equiv \int dx_1 \dots dx_n F_1(x_1) H_1(|x_1 - x_2|) \dots F_n(x_n) H_n(|x_n - x_1|) \tag{29}$$

$$\text{and} \quad J^* \equiv \int dx_1 \dots dx_n F_1^*(x_1) H_1(|x_1 - x_2|) \dots F_n^*(x_n) H_n(|x_n - x_1|) \tag{30}$$

$$\text{Then} \quad J \leq J^* \tag{31}$$

Proof. By the arguments in Sec. 10.14 of [3], it suffices to treat the case in which all the F_i vanish outside some finite interval $[-\frac{1}{2}L, \frac{1}{2}L]$. We let $f_1, \dots, f_n$ be periodic functions (period 2π) satisfying

$$f_i(\vartheta) = F_i(L\vartheta/\pi) \tag{32}$$

for $-\pi < \vartheta < \pi$, and f_i^* be the PSDR of f_i . Then we have also

$$f_i^*(\vartheta) = F_i^*(L\vartheta/\pi) \tag{33}$$

for $-\pi < \vartheta < \pi$. We also define

$$h_i(\cos \vartheta) = H_i(L\vartheta/\pi) \tag{34}$$

for $0 < \vartheta < \pi$. Then (2) is implied by (28). Hence Theorem 1 holds, as embodied in (3), (4) and (5).

Since $F_i(x) = 0$ when $|x| > \frac{1}{2}L$, it follows from (32) that $f_i(\vartheta) = 0$ when $\cos \vartheta < 0$. Therefore, the integral in (3) can just as well be restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The arguments $|\vartheta_1 - \vartheta_2|, \dots, |\vartheta_n - \vartheta_1|$ are then restricted to the interval $[0, \pi]$. Consequently (34) can be used throughout, and (3) is simply related to (29) by

$$I = \left(\frac{\pi}{L}\right)^n J \quad (35)$$

It is easily seen that the F_i^* must also vanish outside $\left[-\frac{1}{2}L, \frac{1}{2}L\right]$, since $\{x | F_i^*(x) \geq F_i^*(x_0)\}$ has measure $> L$ for any x_0 outside this interval. Therefore we may apply the same reasoning to (4), obtaining

$$I^* = \left(\frac{\pi}{L}\right)^n J^* \quad (36)$$

The desired inequality (31) is now a consequence of (5).

Remark 1. This corollary may also be obtained from Theorem 1 of ref. [1] by choosing the H_i there to be even functions satisfying our (28). The full theorem of ref. [1] is related to this special case as Theorem 373 to Theorem 371 in ref. [3].

Remark 2. We might have dispensed with periodic functions and proved this corollary directly by refractions on the F_i . However, we found the circular domain a convenient way of arriving at the next corollary.

To introduce the second corollary, we note that Corollary 1 is "vacuous" if the F_i do not tend (in some sense) to zero at $\pm \infty$, since both (29) and (30) will then diverge. Suppose, in fact, that we have a set of functions $G_1 \dots G_n$, all non-negative and everywhere ≤ 1 , which tend to 1 at $\pm \infty$. The SDR of these

functions do not exist according to the definition we are using. More interesting are the symmetric increasing rearrangements (SIR) $G_1^* \dots G_n^*$ defined by making $1 - G_i^*$ the SDR of $1 - G_i$. One is tempted, then, to guess that Corollary 1 would hold if the F_i and F_i^* were replaced by the G_i and G_i^* . However, both (29) and (30) diverge in that case so that the intent of our guess is lost. But it can be recaptured by subtracting a common term from the integrand in both (29) and (30). We prefer to state the resulting proposition in terms of the functions $1 - G_i$, which we shall call F_i . Thus the F_i , as before, must tend to zero at $\pm \infty$ in order to make the integrals converge.

Corollary 2. Let F_i, F_i^*, H_i ($i = 1, \dots, n$) be defined as in Corollary 1, with the additional condition that $F_i(x) \leq 1$ for all x . Let

$$J \equiv \int dx_1 \dots dx_n \left\{ 1 - \frac{[1 - F_1(x_1)][1 - F_2(x_2)] \dots [1 - F_n(x_n)]}{H_1(|x_1 - x_2|) \dots H_n(|x_n - x_1|)} \right\} \quad (37)$$

$$J^* \equiv \int dx_1 \dots dx_n \left\{ 1 - \frac{[1 - F_1^*(x_1)][1 - F_2^*(x_2)] \dots [1 - F_n^*(x_n)]}{H_1(|x_1 - x_2|) \dots H_n(|x_n - x_1|)} \right\} \quad (38)$$

Then

$$J \geq J^* \quad (39)$$

Proof: Referring again to Sec. 10.14 of [3], we assume that all the F_i vanish outside an interval $[-\frac{1}{2}L, \frac{1}{2}L]$ and that all the H_i vanish for arguments $> K$.

Defining

$$s = (L + nK) / 2\pi \quad (40)$$

we introduce periodic functions f_i (period 2π) defined by

$$f_i(\varnothing) = 1 - F_i(s(\varnothing - \pi)) \quad (41)$$

for $0 < \varnothing < 2\pi$. Then their PSDR f_i^* satisfy

$$f_i^*(\varnothing) = 1 - F_i^*(s(\varnothing - \pi)) \quad (42)$$

for $0 < \varnothing < 2\pi$. Thus, by cutting the circle at 0 instead of at π , we have turned the functions $1 - F_i$ "inside out" so that their SIR look like PSDR on the circle. We also define the functions

$$h_i(\cos \varnothing) = H_i(s \varnothing) \quad (43)$$

for $0 < \varnothing < \pi$; then (2) is satisfied as before. Thus Theorem 1 applies, and (5) holds with (3) and (4).

In (29) and (30), the integrand vanished whenever any of the F_i or F_i^* was zero, and therefore the integral could be restricted to the interval $[-\frac{1}{2}L, \frac{1}{2}L]$ for all the variables. In (37) and (38), we are only able to say that the integrand vanishes when all the F_i or F_i^* are zero. Hence these integrals are unrulier, and we must work harder to restrict them. Because of the factors H_i , the integrand will vanish unless each "step length" $|x_1 - x_2|, |x_2 - x_3|, \dots, |x_n - x_1|$ is $\leq K$. But for any i, j one is able to go from x_i to x_j and back in n steps. Therefore the integrand vanishes unless

$$|x_i - x_j| \leq nK/2 \quad (44)$$

for all i, j . But the factor involving the F_i or F_i^* will vanish unless $|x_j| \leq L/2$ for at least one j . With (44), this implies that

$$|x_i| \leq (L + nK)/2 = s\pi \quad (45)$$

for all i , if the integrand is not to vanish. Therefore we may restrict the integrals in (37) and (38) to the interval $[-s\pi, s\pi]$. Using (41) and (42), we obtain

$$J = s^n I_1 \quad (46)$$

$$J^* = s^n I_1^* \quad (47)$$

where

$$I_1 = \int_0^{2\pi} d\theta_1 \dots d\theta_n \left[1 - f_1(\theta_1) \dots f_n(\theta_n) \right] H_1(s|\theta_1 - \theta_2|) \dots H_n(s|\theta_n - \theta_1|) \quad (48)$$

and

$$I_1^* = \int_0^{2\pi} d\theta_1 \dots d\theta_n \left[1 - f_1^*(\theta_1) \dots f_n^*(\theta_n) \right] H_1(s|\theta_1 - \theta_2|) \dots H_n(s|\theta_n - \theta_1|) \quad (49)$$

For comparison with (3) and (4), we define

$$I_0 = \int_{-\pi}^{\pi} d\theta_1 \dots d\theta_n h_1(\cos(\theta_1 - \theta_2)) \dots h_n(\cos(\theta_n - \theta_1)) \quad (50)$$

so that (using the periodicity of the f_i and f_i^*)

$$I_0 - I = \int_0^{2\pi} d\theta_1 \dots d\theta_n \left[1 - f_1(\theta_1) \dots f_n(\theta_n) \right] h_1(\cos(\theta_1 - \theta_2)) \dots h_n(\cos(\theta_n - \theta_1)) \quad (51)$$

$$I_0 - I^* = \int_0^{2\pi} d\vartheta_1 \dots d\vartheta_n \left[1 - f_1^*(\vartheta_1) \dots f_n^*(\vartheta_n) \right] h_1(\cos(\vartheta_1 - \vartheta_2)) \dots h_n(\cos(\vartheta_n - \vartheta_1)) \quad (52)$$

If we could apply (43) throughout, (51) and (52) would be identical to (48) and (49). Then the desired inequality (39) would be a consequence of (5), (46) and (47). However, (43) holds only when $0 \leq \vartheta \leq \pi$. Therefore, to complete the proof, we must show that the integrand in (51) or (52) vanishes unless

$$|\vartheta_i - \vartheta_{i+1}| \leq \pi \quad (53)$$

for all i .

Let δ_i stand for the shortest arc length between ϑ_i and ϑ_{i+1} regarded as points on a circle. That is,

$$\delta_i = \min(|\vartheta_i - \vartheta_{i+1}|, 2\pi - |\vartheta_i - \vartheta_{i+1}|) \quad (54)$$

where ϑ_{n+1} means ϑ_1 as usual. We can then write

$$h_i(\cos(\vartheta_i - \vartheta_{i+1})) = H_i(s\delta_i) \quad (55)$$

without restriction. Consequently the integrand in (51) or (52) vanishes unless

$$\delta_i \leq K/s \quad (56)$$

for all i . (We observe that $K/s < \pi$ on account of (40), since $n \geq 2$ and $L > 0$.)

Likewise, let ϵ_i be the arc length from ϑ_i to 0 or 2π , whichever is nearer:

$$\epsilon_i = \min(\vartheta, 2\pi - \vartheta) = \pi - |\vartheta - \pi| \quad (57)$$

which leads immediately to the relation

$$2\pi - |\vartheta_i - \vartheta_j| \geq \epsilon_i + \epsilon_j \quad (58)$$

for any i, j .

Now suppose that (53) is violated for a particular i . Then it is easily seen, by visualizing the circle, that

$$\delta_i = \epsilon_i + \epsilon_{i+1} \quad (59)$$

Let j be any index, possibly the same as i or $i+1$. In n "steps" of length $\delta_1, \dots, \delta_n$, it is possible to go from i to $i+1$, thence to j , thence back to i .

Therefore

$$\begin{aligned} \sum_{k=1}^n \delta_k &\geq \delta_i + \min(|\vartheta_{i+1} - \vartheta_j|, 2\pi - |\vartheta_{i+1} - \vartheta_j|) + \min(|\vartheta_j - \vartheta_i|, 2\pi - |\vartheta_j - \vartheta_i|) \\ &= \beta_i + \beta_{i+1} \end{aligned} \quad (60)$$

where

$$\beta_i = \epsilon_i + \min(|\vartheta_i - \vartheta_j|, 2\pi - |\vartheta_i - \vartheta_j|) \quad (61)$$

and similarly for β_{i+1} .

From the triangle inequality we have

$$\begin{aligned} \epsilon_i + |\vartheta_i - \vartheta_j| &= \pi - (|\vartheta_i - \pi| - |\vartheta_i - \vartheta_j|) \\ &\geq \pi - |\vartheta_i - \pi| = \epsilon_j \end{aligned} \quad (62)$$

and combining this with (58) we have

$$\beta_i \geq \epsilon_j \quad (63)$$

in both cases of (61), and likewise for β_{i+1} . Using (60), we now obtain

$$\sum_{k=1}^n \delta_k \geq 2\epsilon_j = 2\pi - 2|\theta_j - \pi| \quad (64)$$

Since this result holds for all j , we find, on referring to (56), that the integrand vanishes unless

$$2|\theta_j - \pi| \geq 2\pi - \frac{nK}{s} = L/s \quad (65)$$

for all j , where we have used (40). But (65) implies, in view of (41) and (42), that all the f_i and f_i^* are 1, so that the integrand in (51) or (52) vanishes anyway. (The case of equality in (65) is of zero measure.)

It follows that the integrands in question vanish when (53) is not satisfied for all i . This completes the proof of Corollary 2.

An intuitive version of the last part of the proof depends on visualizing the circle with a cut separating the adjacent points 0 and 2π . If (53) is violated, the "step" from θ_i to θ_{i+1} is one that jumps the cut. On the other hand, if $f_i(\theta_i) \neq 1$, θ_i must be "near" to π ; then, if all the h 's are nonzero, the "steps" must be short, so that one cannot go from θ_j to the cut and back in n steps.

As a final remark, we put forward the following conjecture.

Conjecture 1. Let $g_1, \dots, g_n$ be periodic functions on the real line, with period 2π . Let $g_1^*, \dots, g_n^*$ be their PSDR. Then Theorem I still holds if $h_i(\cos(\theta_i - \theta_{i+1}))$ is replaced by $g_i(\theta_i - \theta_{i+1})$ in (3) and by $g_i^*(\theta_i - \theta_{i+1})$ in (4).

This conjecture implies Theorem I, and is related to it as is Theorem 373 to Theorem 371 in [3], or the main theorem of [1] to our Corollary 1. As noted in

Sec. 10.8 of [3], the method of proof used in Theorem 1 does not apply to propositions of the type of this conjecture. On the other hand, Conjecture 1 is simply the generalization of the theorem of [1] to the circular domain. However, we have been unable to prove it by the methods of [1]; nor have we found a counterexample.

The corresponding generalization of Corollary 2 is certainly false:

Conjecture 2. Corollary 2 still holds if $H_i (|x_i - x_{i+1}|)$ is replaced in (37) by $H_i (x_i - x_{i+1})$ and in (38) by $H_i^* (x_i - x_{i+1})$, where the H_i are now arbitrary non-negative functions and the H_i^* are their SDR.

A counter-example is obtained by setting all the $F_i = 1$ everywhere, and letting the H_i be arbitrary functions. The inequality then goes the wrong way. However, this example tells us nothing about the validity of Conjecture 1.

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