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Lectures on DUAL RESONANCE MODELS - P. H. Frampton

PART SIX[†]: SYMMETRIC GROUP^{*}

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Introduction

We (study here the problem of changing the Regge intercepts to realistic values.) At the four-particle level, we decide to retain two and reject two of four properties possessed by the unit-intercept Euler B function model. We retain the properties of no odd daughters and of summability over inequivalent permutations; we reject a supplementary condition on the trajectory functions and the Plahte phase identities. The reasons for choosing these properties are given, and it is shown that we are led to introduce new complex singularities into the integrand. The requirement of Regge behaviour dictates that these singularities be at the fixed points of a six-element S_3 symmetric group. A general four-meson amplitude is given and contains a function that is both invariant under the S_3 group and analytic everywhere except for possible poles at the fixed points. A particularly simple example is selected as a physically-interesting model for four pions, and its properties are discussed. A speculative general connection between the fixed-point singularities and the existence of baryons containing precisely three quarks is mentioned.

By re-writing the general four meson amplitude in Koba-Nielsen variables, we are able to extend all factors including these with fixed-point singularities to a multiparticle amplitude. The general N-particle amplitude is given; it contains an integrand which is invariant under an S_N symmetric group. Remarkably, several earlier proposals for ghost-free Born amplitudes correspond to special cases of this formula. This gives confidence in using the method to propose

a multipion amplitude which generalises the specific four-pion model and which can accomodate realistic intercepts. This multipion model possesses a symmetry, spin-lowering symmetry, which is much larger than the symmetric group used in its construction and which plays an important role in decoupling unwanted ancestor states. As a first step towards factorization, it is shown that for general N the amplitude satisfies bootstrap consistency at internal pion poles.

Finally there is a brief summary and outlook.

6a. The Objective: Realistic Intercepts

So far, we have studied¹ two explicit examples of dual theories which satisfy the fundamental requirements of crossing symmetry, Regge behaviour and absence of imaginary coupling constants (ghosts). In both cases, however, the model is quite unrealistic because the trajectory intercepts are unacceptable physically: for the generalized Euler-B function we need $\alpha(0)=1$ to eliminate ghosts; for the Neveu-Schwarz theory we need $\alpha_\rho(0)=1$, $\alpha_\pi(0)=1/2$. On the other hand, experimentally the leading meson intercepts are $\alpha_\rho(0)\approx\alpha_f(0)\approx 1/2$.

To go in the direction of a realistic dual theory, therefore, the first objective is to construct a Born amplitude with the consistency requirements listed above, together with Regge intercepts and a mass spectrum close to the empirical one.

In order to construct such an ideal Born amplitude, the following approaches may be considered:

- i) Operator formalism techniques
- ii) Rubber string approach
- iii) Symmetric group ideas

In the first method (i), which we have discussed most extensively, the procedure is to make a realisation of the generalised projective algebra on some Fock space $\mathcal{F}$ spanned by sets of creation and annihilation operators. One then constructs vertices such that ghost-eliminating gauges are ensured for the model. This approach led to the dual pion model already described, and has been further investigated, particularly by Gervais and Neveu,^{2,3} as we shall indicate briefly below.

As we have seen, the rubber string approach (ii) provides an attractive physical picture for the spectrum of free states in the known models. So far this picture has not led to any new explicit model, although some understanding of strings in interaction has now been achieved.⁴⁻⁸

The symmetric group method⁹⁻¹¹ (iii), which we shall concentrate on here, starts from an integral representation for a four-particle amplitude, and then makes an extension to any number of particles. This is similar to the historical development of the conventional model. The underlying principle is to impose very high symmetry properties on the integrand, similar to those obtaining in the ghost-free earlier models.

Before proceeding we make the two remarks:

- i) Some attempts [see, for example, Bardakci¹²] have been made to introduce spontaneous symmetry breakdown into dual theories, in order that the massless vector state at $\alpha=1$ acquire a mass. So far these potentially important developments are at a preliminary stage.
- ii) We have encountered, in addition to intercepts, another unphysical parameter in the theories, namely the critical space-time dimensionality, d_c . This took the values $d_c=26$ and $d_c=10$ respectively in the two principal models discussed. Because of its model dependence, the value of d_c is clearly intimately related to the intercept problem. Our present philosophy is, therefore, to regard the intercept question as the primary one; at the same time, one hopes that the (secondary) result $d_c=4$ will eventually emerge naturally as the only consistent possibility when everything else is right.

6b. Four-Meson Amplitude; Symmetric Group¹³

In the simplest dual theory, namely the generalised Euler B function model, there is an absence of ghosts only if the leading intercept satisfies $\alpha(0)=1$. To proceed towards realistic intercepts, therefore, we begin by carefully studying the unit-intercept four-particle amplitude for scalar mesons (with momenta $p_1+p_2 \rightarrow (-p_3)+(-p_4)$) of the form

$$T = B_{st} + B_{tu} + B_{us} \quad (6.1)$$

with

$$B_{st} = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} \quad (6.2)$$

where $\alpha_s = \alpha(0) + \alpha's$, etc. and $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, and $u = (p_1 + p_3)^2$.

For the special case $\alpha(0)=1$ we should recall the following four properties of the Euler B function, as we discussed much earlier.

i) The absence of odd daughter trajectories i.e, the Regge trajectories are spaced by two units in angular momentum. To see this we may look at the asymptotic behaviour or, more simply, at the residues in

$$B_{st} = \sum_{n=0}^{\infty} \frac{R_n(\alpha_t)}{n - \alpha_s} \quad (6.3)$$

with

$$R_n(\alpha_t) = \frac{1}{n!} (\alpha_t + 1)(\alpha_t + 2) \dots (\alpha_t + n) \quad (6.4)$$

and hence, since $\alpha_t = -1 - n - \alpha_u$ on mass shell,

$$R_n(\alpha_t) = (-1)^n R_n(\alpha_u) \quad (6.5)$$

This implies that only alternate powers of

$$\cos \theta_s = \frac{\alpha_t - \alpha_u}{2 - (\alpha_t + \alpha_u)} \quad (6.6)$$

are present.

ii) The summability condition that we may add the three terms

$$T = B_{st} + B_{tu} + B_{us} \quad (6.7)$$

$$= \int_{-\infty}^{\infty} dx |x|^{-\alpha_t-1} |1-x|^{-\alpha_s-1} \quad (6.8)$$

as can be easily seen by making the transformations $x \rightarrow (1-x)^{-1} \rightarrow (1-1/x)$.

iii) The supplementary condition on the trajectory functions

$$\alpha_s + \alpha_t + \alpha_u + 1 = \gamma = 0 \quad (6.9)$$

when we identify $\alpha' p_i^2 = -\alpha(0) = -1$.

iv) The phase relations (Plahte¹⁴) of the form

$$B_{st} - e^{\mp i\pi\alpha_t} B_{tu} - e^{\mp i\pi\alpha_s} B_{us} = 0 \quad (6.10)$$

and cyclic permutations thereof.

To obtain these identities we close the contour in the summed form for $(B_{st} + B_{tu} + B_{us})$, and exploit the analyticity of the integrand away from the real axis, taking care of phases due to the branch points at $x=0$ and $x=1$.

These four properties are special to the case $\alpha(0)=1$; in an approach to realistic intercepts we decide to keep the first two properties and reject the third and fourth.

The reason for keeping the summability condition is that, as we have already seen, the earlier ghost-free meson Born amplitudes possessed invariance under the $N!$ permutations of the variables $\{p_i, z_i\}$ in the integrand. In fact for $\alpha(0) \neq 1$ in the Euler B function model we have seen that the factor

$$\prod_{i=1}^N (z_i - z_{i+1})^{\alpha(0)-1} \quad (6.11)$$

is precisely what destroys generalised projective gauge invariance. As we shall see shortly, this invariance (symmetric group invariance) is what is the immediate extension of the summability condition (ii) for case $N=4$. Once the summability condition is satisfied, the absence of odd daughters (i) will follow. Hence we keep both (i) and (ii).

On the other hand physical trajectories do not, in general, satisfy the supplementary condition(iii): for example, in $\pi\pi\rightarrow\pi\pi$ one has $\gamma=\alpha_s+\alpha_t+\alpha_u+1 = 3\alpha_\rho(0)-4\alpha_\pi(0)+1 \approx 5/2$. Therefore this must be relaxed. Concerning the phase identities note that from the linear combination

$$B_{st} - e^{+i\pi\alpha_t} B_{tu} - e^{-i\pi\alpha_s} B_{us} + e^{i\pi\alpha_t} (B_{tu} - e^{i\pi\alpha_u} B_{us} - e^{-i\pi\alpha_t} B_{st}) = 0 \quad (6.12)$$

we deduce that

$$e^{i\pi(\alpha_s + \alpha_t + \alpha_u)} = -1 \quad (6.13)$$

or

$$\gamma = \alpha_s + \alpha_t + \alpha_u + 1 \quad (6.14)$$

$$= 0, \text{ modulo } 2 \quad (6.15)$$

We may refer to this as a generalised supplementary condition. Its deduction implies that we must abandon also the phase identities (iv).

Since the phase relations were previously derived from summability (which we have decided to keep) plus analyticity, it is clear that the analytic properties of the integrand have to be altered in an essential way by the introduction of complex singularities in the Euler x -plane. In the following, we shall be able to specify rather

precisely the nature of these new singularities.

We now modify the integrand by multiplication with an arbitrary function which we choose to write in the form

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; x) \quad (6.16)$$

The factor $(1-x+x^2)^{\gamma/2}$ has been separated out so that the analytic properties of ϕ_4 are simplified, as we shall show later.

It turns out that the necessary and sufficient conditions for property (i) (no odd daughters) and for property (ii) (summability) coincide. For example, we may consider the asymptotic behaviour of A_4 by converting to a Laplace transform through the change of variables

$$x = 1 - e^{-w} \quad (6.17)$$

$$v = \frac{1}{2} (\alpha_s - \alpha_u) \quad (6.18)$$

to arrive at

$$A_4 = \int_0^\infty dw \, w^{-\alpha_t-1} e^{vw} \left[\sum_{n=0}^\infty \frac{1}{(2n+1)!} \left(\frac{w}{2}\right)^{2n} \right]^{-\alpha_t-1} \cdot \left[1 + 2 \sum_{n=1}^\infty \frac{w^{2n}}{(2n)!} \right]^{\gamma/2} \phi_4(\alpha_s, \alpha_t, \alpha_u; 1-e^{-w}) \quad (6.19)$$

Using

$$\int_0^\infty dw \, w^{-\alpha_t-1-r} e^{vw} = (-av)^{\alpha_t-r} \Gamma(-\alpha_t+r) \quad (6.20)$$

we see that odd daughter terms (odd r) are absent provided that ϕ_4 is even in w . This implies that

$$\phi_4(\alpha_s, \alpha_t, \alpha_u; x) = \phi_4(\alpha_u, \alpha_t, \alpha_s; \frac{-x}{1-x}) \quad (6.21)$$

Combining this with the crossing symmetry requirement that

$$\phi_4(\alpha_s, \alpha_t, \alpha_u; x) = \phi_4(\alpha_t, \alpha_s, \alpha_u; 1-x) \quad (6.22)$$

we find that ϕ_4 satisfies invariance under a six element S_3 symmetric group.

This is most clearly expressed by introducing a function $\beta_s(x)$ which is invariant under inversion (we will specify an explicit function in the following section)

$$\beta_s(x) = \beta_s\left(\frac{1}{x}\right) \quad (6.23)$$

Now define the entities

$$\beta_t(x) = \beta_s(1-x) \quad (6.24)$$

$$\beta_u(x) = \beta_s\left(\frac{-x}{1-x}\right) \quad (6.25)$$

With these definitions, the S_3 invariance of ϕ_4 may be expressed by the requirement that

$$\phi_4(\alpha_s, \alpha_t, \alpha_u; x) = \Phi_4\left(\begin{matrix} \alpha_s \\ \beta_s(x) \end{matrix}; \begin{matrix} \alpha_t \\ \beta_t(x) \end{matrix}; \begin{matrix} \alpha_u \\ \beta_u(x) \end{matrix}\right) \quad (6.26)$$

be invariant under arbitrary permutations of the three argument pairs $\{\alpha_i, \beta_i(x)\}$ with $i=s, t, u$.

It is a straightforward exercise to check that we have also ensured the summability condition.

$$\begin{aligned} A_4(-\alpha_s, -\alpha_t) + A_4(-\alpha_t, -\alpha_u) + A_4(-\alpha_u, -\alpha_s) = \\ = \int_{-\infty}^{\infty} dx |x|^{-\alpha_t-1} |1-x|^{-\alpha_s-1} (1-x+x^2)^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; x) \end{aligned} \quad (6.27)$$

by imposing this S_3 -invariance requirement on ϕ_4 . This A_4 is therefore the most general form consistent with our requirements (i) and (ii). To proceed further, we impose the requirement of Regge behaviour on this general amplitude.

6c. Regge Behaviour and Analytic Properties

It is necessary to ensure that the four-particle amplitude satisfy Regge asymptotic behaviour as $|v| \rightarrow \infty$ (at fixed α_t) in any complex direction of the v -plane. By using Watson's lemma¹⁵ applied to the rotation of the contour in the Laplace transform given above, we deduce that ϕ_4 must be analytic in the domain $\text{Re } w > 0$, or equivalently $|1-x| < 1$. For fixed α_s Regge behaviour, we similarly require analyticity in $|x| < 1$. The combination gives an analyticity domain

$$\mathcal{D}(x) = (|x| < 1) \cup (|1-x| < 1) \quad (6.28)$$

in the x -plane, as indicated in Figure 1. The function ϕ_4 must not contain singularities in x within this "opera-glass" shape, $\mathcal{D}(x)$.

This required domain of analyticity can be greatly extended by considering simultaneously the three permutations in the summed form

$$T = A_4(-\alpha_s, -\alpha_t) + A_4(-\alpha_t, -\alpha_u) + A_4(-\alpha_u, -\alpha_s) \quad (6.29)$$

$$= \int_{-\infty}^{\infty} dx |x|^{-\alpha_t-1} |1-x|^{-\alpha_s-1} (1-x+x^2)^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; x) \quad (6.30)$$

$$\begin{aligned} &= \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; x) \\ &+ \int_0^1 d\left(\frac{1}{1-x}\right) \left(\frac{1}{1-x}\right)^{-\alpha_u-1} \left(1-\frac{1}{1-x}\right)^{-\alpha_t-1} \left(\frac{1}{1-x}-e^{i\pi/3}\right)^{\frac{\gamma}{2}} \left(\frac{1}{1-x}-e^{-i\pi/3}\right)^{\frac{\gamma}{2}} \\ &\quad \phi_4(\alpha_s, \alpha_t, \alpha_u; \frac{1}{1-x}) \\ &+ \int_0^1 d\left(1-\frac{1}{x}\right) \left(1-\frac{1}{x}\right)^{-\alpha_t-1} \left(1-\left(1-\frac{1}{x}\right)\right)^{-\alpha_u-1} \left(1-\frac{1}{x}-e^{i\pi/3}\right)^{\frac{\gamma}{2}} \\ &\quad \left(1-\frac{1}{x}-e^{-i\pi/3}\right)^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; 1-\frac{1}{x}) \end{aligned} \quad (6.31)$$

Consideration of all Regge limits¹⁶ involved in the three terms shows that ϕ_4 must be analytic in the extended domain

$$\mathbb{D}(x) \cup \mathbb{D}\left(\frac{1}{1-x}\right) \cup \mathbb{D}\left(1-\frac{1}{x}\right) \quad (6.32)$$

which, remarkably, comprises the whole x -plane, including the point at infinity, except for two points $x=e^{\pm i\pi/3}$ (see Fig.1). That is,

$$\mathbb{D}(x) \cup \mathbb{D}\left(\frac{1}{1-x}\right) \cup \mathbb{D}\left(1-\frac{1}{x}\right) = \{x \mid x \neq e^{\pm i\pi/3}\} \quad (6.33)$$

Since we have already shown that the integrand must contain complex singularities, we now deduce that they are required to be precisely at the two conjugate points $x=e^{\pm i\pi/3}$.

By considerations of the S_3 group alone we can already see that these two points play a special role, as follows. In general, starting from any point x and making the S_3 transformations

$$x \rightarrow 1-x \rightarrow \frac{1}{x} \rightarrow 1-\frac{1}{x} \rightarrow \frac{1}{1-x} \rightarrow \frac{x}{x-1} \quad (6.34)$$

we arrive in general at six-different points (a six-point orbit).

For special cases, only three points are involved, namely

$$x = 0, 1, \infty \quad (6.35)$$

$$x = -1, \frac{1}{2}, 2 \quad (6.36)$$

Finally for only one case there is a two-point orbit, namely

$$x = e^{\pm i\pi/3} \quad (6.37)$$

Thus these special values correspond to the most degenerate orbit of the S_3 group.

6d. Explicit Construction

We have deduced that ϕ_4 must satisfy the two fundamental properties

i) $\phi_4 \left(\begin{matrix} \alpha_s & \alpha_t & \alpha_u \\ \beta_s(x) & \beta_t(x) & \beta_u(x) \end{matrix} \right)$ is S_3 invariant under permutation of the 3 arguments $\{\alpha_i, \beta_i(x)\}$, $i=s,t,u$.

ii) ϕ_4 must be analytic in the entire x -plane, except possibly for poles at $x=e^{\pm i\pi/3}$.

The analyticity properties suggest that we introduce the definition

$$\beta_s(x) = x(x - e^{i\pi/3})^{-1} (x - e^{-i\pi/3})^{-1} \quad (6.38)$$

$$= x(1 - x + x^2)^{-1} = \beta_s\left(\frac{1}{x}\right) \quad (6.39)$$

with simple poles at the points $x=e^{\pm i\pi/3}$. It follows that

$$\beta_t(x) = \frac{1-x}{(1-x+x^2)} \quad (6.40)$$

$$\beta_u(x) = \frac{-x(1-x)}{(1-x+x^2)} \quad (6.41)$$

The analyticity requirement (ii) on ϕ_4 may now be re-expressed by stating that ϕ_4 is analytic in the $\beta_i(x)$; it must also be analytic in the α_i , or equivalent in s,t,u from the usual analyticity requirements of $A_4(-\alpha_s, -\alpha_t)$ as a scattering amplitude.

In order to distinguish between an infinite number of possibilities, we must adopt some simplicity criterion, and we shall assume that the poles in ϕ_4 at $x=e^{\pm i\pi/3}$ are of low order*. Up to double poles we may write, with full generality,

$$\phi_4 = \sum_{i=0}^4 \lambda_i \phi_4^{(i)} \quad (6.42)$$

* This simplicity criterion is an important assumption and it is expected that it may eventually be replaced by, or be shown to be equivalent to, the criterion of minimum degeneracy in the factorisation of the corresponding N -point amplitude.

with

$$\phi_4^{(0)} = 1 \quad (6.43)$$

$$\phi_4^{(1)} = \alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u \quad (6.44)$$

$$\phi_4^{(2)} = \alpha_s \beta_t \beta_u + \alpha_t \beta_u \beta_s + \alpha_u \beta_s \beta_t \quad (6.45)$$

$$\phi_4^{(3)} = \alpha_s \alpha_t \beta_s \beta_t + \alpha_t \alpha_u \beta_t \beta_u + \alpha_u \alpha_s \beta_u \beta_s \quad (6.46)$$

$$\phi_4^{(4)} = \alpha_s^2 \beta_s + \alpha_t^2 \beta_t + \alpha_u^2 \beta_u \quad (6.47)$$

If we are considering, in particular, pion-pion elastic scattering we must impose $\lambda_0=0$ to avoid a tachyon on the positive intercept rho trajectory.

A more general formula for ϕ_4 (see Balachandran and Rupertsberger, Reference 17) is the following

$$\begin{aligned} \phi_4 = & S_1 + \left[\frac{1}{3} (2\alpha_s - \alpha_t - \alpha_u) (2\beta_s - \beta_t - \beta_u) + (\alpha_t - \alpha_u) (\beta_t - \beta_u) \right] S_2 \\ & + \left[\frac{1}{3} (2\alpha_s^2 - \alpha_t^2 - \alpha_u^2) (2\beta_s - \beta_t - \beta_u) + (\alpha_t^2 - \alpha_u^2) (\beta_t - \beta_u) \right] S_3 \\ & + \left[\frac{1}{3} (2\alpha_s - \alpha_t - \alpha_u) (2\beta_s^2 - \beta_t^2 - \beta_u^2) + (\alpha_t - \alpha_u) (\beta_t^2 - \beta_u^2) \right] S_4 \\ & + \left[\frac{1}{3} (2\alpha_s^2 - \alpha_t^2 - \alpha_u^2) (2\beta_s^2 - \beta_t^2 - \beta_u^2) + (\alpha_t^2 - \alpha_u^2) (\beta_t^2 - \beta_u^2) \right] S_5 \\ & + (\alpha_s - \alpha_t) (\alpha_t - \alpha_u) (\alpha_u - \alpha_s) (\beta_s - \beta_t) (\beta_t - \beta_u) (\beta_u - \beta_s) S_6 \end{aligned} \quad (6.48)$$

where the six functions S_i are power series in the three symmetric quantities

$$\bar{\alpha} = \alpha_s \alpha_t + \alpha_t \alpha_u + \alpha_u \alpha_s \quad (6.49)$$

$$\underline{\alpha} = \alpha_s \alpha_t \alpha_u \quad (6.50)$$

$$\bar{\beta} = \beta_s \beta_t \beta_u \quad (6.51)$$

The variable Z (which was discussed already in a paper by Dixon¹⁸ in 1904) is fully invariant under the S_3 group.

To derive¹⁷ this general expression for ϕ_4 we note that the S_3 invariance implies that we may write ϕ_4 as a superposition of functions of the form ϕ where

$$\begin{aligned} \phi(\alpha_s, \alpha_t, \alpha_u; x) = & \psi^{(s)}(\alpha_s, \alpha_t, \alpha_u) \chi^{(s)}(x) + \psi^{(A)}(\alpha_s, \alpha_t, \alpha_u) \chi^{(A)}(x) \\ & + \sum_{\rho, \sigma=1}^2 \psi_{\rho}^{(M)}(\alpha_s, \alpha_t, \alpha_u) g_{\rho\sigma} \chi_{\sigma}^{(M)}(x) \end{aligned} \quad (6.52)$$

Here $\psi^{(s)}$ is fully symmetric in $\alpha_s, \alpha_t, \alpha_u$ and $\chi^{(s)}$ is invariant under $x \rightarrow (1-x)^{-1} \rightarrow (x-1)x^{-1} \rightarrow x^{-1} \rightarrow (1-x) \rightarrow x(x-1)^{-1}$. Similarly $\psi^{(A)}$ and $\chi^{(A)}$ carry the antisymmetric representation of S_3 , while $\psi_{\rho}^{(M)}, \chi_{\rho}^{(M)}$ are the basis for the two-dimensional representation with metric g .

The method of constructing the $\psi^{(i)}$ is already known from work on two-variable expansions of scattering amplitudes. Hence $\psi^{(s)}$ and $\psi^{(A)}$ are of the form

$$\psi^{(s)}(\alpha_s, \alpha_t, \alpha_u) = \sum_{p, q \geq 0} E_{pq}^{(s)} \bar{\alpha}^p \alpha^q \quad (6.53)$$

$$\psi^{(A)}(\alpha_s, \alpha_t, \alpha_u) = (\alpha_s - \alpha_t)(\alpha_t - \alpha_u)(\alpha_u - \alpha_s) \sum_{p, q \geq 0} E_{pq}^{(A)} \bar{\alpha}^p \alpha^q \quad (6.54)$$

while the two-dimensional representations are carried by functions either of the form

$$(\psi_1^{(M)}(\alpha_s, \alpha_t, \alpha_u), \psi_2^{(M)}(\alpha_s, \alpha_t, \alpha_u)) = (2\alpha_s - \alpha_t - \alpha_u, \alpha_t - \alpha_u) \sum_{p, q \geq 0} G_{pq}^{(1)} \bar{\alpha}^p \alpha^q \quad (6.55)$$

or of the form

$$(\psi_1^{(M)}(\alpha_s, \alpha_t, \alpha_u), \psi_2^{(M)}(\alpha_s, \alpha_t, \alpha_u)) = (2\alpha_s^2 - \alpha_t^2 - \alpha_u^2, \alpha_t^2 - \alpha_u^2) \sum_{p, q \geq 0} G_{pq}^{(2)} \bar{\alpha}^p \alpha^q \quad (6.56)$$

From these formulae the metric is checked to be

$$g = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \quad (6.57)$$

To find $\chi^{(s)}$, we can first show that $\chi^{(s)}(x) - \chi^{(s)}(0)$ must have zeros of order at least two at $x=0, 1, \infty$ by considering

$$\chi^{(s)}(x) - \chi^{(s)}(0) = \chi^{(s)}\left(\frac{x}{x-1}\right) - \chi^{(s)}(0) \quad (6.58)$$

$$= \chi^{(s)}\left(\frac{1}{1-x}\right) - \chi^{(s)}(0) \quad (6.59)$$

$$= \chi^{(s)}\left(\frac{1}{x}\right) - \chi^{(s)}(0) \quad (6.60)$$

and the first and second derivatives thereof.

By considering, in addition,

$$[\chi^{(s)}(x) - \chi^{(s)}(0)]^{-1} = [\chi^{(s)}\left(\frac{1}{1-x}\right) - \chi^{(s)}(0)]^{-1} \quad (6.61)$$

and its derivatives, we find that $[\chi^{(s)}(x) - \chi^{(s)}(0)]$ has poles of order at least three at $x=e^{\pm i\pi/3}$. Hence we may write

$$\chi^{(s)}(x) = \chi^{(s)}(0) + z \chi_1^{(s)}(x) \quad (6.62)$$

where

$$z = \beta_s(x) \beta_t(x) \beta_u(x) = -\frac{x^2(1-x)^2}{(1-x+x^2)^3} \quad (6.63)$$

By repeating the process we deduce that $\chi^{(s)}(x)$ is a polynomial

$$\chi^{(s)}(x) = \sum_{p=0}^{\infty} c_p z^p \quad (6.64)$$

in the variable z .

Concerning $\chi^A(x)$, antisymmetry dictates that it change sign under $x \rightarrow 1/x$, $x \rightarrow (1-x)$, $x \rightarrow x(x-1)^{-1}$ and therefore it must have zeros at $x=-1$, 0 , $1/2$, 1 , 2 and ∞ . By considering $\chi^A(x)^{-1}$ we may show that $\chi^A(x)$, like $\chi^{(s)}(x)$, has poles of order at least three at $x=e^{\pm i\pi/3}$.

Therefore

$$\chi^{(A)} = -2(x+1)x(x-\frac{1}{2})(x-1)(x-2)\bar{\chi}^{(s)}(x) \quad (6.65)$$

$$= (\beta_s - \beta_t)(\beta_t - \beta_u)(\beta_u - \beta_s)\bar{\chi}^{(s)}(x) \quad (6.66)$$

where $\chi^{(s)}(x)$ is fully symmetric, and hence a polynomial in Z .

At this stage, we may proceed further deductively but by now it is an obvious guess that the correct mixed-symmetry $\chi^{(M)}(x)$ are either of the form (by analogy with $\psi^{(M)}$)

$$(2\beta_s - \beta_t - \beta_u, \beta_t - \beta_u) \hat{\chi}^{(s)}(x) \quad (6.67)$$

or of the form

$$(2\beta_s^2 - \beta_t^2 - \beta_u^2, \beta_t^2 - \beta_u^2) \hat{\hat{\chi}}^{(s)}(x) \quad (6.68)$$

where $\hat{\chi}^{(s)}, \hat{\hat{\chi}}^{(s)}$ are polynomials in Z . Indeed, it is possible to show that this exhausts all possible forms for $\chi^{(M)}$.

Combining the results for $\psi^{(s)}, \psi^{(A)}, \psi^{(M)}$ with those for $\chi^{(s)}, \chi^{(A)}, \chi^{(M)}$ then demonstrates that the formula for ϕ_4 , given earlier, is the most general representation consistent with the S_3 invariance and the given analytic properties.

6e. Four-Pion Amplitude

To select between the above candidates for the four-pion amplitude, we must bear in mind that the ultimate goal is an N -pion amplitude which factorises on a ghost-free spectrum. A necessary condition is therefore that the four-pion amplitude should have on-shell residues such that the partial-wave projections give rise to a positive-definite imaginary part.

This question has been examined in Ref. 10. The result is that any linear combination

$$\phi_4 = \lambda_2 \phi_4^{(2)} + \lambda_3 \phi_4^{(3)} \quad (6.69)$$

of the double-pole models contains negative-squared coupling constants and is thereby eliminated. Similarly the model corresponding to $\phi_4^{(4)}$ has ghosts for physical intercepts. This leaves only the (simplest) model $\phi_4^{(1)}$.

For this model, corresponding to $\phi_4^{(1)}$, the amplitude is

$$A_4 = \int_0^1 dx \ x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{x}{2}} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) \quad (6.70)$$

Setting $\alpha_\rho(0) - \alpha_\pi(0) = 1/2$ exactly, this amplitude becomes

$$A_4 = \int_0^1 dx \ x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{1-\alpha_\rho(0)}{2}} (\alpha_s x + \alpha_t(1-x) - \alpha_u x(1-x)) \quad (6.71)$$

For a wide range of values of $\alpha_\rho(0)$ (at least $1/3 \leq \alpha_\rho(0) \leq 1$) it can be shown numerically that the partial waves are all positive for the lowest-lying mass levels.

We should note that in the unphysical limit $\alpha_\rho(0) \rightarrow 1$, the amplitude becomes

$$A_4 \xrightarrow{\alpha_\rho(0) \rightarrow 1} - \frac{3 \Gamma(1-\alpha_s) \Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \quad (6.72)$$

which is precisely the four-pion amplitude of the Neveu-Schwarz model, known to be ghost-free.

For other values of $\alpha_\rho(0)$, we can give an analytic proof¹⁹ of the Gribov-Pomeranchuk identities²⁰, namely

$$\frac{\partial^n}{\partial s^n} \text{Im} A_4(s, t) \Big|_{s=0} \geq 0 \quad \text{for all } n \quad (6.73)$$

This condition follows from unitarity, and is a necessary condition for the absence of ghost states.

To prove these inequalities for $1/2 \leq \alpha_p(0) \leq 1$ we re-write

$$A_4 = \int_0^1 dx \ x^{-\alpha_t-1} (1-x)^{-s} \left[\frac{1+x^3}{(1-x)(1-x^2)} \right]^{\frac{1-\alpha_p(0)}{2}} (1-x)^{1-2\alpha_p(0)} \left[\alpha_s \frac{x}{1-x} + \alpha_t - \alpha_u x \right] \quad (6.74)$$

By examining in turn the factors, we deduce that

$$(1-x)^{-s} \quad (6.75)$$

has a positive-definite power series $(1-x)$ for all s . Similarly

$$(1-x)^{1-2\alpha_p(0)} \quad (6.76)$$

is positive definite for $\alpha_p(0) \geq 1/2$. We can re-write

$$\begin{aligned} & \alpha_s \frac{x}{1-x} + \alpha_t - \alpha_u x \\ &= (\alpha_p(0) + t) + (4\alpha_p(0) + 2s + t - 2) + (\alpha_p(0) + s) \sum_{n=2}^{\infty} x^n \end{aligned} \quad (6.77)$$

which is positive for all s , provided $\alpha_p(0) \geq 1/2$. Finally we note that

$$\begin{aligned} & \left[\frac{1+x^3}{(1-x)(1-x^2)} \right]^{\frac{1-\alpha_p(0)}{2}} = \\ &= \exp \left\{ \left(\frac{1-\alpha_p(0)}{2} \right) \left[\ln(1+x^3) - \ln(1-x) - \ln(1-x^2) \right] \right\} \end{aligned} \quad (6.78)$$

$$= \exp \left\{ \left(\frac{1-\alpha_p(0)}{2} \right) \left[- \sum_{n=1}^{\infty} \frac{(-x^3)^n + x^n + (x^2)^n}{n} \right] \right\} \quad (6.79)$$

which is positive-definite for $\alpha_p(0) \leq 1$. This completes the prove of the inequalities for $1/2 < \alpha_p(0) \leq 1$ since we have shown that

$$A_4 = \int_0^1 dx \ x^{-\alpha_t-1} \sum_{n=1}^{\infty} R_n(\alpha_s) x^n \quad (6.80)$$

and

$$\frac{\partial^r}{\partial s^r} R_n(\alpha_s) \Big|_{s=0} \geq 0 \quad (6.81)$$

as required.

To extend this proof to $1/3 \leq \alpha_p(0) \leq 1/2$ we write A_4 in the alternative form

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-s} \left(\frac{1+x^3}{1-x^2} \right)^{\frac{1-\alpha_p(0)}{2}} (1-x)^{\frac{1-3\alpha_p(0)}{2}} \left[\alpha_s \frac{x}{1-x} + \alpha_t - \alpha_u x \right] \quad (6.82)$$

Here the three factors

$$(1-x)^{-s} \quad (6.83)$$

$$(1-x)^{\frac{1-3\alpha_p(0)}{2}} \quad (6.84)$$

$$\left(\alpha_s \frac{x}{1-x} + \alpha_t - \alpha_u x \right) \quad (6.85)$$

have positive powers series in x for all s , provided $\alpha_p(0) \geq 1/3$.

Consider the function

$$F(x) = \left(\frac{1+x^3}{1-x} \right)^{\frac{1-\alpha_p(0)}{2}} = \sum_{n=0}^{\infty} c_n x^n \quad (6.86)$$

Then

$$(1+x^3)(1-x^2) F'(x) = \left(\frac{1-\alpha_p(0)}{2} \right) F(x) x(2-x) \quad (6.87)$$

$$(6.88)$$

Hence (putting $A = \frac{1-\alpha_p(0)}{2}$)

$$(n+1) c_{n+1} = 2n c_n + 2(A+1-n) c_{n-1} + (n-2-A) c_{n-2} \quad (6.89)$$

and from this we deduce that

$$(n+4) c_{n+4} = 2A c_{n+2} + 2A c_n + 3A c_{n-1} + (n-2-A) c_{n-2} \text{ with}$$

asymptotically ($n \rightarrow \infty$) positive coefficients and that the first few Taylor coefficients are

$$C_0 = 1 \quad (6.90)$$

$$C_1 = 0 \quad (6.91)$$

$$C_2 = A \quad (6.92)$$

$$C_3 = A \quad (6.93)$$

$$C_4 = \frac{1}{2} A(A+1) \quad (6.94)$$

$$C_5 = A^2 \quad (6.95)$$

$$C_6 = \frac{1}{6} A(A^2 + 6A - 1) \quad (6.96)$$

From this we deduce that

$$C_6 \geq 0 \quad (6.97)$$

requires $A \geq \sqrt{10} - 3$, or

$$\alpha(0) \leq 7 - 2\sqrt{10} \approx 0.68 \quad (6.98)$$

If this is satisfied, then it follows that $C_n \geq 0$ for all n by using the recursion formula given above. Thus we have positivity for

$$(7 - 2\sqrt{10}) \geq \alpha_p(0) \geq 1/3.$$

Collecting these results, we have demonstrated that A_4 satisfies the Gribov-Pomeranchuk inequalities for at least the range $1/3 \leq \alpha_p(0) \leq 1$ and $\alpha_p(0) - \alpha_\pi(0) = 1/2$.

Next we consider the Adler consistency condition which states that for the four pion amplitude

$$\lim_{p_i^2 \rightarrow 0} [T(s, t, u)] = 0 \quad (6.99)$$

We may re-write A_4 as

$$A_4(-\alpha_s, -\alpha_t) = - \frac{3 \Gamma(1-\alpha_s) \Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s - \alpha_t)} + (\alpha_p(0) - 1) B(1-\alpha_s, 1-\alpha_t) \\ - \sum_{R=0}^{\infty} \left(\frac{1-\alpha_p(0)}{2^{R+1}} \right) (-1)^R \begin{bmatrix} \alpha_s B(R+1-\alpha_s, R+2-\alpha_t) \\ + \alpha_t B(R+2-\alpha_s, R+1-\alpha_t) \\ - \alpha_u B(R+2-\alpha_s, R+2-\alpha_t) \end{bmatrix} \quad (6.100)$$

The leading trajectory contribution is now entirely contained in the first term of the right hand side, and this leading term vanishes at the Adler point $s=t=u=\mu^2$ provided that

$$\alpha_f(\mu^2) = \alpha_f(0) - \alpha_\pi(0) = 1/2 \quad (6.101)$$

Thus at the Adler point, the only contributions are then from the daughter levels which are more distant singularities. This particular intercept constraint is well satisfied physically, and, as we shall see later, it leads also to a simplification of the N-pion generalisation of this N=4 amplitude.

Finally, we may briefly mention some phenomenological implications of the amplitude. For the elastic decay widths, into two pions, of the parent resonances the predictions coincide with those of the Lovelace-Shapiro formula²², which contains only the leading term. At the daughter levels, however, the predictions are quite different (see Fig.2). The s-wave (ϵ) daughter of the ρ is absent; this is consistent with recent phenomenological analysis by, for example, Protopopescu et al.²³ The p-wave (ρ') daughter of the f is absent, in agreement with experiment, but there is an s-wave (ϵ') at this mass with $\Gamma_{\pi\pi} \sim 260$ MeV in agreement with $\pi\pi$ phase shift analysis²⁴. The p-wave (ρ'') daughter of the g has $\Gamma_{\pi\pi} \sim 140$ MeV, consistent with experiment²⁵. (Fig.2a)

By contrast, the Lovelace-Shapiro formula predicts a very broad ϵ resonance, a strongly coupled ρ' and a slightly ghost-like ϵ' resonance (Fig.2b). We conclude, therefore, that the present model is phenomenologically preferable, at the daughter levels, to the Lovelace-Shapiro model.

[Note: In Fig.2, the widths are normalised to $\Gamma_{\rho \rightarrow 2\pi} = 112$ MeV, and the trajectories are $\alpha_\rho(s) = 0.48 + 0.9s$, $\alpha_\pi(s) = -0.02 + 0.9s$]

6f. Possible Connection to Three-Quark Baryons

It is possible to speculate²⁶ that the Euler x-plane fixed-point singularities at $x=e^{\pm i\pi/3}$ may be deeply connected with the three-quark structure of baryons.

We have earlier considered a picture²⁸ in which quarks are distributed on a circular rubber string. Consider a baryon-meson elastic scattering process in which two different quarks (separated by angle θ) are active, as in Fig.3a. Let θ satisfy $0 \leq \theta \leq \pi$ on the string which is of total period 2π , and suppose there is in addition some arbitrary number of spectator quarks.

The amplitude is given by

$$\begin{aligned}
 A_4^{(BM)}(-\alpha_u, -\alpha_s) &= \int_0^1 dy y^{-\alpha_s-1} \cdot \\
 &\quad \cdot \langle 0 | \exp(\sqrt{2} i p_2 \cdot \sum_{n=1}^{\infty} \frac{a^{(n)}}{\sqrt{n}} \cos n\theta) y^R \\
 &\quad \exp(\sqrt{2} i p_4 \cdot \sum_{n=1}^{\infty} \frac{a^{(n)+}}{\sqrt{n}}) | 0 \rangle \\
 &\quad + (p_2 \leftrightarrow p_4)
 \end{aligned} \tag{6.102}$$

$$\begin{aligned}
 &= \int_0^1 dy y^{-\alpha_s-1} \exp \left[p_2 \cdot p_4 \sum_{n=1}^{\infty} \frac{y^n}{n} (e^{in\theta} + e^{-in\theta}) \right] \\
 &\quad + (p_2 \leftrightarrow p_4)
 \end{aligned} \tag{6.103}$$

$$\begin{aligned}
 &= \int_0^1 dy y^{-\alpha_s-1} (y^2 - 2y \cos \theta + 1)^{-\frac{1}{2}(\alpha_u + \alpha_s)} + (\alpha_s \leftrightarrow \alpha_u)
 \end{aligned} \tag{6.104}$$

To combine the two terms in $A_4(-\alpha_u, -\alpha_s)$ one puts

$$u = y (y^2 - 2y \cos \theta + 1)^{-1/2} \quad (6.105)$$

$$v = u/y \quad (6.106)$$

whereupon

$$\begin{aligned} A_4^{(BM)}(-\alpha_u, -\alpha_s) &= \int_0^{1/2\sin\theta/2} \frac{du}{v - u\cos\theta} u^{-\alpha_s-1} v^{-\alpha_u-1} \\ &+ \int_{1/2\sin\theta/2}^1 \frac{dv}{u - v\cos\theta} v^{-\alpha_s-1} u^{-\alpha_u-1} \end{aligned} \quad (6.107)$$

$$= 2 \int_0^1 du \int_0^1 dv u^{-\alpha_s-1} v^{-\alpha_u-1} \delta(u^2 + v^2 - 2uv\cos\theta - 1) \quad (6.108)$$

Now put

$$u = x [x^2 + (1-x)^2 - 2x(1-x)\cos\theta]^{1/2} \quad (6.109)$$

to obtain

$$A_4^{(BM)}(-\alpha_u, -\alpha_s) = \int_0^1 dx x^{-\alpha_s-1} (1-x)^{-\alpha_u-1} \left[1 - 4x(1-x)\cos^2 \frac{\theta}{2} \right]^{\frac{\alpha_s + \alpha_u + 2}{2}} \quad (6.110)$$

This is remarkably similar to the forms we have considered for meson-meson scattering. In particular, the integrand of $A_4^{(BM)}$ has branch points situated symmetrically about the real axis at

$$x_0 = \frac{1}{2} \pm \frac{i}{2} \left(\sec^2 \frac{\theta}{2} - 1 \right)^{1/2} \quad (6.111)$$

Consider now what happens as we change continuously the angular separation, θ , of the active quarks. For $\theta = \pi$, the branch points are absent (i.e. are at infinity); for decreasing θ the branch points move to the finite x -plane and eventually reach the special points $x = e^{\pm i\pi/3}$ when $\theta = 2\pi/3$. (Fig. 3b)

Now assume that our earlier result, that the only singularities are at $x=0, 1, \infty$ and $e^{\pm i\pi/3}$, extends to this baryonic case. It follows that the angular separation must be $\theta=2\pi/3$ and hence that, within such dual schemes, the number of quarks in the (symmetric) baryon should be a precise multiple of three.

To summarise, we may conjecture that the singularities at $x=e^{\pm i\pi/3}$ that we have introduced to change the intercept of the meson amplitude, could have a much deeper significance: it may ultimately be shown that such singularities play a central role in a consistent bootstrap solution for mesons and baryons together.

6g. Multiparticle Extension

We have written the four-meson amplitude in the general form

$$A_4(-\alpha_s, -\alpha_t) = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4 \quad (6.112)$$

where $\phi_4(\alpha_s, \alpha_t, \alpha_u; x)$ is invariant under an S_3 group acting on the objects $\{\alpha_i, \beta_i(x)\}$, $i=s, t, u$.

We would now like to extend this formula to any (even) number of external pions. In particular, we wish to maintain the symmetric group properties of the integrand in the multiparticle extension; as we shall see, this greatly resolves the ambiguity inherent in the generalisation from the four-particle amplitude.

For A_4 , we introduce the variables^{29,30} z_i ($i=1,2,3,4$) lying on a circle in the complex z -plane, and make the identification

$$\chi = (z_2, z_1; z_3, z_4) = \frac{(23)}{(13)} \quad (6.113)$$

$$1-\chi = (z_1, z_4; z_2, z_3) = \frac{(12)}{(13)} \quad (6.114)$$

where the notation for an anharmonic ratio is

$$(a, b; c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)} \quad (6.115)$$

and we have introduced the convenient shorthand, to make the later development more manageable.

$$(12) = (z_1 - z_2)(z_3 - z_4) \quad (6.116)$$

$$(23) = (z_2 - z_3)(z_1 - z_4) \quad (6.117)$$

$$(13) = (z_1 - z_3)(z_2 - z_4) \quad (6.118)$$

First we note that the Euler B function may be written (for $\alpha_\rho(0) \neq \alpha_\pi(0)$) in the form

$$B(-\alpha_s, -\alpha_t) = \int \prod_{i=1}^4 dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \\ \prod_{i < j} (z_i - z_j)^{-2p_i p_j} \prod_{i=1}^4 (z_i - z_{i+1})^{\alpha_\rho(0)-1} \left[\prod_{i < j} (z_i - z_j)^{(-1)^{i-j}} \right]^{2(\alpha_\rho(0) - \alpha_\pi(0))} \quad (6.119)$$

where z_a, z_b, z_c are fixed values for three out of the four z_i (arbitrarily chosen) and the integration domain is the boundary of the circle with the restriction that the cyclic ordering of the points z_1, z_2, z_3, z_4 is preserved.

It is useful to note the identity

$$\frac{1}{2} \gamma = -\frac{1}{2} (\alpha_\rho(0) - 1) + 2 [\alpha_\rho(0) - \alpha_\pi(0)] \quad (6.120)$$

This identity motivates us to establish two fundamental identities

which enable us to extend the factor $(1-x+x^2)^{\gamma/2}$ containing the branch point singularities at $x=e^{\pm i\pi/3}$.

First we note that

$$\prod_{i=1}^4 (z_i - z_{i+1})^2 = (12)^2 (23)^2 \quad (6.121)$$

and then consider the symmetric expression

$$\sum_{\{q_1, q_2, q_3, q_4\}} \prod_{i=1}^4 (z_{q_i} - z_{q_{i+1}})^{-2} \quad (6.122)$$

where the sum is over the 4! permutations of $\{q_1, q_2, q_3, q_4\} = \{1, 2, 3, 4\}$.

This is equal to

$$\sum_{\{q_1, q_2, q_3, q_4\}} \prod_{i=1}^4 (z_{q_i} - z_{q_{i+1}})^{-2} = 8 \left[\frac{1}{(12)^2 (23)^2} + \frac{1}{(12)^2 (13)^2} + \frac{1}{(23)^2 (13)^2} \right] \quad (6.123)$$

$$= 16 \prod_{i=1}^4 (z_i - z_{i+1})^{-2} (x - e^{i\pi/3}) (x - e^{-i\pi/3}) \quad (6.124)$$

In other words, there is the identity

$$(x - e^{i\pi/3})(x - e^{-i\pi/3}) = \frac{1}{16} \prod (z_i - z_{i+1})^2 \sum_{\{q_1, q_2, q_3, q_4\}} \prod_{i=1}^4 (z_{q_i} - z_{q_{i+1}})^{-2} \quad (6.125)$$

The second identity concerns the expression

$$\prod_{i < j} (z_i - z_j)^{(-1)^{i-j}} = \frac{(13)}{(12)(23)} \quad (6.126)$$

The corresponding symmetric form is

$$\sum_{\{q_1, q_2, q_3, q_4\}} (-1)^p \prod_{k < l} (z_{q_k} - z_{q_l})^{(-1)^{k-l}} = 8 \left[\frac{(13)}{(12)(23)} + \frac{(23)}{(12)(13)} + \frac{(12)}{(23)(13)} \right] \quad (6.127)$$

which leads to the identity

$$(x - e^{i\pi/3})(x - e^{-i\pi/3}) = \frac{1}{16} \prod_{i < j} (z_i - z_j)^{(-1)^{i-j}} \sum_{\{q_1, q_2, q_3, q_4\}} (-1)^p \prod_{k < l} (z_{q_k} - z_{q_l})^{(-1)^{k-l}} \quad (6.128)$$

In these expressions, P is the parity of the permutation.

Collecting these results together leads to

$$\begin{aligned}
 & (\chi - e^{i\pi/3})^{\gamma/2} (\chi - e^{-i\pi/3})^{\delta/2} = \\
 & = 4^{-\gamma} \prod_{i=1}^4 (z_i - z_{i+1})^{1-\alpha_f(o)} \left[\prod_{i<j} (z_i - z_j)^{(-1)^{i-j}} \right]^{-2(\alpha_f(o) - \alpha_\pi(o))} \\
 & \quad \left[\sum_{\{q_1, q_2, q_3, q_4\}} \prod_{k=1}^4 (z_{q_k} - z_{q_{k+1}})^{-2} \right]^{\frac{1-\alpha_f(o)}{2}} \\
 & \quad \left[\sum_{\{q_1, q_2, q_3, q_4\}} (-1)^P \prod_{k \neq l} (z_{q_k} - z_{q_l})^{(-1)^{k-l}} \right]^{2(\alpha_f(o) - \alpha_\pi(o))} \quad (6.129)
 \end{aligned}$$

We may insert such an expression directly into A_4 . We see that all asymmetric factors are thereby cancelled, and replaced by symmetric ones.

The remaining question is to deal with ϕ_4 . It is not difficult to see that the S_3 invariance of

$$\phi_4 \left(\begin{matrix} \alpha_s & \alpha_t & \alpha_u \\ \beta_s(x) & \beta_t(x) & \beta_u(x) \end{matrix} \right) \quad (6.130)$$

becomes, when re-written in terms of Koba-Nielsen variables, the S_4 invariance of

$$\Phi_4 \left(\begin{matrix} p_1 & p_2 & p_3 & p_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right) \quad (6.131)$$

under the $4!$ permutations of the argument pairs $\{p_i, z_i\}$, $i=1,2,3,4$.

Collecting results, we arrive at an expression for A_4 in a form immediately extendable to A_N , $N>4$. The result is

$$\begin{aligned}
 A_N = & \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i<j} (z_i - z_j)^{-2p_i \cdot p_j} \left(S_N^{(1)}(z) \right)^{\frac{1-\alpha_f(o)}{2}} \\
 & \left(S_N^{(2)}(z) \right)^{2(\alpha_f(o) - \alpha_\pi(o))} \Phi_N \left(\begin{matrix} p_1 & p_2 & \dots & p_N \\ z_1 & z_2 & \dots & z_N \end{matrix} \right) \quad (6.132)
 \end{aligned}$$

where the whole integrand, including the final factor ϕ_N , is invariant under an S_N symmetric group acting on the N argument pairs $\{p_i, z_i\}, i=1,2,3,\dots,N$. We have introduced the notations

$$S_N^{(1)}(z) = \sum_{\{q_1, q_2, \dots, q_N\}} \prod_{k=1}^N (z_{q_k} - z_{q_{k+1}})^{-2} \quad (6.133)$$

$$S_N^{(2)}(z) = \sum_{\{q_1, q_2, \dots, q_N\}} (-1)^P \prod_{k < l} (z_{q_k} - z_{q_l})^{(-1)^{k-l}} \quad (6.134)$$

The crucial point is that there is little freedom in making the multiparticle extension, when we insist on the symmetric group invariance. This is a very strong result.

6h. Classification of Earlier Models

The advantages of the symmetric group become clear, when we realise that the two classical dual models, together with new ones proposed by Gervais and Neveu^{2,3} from an operator formalism approach, can be readily accommodated by the general formula.

This fact will then give more confidence in using the method to propose a multipion amplitude for realistic intercepts.

Let us deal with the earlier models in turn:

- i) The conventional (generalised Euler B function) model corresponds to putting $\alpha_\rho(0)=\alpha_\pi(0)=1$ and $\phi_N=1$ in our general formula. This proposal was important historically as the first closed-form Born amplitude. Indeed the mechanism for producing the resonance poles is unaltered, in general, by the new singularities of the present integrand.
- ii) The Neveu-Schwarz dual pion model, which has $\alpha_\rho(0)=1$ and $\alpha_\pi(0)=\frac{1}{2}$

may be written

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i<j} (z_i - z_j)^{-2p_i \cdot p_j} \sum_{\{q_1, q_2, \dots, q_N\}} (-1)^P \frac{(p_{q_1} \cdot p_{q_2})}{(z_{q_1} - z_{q_2})} \frac{(p_{q_3} \cdot p_{q_4})}{(z_{q_3} - z_{q_4})} \dots \frac{(p_{q_{N-1}} \cdot p_{q_N})}{(z_{q_{N-1}} - z_{q_N})} \quad (6.135)$$

and therefore corresponds to the specific choice

$$\phi_N = (S_N^{(2)})^{-1} \sum_{\{q_1, q_2, \dots, q_N\}} (-1)^P \frac{(p_{q_1} \cdot p_{q_2})}{(z_{q_1} - z_{q_2})} \frac{(p_{q_3} \cdot p_{q_4})}{(z_{q_3} - z_{q_4})} \dots \frac{(p_{q_{N-1}} \cdot p_{q_N})}{(z_{q_{N-1}} - z_{q_N})} \quad (6.136)$$

An alternative form for A_N is (treating all z -differences as independent)

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \sum_{\{q_1, q_2, \dots, q_N\}} (-1)^P \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \dots \frac{\partial}{\partial(z_{q_{N-1}} - z_{q_N})} \left[\prod_{i<j} (z_i - z_j)^{-2p_i \cdot p_j} \right] \quad (6.137)$$

and corresponds to a generalised logarithmic derivative choice for ϕ_N

$$\phi_N = (S_N^{(2)}(z))^{-1} \left(\prod_{i<j} (z_i - z_j)^{-2p_i \cdot p_j} \right)^{-1} \sum_{\{q_1, q_2, \dots, q_N\}} (-1)^P \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \dots \frac{\partial}{\partial(z_{q_{N-1}} - z_{q_N})} \left[\prod_{i<j} (z_i - z_j)^{-2p_i \cdot p_j} \right] \quad (6.138)$$

We have seen earlier how the second-Fock-space $\mathcal{F}_2$ reformulation of the Neveu-Schwarz model corresponds to re-writing A_N in the forms

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i<j} (z_i - z_j)^{-2p_i \cdot p_j} \frac{1}{(z_a - z_b)} \sum'_{\{q_1, q_2, \dots, q_N\}} (-1)^P \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \dots \frac{\partial}{\partial(z_{q_{N-3}} - z_{q_{N-2}})} \left[\prod_{i<j} (z_i - z_j)^{-2p_i \cdot p_j} \right] \quad (6.139)$$

or, equivalently,

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} \frac{1}{(z_a - z_b)} \sum'_{\{q_1, q_2, \dots, q_{N-2}\}} (-1)^P \frac{(p_{q_1} \cdot p_{q_2})}{(z_{q_1} - z_{q_2})} \dots \frac{(p_{q_{N-3}} \cdot p_{q_{N-2}})}{(z_{q_{N-3}} - z_{q_{N-2}})} \quad (6.140)$$

These correspond to

$$\phi_N = (S_N^{(2)})^{-1} \left(\prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} \right)^{-1} \frac{1}{(z_a - z_b)} \sum'_{\{q_1, \dots, q_{N-2}\}} (-1)^P \frac{\partial}{\partial (z_{q_1} - z_{q_2})} \dots \frac{\partial}{\partial (z_{q_{N-3}} - z_{q_{N-2}})} \left[\prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} \right] \quad (6.141)$$

and

$$\phi_N = (S_N^{(2)})^{-1} \frac{1}{(z_a - z_b)} \sum'_{\{q_1, \dots, q_{N-2}\}} (-1)^P \frac{p_{q_1} \cdot p_{q_2}}{(z_{q_1} - z_{q_2})} \dots \frac{p_{q_{N-3}} \cdot p_{q_{N-2}}}{(z_{q_{N-3}} - z_{q_{N-2}})} \quad (6.142)$$

respectively.

Here the primed summations are over all permutations of $\{q_1 \dots q_{N-2}\} = 1, 2, 3, \dots, N$ (excluding a, b -which are chosen arbitrarily).

At the level $N=4$, the two forms are easily identifiable to be (original formulation)

$$\phi_4^{(4)} = \alpha_s^2 \beta_s + \alpha_t^2 \beta_t + \alpha_u^2 \beta_u \quad (6.143)$$

and ($\mathcal{M}_2$ reformulation)

$$\phi_4^{(1)} = \alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u \quad (6.144)$$

respectively.

Thus, from the symmetric group standpoint, we can understand both the Neveu-Schwarz model and its $\mathcal{M}_2$ reformulation in an illuminating way.

iii) Gervais and Neveu² have obtained an operational factorisation of the amplitude

$$A_N = \int \prod_{i=1}^N dz_i (dv_3)^{-1} \prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} \left(S_N^{(1)}(z) \right)^{\frac{1-\alpha_p(6)}{2}} \quad (6.145)$$

which corresponds to $\alpha_p(0) = \alpha_\pi(0)$ and $\phi_N = 1$ in our general formula.

For $N=4$ this reduces to

$$A_4 = \int_0^1 dx \cdot x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{8/2} \quad (6.146)$$

as expected, and as was proposed by Mandelstam¹³ some time ago.

Actually the factorisation was obtained for $\alpha(0) = -1$, where there was a generalised projective gauge algebra. Unfortunately, for this low intercept the model contained ghosts³¹.

Full factorisation for $\alpha_p(0)$ in the range near $\alpha_p(0) \approx 1/2$ has not yet been exhibited, although the importance of doing so for this amplitude will become evident in the later development.

iv) In a subsequent work, Gervais and Neveu³ have investigated the possibility of constructing dual vertices, from the oscillators $a_\mu^{(n)}, a_\mu^{(n)\dagger}$ of the conventional model, that will lead to ghost-free amplitudes for general $\alpha(0) < 1$. They do this by multiplying the conventional vertex by a function of the gauge operators L_n .

The solution is found, only for special cases, in closed form. In particular a model with leading intercept $\alpha(0) = 0$ and critical dimension $d_c = 25$ was constructed. For this model the four-particle amplitude is

$$A_4 = \int_0^1 dx \cdot x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{1}{2}(1-\alpha(6))} \\ \left[2 {}_2F_1\left(\frac{1}{6}, -\frac{1}{6}; \frac{3}{2}; y\right) - {}_2F_1\left(\frac{1}{6}, -\frac{1}{6}; \frac{1}{2}; y\right) \right] \quad (6.147)$$

with

$$y = \frac{27}{4} \frac{x^2(1-x)^2}{(1-x+x^2)^3} \quad (6.148)$$

This is a special case of our general formula for A_4 , corresponding to a particular non-polynomial choice for the function S_1 .

6i. Multipion Amplitude

We have selected as a new four-pion amplitude the formula

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\delta/2} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) \quad (6.149)$$

If we put $\alpha_\rho(0) - \alpha_\pi(0) = 1/2$, as suggested by soft-pion consistency at leading order, this amplitude becomes

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{1-\alpha_\rho(0)}{2}} [\alpha_s x + \alpha_t (1-x) - \alpha_u x(1-x)] \quad (6.150)$$

Now we may re-write

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\delta/2} (\hat{\alpha}_s \beta_s + \hat{\alpha}_t \beta_t + \hat{\alpha}_u \beta_u) \quad (6.151)$$

where (putting $a = 1 - \alpha_\rho(0)$, $\alpha_\rho(0) - \alpha_\pi(0) = 1/2$)

$$\hat{\alpha}_s = \alpha_s + \frac{a \beta_t \beta_u}{2 \beta_s} \quad (6.152)$$

$$\hat{\alpha}_t = \alpha_t + \frac{a \beta_u \beta_s}{2 \beta_t} \quad (6.153)$$

$$\hat{\alpha}_u = \alpha_u + \frac{a \beta_s \beta_t}{2 \beta_u} \quad (6.154)$$

since we have added to the original amplitude a quantity which vanishes identically, namely

$$\frac{a}{2} (\beta_s \beta_t + \beta_t \beta_u + \beta_u \beta_s) = 0 \quad (6.155)$$

More generally it is useful to define, for an N-point function,

$$\hat{\alpha}_{ij} = \alpha_{ij} + \frac{a S_N^{(1)} \{ij\}}{S_N^{(1)}} \quad (6.156)$$

in which $S_N^{(1)} \{ij\}$ contains only those terms of the sum over permutations $\{q_1 q_2 \dots q_N\} = 1, 2, 3, \dots, N$ in which i, j are adjacent. These quantities α_{ij} possess many intercept-independent properties, and are easily checked to coincide with the definitions of $\hat{\alpha}_s, \hat{\alpha}_t, \hat{\alpha}_u$ given above for $N=4$. We also may check, for example, that

$$\hat{\gamma} = \hat{\alpha}_s + \hat{\alpha}_t + \hat{\alpha}_u + 1 \quad (6.157)$$

$$= 0 \quad (6.158)$$

for any value of a .

We may also write

$$A_4 = 3 \int_0^1 dx \ x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\gamma/2} \left[\hat{\alpha}_s^2 \beta_s + \hat{\alpha}_t^2 \beta_t + \hat{\alpha}_u^2 \beta_u - \frac{3a}{2} \beta_s \beta_t \beta_u \right] \quad (6.159)$$

Notice that in the unphysical limit $\gamma \rightarrow 0$, $a \rightarrow 0$, $\hat{\alpha}_{ij} \rightarrow \alpha_{ij}$ this formula becomes that corresponding to $\phi_4^{(4)}$ given earlier. The reason for re-writing the $a \neq 0$ case in this way is that it makes it considerably easier to write the multiparticle amplitude, to which we now turn.

In keeping with the general method of extension we write the multiparticle amplitude as $(\alpha_\rho(0) - \alpha_\pi(0) = 1/2)$

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \sum_{\{q_1 q_2 \dots q_N\}} (-1)^P \frac{\partial}{\partial (z_{q_1} - z_{q_2})} \dots \frac{\partial}{\partial (z_{q_{N-1}} - z_{q_N})} \cdot \left[\prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} \left(S_N^{(1)}(z) \right)^{a/2} \right] \quad (6.160)$$

Here the integrand is S_N -invariant as required.

We should first note that for $a \rightarrow 0$ this becomes the Neveu-Schwarz dual pion model, for which a no-ghost theorem is well established.

Let us check that for $N=4$, this coincides with our earlier A_4 formula. We find

$$A_4 = \int \prod_{i=1}^4 dz_i (dv_3)^{-1} \prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} (S_4^{(1)})^{a/2} \sum_{\{q_1, q_2, q_3, q_4\}} (-1)^P \frac{\Theta_{q_1, q_2, q_3, q_4}}{(z_{q_1} - z_{q_2})(z_{q_3} - z_{q_4})} \quad (6.161)$$

where $\Theta_{q_1, q_2, q_3, q_4}$ is given by

$$\begin{aligned} \Theta_{q_1, q_2, q_3, q_4} = & (-2p_{q_1} \cdot p_{q_2})(-2p_{q_3} \cdot p_{q_4}) \\ & + (-2p_{q_1} \cdot p_{q_2})(S_4^{(1)})^{-\frac{a}{2}} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} (S_4^{(1)})^{a/2} \\ & + (-2p_{q_3} \cdot p_{q_4})(S_4^{(1)})^{-\frac{a}{2}} \frac{\partial}{\partial(z_{q_1} - z_{q_2})} (S_4^{(1)})^{a/2} \\ & + (S_4^{(1)})^{-a/2} \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} (S_4^{(1)})^{a/2} \end{aligned} \quad (6.162)$$

Now

$$(S_N^{(1)})^{-\frac{a}{2}} \frac{\partial}{\partial(z_{q_1} - z_{q_2})} (S_N^{(1)})^{a/2} = - \frac{a S_N^{(1)} \{q_1, q_2\}}{S_N^{(1)}} \quad (6.163)$$

and

$$\begin{aligned} (S_N^{(1)})^{-\frac{a}{2}} \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} (S_N^{(1)})^{a/2} = \\ = \frac{2a S_N^{(1)} \{q_1, q_2, q_3, q_4\}}{S_N^{(1)}} + \frac{a(a-2) S_N^{(1)} \{q_1, q_2\} S_N^{(1)} \{q_3, q_4\}}{(S_N^{(1)})^2} \end{aligned} \quad (6.164)$$

where $S_N^{(1)}\{q_1, q_2, q_3, q_4\}$ contains only those terms with (q_1, q_2) and (q_3, q_4) as adjacent pairs.

We can now write

$$\theta_{q_1 q_2, q_3 q_4} = \hat{\alpha}_{q_1 q_2} \hat{\alpha}_{q_3 q_4} + \frac{2a}{S_4^{(1)}} \left[S_4^{(1)}\{q_1, q_2, q_3, q_4\} - \frac{S_4^{(1)}\{q_1, q_2\} S_4^{(1)}\{q_3, q_4\}}{S_4^{(1)}} \right] \quad (6.165)$$

Bearing in mind that (in our earlier shorthand)

$$S_4^{(1)} = 8 \left[\frac{1}{(12)^2 (23)^2} + \frac{1}{(12)^2 (13)^2} + \frac{1}{(23)^2 (13)^2} \right] \quad (6.166)$$

$$= \frac{8}{(12)^2 (23)^2} \left[1 + x^2 + (1-x)^2 \right] \quad (6.167)$$

we find that A_4 , as derived from the general A_N , corresponds to a choice

$$\begin{aligned} \phi_4 = & \hat{\alpha}_s^2 \beta_s + \hat{\alpha}_t^2 \beta_t + \hat{\alpha}_u^2 \beta_u + 2a \left[\left(1 - \frac{\beta_t \beta_u}{2\beta_s}\right) \frac{\beta_t \beta_u}{2} \right. \\ & \left. + \left(1 - \frac{\beta_u \beta_s}{2\beta_t}\right) \frac{\beta_u \beta_t}{2} + \left(1 - \frac{\beta_s \beta_t}{2\beta_u}\right) \frac{\beta_s \beta_t}{2} \right] \end{aligned} \quad (6.168)$$

$$= \hat{\alpha}_s^2 \beta_s + \hat{\alpha}_t^2 \beta_t + \hat{\alpha}_u^2 \beta_u - \frac{3a}{2} \beta_s \beta_t \beta_u \quad (6.169)$$

where we used

$$\beta_s \beta_t + \beta_t \beta_u + \beta_u \beta_s = 0 \quad (6.170)$$

together with

$$\frac{\beta_t^2 \beta_u^2}{\beta_s} + \frac{\beta_u^2 \beta_s^2}{\beta_t} + \frac{\beta_s^2 \beta_t^2}{\beta_u} = 3 \beta_s \beta_t \beta_u \quad (6.171)$$

This completes the proof that

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{1+\alpha}{2}} (\hat{\alpha}_s^2 \beta_s + \hat{\alpha}_t^2 \beta_t + \hat{\alpha}_u^2 \beta_u - \frac{3\alpha}{2} \beta_s \beta_t \beta_u) \quad (6.172)$$

$$= \frac{1}{3} \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{1+\alpha}{2}} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) \quad (6.173)$$

is a special case of our general multipion amplitude, as required.

6j. Spin-Lowering Symmetry

The basic conjecture underlying the present approach is that the insistence on the symmetric group invariance will ensure for some $\alpha_p(0)$, other than the known case $\alpha_p(0)=1$, that there will be a generalised projective gauge group, to allow ghost elimination. Actually, since there is extra momentum-dependence in the integrand, as is essential to decouple tachyons, we expect to need an enlarged gauge group, analogous to the G-gauges of the Neveu-Schwarz dual pion model.

In the integral representation, we know that the existence of these extra gauges should be reflected by a higher symmetry, which we shall denote as spin-lowering symmetry^{32,33} (for reasons to become clear later). To be explicit, we expect to be able to rewrite the amplitude as

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \frac{1}{(z_a - z_b)} \sum_{\{q_1, q_2, \dots, q_{N-2}\}} (-1)^p \frac{\partial}{\partial (z_{q_1} - z_{q_2})} \dots \frac{\partial}{\partial (z_{q_{N-3}} - z_{q_{N-2}})} \left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} (S_N^{(1)})^{q/2} \right] \quad (6.174)$$

where a,b are chosen arbitrarily and the primed summation is over all permutations of $\{q_1, q_2, \dots, q_{N-2}\} = 1, 2, 3, \dots, N$ (excluding a,b).

While there is no rigorous proof of this re-writing for general N we shall convince the reader that it is valid by considering specific cases. The validity appears to hold for general a , and therefore the new A_N possesses all of the symmetry properties of the integrand (symmetric group invariance plus spin-lowering symmetry) of the Neveu-Schwarz amplitude.

Firstly we consider $N=4$. There are three inequivalent choices of a, b which we may take as $(a, b) = (1, 2)$, $(1, 3)$ and $(1, 4)$. For these three choices we obtain respectively the three formulae

$$A_4 = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{a}{2}+1} \hat{\alpha}_s \beta_s \quad (6.175)$$

$$A_4 = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{a}{2}+1} \hat{\alpha}_t \beta_t \quad (6.176)$$

$$A_4 = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{a}{2}+1} \hat{\alpha}_u \beta_u \quad (6.177)$$

Now consider the perfect differential

$$0 = \int_0^1 dx \frac{d}{dx} \left[x^{-\alpha_t} (1-x)^{-\alpha_s} (1-x+x^2)^{\frac{a}{2}} \right] \quad (6.178)$$

$$= \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{a}{2}+1} \left[\alpha_s \beta_s - \alpha_t \beta_t + \frac{a(2x-1)x(1-x)}{2(1-x+x^2)^2} \right] \quad (6.179)$$

$$= \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{a}{2}+1} (\hat{\alpha}_s \beta_s - \hat{\alpha}_t \beta_t) \quad (6.180)$$

to see that the first two forms for A_4 are equal. The equality of the third form, involving $\hat{\alpha}_u \beta_u$, follows from the symmetric group property. Finally note that, using the symbol $\equiv$ to denote equality after integration with

$$\int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{a}{2}+1} \quad \text{-----} \quad (6.181)$$

we have the equivalences

$$\hat{\alpha}_s \beta_s \equiv \hat{\alpha}_t \beta_t \equiv \hat{\alpha}_u \beta_u \equiv \frac{1}{3} (\hat{\alpha}_s \beta_s + \hat{\alpha}_t \beta_t + \hat{\alpha}_u \beta_u) \quad (6.182)$$

$$= \frac{1}{3} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) + \frac{1}{3} (\beta_s \beta_t + \beta_t \beta_u + \beta_u \beta_s) \quad (6.183)$$

$$= \frac{1}{3} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) \quad (6.184)$$

to see that all three choices $(a,b)=(1,2), (1,3), (1,4)$ lead to the original amplitude A_4 . This is consistent with the proposed spin-lowering symmetry.

When we discuss the bootstrap condition at $N=6$, in the following sub-section, we shall show how spin-lowering symmetry decouples unwanted ancestor and tachyon states, similar to the way they are decoupled by G-gauges in the $(a=0)$ Neveu-Schwarz model.

6k. Bootstrap Consistency

We now consider factorisation at the internal pion pole of A_6 .

That is, we consider

$$\lim_{\alpha_{123} \rightarrow 0} (\alpha_{123} A_6) \quad (6.185)$$

Later we shall exhibit bootstrap consistency for general N , but it is instructive to study $N=6$ in detail since the important role of spin-lowering symmetry becomes evident.

We can re-write A_6 in the form.

$$A_6 = \int \prod_{i=1}^6 dz_i (dV_3)^{-1} \left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \prod_{i=1}^6 (z_i - z_{i+1})^{-a} \prod_{i < j} (z_i - z_j)^{(-1)^{i-j}} \right] \left[\prod_{i=1}^6 (z_i - z_{i+1})^2 (S_6^{(1)})^{q_2} \right] \left[\prod_{i < j} (z_i - z_j)^{-(-1)^{i-j}} \sum_{\{q_1, \dots, q_6\}} (-1)^P \frac{\theta_{q_1 q_2, q_3 q_4, q_5 q_6}}{(z_{q_1} - z_{q_2})(z_{q_3} - z_{q_4})(z_{q_5} - z_{q_6})} \right] \quad (6.186)$$

which is convenient for the discussion of factorisation.

The first of the three square brackets is simply the integrand for the generalised Euler B function and has, therefore, well-known factorisation properties.

To study the factorisation properties of the second and third square brackets in A_6 we introduce integration variables

$$x = (z_1, z_6; z_2, z_3) \quad (6.187)$$

$$w = (z_1, z_6; z_7, z_4) \quad (6.188)$$

$$y = (z_1, z_6; z_4, z_5) \quad (6.189)$$

satisfying $0 \leq x, w, y \leq 1$ (see Figure 4), and take the limit $w \rightarrow 0$. For the second square bracket we find

$$\lim_{w \rightarrow 0} \left[\prod_{i=1}^6 (z_i - z_{i+1})^2 S_6^{(1)}(z) \right] = 96(1-x+x^2)(1-y+y^2) \quad (6.190)$$

consistent with factorisation, when raised to an arbitrary (non-integer) power $(a/2)$.

The factorisation of the third and final bracket is more subtle.

Let us define, in analogy with the $N=4$ notations,

$$\beta_{12}(x) = (1-x)(1-x+x^2)^{-1} \quad (6.191)$$

$$\beta_{23}(x) = \beta_{12}(1-x) \quad (6.192)$$

$$\beta_{13}(x) = \beta_{12}(1-1/2x) \quad (6.193)$$

and

$$\beta_{56}(y) = (1-y)(1-y+y^2)^{-1} \quad (6.194)$$

$$\beta_{45}(y) = y(1-y+y^2)^{-1} \quad (6.195)$$

$$\beta_{46}(y) = -y(1-y)(1-y+y^2)^{-1} \quad (6.196)$$

Then we find that

$$\lim_{\alpha_{123} \rightarrow 0} (\alpha_{123} A_6) = \int_0^1 dx x^{-\alpha_{12}-1} (1-x)^{-\alpha_{23}-1} (1-x+x^2)^{\frac{\alpha}{2}+1} \int_0^1 dy y^{-\alpha_{56}-1} (1-y)^{-\alpha_{45}-1} (1-y+y^2)^{\frac{\alpha}{2}+1} f(x,y) \quad (6.197)$$

where

$$\begin{aligned} f(x,y) = & \beta_{12}(x) \beta_{56}(y) \bar{\Theta}_{12,34,56} + \beta_{12}(x) \beta_{45}(y) \bar{\Theta}_{12,36,45} \\ & + \beta_{12}(x) \beta_{46}(y) \bar{\Theta}_{12,35,46} + \beta_{23}(x) \beta_{56}(y) \bar{\Theta}_{14,23,56} \\ & + \beta_{23}(x) \beta_{45}(y) \bar{\Theta}_{16,23,45} + \beta_{23}(x) \beta_{46}(y) \bar{\Theta}_{15,23,46} \\ & + \beta_{13}(x) \beta_{56}(y) \bar{\Theta}_{13,24,56} + \beta_{13}(x) \beta_{45}(y) \bar{\Theta}_{13,26,45} \\ & + \beta_{13}(x) \beta_{46}(y) \bar{\Theta}_{13,25,46} \end{aligned} \quad (6.198)$$

in which

$$\bar{\Theta}_{9,92,9394,9596} = \lim_{N \rightarrow 0} [\Theta_{9,92,9394,9596}] \quad (6.199)$$

It is convenient now to define

$$f_{12}(x) = 1 + \frac{\beta_{23} \beta_{13}}{2\beta_{12}} \quad (6.200)$$

$$f_{23}(x) = 1 + \frac{\beta_{12} \beta_{13}}{2\beta_{23}} \quad (6.201)$$

$$f_{13}(x) = 1 + \frac{\beta_{12} \beta_{23}}{2\beta_{13}} \quad (6.202)$$

and, similarly

$$f_{56}(y) = 1 + \frac{\beta_{45} \beta_{46}}{2\beta_{56}} \quad (6.203)$$

$$f_{45}(y) = 1 + \frac{\beta_{46} \beta_{56}}{2\beta_{45}} \quad (6.204)$$

$$f_{46}(y) = 1 + \frac{\beta_{45} \beta_{56}}{2\beta_{46}} \quad (6.205)$$

We now find, after some algebra, that

$$\begin{aligned} \bar{\Theta}_{12,34,56} = & \hat{\alpha}_{12} \hat{\alpha}_{34} \hat{\alpha}_{56} + 2p_1 \cdot p_2 a f_{12} f_{56} (1-f_{56}) \\ & + 2p_5 \cdot p_6 a f_{12} f_{56} (1-f_{12}) + 2a f_{12} f_{56} \\ & + a(a-2) f_{12} f_{56} (f_{12} + f_{56}) + 2a(1-a) f_{12}^2 f_{56}^2 \end{aligned} \quad (6.206)$$

with

$$\hat{\alpha}_{34} = 2p_3 \cdot p_4 + \frac{a}{2} f_{12} f_{56} \quad (6.207)$$

Here we notice that $\rho(x,y)$ will contain terms linear in the momentum transfer, corresponding to a spin-one ancestor, unless there is some delicate decoupling mechanism at work. Such a decoupling does occur, due to spin-lowering symmetry in the form of the equivalences.

$$\hat{\alpha}_{12} \beta_{12} \equiv \hat{\alpha}_{23} \beta_{23} \equiv \hat{\alpha}_{13} \beta_{13} \quad (6.208)$$

and

$$\hat{\alpha}_{45} \beta_{45} \equiv \hat{\alpha}_{56} \beta_{56} \equiv \hat{\alpha}_{46} \beta_{46} \quad (6.209)$$

This enables us to recombine the ancestor terms in $\rho(x,y)$ according to

$$\begin{aligned} & \hat{\alpha}_{12} \beta_{12} (2p_3 \cdot p_4) \hat{\alpha}_{56} \beta_{56} + \hat{\alpha}_{12} \beta_{12} (2p_7 \cdot p_6) \hat{\alpha}_{45} \beta_{45} \\ & + \hat{\alpha}_{12} \beta_{12} (2p_3 \cdot p_5) \hat{\alpha}_{46} \beta_{46} + \hat{\alpha}_{23} \beta_{23} (2p_1 \cdot p_4) \hat{\alpha}_{56} \beta_{56} \\ & + \hat{\alpha}_{23} \beta_{23} (2p_1 \cdot p_6) \hat{\alpha}_{45} \beta_{45} + \hat{\alpha}_{23} \beta_{23} (2p_1 \cdot p_5) \hat{\alpha}_{46} \beta_{46} \\ & + \hat{\alpha}_{13} \beta_{13} (2p_2 \cdot p_4) \hat{\alpha}_{56} \beta_{56} + \hat{\alpha}_{13} \beta_{13} (2p_2 \cdot p_6) \hat{\alpha}_{45} \beta_{45} \\ & + \hat{\alpha}_{13} \beta_{13} (2p_2 \cdot p_5) \hat{\alpha}_{46} \beta_{46} \\ & \equiv (1-2a) \hat{\alpha}_{12} \beta_{12} \hat{\alpha}_{56} \beta_{56} \end{aligned} \quad (6.210)$$

By using this relationship we find that

$$\begin{aligned}
 \rho(x,y) = & \hat{\alpha}_{12} \beta_{12} \hat{\alpha}_{56} \beta_{56} (1 + 5a/2) \\
 & + \frac{3a}{4} (\hat{\alpha}_{12} \beta_{12} \hat{\phi}_4^{(2)}(456) + \hat{\phi}_4^{(2)}(123) \hat{\alpha}_{56} \beta_{56}) \\
 & - \frac{9a}{4} (\hat{\alpha}_{12} \beta_{12} \beta_{45} \beta_{56} \beta_{46} + \beta_{12} \beta_{23} \beta_{13} \hat{\alpha}_{56} \beta_{56}) \\
 & + \frac{a}{8} \hat{\phi}_4^{(2)}(123) \hat{\phi}_4^{(2)}(456) + \frac{9a}{8} \beta_{12} \beta_{23} \beta_{13} \beta_{45} \beta_{56} \beta_{46} \\
 & - \frac{3a}{8} (\hat{\phi}_4^{(2)}(123) \beta_{45} \beta_{56} \beta_{46} + \beta_{12} \beta_{23} \beta_{13} \hat{\phi}_4^{(2)}(456))
 \end{aligned} \tag{6.211}$$

where we defined

$$\hat{\phi}_4^{(2)}(123) = \hat{\alpha}_{12} \beta_{23} \beta_{13} + \hat{\alpha}_{23} \beta_{12} \beta_{13} + \hat{\alpha}_{13} \beta_{12} \beta_{23} \tag{6.212}$$

$$\hat{\phi}_4^{(2)}(456) = \hat{\alpha}_{56} \beta_{45} \beta_{56} + \hat{\alpha}_{45} \beta_{56} \beta_{46} + \hat{\alpha}_{46} \beta_{45} \beta_{56} \tag{6.213}$$

Now we may use the perfect differentials that give vanishing contribution under the x and y integrals respectively, namely

$$\hat{\alpha}_{12} \beta_{12} + \frac{1}{2} \hat{\phi}_4^{(2)}(123) - \frac{3}{2} \beta_{12} \beta_{23} \beta_{13} \equiv 0 \tag{6.214}$$

and

$$\hat{\alpha}_{56} \beta_{56} + \frac{1}{2} \hat{\phi}_4^{(2)}(456) - \frac{3}{2} \beta_{45} \beta_{56} \beta_{46} \equiv 0 \tag{6.215}$$

and thereby deduce that

$$\begin{aligned}
 \rho(x,y) = & ((1+a) \hat{\alpha}_{12} \beta_{12} + \frac{a}{2} \hat{\phi}_4^{(2)}(123) - \frac{3a}{2} \beta_{12} \beta_{23} \beta_{13}) \\
 & \cdot ((1+a) \hat{\alpha}_{56} \beta_{56} + \frac{a}{2} \hat{\phi}_4^{(2)}(456) - \frac{3a}{2} \beta_{45} \beta_{56} \beta_{46}) \\
 = & (\hat{\alpha}_{12} \beta_{12}) (\hat{\alpha}_{56} \beta_{56})
 \end{aligned} \tag{6.216}$$

$$\tag{6.217}$$

as required for bootstrap consistency.

This result holds for any value of $a=1-\alpha_\rho(0)$. It is intriguing to notice, however, that if we do not exploit the vanishing perfect differentials bootstrap consistency gives non-linear constraints on a , namely the six equations

$$(1+a)^2 = 1 + 5a/2 \quad (6.218)$$

$$\frac{a}{2}(1+a) = 3a/4 \quad (6.219)$$

$$a^2/4 = a/8 \quad (6.220)$$

$$9a^2/4 = 9a/8 \quad (6.221)$$

$$-3a^2/4 = -3a/8 \quad (6.222)$$

$$-\frac{3a}{2}(1+a) = -9a/4 \quad (6.223)$$

These equations have two solutions: (I) $a=0$ corresponding to $\alpha_\rho(0)=\alpha_\pi(0)+1/2=1$ and, more interesting, (II) $a=1/2$ corresponding to $\alpha_\rho(0)=\alpha_\pi(0)+1/2=1/2$. This may indicate to us that the intercept $\alpha_\rho(0)=1/2$ which is, of course, physically acceptable may play a special role in the full factorisation.

We can now examine further how the spin-lowering symmetry works at $N=6$ when we write the alternative form

$$A'_6 = \int \prod_{i=1}^6 dz_i (dV_3)^{-1} \frac{1}{(z_a - z_b)} \sum_{\{q_1 q_2 q_3 q_4\}} (-1)^P \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} \left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \left(S_6^0 \right)^{\frac{a}{2}} \right] \quad (6.224)$$

where the sum is over $\{q_1 q_2 q_3 q_4\} = 1, 2, 3, 4, 5, 6$ (excluding a, b). For the factorisation at $\alpha_{123}=0$ there are six inequivalent ways of choosing a, b which we may take as $(a, b) = (1, 4); (1, 5); (1, 6); (2, 5); (5, 6); (4, 6)$.

Defining $\rho'(x,y)$ by

$$\lim_{\alpha_{12} \rightarrow 0} (\alpha_{123} A'_6) = \int_0^1 dx x^{-\alpha_{12}-1} (1-x)^{-\alpha_{23}-1} (1-x+x^2)^{\frac{a}{2}+1} \int_0^1 dy y^{-\alpha_{56}-1} (1-y)^{-\alpha_{45}-1} (1-y+y^2)^{\frac{a}{2}+1} \quad (6.225)$$

we find for the first four cases, respectively

$$\rho'(x,y) = \hat{\alpha}_{23} \beta_{23}(x) \hat{\alpha}_{56} \beta_{56}(y) \quad \text{for } a,b = 1,4 \quad (6.226)$$

$$\rho'(x,y) = \hat{\alpha}_{23} \beta_{23}(x) \hat{\alpha}_{46} \beta_{46}(y) \quad \text{for } a,b = 1,5 \quad (6.227)$$

$$\rho'(x,y) = \hat{\alpha}_{23} \beta_{23}(x) \hat{\alpha}_{45} \beta_{45}(y) \quad \text{for } a,b = 1,6 \quad (6.228)$$

$$\rho'(x,y) = \hat{\alpha}_{13} \beta_{13}(x) \hat{\alpha}_{46} \beta_{46}(y) \quad \text{for } a,b = 2,5 \quad (6.229)$$

and bootstrap consistency follows at once.

Taking $a,b=5,6$ one finds

$$\begin{aligned} \rho'(x,y) = & \beta_{12}(x) \beta_{56}(y) \left[\hat{\alpha}_{12} \hat{\alpha}_{34} + \frac{2a}{S_6^{(1)}} \left\{ S_6^{(1)} \{12,34\} - \frac{S_6^{(1)} \{12\} S_6^{(1)} \{34\}}{S_6^{(1)}} \right\} \right] \\ & + \beta_{23}(x) \beta_{56}(y) \left[\hat{\alpha}_{23} \hat{\alpha}_{56} + \frac{2a}{S_6^{(1)}} \left\{ S_6^{(1)} \{23,14\} - \frac{S_6^{(1)} \{23\} S_6^{(1)} \{14\}}{S_6^{(1)}} \right\} \right] \\ & + \beta_{13}(x) \beta_{56}(y) \left[\hat{\alpha}_{13} \hat{\alpha}_{56} + \frac{2a}{S_6^{(1)}} \left\{ S_6^{(1)} \{13,24\} - \frac{S_6^{(1)} \{13\} S_6^{(1)} \{24\}}{S_6^{(1)}} \right\} \right] \end{aligned} \quad (6.230)$$

$$\begin{aligned} = & \beta_{56}(y) \hat{\alpha}_{12} \beta_{12}(x) 2p_5 \cdot p_6 \\ & + a \beta_{56}(y) f_{56}(y) \left[\beta_{12}(x) \left(\frac{1}{2} \hat{\alpha}_{12} f_{12} + f_{12} (1-f_{12}) \right) \right. \\ & \quad + \beta_{23}(x) \left(\frac{1}{2} \hat{\alpha}_{23} f_{23} + f_{23} (1-f_{23}) \right) \\ & \quad \left. + \beta_{13}(x) \left(\frac{1}{2} \hat{\alpha}_{13} f_{13} + f_{13} (1-f_{13}) \right) \right] \end{aligned} \quad (6.231)$$

$$= \hat{\alpha}_{12} \beta_{12}(x) \hat{\alpha}_{56} \beta_{56}(y) \quad (6.232)$$

as required. By a similar calculation, one finds the same result $\rho'(x,y)$ corresponding to the choice $(a,b)=(4,6)$.

Thus the residue at the internal pion pole of A'_6 satisfies bootstrap consistency and further it is independent of the choice of a,b as expected; these results are consistent with the spin-lowering symmetry defined by $A'_6=A_6$.

Note that the higher symmetry is clearly associated with the tachyon-killing mechanism for the rho trajectory, since the latter is the primary function of ϕ_N , which the higher symmetry describes. We have also seen how this symmetry decouples unwanted ancestor states. Both properties support the suggestion that spin-lowering symmetry provides an extension, of the G-gauges for the $\alpha_\rho(0)=1$ dual pion model, to other intercepts $\alpha_\rho(0) \neq 1$.

We have studied $N=6$ in some detail to learn about spin-lowering symmetry. Before proceeding to arbitrary N , it is worth noting some intercept-independent properties of the $\hat{\alpha}_{ij}$ introduced earlier. for $N=6$, in the $w \rightarrow 0$ limit we see that

$$\hat{\alpha}_{34} + \hat{\alpha}_{35} + \hat{\alpha}_{36} = \hat{\alpha}_{12} \quad (6.233)$$

$$\hat{\alpha}_{24} + \hat{\alpha}_{25} + \hat{\alpha}_{26} = \hat{\alpha}_{13} \quad (6.234)$$

$$\hat{\alpha}_{14} + \hat{\alpha}_{15} + \hat{\alpha}_{16} = \hat{\alpha}_{12} \quad (6.235)$$

and similarly

$$\hat{\alpha}_{14} + \hat{\alpha}_{24} + \hat{\alpha}_{34} = \hat{\alpha}_{56} \quad (6.236)$$

$$\hat{\alpha}_{15} + \hat{\alpha}_{25} + \hat{\alpha}_{35} = \hat{\alpha}_{46} \quad (6.237)$$

$$\hat{\alpha}_{16} + \hat{\alpha}_{26} + \hat{\alpha}_{36} = \hat{\alpha}_{45} \quad (6.238)$$

for all intercepts $\alpha_\rho(0)$.

More generally, for any N, we may check, for example, that

$$\sum_{\substack{j \\ j \neq i}} \hat{\alpha}_{ij} = \sum_{\substack{j \\ j \neq i}} \left(2p_i \cdot p_j + a \frac{S_N^{(1)} \{ij\}}{S_N^{(1)}} \right) \quad (6.239)$$

$$= -2p_i^2 + 2a \quad (6.240)$$

$$= 1 \quad (6.241)$$

for any intercept $\alpha_p(0)$.

Finally, to provide the promised proof³³ of bootstrap consistency for general N, it is simplest to work directly in Koba-Nielsen variables, as follows.

Consider the pion pole in $(p_1 + p_2 + \dots + p_\ell)^2$ of A_N and define

$$\alpha_{1\ell} = \alpha_\pi \left((p_1 + p_2 + \dots + p_\ell)^2 \right) = \alpha_\pi (s_{1\ell}) \quad (6.242)$$

where ℓ is odd and N is even. We wish to consider the limit

$$\lim_{\alpha_{1\ell} \rightarrow 0} (\alpha_{1\ell} A_N) \quad (6.243)$$

First note that the channel variable $u_{1\ell}$ is given by

$$u_{1\ell} = (z_1, z_N; z_\ell, z_{\ell+1}) \quad (6.244)$$

$$= \frac{(z_\ell - z_1)(z_{\ell+1} - z_N)}{(z_\ell - z_N)(z_{\ell+1} - z_1)} \quad (6.245)$$

Consequently if we take as the three fixed values

$$z_1 = 0, \quad z_{\ell+1} = 1, \quad z_N = \infty \quad (6.246)$$

and define $z_\ell = z$ then we have the simple relationship

$$u_{1\ell} = z \quad (6.247)$$

Now we define Koba-Nielsen variables for the two sub-systems

by writing

$$z_i = x_i z \quad (1 \leq i \leq \ell) \quad (6.248)$$

$$z_i = 1/y_i \quad (\ell+1 \leq i \leq N) \quad (6.249)$$

Introducing new variables x_I, y_I corresponding to the intermediate state, the sub-systems have fixed points

$$x_1 = 0, \quad x_\ell = 1, \quad x_I = \infty \quad (6.250)$$

$$y_N = 0, \quad y_{\ell+1} = 1, \quad y_I = \infty \quad (6.251)$$

respectively.

Now consider the full integral

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \sum_{\{q_1, q_2, \dots, q_N\}} (-1)^P \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} \dots \frac{\partial}{\partial(z_{q_{N-1}} - z_{q_N})} \left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} (S_N^{(1)})^{q/2} \right] \quad (6.252)$$

under the change of variables to the x_i and y_i defined above.

The Jacobian is

$$\prod_{i=1}^N dz_i (dV_3)^{-1} = \prod_{i=1}^{\ell+1} dx_i (dV_3(x))^{-1} \prod_{j=\ell}^N dy_j (dV_3(y))^{-1} \prod_{i=\ell+1}^N y_i^{-2} dz_i z^{\ell-2} \quad (6.253)$$

where we have identified $x_{\ell+1} = x_I$ and $y_\ell = y_I$.

Note also that, under this change of variables

$$\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} = \prod_{1 \leq i < j \leq \ell+1} (x_i - x_j)^{-2p_i \cdot p_j} \prod_{\ell \leq i < j \leq N} (y_i - y_j)^{-2p_i \cdot p_j} \prod_{\substack{1 \leq m \leq \ell \\ \ell+1 \leq n \leq N}} (1 - x_m z_n)^{-p_m \cdot p_n} \prod_{i=\ell+1}^N y_i^{-2\mu^2} z^{-S_{2\ell} + \ell\mu^2} \quad (6.254)$$

where $\mu^2 = p_i^2$.

Next we calculate the leading term (in z) of the factor $(S_N^{(1)}(z))^{\frac{a}{2}}$.

After some simple algebra we find that

$$\begin{aligned} (S_N^{(1)}(z))^{\frac{a}{2}} &= \left(\sum_{\{q_1 \dots q_{\ell+1}\}} \prod_{k=1}^{\ell+1} (x_{q_k} - x_{q_{k+1}})^{-2} \right)^{\frac{a}{2}} \\ &\quad \left(\sum_{\{q_{\ell} \dots q_N\}} \prod_{k=\ell}^N (y_{q_k} - y_{q_{k+1}})^{-2} \right)^{\frac{a}{2}} \\ &\quad \prod_{i=\ell+1}^N y_i^{2a} z^{-a(\ell-1)} (1 + O(z)) \end{aligned} \quad (6.255)$$

as required for factorisation.

Finally we must deal with the partial differential operators.

The leading term in z occurs when we take $\frac{1}{2}(\ell-1)$ differentiations with respect to x -differences $(x_i - x_j)$. To see this, we re-write

$$\frac{\partial}{\partial(z_i - z_j)} \equiv \frac{1}{z} \frac{\partial}{\partial(x_i - x_j)} \quad \text{for } 1 \leq i, j \leq \ell \quad (6.256)$$

and

$$\frac{\partial}{\partial(z_i - z_j)} \equiv y_i y_j \frac{\partial}{\partial(y_i - y_j)} \quad \text{for } (\ell+1) \leq i, j \leq N \quad (6.257)$$

Collecting results, we find that after these differentiations the leading term in z for the complete expression is given by

$$\begin{aligned} A_N &= \int \prod_{i=1}^{\ell+1} dx_i (dV_3(x))^{-1} \sum_{\{q_1 \dots q_{\ell+1}\}} (-1)^p \frac{\partial}{\partial(x_{q_1} - x_{q_2})} \dots \frac{\partial}{\partial(x_{q_{\ell}} - x_{q_{\ell+1}})} \\ &\quad \left[\prod_{i \neq j} (x_i - x_j)^{-p_i p_j} S_{\ell+1}^{(1)}(x) \right]^{\frac{a}{2}} \\ &\quad \int \prod_{j=\ell}^N dy_j (dV_3(y))^{-1} \sum_{\{q_{\ell} \dots q_N\}} (-1)^p \frac{\partial}{\partial(y_{q_{\ell}} - y_{q_{\ell+1}})} \dots \frac{\partial}{\partial(y_{q_{N-1}} - y_{q_N})} \end{aligned}$$

cont'd.

$$\cdot \left[\prod_{i \neq j} (y_i - y_j)^{-p_i \cdot p_j} \left(S_{N-l+1}^{(1)}(y) \right)^{a/2} \right] \prod_{i=l+1}^N y_i^{2a-2\mu^2-1}$$

$$\int_0^1 dz \quad z^{-1+\mu^2-S_{1l}} \quad z^{(l-1)(\frac{1}{2}-a+\mu^2)} \quad (1+O(z))$$

(6.258)

Now the amplitude was constructed under the constraint

$$\alpha_\rho(0) - \alpha_\pi(0) - \frac{1}{2} = \frac{1}{2} - a + \mu^2 = 0 \quad (6.259)$$

and hence we have

$$\prod_{i=1}^{l+1} y_i^{2a-2\mu^2-1} = 1 \quad (6.260)$$

and

$$z^{(l-1)(\frac{1}{2}-a+\mu^2)} = 1 \quad (6.261)$$

This leaves a simple z integration, which gives

$$A_N \underset{\alpha_{1l} \rightarrow 0}{\sim} A_{l+1} \frac{1}{\alpha_{1l}} A_{N-l+1} \quad (6.262)$$

as required for bootstrap consistency. Note that the leading intercept $\alpha_\rho(0) = \alpha_\pi(0) + 1/2$ is not constrained by this requirement.

Summary and Outlook

By the present approach we have been able to understand from a new viewpoint the structure of the ghost-free Born amplitudes, and to make some progress towards obtaining a realistic model.

It is quite impossible to foresee how any subject will develop in the future but we may recklessly speculate and imagine progress towards a better theory in three steps, as follows.

- i) The present multipion amplitude has been shown to satisfy

bootstrap consistency and appears to share all the symmetries of the integrand of the unit-intercept dual pion model of Neveu and Schwarz. The main problem is to exhibit full factorisation and to examine whether the symmetries are sufficient to ensure ghost elimination for at least some value of the leading intercept other than $\alpha_\rho(0)=1$. To study full factorisation, it may be possible to exploit further the symmetric group itself but the development of an operator formalism or a string picture for the model would provide the most rapid solution to this question.

ii) Because of the similarity to the meson sector of the RNS theory, we may expect the existence of a fermion quark sector to that of Ramond³⁵. If this could be established, we would like to construct baryons as three-quark systems, in order to suggest a Born amplitude for a process as complex as, say, pion-nucleon elastic scattering. In this connection, we have discussed above one possible link (the fixed-point singularities) between meson amplitudes and three-quark baryon amplitudes.

iii) A third step would be to incorporate strangeness, and SU_3 . In the meson sector, we would like to understand, for example, why in addition to the intercept quantisation $\alpha_\rho(0)=\alpha_\pi(0)+1/2 \approx 1/2$ the leading intercepts for K and $K^*(890)$ satisfy $\alpha_{K^*}(0)=\alpha_K(0)+1/2 \approx 1/4$.

The success of the first of these three steps appears as a prerequisite to the success of the subsequent enlargement of the theory to include more types of hadron. It is worth stressing that, although we have specified an explicit form for the multipion amplitude here based on a criterion of simplicity, it is clear that this amplitude is

the first of an infinite family that can be constructed in a similar way. A sounder basis for the simplicity assumption is badly needed; hopefully it will be provided by a further study of the factorisation properties.

To conclude, the very worthy and exciting objective of a Born amplitude programme such as that just outlined is to obtain a fuller understanding, and in principle even a complete theory on a footing with quantum electrodynamics, of the (mysterious) strong interactions.

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FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4