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MAGNETOHYDRODYNAMICS OF HEMP

Mark L. Wilkins

March 1973

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Notation

ρ	= actual density
ρ_0	= reference density
V	= relative volume = ρ_0/ρ
$\vec{W}$	= velocity vector
$\dot{X}, \dot{Y}$	= components of velocity vector
$\vec{\Sigma}$	= stress tensor
t	= time
X, Y	= space coordinate with cylindrical symmetry about the X axis
$\hat{O}$	= direction perpendicular to the X-Y plane
$\vec{H}$	= magnetic field
μ_m^{Φ}	= magnetic flux
μ_m	= magnetic permeability
P	= material pressure + radiation pressure
q	= artificial viscosity
E	= internal energy (per original volume) = $c_v T + a_R V T^4$
m_e	= material internal energy (per original volume) = $c_v T$
dZ	= change in internal energy by distortion and magnetic pressure stresses
S	= surface
ds	= element of surface
$\vec{n}$	= outward normal to surface
$d\vec{\ell}$	= line element of contour (C)
A	= zone area
M	= zone mass
$\dot{\epsilon}_{xy}, \dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\epsilon}_{\theta\theta}$	components of the strain rate tensor
$T_{xy}, s_{xx}, s_{yy}, s_{\theta\theta}$	components of the deviatoric stress tensor
T	= temperature
K	= thermal conductivity
λ	= Rosseland mean free path
Λ	= $K + 4/3 a_R c \lambda T^3$ = transmissivity
$\vec{E}$	= electric field
$\vec{j}$	= current density
C	= electrical conductivity
c_v	= specific heat
a_R	= Stefan-Boltzmann constant
c	= velocity of light

Units

Symbol	Quantity	Units	SI equivalent
eu	energy unit	10^{12} ergs	100 kJ
ρ	density	gm/cc	Mg/m ³
V	relative volume	dimensionless	—
E	internal energy per original volume	$\frac{\text{eu}}{\text{gm}} \times \rho^0 = \frac{\text{eu}}{\text{cm}^3}$	$\frac{100 \text{ GJ}}{\text{m}^3}$
t	time, μs	$\mu\text{s} = 10^{-6}$ sec	μs
X, Y	space coordinates	cm	10 mm
$\dot{X}, \dot{Y}$	velocity	cm/ μs	10 km/s
T	temperature	K	K
P	pressure (and stress)	$\text{Mb} = 10^{12} \frac{\text{dynes}}{\text{cm}^2}$	100 GPa
$\vec{H}$	magnetic field	10^6 gauss	100 T
$\vec{E}$	electrical field	$\frac{10^4 \text{ volts}}{\text{cm}}$	MV/m
$\vec{j}$	current density	$\frac{10^7 \text{ amps}}{\text{cm}^2}$	100 GA/m ²
C	electrical conductivity	$\frac{10^3 \text{ mho}}{\text{cm}}$	100 kS/m
λ	Rosseland mean free path	cm	10 mm
K	coefficient of thermal conduction	$\frac{\text{eu}}{(\text{cm})(^\circ\text{K})(\mu\text{s})}$	10 TJ/m ² ·K·s
c_v	specific heat	$\frac{\text{eu} \times \rho^0}{(\text{gm})(^\circ\text{K})} = \frac{\text{eu}}{(\text{cm}^3)(^\circ\text{K})}$	100 GJ/k·m ³
σ_R	Stefan-Boltzmann radiation constant	$\frac{7.56 \times 10^{-27} \text{ eu}}{(\text{cm}^3)(^\circ\text{K}^4)}$	756 aJ/K ⁴ ·m ³
c	velocity of light	$3 \times 10^4 \frac{\text{cm}}{\mu\text{s}}$	300 Mm/s
μ_m	magnetic permeability	1	1

MAGNETOHYDRODYNAMICS OF HEMP

Abstract

Using the continuum mechanics approach, details are presented for solving the equations of magnetohydrodynamics in two space dimensions and time. The problem considers cylindrical symmetry, in that only the H_θ component of a magnetic field is present. The

problem is formulated so that the stress contributions resulting from a magnetic field are incorporated into the stress tensor as an elastic-plastic computer program. In addition to the Lorentz force and magnetic diffusion, thermal and radiation diffusion are also treated.

Background

The HEMP code¹ solves problems in dynamic plasticity in Lagrange coordinates with two space dimensions and cylindrical symmetry. The computer program was originally formulated to include the equations appropriate to magnetohydrodynamics, HEMP being an acronym of the words Hydrodynamic Elastic Magneto Plastic. Here we present the magnetohydrodynamics portion of the code, including thermal and radiation diffusion. Details of the calculational procedure are given for the case where only the $\vec{H}_\theta$ component of an applied magnetic field, $\vec{H}$, is included. The problem consists of developing a finite difference approximation to the double operator, $\nabla \times \nabla \times \vec{V}$, where $\vec{V}$ is a vector function. The mathematical problem is similar to that for approximating the double operator, $\nabla \cdot \nabla V$, where V is a scalar function. For the problem at hand, $\vec{V}$ is the magnetic field, $\vec{H}$, and V is the temperature, T . It is seen from the vector identity:

$$\begin{aligned}\nabla \times \nabla \times \vec{H} &= \nabla(\nabla \cdot \vec{H}) - \nabla \cdot \nabla \vec{H} \text{ that} \\ \nabla \times \nabla \times \vec{H} &= -\nabla \cdot \nabla \vec{H} = -\nabla^2 \vec{H}, \text{ since} \\ \nabla \cdot \vec{H} &= 0.\end{aligned}$$

In the calculation of thermal diffusion of the temperature, T , a simple change of $\vec{H}$ for T cannot be made in

the difference equations. The reason is that in cylindrical coordinates the Laplacian operation, ∇^2 , yields a different result when applied to a vector than it does when applied to a scalar quantity.

For the case considered here, where $\vec{H} = \vec{H}_\theta$, the relationship between the Laplacian applied to the vector, $\vec{H}_\theta$ and to a scalar, H_θ is:

$$\nabla^2 \vec{H}_\theta = \nabla^2 H_\theta - \frac{H_\theta}{R^2}.$$

Here R is the radial space coordinate.

An important consideration in calculating magnetic diffusion or thermal diffusion is the manner in which the constitutive relations are introduced into the difference equations. The constitutive relations describing the electrical conductivity in the calculation of magnetic diffusion can be included in a more physical way if the curl-curl formulation is used. Thus for the difference equations given here approximations are made to the operator, $\nabla \times \nabla \times$, acting on the vector, $\vec{H}_\theta$, and to ∇^2 acting on the scalar, T .

Method

The fundamental equations are organized so that changes of variables associated with a mass point can be interpreted as due to a flux through a surface surrounding the mass point. The difference operators are organized in the same way, i.e., in the spirit of the divergence theorem. Space derivatives in two dimensions are defined as the summation of the normal components of the flux around an enclosed area. Thus, for a vector, F , representing a flux (e.g., magnetic, heat flow, velocity, etc.), the following integral definitions of the partial derivatives are used. See Fig. 1.

$$\frac{\partial F}{\partial X} = \lim_{A \rightarrow 0} \int_{(C)} \frac{F(\vec{n} \cdot \vec{i})}{A} d\ell \quad (1)$$

$$\frac{\partial F}{\partial Y} = \lim_{A \rightarrow 0} \int_{(C)} \frac{F(\vec{n} \cdot \vec{j})}{A} d\ell$$

where

$d\ell$ = arc length

(C) = boundary of area A

$\vec{n}$ = normal vector

$\vec{t}$ = tangent vector

The material is divided into a Lagrange grid that moves with the flow. The intersection of the grid lines will be called zone node points, and the interior space of the mesh called zones. The integration path for evaluating the partial derivatives is defined in the grid in two ways. Figure 2 shows the integration path for evaluating components of the vector, $\nabla \times \vec{H}$. Referring to Fig. 2, if F is a function (in this case, $F = H_y$) defined at zone centers, then the difference equations for

evaluating $\frac{\partial F}{\partial X}$ and $\frac{\partial F}{\partial Y}$ centered at point 1 are:

$$\frac{\partial F}{\partial X} \Big|_{\odot} = \frac{\int_{(C)} F(\vec{t} \cdot \vec{i}) d\ell}{A}$$

where

$$\int_{(C)} F(\vec{t} \cdot \vec{i}) d\ell = -[F_{\odot}(Y_{II} - Y_{III}) + F_{\odot}(Y_{III} - Y_{IV}) + F_{\odot}(Y_{IV} - Y_I) + F_{\odot}(Y_I - Y_{II})],$$

and

$$\frac{\partial F}{\partial Y} \Big|_{\odot} = - \frac{\int_{(C)} F(\vec{n} \cdot \vec{j}) d\ell}{A}$$

where

$$\int_{(C)} F(\vec{n} \cdot \vec{j}) d\ell = +[F_{\odot}(X_{II} - X_{III}) + F_{\odot}(X_{III} - X_{IV}) + F_{\odot}(X_{IV} - X_I) + F_{\odot}(X_I - X_{II})].$$

A is the area enclosed by the integration path, I, II, III, IV (Fig. 2), and is taken as the average of the area contributions of the four zones surrounding point 1 (Fig. 2).

Figure 3 shows the path for evaluating $\nabla \times \vec{H}$. If F is a function (in this case a component of the vector $\nabla \times \vec{H}$) defined at zone nodes, then the difference equations for evaluating $\frac{\partial F}{\partial X}$ and $\frac{\partial F}{\partial Y}$, defined at a zone center, are:

$$\begin{aligned} \frac{\partial F}{\partial X} \Big|_{\odot} &= \frac{\int_{(C)} F(\vec{t} \cdot \vec{i}) d\ell}{A} \\ &= -\frac{1}{A} [F_{23}(Y_2 - Y_3) + F_{34}(Y_3 - Y_4) \\ &\quad + F_{41}(Y_4 - Y_1) + F_{12}(Y_1 - Y_2)] \\ &= +\frac{1}{2A} [(F_2 - F_4)(Y_3 - Y_1) - (Y_2 - Y_4)(F_3 - F_1)]. \end{aligned}$$

Similarly,

$$\frac{\partial F}{\partial Y} \Big|_{\odot} = -\frac{1}{2A} [(F_2 - F_4)(X_3 - X_1) - (X_2 - X_4)(F_3 - F_1)].$$

Here, $F_{23} = (F_2 + F_3)/2$, etc., and A is the area of zone $\odot$ defined by points 1, 2, 3, 4.

Fig. 1. Integration scheme.

Fig. 2. Grid for calculating $\nabla \times \vec{H}$, where $\vec{H}$ is given in a zone center. Dotted line shows integration path for evaluating components of $\nabla \times \vec{H}$, centered at zone node point 1: similar integration paths surround points 2, 3, and 4.

Fig. 3. Grid for calculating $\nabla \times \nabla \times \vec{H}$, where $\nabla \times \vec{H}$ is given at node points. Arrows show integration path where $\nabla \times \vec{H}$ is centered at $j+1/2, k+1/2$ (zone center).

Fundamental Equations of Magnetohydrodynamics

The following equations are solved in Lagrange coordinates with 2-D cylindrical symmetry, using the structure of the HEMP code:

EQUATION OF MOTION

$$\rho \frac{d\vec{W}}{dt} = \nabla \vec{\Sigma} + \left\{ \mu_m \vec{J} \times \vec{H} \right\}$$

As shown in the following section, the term in braces [Lorentz force] is incorporated into the total stress tensor, $\vec{\Sigma}$, of the HEMP program.

ELECTROMAGNETIC FIELD EQUATIONS

Faraday's Law

$$\mu_m \frac{d}{dt} \iint_S \vec{H} \cdot \vec{n} \, ds = - \int_{(C)} \vec{E} \cdot d\vec{\ell} \quad (2)$$

Ampère's Law

$$\nabla \times \vec{H} = 4\pi \vec{J} \quad (3)$$

Ohm's Law

$$\vec{E} = \frac{\vec{J}}{C} \quad (4)$$

By Stoke's theorem and substitution of Eqs. (3) and (4), Eq. 2 can be written as:

$$\begin{aligned} \mu_m \frac{d}{dt} \iint_S \vec{H} \cdot \vec{n} \, ds &= \int_{(C)} \vec{E} \cdot d\vec{\ell} = - \iint_S \nabla \times \vec{E} \cdot \vec{n} \, ds \\ &= - \iint_S \nabla \times \left(\frac{1}{4\pi C} \nabla \times \vec{H} \right) \cdot \vec{n} \, ds \end{aligned} \quad (5)$$

The above relations apply to a local region of the fluid that moves with the fluid. Applying Eq. (5) to the Lagrange region and considering an H_θ field only gives the magnetic flux diffusion equation:

$$\mu_m \frac{d\Phi}{dt} = - \nabla \times \left(\frac{1}{4\pi C} \nabla \times H_\theta \right) A \quad (6)$$

$$\mu_m \Phi = \mu_m H_\theta A = \text{magnetic flux}$$

$$H_\theta = \text{theta component of magnetic field, } \vec{H}$$

$$A = \text{surface element, defined later as the area of a Lagrange zone.}$$

It is in the form of Eq. (6) that Faraday's law, Ampere's law, and Ohm's law are used in the calculation. It is noted that the displacement current has

been omitted and that $\nabla \cdot \mu_m \vec{H} = 0$ is identically satisfied for the case considered here with $\vec{H} = H_\theta$ only and symmetry in the θ direction.

ENERGY EQUATION

The total change in interval energy, dE , in a given volume is:

$$dE = -(P + q)dV + V \left\{ \left[\left(K + \frac{4}{3} a_R c \lambda T^3 \right) \nabla T \right] \right\} dt + V \frac{J^2}{C} dt + dZ, \quad (7)$$

where:

$$dZ = V T_{xy} \dot{\epsilon}_{xy} dt + V \left(\epsilon_{xx} - \frac{\mu_m}{8\pi} H^2 \right) \dot{\epsilon}_{xx} dt + V \left(\epsilon_{yy} - \frac{\mu_m}{8\pi} H^2 \right) \dot{\epsilon}_{yy} dt + V \left(\epsilon_{\theta\theta} + \frac{\mu_m}{8\pi} H^2 \right) \dot{\epsilon}_{\theta\theta} dt.$$

The first term on the right in Eq. (7) is the change in internal energy from pressure forces; the second term is the energy change from thermal conduction and radiation flow; the third term is the energy change from ohmic heating; and the fourth term, from distortion stresses.

The following definitions are used:

$$E = c_v T + a_R V T^4 \approx \text{material energy} + \text{radiation energy} \quad (8)$$

$$dE = \frac{\partial E}{\partial T} \Big|_V dT + \frac{\partial E}{\partial V} \Big|_T dV \quad (9)$$

$$(i) \quad \frac{\partial E}{\partial T} \Big|_V = \frac{\partial m_e}{\partial T} \Big|_V + 4 a_R V T^3$$

$$(ii) \quad \frac{\partial E}{\partial V} \Big|_T = \frac{\partial m_e}{\partial V} \Big|_T + a_R T^4$$

Here, m_e = material energy = $c_v T$.

Substitution of Eq. (9) into Eq. (8) gives:

$$dT \frac{\partial E}{\partial T} \Big|_V = - (P + \frac{\partial E}{\partial V} \Big|_T + q) dV + (dV) \nabla \cdot (A \nabla T) + \frac{V J^2}{C} dt + dZ \quad (10)$$

where:

$$A = K + \frac{4}{3} a_R c \lambda T^3$$

dZ = change in internal energy from distortion stresses.

Rewriting Eq. (10):

$$dT \frac{\partial E}{\partial T} \Big|_V = V [\nabla \cdot (A \nabla T)] dt + W \quad (11)$$

where:

$$W = - \left(P + \frac{\partial E}{\partial V} \Big|_T + q \right) dV + \frac{V J^2}{C} dt + dZ$$

$$P = \text{material pressure} + \frac{1}{3} a_R T^4.$$

It is in the form of Eq. (11) that changes in the internal energy are calculated. Equation (11) is a diffusion equation for the temperature, T .

CONTINUITY EQUATION

$$\nabla \cdot \vec{W} = \frac{\dot{V}}{V}$$

CONSTITUTIVE RELATIONS

P	$= P_m + P_R$	= total pressure
P_m	$= P_m(T, V)$	= material pressure
P_R	$= \frac{1}{3} a_R T^4$	= radiation pressure
K	$= K(T, V, H)$	= thermal conductivity
λ	$= \lambda(T, V)$	= Rosseland mean free path
c_v	$= c_v(T, V)$	= specific heat
C	$= C(T, V)$	= electrical conductivity
m_e	$= c_v T$	= material internal energy
$\frac{\partial E}{\partial T} \Big _V$	$= \frac{\partial m_e}{\partial T} \Big _V + 4 a_R T^3$	= change in internal energy with respect to temperature
$\frac{\partial E}{\partial V} \Big _T$	$= \frac{\partial m_e}{\partial V} \Big _T + a_R T^4$	= change in internal energy with respect to volume

The above relationships must be given in advance in the form of explicit functions or table look-ups.

Difference Equations for Magnetohydrodynamics

EQUATIONS OF MOTION

$$\rho \frac{d\vec{W}}{dt} = \nabla \vec{\Sigma} + \mu_m \vec{J} \times \vec{H} \quad (12)$$

Lorentz force (effect of the magnetic field on the fluid motion):

$$\mu_m \vec{J} \times \vec{H} = \frac{\mu_m}{4\pi} (\nabla \times \vec{H}) \times \vec{H}.$$

Using X-Y coordinates with cylindrical symmetry about the X-axis, and considering a field, H , in the theta direction H_θ only, Eq. (12) becomes:

$$(a) \begin{cases} \rho \ddot{X} = \frac{\partial \Sigma_{xx}}{\partial X} + \frac{\partial T_{xy}}{\partial Y} + \frac{T_{xy}}{Y} + \left(-\frac{\mu_m}{8\pi} \frac{\partial H^2}{\partial X} \right) \\ \rho \ddot{Y} = \frac{\partial T_{xy}}{\partial X} + \frac{\partial \Sigma_{yy}}{\partial Y} + \frac{\Sigma_{yy}}{Y} + \left(-\frac{\mu_m}{8\pi} \frac{\partial H^2}{\partial Y} - \frac{\mu_m}{4\pi} \frac{H^2}{Y} \right) \end{cases} \quad (13)$$

or:

$$(b) \begin{cases} \rho \ddot{X} = \frac{\partial \Sigma_{xx}}{\partial X} + \frac{\partial T_{xy}}{\partial Y} + \frac{T_{xy}}{Y} \\ \rho \ddot{Y} = \frac{\partial T_{xy}}{\partial X} + \frac{\partial \Sigma_{yy}}{\partial Y} + \frac{\Sigma_{yy}}{Y} - \Sigma_{\theta\theta} \end{cases} \quad (14)$$

where:

$$\Sigma_{xx} = -P + s_{xx} - \frac{\mu_m}{8\pi} H^2$$

$$\Sigma_{yy} = -P + s_{yy} - \frac{\mu_m}{8\pi} H^2$$

$$\Sigma_{\theta\theta} = -P + s_{\theta\theta} + \frac{\mu_m}{8\pi} H^2$$

It is seen that the magnetic force is like a pressure in the X and Y directions, and like a tension in the theta direction. With the addition of the above pressure-like terms from the magnetic field, there are no other changes to the finite difference equations of motion of the HEMP code.

MAGNETIC DIFFUSION

Calculation of $d\Phi/dt$ centered in space at $j + 1/2$, $k + 1/2$, (see Fig. 4) and in time at $t^{n+1/2}$:

$$\mu_m \frac{d\Phi}{dt} = -\nabla \times \left(\frac{1}{4\pi C} \nabla \times \vec{H} \right) A \quad (15)$$

Fig. 4. Lagrange zone defined by points 1, 2, 3, and 4.

$\vec{H} = H_\theta$ only
 A = zone area
 C = conductivity
 $\Phi = HA$
 μ_m = magnetic permeability

Procedure

a. The vector $\nabla \times \vec{H}$ is evaluated at zone node points 1, 2, 3 and 4, using the difference scheme of the preceding section, where $\vec{H}$ is considered defined at a zone center (see Fig. 4).

b. The required value of $\nabla \times \left(\frac{1}{4\pi C} \nabla \times \vec{H} \right)$ defined at $j + 1/2$, $k + 1/2$ (zone center) is obtained by averaging the components of the vector $\nabla \times \vec{H}$, so that they are defined at the middle of the line connecting consecutive j, k points and integrating around the area A , as discussed in the preceding section. A representative value of the conductivity, C , for each leg of the integration path is obtained by an averaging scheme based on current flow in parallel circuits.

c. A forward difference scheme is used which means that the desired value of H^{n+1} at time, t^{n+1} , is an implicit function of H^{n+1} , i.e.,

$$\frac{[(HA)^{n+1} - (HA)^n]}{t^{n+1} - t^n} = f(H^{n+1}, H^n) \quad (15a)$$

For a given j -line, the value of $H_{j+1/2,k+1/2}^{n+1}$ can be found by solving a set of linear equations if we first evaluate $\nabla \times \vec{H}$ at points 1, 2, 3 and 4, using values of H^{n+1} for all k -lines corresponding to the given j -line (Fig. 5). The change of flux, $\mu_m d\Phi$ is then determined by using only sides 1-2 and 3-4 in step (b) above. The conductivity, C , associated with a side is obtained from the conductivity of the zones on either side of the line being considered. The calculated value of $\mu_m d\Phi$ will be the change in flux in the

k -direction. Details of the calculation are given in Appendix 1.a.

d. The calculation of $\nabla \times \vec{H}$ is repeated for points 1, 2, 3 and 4, but this time using values of H^{n+1} for all j -lines corresponding to a given k -line (Fig. 6). The change in flux, $\mu_m d\Phi$ is determined as in step (c), but this time using only sides 2-3 and 4-1. This will be the change in flux in the j -direction. See Appendix 1.b.

Fig. 5. Grid showing the time centering of $\vec{H}$ used to evaluate the components of $\nabla \times \vec{H}$ for calculating the flux change in the k -direction, where j and k are Lagrange coordinates.

Fig. 6. Grid showing time centering of $\vec{H}$ used to evaluate the components of $\nabla \times \vec{H}$ for calculating the flux change in the j -direction.

e. The total change in flux, $\mu_m \Phi$, for the zone $j = 1/2$, $k + 1/2$ is obtained by adding together the flux change in the two directions; i.e., total flux change, $\mu_m d\Phi = \mu_m (d\Phi)^k + \mu_m (d\Phi)^j$, where $(d\Phi)^k$ is obtained from step (c), and $(d\Phi)^j$ is obtained from step (d).

f. The desired value of H^{n+1} is obtained from step (e):

$$\begin{aligned} (HA)^{n+1} - (HA)^n &= [(HA)^{n+1} - (HA)^n]^k \\ &+ [(HA)^{n+1} - (HA)^n]^j \\ [(HA)^n]^k &= [(HA)^n]^j = (HA)^n \\ H^{n+1} &= (H^{n+1})^k + (H^{n+1})^j - \left(\frac{H^n A^n}{A^{n+1}} \right) \end{aligned}$$

When specified to one space dimension, the space and time-centering of the difference equations correspond to the Crank-Nicholson differencing scheme.²

ENERGY EQUATIONS

Temperature Diffusion

Rewriting Eq. (11) with the quantities at times n and $n + 1/2$ considered to be known, where the unknown quantity is T^{n+1} , gives:

$$\left(T^{n+1} - T^n \right) \left(\frac{\partial E}{\partial T} \right)_V^n = \Delta t^{n+1/2} \nabla^{n+1/2} \left(\nabla \cdot \Lambda^n \nabla T \right) + W^n \quad (16)$$

where ∇T is calculated with T^n and T^{n+1} . The temperature, T , is centered at $j + 1/2$, $k + 1/2$ (zone center).

a. Equation (16) is solved in the same manner as the magnetic diffusion equation, Eq. (15), to obtain new temperatures $(T^{n+1})^k$ and $(T^{n+1})^j$ from the temperature diffusion in the k and j directions, respectively. Details of these calculations are given in Appendix B.

b. The total energy charge is the sum of contributions from the k plus the j directions. The value

of $\partial E / \partial T$ is considered constant during the time interval, $t^n \rightarrow t^{n+1}$. The new temperature for a zone is obtained as follows:

$$\begin{aligned} T^{n+1} - T^n &= (T^{n+1})^k - T^n + (T^{n+1})^j - T^n \\ \therefore T^{n+1} &= (T^{n+1})^k + (T^{n+1})^j - T^n \end{aligned}$$

This temperature is considered to be the first approximation to the desired temperature and will be designated as $\tilde{T}^{n+1}$.

c. The pressure, $\tilde{p}^{n+1}$, and $(\partial \tilde{E} / \partial V)^{n+1}$ are calculated from the given constitutive relations using $\tilde{T}^{n+1}$ and ∇^{n+1} . The quantity, W , is then recalculated using $p^{n+1/2}$ and $(\partial E / \partial V)^{n+1/2}$, with the other quantities remaining the same; where,

$$\begin{aligned} p^{n+1/2} &= 1/2 (\tilde{p}^{n+1} + p^n), \\ (\partial E / \partial V)^{n+1/2} &= 1/2 [(\partial \tilde{E} / \partial V)^{n+1} + (\partial E / \partial V)^n] \end{aligned}$$

d. The quantity $(\partial E / \partial T)^{n+1/2}$ is calculated from the given constitutive relation. The time-centering of T is given by:

$$\begin{aligned} \text{(i)} \quad T^{n+1/2} &= 1/2 (\tilde{T}^{n+1} + T^n) \\ \text{(ii)} \quad (T^{n+1/2})^3 &= \frac{1}{4} \left\{ (\tilde{T}^{n+1})^3 + (\tilde{T}^{n+1})^2 \cdot (T^n) \right. \\ &\quad \left. + (\tilde{T}^{n+1}) (T^n)^2 + (T^n)^3 \right\} \end{aligned}$$

This definition of T^3 assures that

$$dE \approx (c_v + 4a_R VT^3) dT,$$

if c_v and V are constant in the time interval between n and $n+1$.

e. Steps (a) and (b) are repeated using up-dated values of $\partial E / \partial T$, Λ , and W obtained from the time-centering given in steps (c) and (d), and the constitutive relations.

Space-Centering of Parameters at the Interfaces of Consecutive Zones.

In the integration scheme, representative values of parameters calculated from the constitutive relations

are required for each leg of the integration path. The averaging technique used is given below.

Consider the leg 1-2 of the integration path 1, 2, 3, 4, 1 around zone ⑤ in Fig. 7.

Coefficient of thermal conduction, K -

$$K_{1-2} = \frac{A_{⑤} + A_{②}}{\frac{A_{⑤}}{K_{⑤}} + \frac{A_{②}}{K_{②}}} \quad A = \text{zone area}$$

Rosseland mean free path, λ -

$$\lambda_{1-2} = \frac{A_{⑤} + A_{②}}{\frac{A_{⑤}}{\lambda_{⑤}^*} + \frac{A_{②}}{\lambda_{②}^*}}$$

To insure the thermal flow is from high to low temperature zones, $\lambda_{⑤}^*$ and $\lambda_{②}^*$ are calculated using the maximum temperature of the zones ② and ⑤ (Ref. 3).

Transmissivity, Λ -

$$\Lambda_{1-2} = K_{1-2} \left(\frac{4}{3} a R^c \right) \cdot \lambda_{1-2} \cdot T_{1-2}^3$$

where

$$T_{1-2}^3 = \frac{1}{4} (T_{⑤}^2 + T_{②}^2) (T_{⑤} + T_{②})$$

Electrical conductivity C -

$$C_{1-2} = \frac{\left(CV \frac{M}{\rho_0} \right)_{⑤} + \left(CV \frac{M}{\rho_0} \right)_{②}}{\left(V \frac{M}{\rho_0} \right)_{⑤} + \left(V \frac{M}{\rho_0} \right)_{②}}$$

where M = zone mass.

Similar expressions are written for legs 2-3, 3-4 and 4-1.

Ohmic Heating, $\frac{j^2}{C}$

$$\begin{aligned} \frac{j^2}{C} &= \frac{1}{16\pi^2 C} (\nabla \times \vec{H})^2 \\ &= \frac{1}{16\pi^2 C} (a^2 + b^2) \end{aligned}$$

where $\nabla \times \vec{H} = a\vec{i} + b\vec{j}$.

Referring to Fig. 7, the ohmic heating for zone ⑤ is given by:

$$\left(\frac{j^2}{C} \right)_{⑤} = \left[\frac{1}{2} \left(\frac{j^2}{C} \right)_{③}^k + \left(\frac{j^2}{C} \right)_{④}^j \right] \quad (17)$$

Fig. 7. Nodes 1, 2, 3, and 4 and zones ②, ③, ④, and ⑤ surrounding zone ⑤.

where:

$$\left(\frac{J^2}{C}\right)_{\oplus}^k = \frac{C_{\oplus}}{2 \cdot 16\pi^2} \left[\left(\frac{a_1^2 + a_2^2}{2} + \frac{b_1^2 + b_2^2}{2} \right) \left(\frac{1}{C_{1-2}} \right)^2 + \left(\frac{a_3^2 + a_4^2}{2} + \frac{b_3^2 + b_4^2}{2} \right) \left(\frac{1}{C_{3-4}} \right)^2 \right]$$

$$\left(\frac{J^2}{C}\right)_{\oplus}^j = \frac{C_{\oplus}}{2 \cdot 16\pi^2} \left[\left(\frac{a_2^2 + a_3^2}{2} + \frac{b_2^2 + b_3^2}{2} \right) \left(\frac{1}{C_{2-3}} \right)^2 + \left(\frac{a_4^2 + a_1^2}{2} + \frac{b_4^2 + b_1^2}{2} \right) \left(\frac{1}{C_{4-1}} \right)^2 \right]$$

The superscripts k and j refer to the k and j sweeps described earlier.

CONTINUITY EQUATION

Mass is conserved explicitly in the Lagrange formulation of the fundamental equations.

The continuity equation

$$\nabla \cdot \vec{W} = \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\dot{V}}{V}$$

is used to describe components of the strain tensor and is explained in Ref. 1.

TIME-STEP CONTROL

The time step is taken as the minimum over the grid mesh of the mesh size divided by the largest wave speed (Courant condition). This choice of time-step can insure stability of the difference equations, but does not insure accuracy of the diffusion calculations. In the diffusion calculations a much smaller time-step is required in order to provide communication of the boundary conditions throughout the grid.

An extrapolation method devised by Julius Chang and Jim LeBlanc permits a large time-step to be used, but still maintains the accuracy of a much smaller time-step. The application specialized to one dimension is discussed in Ref. 4. This scheme provides a considerable savings in computation time in the solution of diffusion equations by the double sweep technique described here.

With an equation of the form

$$\frac{F^{n+1} - F^n}{\Delta t} = f(F^{n+1}, F^n)$$

this extrapolation method advances in time the parameter F from F^n to F^{n+1} . Using the double sweep technique, solve the implicit equation three times, as follows:

$$(a) \quad \frac{F^i - F^n}{\Delta t} = f(F^i, F^n)$$

$$(b) \quad \frac{F^{ii} - F^n}{\frac{\Delta t}{2}} = f(F^{ii}, F^n)$$

$$(c) \quad \frac{F^{iii} - F^{ii}}{\frac{\Delta t}{2}} = f(F^{iii}, F^{ii})$$

Then $F^{n+1} = 2F^{iii} - F^i$.

BOUNDARY CONDITIONS

The magnetic field, temperature and pressure must be specified for all times on the exterior boundaries of the material region being considered. Usually the pressure is assumed zero on exterior surfaces. For problems where the material occupies a region containing the axis of symmetry, the boundary conditions for the axis of symmetry are:

$$H_{\theta} = 0$$

$$\frac{\partial P}{\partial Y} = \frac{\partial T}{\partial Y} = 0,$$

where Y is the radial coordinate. For symmetry planes perpendicular to the X-axis:

$$\frac{\partial H_{\theta}}{\partial X} = \frac{\partial P}{\partial X} = \frac{\partial T}{\partial X} = 0.$$

For the application of boundary conditions, it is convenient to provide phantom zones around the region of interest. The zero gradient conditions above are achieved by reflecting the values of the interior parameters into the phantom zones.

SLIDING INTERFACES

For most problems of practical interest, it is important to allow for one material to slide upon another. The details of the computer logic for sliding interfaces in the framework of elastic-plastic flow are given in Ref. 1. In the solutions of the diffusion equations presented here, it is seen that the zones must be continuous in the Lagrange zone directions, j and k . This is accomplished by subdividing the grid just before the diffusion calculations are made.

Figure 8 shows a sliding interface that separates regions A and B. The slide line logic of the HEMP program that is required to develop in the computer the geometry corresponding to Fig. 8 is described in Ref. 1. As shown by the dotted lines in Fig. 8 the grid of region B is extended through region A. Thus, region A is now considered to be mapped by the original k lines of region A and the j lines of region B. The new j grid of region A is generated so that it intersects the k lines in the same proportions to the original j lines as occurred on the slide line. The new grid will overlay one or more zones of the old grid of region A. A single value of the parameters H or T

is obtained for each zone of the new grid by an area weighting scheme. For example if a zone of the new grid is composed of parts of zone a and zone b , see Fig. 8, then the appropriate value of the parameter H for the new grid is

$$H = \frac{H_a A_a + H_b A_b}{A_a + A_b} \quad (\text{similar for } T)$$

Here A_a and A_b refer to the area of the new grid that overlay zones a and b respectively of the old grid.

Thus, the form of the original calculational scheme is established and the same method can be used to solve the implicit equation for new values of the magnetic field H and the temperature T . After the diffusion calculations are completed, the updated parameters are reassigned to the solid line grid by an area weighting scheme similar to the above.

In this example the zones of region B have been used to define zones in region A. The same logic could have been applied using region A to define zones in region B.

It is seen in Fig. 8 that region A extends beyond region B. The problem requires that the boundary conditions are always known in advance of a time step calculation. This information is carried in the

Fig. 8. Grid for calculating field and temperature diffusion across a sliding interface.
 (a) Solid lines — Lagrange grid at end of equation of motion calculation;
 (b) An ordered grid is created by the dotted lines for the diffusion calculations.

computer logic by pseudo zones that surround the entire calculational grid. The use of pseudo zones permits parameters on the grid boundaries, including those of the overhang of region A, Fig. 8 to be calculated in the same manner as any other point in the grid. (A slide line extension is required if points associated with region B on the slide line k_s move beyond the ends of region A.)

The example shown here has considered a single sliding surface along a k line. The same approach is used if more than one k line is a slide line. The region that is to control the zoning for the diffusion calculations must be stated in advance.

Figure 9 shows a hydrodynamic calculation set up with slide lines in two directions (j and k slide lines).

This provision in the computer program is very useful for permitting a fine zoned region to be defined in the two dimensional grid. The same mapping scheme for calculating the temperature and σ magnetic diffusion is used, the only difference being that additional mapping is required so as to include all material regions surrounding a given region, i.e., both the k-lines and the j-lines are extended from the region that controls the zoning.

CHECK PROBLEMS

The accuracy of the numerical technique that calculates the diffusion of T^* in one space dimension

Fig. 9. Application of slide lines in two directions. The calculation permits sliding between each pair of adjacent materials.

can be checked by a problem suggested by W. Schultz of L³L. A material with the special properties

$K \equiv 0$, $P \equiv 0$, $c_v = \rho_0 T$ and $\lambda = \frac{10^6 T^2}{a_R \rho_0}$ is considered.

An extreme density, $\rho_0 = 10^5 \text{ Mg/m}^3$, is used to insure that the radiation energy density is negligible compared to the material energy density.

Specializing to one space dimension, the energy equation (7) with the above material model can be written:

$$\frac{dE}{dt} = V \left[\frac{\partial}{\partial X} \left(\frac{1}{3} a_R c \lambda \right) \frac{\partial T^4}{\partial X} \right]$$

Substituting for λ , c_v and ρ_0 where $E = c_v T = \rho_0 T^2$, $V = 1$ and $c = 300 \text{ Mm/s}$ gives:

$$\frac{dT^4}{dt} = \frac{\partial}{\partial X} \left(T^2 \frac{\partial T^4}{\partial X} \right)$$

This equation has $T^4 = t - X$ as a solution.

A calculation was done using the zoning shown in Fig. 10 with boundary conditions $T^4 = t$ at the left end and $T = 0$ at the right end.

The results of the calculation are shown in Fig. 11. The calculated quantity, T^4 , is plotted at zone centers.

Fig. 10. Computational grid to test T^4 diffusion.

Fig. 11. Comparison of the finite difference calculation with a theoretical calculation for the diffusion of T^4 .

It can be seen that all points of the calculated results fall on the straight lines of the theoretical solution, $T^* = t - X$.

A test of the accuracy for solving the thermal diffusion equations in two space dimensions and time can be obtained by imposing the first Fourier harmonic to the general solution of the heat equations on a unit square, as an initial condition. The theoretical result for this initial condition is an exponential decay in temperature T as time t progresses. Material properties are assumed such that the energy equation, Eq. (7), becomes:

$$\frac{dT}{dt} = \nabla^2 T$$

A calculation was done using a unit square with a 20×20 grid. Initial conditions:

$$T(X, Y, 0) = \sin \pi X \sin \pi Y.$$

Boundary conditions on all four boundaries: $T = 0$ for all time.

The theoretical solution to this problem is

$$T(X, Y, t) = \sin \pi X \sin \pi Y e^{-2\pi^2 t}$$

Figure 12 a, b, c shows temperature distribution for various times. Figure 12 d compares the calculated and theoretical solutions.

A check of the program for calculating thermal diffusion with different grids is shown in Fig. 13. The initial conditions are $T = 200^\circ$ for the spherical region and $T = 100^\circ$ for the region outside the sphere. Reflection boundary conditions are used for all four boundaries. It is seen that the calculated temperature contours at $t = 1.02 \mu s$ are essentially the same.

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interfaces. Check problems were done by Jonathon Katz and Paul Grantham. The assistance of R. Barton and R. Harding in locating errors in the program is gratefully acknowledged.

Fig. 12. Calculation of thermal diffusion in two space dimensions and time.

a, b, and c) Temperature distributions at $t = 0$ and at later times

d) Comparisons between the finite difference calculation (points) and the theoretical solution (curves).

Temperature is shown as a function of distance (X) along a diagonal of the square.

Fig. 13. Calculated temperature contours at $t = 1.02 \mu s$ for three calculational grids.

- a) Undistorted grid
- b) Distorted grid
- c) Grid with slide line.

References

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APPENDIX A

MAGNETIC FLUX CALCULATION

Calculation Scheme for k-sweep

First we calculate $\nabla \times \vec{H}$ to be used for the flux change in the k-direction. Subscripts in the equations refer to nodes and zones indicated in Fig. A-1.

$$\nabla \times \vec{H} = \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right) \vec{i} - \frac{\partial H}{\partial X} \vec{j}$$

Note: Any quantity shown without a time index is centered at time $t^{n+1/2}$.

$$a_1 \frac{\partial H}{\partial Y} \Big|_1^k = + \frac{1}{2A_1^{n+1/2}} \left[H_{\odot}^{n+1} (X_2 - X_4) + H_{\odot}^n (X_4 - X_{12}) + H_{\odot}^n (X_{12} - X_5) + H_{\odot}^{n+1} (X_5 - X_2) \right]$$

$$\frac{H}{Y} \Big|_1^k = + \frac{(HA)_{\odot}^{n+1} + (HA)_{\odot}^n + (HA)_{\odot}^n + (HA)_{\odot}^{n+1}}{4\bar{Y}_1 A_1^{n+1/2}}$$

$$\frac{\partial H}{\partial X} \Big|_1^k = - \frac{1}{2A_1^{n+1/2}} \left[H_{\odot}^{n+1} (Y_2 - Y_4) + H_{\odot}^n (Y_4 - Y_{12}) + H_{\odot}^n (Y_{12} - Y_5) + H_{\odot}^{n+1} (Y_5 - Y_2) \right]$$

$$a_1^k \equiv \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right)_1^k; b_1^k \equiv - \left(\frac{\partial H}{\partial X} \right)_1^k$$

$$A_1^{n+1/2} = \frac{1}{4} \left(A_{\odot}^{n+1/2} + A_{\odot}^{n+1/2} + A_{\odot}^{n+1/2} + A_{\odot}^{n+1/2} \right); \bar{Y}_1 = \frac{1}{4} \left(Y_{\odot}^{n+1/2} + Y_{\odot}^{n+1/2} + Y_{\odot}^{n+1/2} + Y_{\odot}^{n+1/2} \right)$$

$Y_{\odot}^{n+1/2}$, etc., as calculated in HEMP. (If k is the X-axis, $Y_{\odot}$, $Y_{\odot}$, $A_{\odot}$, and $A_{\odot}$ are zero.)

Fig. A-1. Calculational grid for k-sweep. Plain integers designate a quantity at a node (j-k line intersection). Integers in circles designate a quantity in the zone center.

$$\begin{aligned}
 & \left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} = \varepsilon_{\lambda} \\
 & \left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} = \lambda^{\otimes_{n+1/2}} \\
 & \left(\frac{\partial \lambda}{\partial \lambda} \right) - \equiv \left(\frac{\lambda}{\lambda} + \frac{\lambda}{\lambda} \right) \equiv \varepsilon_{\lambda} \\
 & \left[\left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} \right] \frac{\lambda^{\otimes_{n+1/2}}}{1} = \left(\frac{\partial \lambda}{\partial \lambda} \right) \\
 & \frac{\lambda^{\otimes_{n+1/2}}}{\left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right)} + = \left(\frac{\lambda}{\lambda} \right) \\
 & \left[\left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} \right] \frac{\lambda^{\otimes_{n+1/2}}}{1} = \left(\frac{\partial \lambda}{\partial \lambda} \right) \\
 & \left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} = \lambda^{\otimes_{n+1/2}} \\
 & \left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} = \lambda^{\otimes_{n+1/2}} \\
 & \left(\frac{\partial \lambda}{\partial \lambda} \right) - \equiv \left(\frac{\lambda}{\lambda} + \frac{\lambda}{\lambda} \right) \equiv \varepsilon_{\lambda} \\
 & \left[\left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} \right] \frac{\lambda^{\otimes_{n+1/2}}}{1} = \left(\frac{\partial \lambda}{\partial \lambda} \right) \\
 & \frac{\lambda^{\otimes_{n+1/2}}}{\left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right)} + = \left(\frac{\lambda}{\lambda} \right) \\
 & \left[\left(\lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} + \lambda^{\otimes_{n+1/2}} \right) \frac{1}{1} \right] \frac{\lambda^{\otimes_{n+1/2}}}{1} = \left(\frac{\partial \lambda}{\partial \lambda} \right)
 \end{aligned}$$

$$d: \left. \frac{\partial H}{\partial Y} \right|_4^k = + \frac{1}{2A_4^{n+1/2}} \left[H_{\textcircled{8}}^{n+1} (X_3 - X_{10}) + H_{\textcircled{2}}^n (X_{10} - X_{11}) + H_{\textcircled{6}}^n (X_{11} - X_1) + H_{\textcircled{3}}^{n+1} (X_1 - X_3) \right]$$

$$\left. \frac{H}{Y} \right|_4^k = + \frac{(HA)_{\textcircled{8}}^{n+1} + (HA)_{\textcircled{2}}^n + (HA)_{\textcircled{6}}^n + (HA)_{\textcircled{3}}^{n+1}}{4\bar{Y}_4 A_4^{n+1/2}}$$

$$\left. \frac{\partial H}{\partial X} \right|_4^k = - \frac{1}{2A_4^{n+1/2}} \left[H_{\textcircled{8}}^{n+1} (Y_3 - Y_{10}) + H_{\textcircled{2}}^n (Y_{10} - Y_{11}) + H_{\textcircled{6}}^n (Y_{11} - Y_1) + H_{\textcircled{3}}^{n+1} (Y_1 - Y_3) \right]$$

$$a_4^k \equiv \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right)_4^k ; b_4^k \equiv - \left(\frac{\partial H}{\partial X} \right)_4^k$$

$$A_4^{n+1/2} = \frac{1}{4} \left(A_{\textcircled{8}}^{n+1/2} + A_{\textcircled{2}}^{n+1/2} + A_{\textcircled{6}}^{n+1/2} + A_{\textcircled{3}}^{n+1/2} \right)$$

$$\bar{Y}_4 = \frac{1}{4} \left(Y_{\textcircled{8}}^{n+1/2} + Y_{\textcircled{2}}^{n+1/2} + Y_{\textcircled{6}}^{n+1/2} + Y_{\textcircled{3}}^{n+1/2} \right)$$

Having computed $\nabla \times \vec{H}$ we now can calculate the change in flux in the k direction, (i.e. across 1-2 and 3-4 in Fig. A-2).

Fig. A-2. Calculational Grid.

$$4\pi\mu_m \frac{d\Phi}{dt} = - \nabla \times \left(\frac{1}{C} \nabla \times \vec{H} \right) A$$

$$\nabla \times \vec{H} = a \vec{i} + b \vec{j}$$

$$a = \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right)$$

$$b = - \left(\frac{\partial H}{\partial X} \right)$$

C = electrical conductivity

A = zone area

$$4\pi\mu_m \frac{d\Phi}{dt} = - A_{\textcircled{5}} \left(\frac{1}{C} \frac{\partial b}{\partial X} - \frac{1}{C} \frac{\partial a}{\partial Y} \right) \vec{\theta}$$

where

$$\frac{1}{C} \frac{\partial b}{\partial X} = -\frac{1}{2A_{\oplus}} \left[\frac{1}{C_{1-2}} (b_1^k + b_2^k) (Y_1 - Y_2) + \frac{1}{C_{3-4}} (b_3^k + b_4^k) (Y_3 - Y_4) \right]$$

and

$$\frac{1}{C} \frac{\partial a}{\partial Y} = +\frac{1}{2A_{\oplus}} \left[\frac{1}{C_{1-2}} (a_1^k + a_2^k) (X_1 - X_2) + \frac{1}{C_{3-4}} (a_3^k + a_4^k) (X_3 - X_4) \right].$$

Therefore

$$4\pi\mu_m \frac{[H^{n+1}A^{n+1} - H^nA^n]_{\oplus}}{\Delta t^{n+1/2}} \Big| ^k = +\frac{1}{2} \left[\frac{1}{C_{1-2}} (b_1^k + b_2^k) (Y_1 - Y_2) + \frac{1}{C_{3-4}} (b_3^k + b_4^k) (Y_3 - Y_4) \right. \\ \left. + \frac{1}{C_{1-2}} (a_1^k + a_2^k) (X_1 - X_2) + \frac{1}{C_{3-4}} (a_3^k + a_4^k) (X_3 - X_4) \right]$$

where

$$C_{1-2} = \frac{\left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}{\left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}$$

and

$$C_{3-4} = \frac{\left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}{\left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}.$$

Rewriting the above, we have:

$$4\pi\mu_m \frac{(H^{n+1}A^{n+1} - H^nA^n)_{\oplus}}{\Delta t^{n+1/2}} \Big| ^k \\ = \frac{1}{4} H_{\oplus}^{n+1} \left[\left(\frac{Y_2 - Y_3}{A_1} + \frac{Y_3 - Y_1}{A_2} \right) \left(\frac{Y_1 - Y_2}{C_{1-2}} \right) + \left(\frac{Y_3 - Y_2}{A_3} + \frac{Y_1 - Y_3}{A_4} \right) \left(\frac{Y_3 - Y_4}{C_{3-4}} \right) \right. \\ \left. + \left(\frac{X_1 - X_2}{A_1} + \frac{X_2 - X_1}{A_2} + \frac{A_{\oplus}^{n+1}}{2Y_1A_1} + \frac{A_{\oplus}^{n+1}}{2Y_2A_2} \right) \left(\frac{X_1 - X_2}{C_{1-2}} \right) \right. \\ \left. + \left(\frac{X_3 - X_4}{A_3} + \frac{X_4 - X_3}{A_4} + \frac{A_{\oplus}^{n+1}}{2Y_3A_3} + \frac{A_{\oplus}^{n+1}}{2Y_4A_4} \right) \left(\frac{X_3 - X_4}{C_{3-4}} \right) \right]$$

$$\begin{aligned}
& + \frac{1}{4} H_{\oplus}^{n+1} \left[\left(\frac{Y_2 - Y_2}{A_1} + \frac{Y_1 - Y_6}{A_3} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{X_1 - X_2}{A_1} + \frac{X_1 - X_6}{A_2} + \frac{A_{\oplus}^{n+1}}{2\bar{Y}_1 A_1} + \frac{A_{\oplus}^{n+1}}{2\bar{Y}_2 A_2} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^{n+1} \left[\left(\frac{Y_2 - Y_2}{A_3} + \frac{Y_1 - Y_{10}}{A_4} \right) \left(\frac{Y_2 - Y_4}{C_{3,4}} \right) + \left(\frac{X_2 - X_4}{A_3} + \frac{X_2 - X_{10}}{A_4} + \frac{A_{\oplus}^{n+1}}{2\bar{Y}_3 A_3} + \frac{A_{\oplus}^{n+1}}{2\bar{Y}_4 A_4} \right) \left(\frac{X_2 - X_4}{C_{3,4}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_4 - Y_{12}}{A_1} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{Y_{11} - Y_1}{A_4} \right) \left(\frac{Y_2 - Y_4}{C_{3,4}} \right) \right. \\
& \quad \left. + \left(\frac{X_4 - X_{12}}{A_1} + \frac{A_{\oplus}^n}{2\bar{Y}_1 A_1} \right) \left(\frac{X_2 - X_2}{C_{1,2}} \right) + \left(\frac{X_{11} - X_1}{A_4} + \frac{A_{\oplus}^n}{2\bar{Y}_4 A_4} \right) \left(\frac{X_2 - X_2}{C_{3,4}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_{10} - Y_{11}}{A_4} \right) \left(\frac{Y_2 - Y_4}{C_{3,4}} \right) + \left(\frac{X_{10} - X_{11}}{A_4} + \frac{A_{\oplus}^n}{2\bar{Y}_4 A_4} \right) \left(\frac{X_2 - X_4}{C_{3,4}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_{12} - Y_2}{A_1} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{X_{12} - X_2}{A_1} + \frac{A_{\oplus}^n}{2\bar{Y}_1 A_1} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_7 - Y_2}{A_2} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{Y_2 - Y_6}{A_3} \right) \left(\frac{Y_2 - Y_4}{C_{3,4}} \right) \right. \\
& \quad \left. + \left(\frac{X_1 - X_2}{A_2} + \frac{A_{\oplus}^n}{2\bar{Y}_2 A_2} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) + \left(\frac{X_2 - X_6}{A_3} + \frac{A_{\oplus}^n}{2\bar{Y}_3 A_3} \right) \left(\frac{X_2 - X_4}{C_{3,4}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_6 - Y_7}{A_2} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{X_6 - X_7}{A_2} + \frac{A_{\oplus}^n}{2\bar{Y}_2 A_2} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_8 - Y_9}{A_3} \right) \left(\frac{Y_2 - Y_4}{C_{3,4}} \right) + \left(\frac{X_8 - X_9}{A_3} + \frac{A_{\oplus}^n}{2\bar{Y}_3 A_3} \right) \left(\frac{X_2 - X_4}{C_{3,4}} \right) \right] \quad (A-1)
\end{aligned}$$

Now we arrange the equations for $H_{\oplus}^{n+1}$ into a group of linear equations. Referring to Fig. A-3, rewrite Eq. (A-1), which expresses the change in flux in the k-direction of zone $j+1/2$, $k+1/2$ (zone $\oplus$), as a linear equation in $H_{\oplus}^{n+1}$.

Fig. A-3. Grid for calculating flux change in zone Φ from contributions in the k-direction.

$$A_j H_{j+1} + B_j H_j + C_j H_{j-1} = D_j$$

$$H_{j+1} = H_{\Phi}^{n+1} = H_{j+1/2, k+3/2}^{n+1}$$

$$H_j = H_{\Phi}^{n+1} = H_{j+1/2, k+1/2}^{n+1}$$

$$H_{j-1} = H_{\Phi}^{n+1} = H_{j+1/2, k-1/2}^{n+1}$$

$$A_j = -\frac{1}{4} \frac{\Delta t^{n+1/2}}{4\pi\mu_m} \left[\left(\frac{Y_3 - Y_4}{A_3} + \frac{Y_3 - Y_{1,0}}{A_4} \right) \left(\frac{Y_3 - Y_4}{C_{3,4}} \right) \right.$$

$$\left. + \left(\frac{X_3 - X_4}{A_3} + \frac{X_3 - X_{1,0}}{A_4} + \frac{A_{\Phi}^{n+1}}{2\bar{V}_3 A_3} + \frac{A_{\Phi}^{n+1}}{2\bar{V}_4 A_4} \right) \left(\frac{X_3 - X_4}{C_{3,4}} \right) \right]$$

$$B_j = A_{\Phi}^{n+1} - \frac{1}{4} \frac{\Delta t^{n+1/2}}{4\pi\mu_m} \left[\left(\frac{Y_3 - Y_4}{A_1} + \frac{Y_3 - Y_1}{A_2} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{Y_4 - Y_2}{A_3} + \frac{Y_1 - Y_2}{A_4} \right) \left(\frac{Y_3 - Y_4}{C_{3,4}} \right) \right.$$

$$\left. + \left(\frac{X_3 - X_4}{A_1} + \frac{X_3 - X_1}{A_2} + \frac{A_{\Phi}^{n+1}}{2\bar{V}_1 A_1} + \frac{A_{\Phi}^{n+1}}{2\bar{V}_2 A_2} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) \right.$$

$$\left. + \left(\frac{X_4 - X_2}{A_3} + \frac{X_1 - X_2}{A_4} + \frac{A_{\Phi}^{n+1}}{2\bar{V}_3 A_3} + \frac{A_{\Phi}^{n+1}}{2\bar{V}_4 A_4} \right) \left(\frac{X_3 - X_4}{C_{3,4}} \right) \right]$$

$$\begin{aligned}
C_i = & -\frac{1}{4} \frac{\Delta_1^{n+1}}{4\pi\mu_m} \left[\left(\frac{Y_3 - Y_2}{A_1} + \frac{Y_1 - Y_6}{A_2} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{X_5 - X_2}{A_1} + \frac{X_1 - X_6}{A_2} \right) \right. \\
& \left. + \frac{A_1^{n+1}}{2\bar{Y}_1 A_1} + \frac{A_2^{n+1}}{2\bar{Y}_2 A_2} \right] \left(\frac{X_1 - X_2}{C_{1,2}} \right) \\
D_i = & (HA)_{\oplus}^n + \frac{1}{4} \frac{\Delta_1^{n+1/2}}{4\pi\mu_m} \left\{ H_{\oplus}^n \left[\left(\frac{Y_4 - Y_{12}}{A_1} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{Y_{13} - Y_1}{A_4} \right) \left(\frac{Y_3 - Y_4}{C_{3,4}} \right) \right. \right. \\
& + \left(\frac{X_4 - X_{12}}{A_1} + \frac{A_{\oplus}^n}{2\bar{Y}_1 A_1} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) + \left(\frac{X_{11} - X_1}{A_4} + \frac{A_{\oplus}^n}{2\bar{Y}_4 A_4} \right) \left(\frac{X_3 - X_4}{C_{3,4}} \right) \\
& + H_{\oplus}^n \left[\left(\frac{Y_{10} - Y_{11}}{A_4} \right) \left(\frac{Y_3 - Y_4}{C_{3,4}} \right) + \left(\frac{X_{10} - X_{11}}{A_4} + \frac{A_{\oplus}^n}{2\bar{Y}_4 A_4} \right) \left(\frac{X_3 - X_4}{C_{3,4}} \right) \right] \\
& + H_{\oplus}^n \left[\left(\frac{Y_{12} - Y_5}{A_1} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{X_{12} - X_5}{A_1} + \frac{A_{\oplus}^n}{2\bar{Y}_1 A_1} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) \right] \\
& + H_{\oplus}^n \left[\left(\frac{Y_7 - Y_3}{A_2} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{Y_2 - Y_8}{A_3} \right) \left(\frac{Y_3 - Y_4}{C_{3,4}} \right) \right. \\
& + \left(\frac{X_7 - X_3}{A_2} + \frac{A_{\oplus}^n}{2\bar{Y}_2 A_2} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) + \left(\frac{X_2 - X_8}{A_3} + \frac{A_{\oplus}^n}{2\bar{Y}_3 A_3} \right) \left(\frac{X_3 - X_4}{C_{3,4}} \right) \\
& + H_{\oplus}^n \left[\left(\frac{Y_6 - Y_7}{A_2} \right) \left(\frac{Y_1 - Y_2}{C_{1,2}} \right) + \left(\frac{X_6 - X_7}{A_2} + \frac{A_{\oplus}^n}{2\bar{Y}_2 A_2} \right) \left(\frac{X_1 - X_2}{C_{1,2}} \right) \right] \\
& \left. + H_{\oplus}^n \left[\left(\frac{Y_8 - Y_9}{A_3} \right) \left(\frac{Y_3 - Y_4}{C_{3,4}} \right) + \left(\frac{X_8 - X_9}{A_3} + \frac{A_{\oplus}^n}{2\bar{Y}_3 A_3} \right) \left(\frac{X_3 - X_4}{C_{3,4}} \right) \right] \right\}
\end{aligned}$$

The method of solution for a set of linear equations of this form is given in Appendix C.

Calculation Scheme for j-sweep.

Fig. A-4. Calculational grid for j-sweep.

Next, referring to Fig. A-4 for the numbering of nodes, zones, and values of H , we execute an analogous calculation of $\nabla \times \vec{H}$ to be used for the flux change in the j -direction.

$$\nabla \times \vec{H} = \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right) \vec{i} - \frac{\partial H}{\partial X} \vec{j}$$

$$a. \quad \frac{\partial H}{\partial Y} \Big|_1^j = + \frac{1}{2A_1^{n+1/2}} \left[H_{10}^{n+1} (X_2 - X_4) + H_{10}^{n+1} (X_4 - X_{12}) + H_{10}^n (X_{12} - X_4) + H_{10}^n (X_4 - X_2) \right]$$

$$\frac{H}{Y} \Big|_1^j = + \frac{(HA)_{10}^{n+1} + (HA)_{10}^{n+1} + (HA)_{10}^n + (HA)_{10}^n}{4\bar{Y}_1 A_1^{n+1/2}}$$

$$\frac{\partial H}{\partial X} \Big|_1^j = - \frac{1}{2A_1^{n+1/2}} \left[H_{10}^{n+1} (Y_2 - Y_4) + H_{10}^{n+1} (Y_4 - Y_{12}) + H_{10}^n (Y_{12} - Y_4) + H_{10}^n (Y_4 - Y_2) \right]$$

$$a_1^j \equiv \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right) \Big|_1^j; \quad b_1^j \equiv - \left(\frac{\partial H}{\partial X} \right) \Big|_1^j$$

$$A_1^{n+1/2} = \frac{1}{4} \left(A_0^{n+1/2} + A_0^{n+1/2} + A_0^{n+1/2} + A_0^{n+1/2} \right)$$

$$Y_1 = \frac{1}{4} \left(Y_0^{n+1/2} + Y_0^{n+1/2} + Y_0^{n+1/2} + Y_0^{n+1/2} \right)$$

$$b. \quad \frac{\partial H}{\partial Y} \Big|_2^j = + \frac{1}{2A_1^{n+1/2}} \left[H_0^{n+1} (X_7 - X_2) + H_0^{n+1} (X_3 - X_1) + H_0^n (X_1 - X_4) + H_0^n (X_4 - X_7) \right]$$

$$\frac{H}{Y} \Big|_2^j = + \frac{(HA)_0^{n+1} + (HA)_0^{n+1} + (HA)_0^n + (HA)_0^n}{4Y_2 A_2^{n+1/2}}$$

$$\frac{\partial H}{\partial X} \Big|_2^j = - \frac{1}{2A_1^{n+1/2}} \left[H_0^{n+1} (Y_7 - Y_2) + H_0^{n+1} (Y_3 - Y_1) + H_0^n (Y_1 - Y_4) + H_0^n (Y_4 - Y_7) \right]$$

$$a_2^j = \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right)_2^j; \quad b_2^j = - \left(\frac{\partial H}{\partial X} \right)_2^j$$

$$A_2^{n+1/2} = \frac{1}{4} \left(A_0^{n+1/2} + A_0^{n+1/2} + A_0^{n+1/2} + A_0^{n+1/2} \right)$$

$$Y = \frac{1}{4} \left(Y_0^{n+1/2} + Y_0^{n+1/2} + Y_0^{n+1/2} + Y_0^{n+1/2} \right)$$

$$c. \quad \frac{\partial H}{\partial Y} \Big|_3^j = + \frac{1}{2A_3^{n+1/2}} \left[H_0^n (X_8 - X_9) + H_0^n (X_9 - X_4) + H_0^{n+1} (X_4 - X_2) + H_0^{n+1} (X_2 - X_8) \right]$$

$$\frac{H}{Y} \Big|_3^j = + \frac{(HA)_0^n + (HA)_0^n + (HA)_0^{n+1} + (HA)_0^{n+1}}{4Y_3 A_3^{n+1/2}}$$

$$\frac{\partial H}{\partial X} \Big|_3^j = - \frac{1}{2A_3^{n+1/2}} \left[H_0^n (Y_8 - Y_9) + H_0^n (Y_9 - Y_4) + H_0^{n+1} (Y_4 - Y_2) + H_0^{n+1} (Y_2 - Y_8) \right]$$

$$a_3^j = \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right)_3^j; \quad b_3^j = - \left(\frac{\partial H}{\partial X} \right)_3^j$$

$$A_2^{n+1/2} = \frac{1}{4} \left(A_{\odot}^{n+1/2} + A_{\ominus}^{n+1/2} + A_{\oplus}^{n+1/2} + A_{\ominus}^{n+1/2} \right)$$

$$Y_3 = \frac{1}{4} \left(Y_{\odot}^{n+1/2} + Y_{\ominus}^{n+1/2} + Y_{\oplus}^{n+1/2} + Y_{\ominus}^{n+1/2} \right)$$

$$d. \quad \left. \frac{\partial H}{\partial Y} \right|_4^j = + \frac{1}{2A_4^{n+1/2}} \left[H_{\odot}^n (X_3 - X_{10}) + H_{\ominus}^n (X_{10} - X_{11}) + H_{\oplus}^{n+1} (X_{11} - X_1) + H_{\ominus}^{n+1} (Y_1 - Y_3) \right]$$

$$\left. \frac{H}{Y} \right|_4^j = + \frac{(HA)_{\odot}^n + (HA)_{\ominus}^n + (HA)_{\oplus}^{n+1} + (HA)_{\ominus}^{n+1}}{4Y_4 A_4^{n+1/2}}$$

$$\left. \frac{\partial H}{\partial X} \right|_4^j = - \frac{1}{2A_4^{n+1/2}} \left[H_{\odot}^n (Y_3 - Y_{10}) + H_{\ominus}^n (Y_{10} - Y_{11}) + H_{\oplus}^{n+1} (Y_{11} - Y_1) + H_{\ominus}^{n+1} (Y_1 - Y_3) \right]$$

$$a_4^j = \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right)_4^j ; \quad b_4^j = - \left(\frac{\partial H}{\partial X} \right)_4^j$$

$$A_4^{n+1/2} = \frac{1}{4} \left(A_{\odot}^{n+1/2} + A_{\ominus}^{n+1/2} + A_{\oplus}^{n+1/2} + A_{\ominus}^{n+1/2} \right)$$

$$Y_4 = \frac{1}{4} \left(Y_{\odot}^{n+1/2} + Y_{\ominus}^{n+1/2} + Y_{\oplus}^{n+1/2} + Y_{\ominus}^{n+1/2} \right)$$

Using this value of $\nabla \times \vec{H}$ we calculate the change in flux in the j-direction, i.e. across 2-3 and 4-1 in Fig. A-5.

$$4\pi\mu_m \frac{d\phi}{dt} = - \nabla \times \left(\frac{1}{C} \nabla \times \vec{H} \right) A$$

$$\nabla \times \vec{H} = a \vec{i} + b \vec{j}$$

Fig. A-5. Calculation grid.

$$a = \left(\frac{\partial H}{\partial Y} + \frac{H}{Y} \right)$$

$$b = - \left(\frac{\partial H}{\partial X} \right)$$

C = electrical conductivity

$$4\pi\mu_m \frac{d\phi}{dt} = -A_0 \left[\frac{1}{C} \frac{\partial b}{\partial X} - \frac{1}{C} \frac{\partial a}{\partial Y} \right] \vec{\phi}$$

where

$$\frac{1}{C} \frac{\partial b}{\partial X} = -\frac{1}{2A_0} \left[\frac{1}{C_{2,3}} (b_2^j + b_3^j) (Y_2 - Y_3) + \frac{1}{C_{4,1}} (b_4^j + b_1^j) (Y_4 - Y_1) \right]$$

and

$$\frac{1}{C} \frac{\partial a}{\partial Y} = +\frac{1}{2A_0} \left[\frac{1}{C_{2,3}} (a_2^j + a_3^j) (X_2 - X_3) + \frac{1}{C_{4,1}} (a_4^j + a_1^j) (X_4 - X_1) \right]$$

Therefore

$$\begin{aligned} \frac{4\pi\mu_m [(HA)^{n+1} - (HA)^n]_0}{\Delta t^{n+1/2}} &= +\frac{1}{2} \left[\frac{1}{C_{2,3}} (b_2^j + b_3^j) (Y_2 - Y_3) + \frac{1}{C_{4,1}} (b_4^j + b_1^j) (Y_4 - Y_1) \right. \\ &\quad \left. + \frac{1}{C_{2,3}} (a_2^j + a_3^j) (X_2 - X_3) + \frac{1}{C_{4,1}} (a_4^j + a_1^j) (X_4 - X_1) \right] \end{aligned}$$

where

$$C_{2,3} = \frac{\left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}{\left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}$$

and

$$C_{4,1} = \frac{\left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(C^n v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}{\left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\oplus} + \left(v^{n+1/2} \frac{M}{\rho_0} \right)_{\ominus}}$$

Rewriting the above gives

$$\begin{aligned} & 4\pi\mu_m \frac{|(HA)^{n+1} - (HA)^n|_{\oplus}}{\Delta_1^{n+1/2}} \Bigg| \\ &= \frac{1}{4} H_{\oplus}^{n+1} \left[\left(\frac{Y_3 - Y_1}{A_2} + \frac{Y_4 - Y_2}{A_3} \right) \left(\frac{Y_2 - Y_3}{C_{2,3}} \right) + \left(\frac{Y_1 - Y_3}{A_4} + \frac{Y_2 - Y_4}{A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4,1}} \right) \right. \\ & \quad + \left(\frac{X_3 - X_1}{A_2} + \frac{X_4 - X_2}{A_3} + \frac{A_{\oplus}^{n+1}}{2Y_3 A_2} + \frac{A_{\oplus}^{n+1}}{2Y_3 A_3} \right) \left(\frac{X_2 - X_3}{C_{2,3}} \right) \\ & \quad + \left. \left(\frac{X_1 - X_3}{A_4} + \frac{X_2 - X_4}{A_1} + \frac{A_{\oplus}^{n+1}}{2Y_1 A_4} + \frac{A_{\oplus}^{n+1}}{2Y_1 A_1} \right) \left(\frac{X_4 - X_1}{C_{4,1}} \right) \right] \\ & \quad + \frac{1}{4} H_{\oplus}^{n+1} \left[\left(\frac{Y_2 - Y_3}{A_2} + \frac{Y_2 - Y_6}{A_3} \right) \left(\frac{Y_2 - Y_3}{C_{2,3}} \right) + \left(\frac{X_7 - X_3}{A_2} + \frac{X_2 - X_6}{A_3} + \frac{A_{\oplus}^{n+1}}{2Y_2 A_2} + \frac{A_{\oplus}^{n+1}}{2Y_3 A_3} \right) \left(\frac{X_2 - X_3}{C_{2,3}} \right) \right] \\ & \quad + \frac{1}{4} H_{\oplus}^{n+1} \left[\left(\frac{Y_{11} - Y_1}{A_4} + \frac{Y_4 - Y_{12}}{A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4,1}} \right) \right. \\ & \quad + \left. \left(\frac{X_{11} - X_1}{A_4} + \frac{X_4 - X_{12}}{A_1} + \frac{A_{\oplus}^{n+1}}{2Y_4 A_4} + \frac{A_{\oplus}^{n+1}}{2Y_1 A_1} \right) \left(\frac{X_4 - X_1}{C_{4,1}} \right) \right] \\ & \quad + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_9 - Y_4}{A_3} \right) \left(\frac{Y_2 - Y_3}{C_{2,3}} \right) + \left(\frac{Y_3 - Y_{10}}{A_4} \right) \left(\frac{Y_4 - Y_1}{C_{4,1}} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{X_9 - X_4}{A_3} + \frac{A_{\oplus}^n}{2\bar{Y}_3 A_3} \right) \left(\frac{X_2 - X_3}{C_{2,3}} \right) + \left(\frac{X_3 - X_{10}}{A_4} + \frac{A_{\oplus}^n}{2\bar{Y}_4 A_4} \right) \left(\frac{X_4 - X_1}{C_{4,1}} \right) \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_1 - Y_6}{A_2} \right) \left(\frac{Y_1 - Y_3}{C_{2,3}} \right) + \left(\frac{Y_5 - Y_2}{A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4,1}} \right) \right. \\
& + \left(\frac{X_1 - X_6}{A_2} + \frac{A_{\oplus}^n}{2\bar{Y}_2 A_2} \right) \left(\frac{X_2 - X_3}{C_{2,3}} \right) + \left(\frac{X_5 - X_2}{A_1} + \frac{A_{\oplus}^n}{2\bar{Y}_1 A_1} \right) \left(\frac{X_4 - X_1}{C_{4,1}} \right) \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_{12} - Y_5}{A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4,1}} \right) + \left(\frac{X_{12} - X_5}{A_1} + \frac{A_{\oplus}^n}{2\bar{Y}_1 A_1} \right) \left(\frac{X_4 - X_1}{C_{4,1}} \right) \right. \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_6 - Y_7}{A_2} \right) \left(\frac{Y_2 - Y_3}{C_{2,3}} \right) + \left(\frac{X_6 - X_7}{A_2} + \frac{A_{\oplus}^n}{2\bar{Y}_2 A_2} \right) \left(\frac{X_2 - X_3}{C_{2,3}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_8 - Y_9}{A_3} \right) \left(\frac{Y_2 - Y_3}{C_{2,3}} \right) + \left(\frac{X_8 - X_9}{A_3} + \frac{A_{\oplus}^n}{2\bar{Y}_3 A_3} \right) \left(\frac{X_2 - X_3}{C_{2,3}} \right) \right] \\
& + \frac{1}{4} H_{\oplus}^n \left[\left(\frac{Y_{10} - Y_{11}}{A_4} \right) \left(\frac{Y_4 - Y_1}{C_{4,1}} \right) + \left(\frac{X_{10} - X_{11}}{A_4} + \frac{A_{\oplus}^n}{2\bar{Y}_4 A_4} \right) \left(\frac{X_4 - X_1}{C_{4,1}} \right) \right]. \tag{A-2}
\end{aligned}$$

Arranging the equation for H^{n+1} into a group of linear equations for a given k-line (see Fig. A-6):

$$H_{i+1} \equiv H_{\oplus}^{n+1} = H_{j+3/2, k+1/2}^{n+1}$$

$$H_i \equiv H_{\oplus}^{n+1} = H_{j+1/2, k+1/2}^{n+1}$$

$$H_{i-1} \equiv H_{\oplus}^{n+1} = H_{j-1/2, k+1/2}^{n+1}$$

Fig. A-6. Grid for calculating flux change in zone ⑥ from contributions in the j -direction.

Rewriting again Eq. (A-2), the change in flux in the j -direction

$$A_1 H_{j+1} + B_1 H_j + C_1 H_{j-1} = D_1$$

where

$$A_1 = -\frac{1}{4} \frac{\Delta t^{n+1/2}}{4\pi\mu_m} \left[\left(\frac{Y_7 - Y_2}{A_2} + \frac{Y_2 - Y_8}{A_3} \right) \left(\frac{Y_2 - Y_3}{C_{2,3}} \right) + \left(\frac{X_7 - X_3}{A_2} + \frac{X_2 - X_8}{A_3} \right) + \frac{A_{\text{⑥}}^{n+1}}{2\bar{Y}_2 A_2} + \frac{A_{\text{⑥}}^{n+1}}{2\bar{Y}_3 A_3} \right] \left(\frac{X_2 - X_3}{C_{2,3}} \right)$$

$$B_1 = A_{\text{⑥}}^{n+1} - \frac{1}{4} \frac{\Delta t^{n+1/2}}{4\pi\mu_m} \left[\left(\frac{Y_3 - Y_1}{A_2} + \frac{Y_4 - Y_2}{A_1} \right) \left(\frac{Y_2 - Y_3}{C_{2,3}} \right) + \left(\frac{Y_1 - Y_3}{A_4} + \frac{Y_2 - Y_4}{A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4,1}} \right) + \left(\frac{X_3 - X_1}{A_2} + \frac{X_4 - X_2}{A_3} + \frac{A_{\text{⑥}}^{n+1}}{2\bar{Y}_2 A_2} + \frac{A_{\text{⑥}}^{n+1}}{2\bar{Y}_3 A_3} \right) \left(\frac{X_2 - X_3}{C_{2,3}} \right) + \left(\frac{X_1 - X_3}{A_4} + \frac{X_2 - X_4}{A_1} + \frac{A_{\text{⑥}}^{n+1}}{2\bar{Y}_4 A_4} + \frac{A_{\text{⑥}}^{n+1}}{2\bar{Y}_1 A_1} \right) \left(\frac{X_4 - X_1}{C_{4,1}} \right) \right]$$

$$C_i = -\frac{1}{4} \frac{\Delta t^{n+1/2}}{4\pi\mu_m} \left[\left(\frac{Y_{11} - Y_1}{A_4} + \frac{Y_4 - Y_{12}}{A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4-1}} \right) + \left(\frac{X_{11} - X_1}{A_4} \right. \right. \\ \left. \left. + \frac{X_4 - X_{12}}{A_1} + \frac{A_{\oplus}^{n+1}}{2\bar{V}_4 A_4} + \frac{A_{\oplus}^{n+1}}{2\bar{V}_1 A_1} \right) \left(\frac{X_4 - X_1}{C_{4-1}} \right) \right],$$

$$D_i = (HA)_{\oplus}^n + \frac{1}{4} \frac{\Delta t^{n+1/2}}{4\pi\mu_m} \left[H_{\oplus}^n \left[\left(\frac{Y_9 - Y_4}{A_3} \right) \left(\frac{Y_2 - Y_3}{C_{2-3}} \right) + \left(\frac{Y_3 - Y_{10}}{A_4} \right) \left(\frac{Y_4 - Y_1}{C_{4-1}} \right) \right. \right. \\ \left. \left. + \left(\frac{X_2 - X_4}{A_3} + \frac{A_{\oplus}^n}{2\bar{V}_3 A_3} \right) \left(\frac{X_2 - X_3}{C_{2-3}} \right) + \left(\frac{X_3 - X_{10}}{A_4} + \frac{A_{\oplus}^n}{2\bar{V}_4 A_4} \right) \left(\frac{X_4 - X_1}{C_{4-1}} \right) \right] \right. \\ \left. + H_{\oplus}^n \left[\left(\frac{Y_1 - Y_6}{A_2} \right) \left(\frac{Y_2 - Y_3}{C_{2-3}} \right) + \left(\frac{Y_3 - Y_2}{A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4-1}} \right) \right. \right. \\ \left. \left. + \left(\frac{X_1 - X_6}{A_2} + \frac{A_{\oplus}^n}{2\bar{V}_2 A_2} \right) \left(\frac{X_2 - X_3}{C_{2-3}} \right) + \left(\frac{X_3 - X_2}{A_1} + \frac{A_{\oplus}^n}{2\bar{V}_1 A_1} \right) \left(\frac{X_4 - X_1}{C_{4-1}} \right) \right] \right] \\ \left. + H_{\oplus}^n \left[\left(\frac{Y_{12} - Y_5}{A_1} \right) \left(\frac{Y_3 - Y_1}{C_{4-1}} \right) + \left(\frac{X_{12} - X_5}{A_1} + \frac{A_{\oplus}^n}{2\bar{V}_1 A_1} \right) \left(\frac{Y_4 - Y_1}{C_{4-1}} \right) \right] \right] \\ \left. + H_{\oplus}^n \left[\left(\frac{Y_6 - Y_7}{A_2} \right) \left(\frac{Y_2 - Y_3}{C_{2-3}} \right) + \left(\frac{X_6 - X_7}{A_2} + \frac{A_{\oplus}^n}{2\bar{V}_2 A_2} \right) \left(\frac{X_2 - X_3}{C_{2-3}} \right) \right] \right] \\ \left. + H_{\oplus}^n \left[\left(\frac{Y_8 - Y_9}{A_3} \right) \left(\frac{Y_2 - Y_3}{C_{2-3}} \right) + \left(\frac{X_8 - X_9}{A_3} + \frac{A_{\oplus}^n}{2\bar{V}_3 A_3} \right) \left(\frac{X_2 - X_3}{C_{2-3}} \right) \right] \right] \\ \left. + H_{\oplus}^n \left[\left(\frac{Y_{10} - Y_{11}}{A_4} \right) \left(\frac{Y_4 - Y_1}{C_{4-1}} \right) + \left(\frac{X_{10} - X_{11}}{A_4} + \frac{A_{\oplus}^n}{2\bar{V}_4 A_4} \right) \left(\frac{X_4 - X_1}{C_{4-1}} \right) \right] \right] \right]$$

APPENDIX II

THERMAL DIFFUSION CALCULATION

Calculation Scheme for k-Sweep.

First we must calculate ∇T to be used for the thermal flux change in the k-direction. Refer to Fig. B-1 for numbering of nodes and zones.

$$\nabla T = \frac{\partial T}{\partial X} \vec{i} + \frac{\partial T}{\partial Y} \vec{j}$$

Fig. 4-1. Calculation scheme for k-sweep.

$$a. \quad \left. \frac{\partial T}{\partial Y} \right|_1^k = \frac{+1}{2\Delta_1^{n+1/2}} \left[T_{\oplus}^{n+1} (X_2 - X_3) + T_{\oplus}^n (X_4 - X_{12}) + T_{\oplus}^n (X_{12} - X_5) + T_{\oplus}^{n+1} (X_5 - X_3) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_1^k = \frac{-1}{2\Delta_1^{n+1/2}} \left[T_{\oplus}^{n+1} (Y_2 - Y_4) + T_{\oplus}^n (Y_4 - Y_{12}) + T_{\oplus}^n (Y_{12} - Y_5) + T_{\oplus}^{n+1} (Y_5 - Y_2) \right]$$

$$e_1^k = \left. \frac{\partial T}{\partial X} \right|_1^k ; m_1^k = \left. \frac{\partial T}{\partial Y} \right|_1^k$$

$$b. \quad \left. \frac{\partial T}{\partial Y} \right|_2^k = \frac{+1}{2\Delta_2^{n+1/2}} \left[T_{\oplus}^n (X_7 - X_3) + T_{\oplus}^{n+1} (X_3 - X_1) + T_{\oplus}^{n+1} (X_1 - X_6) + T_{\oplus}^n (X_6 - X_7) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_2^k = \frac{-1}{2\Delta_2^{n+1/2}} \left[T_{\oplus}^n (Y_7 - Y_3) + T_{\oplus}^{n+1} (Y_3 - Y_1) + T_{\oplus}^{n+1} (Y_1 - Y_6) + T_{\oplus}^n (Y_6 - Y_7) \right]$$

$$\ell_2^k = \left. \frac{\partial T}{\partial X} \right|_2^k ; m_2^k = \left. \frac{\partial T}{\partial Y} \right|_2^k$$

$$c. \quad \left. \frac{\partial T}{\partial Y} \right|_3^k = \frac{+1}{2A_3^{n+1/2}} \left[T_{\odot}^n (X_8 - X_9) + T_{\odot}^{n+1} (X_9 - X_4) + T_{\odot}^{n+1} (X_4 - X_2) + T_{\odot}^n (X_2 - X_8) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_3^k = \frac{-1}{2A_3^{n+1/2}} \left[T_{\odot}^n (Y_8 - Y_9) + T_{\odot}^{n+1} (Y_9 - Y_4) + T_{\odot}^{n+1} (Y_4 - Y_2) + T_{\odot}^n (Y_2 - Y_8) \right]$$

$$\ell_3^k = \left. \frac{\partial T}{\partial X} \right|_3^k ; m_3^k = \left. \frac{\partial T}{\partial Y} \right|_3^k$$

$$d. \quad \left. \frac{\partial T}{\partial Y} \right|_4^k = \frac{+1}{2A_4^{n+1/2}} \left[T_{\odot}^{n+1} (X_3 - X_{10}) + T_{\odot}^n (X_{10} - X_{11}) + T_{\odot}^n (X_{11} - X_1) + T_{\odot}^{n+1} (X_1 - X_3) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_4^k = \frac{-1}{2A_4^{n+1/2}} \left[T_{\odot}^{n+1} (Y_3 - Y_{10}) + T_{\odot}^n (Y_{10} - Y_{11}) + T_{\odot}^n (Y_{11} - Y_1) + T_{\odot}^{n+1} (Y_1 - Y_3) \right]$$

$$\ell_4^k = \left. \frac{\partial T}{\partial X} \right|_4^k ; m_4^k = \left. \frac{\partial T}{\partial Y} \right|_4^k$$

Having VT enables us to calculate the change in internal energy from fluxes in the k-direction (i.e. across 1-2 and 3-4 in Fig. B-2).

$$\ell = \frac{\partial T}{\partial X}$$

$$m = \frac{\partial T}{\partial Y}$$

$$\Lambda = K + \left(\frac{4}{3} a_{R^c} \right) T^3 \lambda$$

Fig. B-2. Computational grid.

$$\begin{aligned}
 \text{a. } & \left\{ \begin{aligned} dT \left(\frac{\partial E}{\partial T} \right) \bigg|_V &= V \nabla \cdot (\Lambda \nabla T) dt + W \\ \bullet W &= \frac{1}{2} \left[\left(-P + Q + \frac{\partial E}{\partial V} \right) \bigg|_T dV + V \frac{J^3}{C} dt + dZ \right] \end{aligned} \right. \\
 \text{b. } & \left\{ \begin{aligned} \nabla T &= \ell \hat{i} + m \hat{j} \\ \nabla \cdot (\Lambda \nabla T) &= \Lambda \left(\frac{\partial \ell}{\partial X} + \frac{1}{Y} \frac{\partial Ym}{\partial Y} \right) \end{aligned} \right. \\
 \text{c. } & \left\{ \begin{aligned} \Lambda \frac{\partial \ell}{\partial X} &= -\frac{1}{2A_{\oplus}} \left[(\Lambda)_{1-2} (\ell_1^k + \ell_2^k) (Y_1 - Y_2) + (\Lambda)_{3-4} (\ell_3^k + \ell_4^k) (Y_3 - Y_4) \right] \\ \frac{\Lambda_1}{Y} \frac{\partial Ym}{\partial Y} &= \frac{1}{A_{\oplus} (Y_1 + Y_2 + Y_3 + Y_4)} \left[(\Lambda)_{1-2} (m_1^k + m_2^k) (X_1 - X_2) (Y_1 + Y_2) \right. \\ &\quad \left. + (\Lambda)_{3-4} (m_3^k + m_4^k) (X_3 - X_4) (Y_3 + Y_4) \right] \end{aligned} \right.
 \end{aligned}$$

Rewriting Eq. (B-1):

$$\begin{aligned}
 \text{a. } & (T^{n+1} - T^n)_{\oplus} \left(\frac{\partial E}{\partial T} \right) \bigg|_V_{\oplus}^n \\
 &= \frac{\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} V_{\oplus}^{n+1/2} \left\{ T_{\oplus}^{n+1} \left[\left(\frac{Y_2 - Y_4}{A_1} + \frac{Y_3 - Y_1}{A_2} \right) (Y_1 - Y_2) (\Lambda)_{1-2}^n \right. \right. \\
 &\quad \left. \left. + \left(\frac{Y_4 - Y_2}{A_3} + \frac{Y_1 - Y_3}{A_4} \right) (Y_3 - Y_4) (\Lambda)_{3-4}^n \right. \right. \\
 &\quad \left. \left. + 2 \left(\frac{X_2 - X_4}{A_1} + \frac{X_3 - X_1}{A_2} \right) (X_1 - X_2) \left(\frac{Y_1 + Y_2}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{1-2}^n \right. \right. \\
 &\quad \left. \left. + 2 \left(\frac{X_4 - X_2}{A_3} + \frac{X_1 - X_3}{A_4} \right) (X_3 - X_4) \left(\frac{Y_3 + Y_4}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{3-4}^n \right] \right\}
 \end{aligned}$$

*NOTE: The factor 1/2 is because we are going to calculate the thermal diffusion energy change by adding the flux changes from two directions to obtain the total change; hence, the W terms will appear twice.

$$+ T_{\odot}^n \left[\left(\frac{Y_3 - Y_9}{A_3} \right) (Y_1 - Y_4) (\Lambda)_{3-4} + \frac{2(X_3 - X_4)}{A_3} \left(\frac{X_3 - X_4}{(Y_1 + Y_2 + Y_3 + Y_4)} \right) (\Lambda)_{3-4}^n \right] \Bigg\} + W_{\odot}^n. \quad (B-2)$$

b. Where:

$$W_{\odot}^n = \frac{1}{2} \left\{ - \left[p^n + q^{n+1/2} + \left(\frac{\partial E}{\partial V} \right)_T^n \right]_{\odot} [V^{n+1} - V^n]_{\odot} + V_{\odot}^{n+1/2} \left(\frac{j^2}{c} \right)_{\odot} \Delta t^{n+1/2} \right\}$$

See Eq. (17) for J^2/C .

Referring to Fig. B-3 we see that for a given j-line:

$$T_{i+1} \equiv T_{\odot}^{n+1} = T_{j+1/2, k+3/2}^{n+1}$$

$$T_i \equiv T_{\odot}^{n+1} = T_{j+1/2, k+1/2}^{n+1}$$

$$T_{i-1} \equiv T_{\odot}^{n+1} = T_{j+1/2, k-1/2}^{n+1}$$

Fig. B-3. Grid for calculating temperature diffusion from contributions in the k-direction.

Rewriting again Eq. (B-2):

$$A_i T_{i+1} + B_i T_i + C_i T_{i-1} = D_i$$

where:

$$A_i = \frac{-\Delta t^{n+1/2}}{4A_{\odot}^{n+1/2}} V_{\odot}^{n+1/2} \left[\left(\frac{Y_9 - Y_4}{A_3} + \frac{Y_3 - Y_{10}}{A_4} \right) (Y_3 - Y_4) (\Lambda)_{3-4}^n + 2 \left(\frac{X_9 - X_4}{A_3} + \frac{X_3 - X_{10}}{A_4} \right) \left(\frac{X_3 - X_4}{(Y_1 + Y_2 + Y_3 + Y_4)} \right) (\Lambda)_{3-4}^n \right]$$

$$\begin{aligned}
B_1 = & \left(\frac{\partial E}{\partial T} \right)_V \Big|_{\oplus}^n \frac{-\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} v_{\oplus}^{n+1/2} \left[\left(\frac{Y_2 - Y_4}{A_1} + \frac{Y_3 - Y_1}{A_2} \right) (Y_1 - Y_2) (\Lambda)_{1-2}^n \right. \\
& + \left(\frac{Y_4 - Y_2}{A_3} + \frac{Y_1 - Y_3}{A_4} \right) (Y_3 - Y_4) (\Lambda)_{3-4}^n \\
& + 2 \left(\frac{X_2 - X_4}{A_1} + \frac{X_3 - X_1}{A_2} \right) \left(\frac{(X_1 - X_2)(Y_1 + Y_2)}{(Y_1 + Y_2) + (Y_3 + Y_4)} \right) (\Lambda)_{1-2}^n \\
& \left. + 2 \left(\frac{X_4 - X_2}{A_3} + \frac{X_1 - X_3}{A_4} \right) \left(\frac{(X_3 - X_4)(Y_3 + Y_4)}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{3-4}^n \right],
\end{aligned}$$

$$\begin{aligned}
C_1 = & \frac{-\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} v_{\oplus}^{n+1/2} \left[\left(\frac{Y_5 - Y_2}{A_1} + \frac{Y_1 - Y_6}{A_2} \right) (Y_1 - Y_2) (\Lambda)_{1-2}^n \right. \\
& \left. + 2 \left(\frac{X_5 - X_2}{A_1} + \frac{X_1 - X_6}{A_2} \right) (X_1 - X_2) \left(\frac{Y_1 + Y_2}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{1-2}^n \right],
\end{aligned}$$

$$\begin{aligned}
D_1 = & T_{\oplus}^n \left(\frac{\partial E}{\partial T} \right)_V \Big|_{\oplus}^n + \frac{\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} v_{\oplus}^{n+1/2} \left\{ T_{\oplus}^n \left[\left(\frac{Y_4 - Y_{12}}{A_4} \right) (Y_1 - Y_2) (\Lambda)_{1-2}^n \right. \right. \\
& + \left(\frac{Y_{11} - Y_1}{A_4} \right) (Y_3 - Y_4) (\Lambda)_{3-4}^n \\
& + 2 \left(\frac{X_4 - X_{12}}{A_1} \right) \frac{(X_1 - X_2)(Y_1 + Y_2)}{(Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{1-2}^n + 2 \frac{(X_{11} - X_1)}{A_4} \frac{(X_3 - X_4)(Y_3 + Y_4)}{(Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{3-4}^n \left. \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_{10} - Y_{11}}{A_4} \right) (Y_3 - Y_4) (\Lambda)_{3-4}^n + 2 \left(\frac{X_{10} - X_{11}}{A_4} \right) \frac{(X_3 - X_4)(Y_3 + Y_4)}{(Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{3-4}^n \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_{12} - Y_5}{A_1} \right) (Y_1 - Y_2) (\Lambda)_{1-2}^n + 2 \left(\frac{X_{12} - X_5}{A_1} \right) \frac{(X_1 - X_2)(Y_1 + Y_2)}{(Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{1-2}^n \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_7 - Y_3}{A_2} \right) (Y_1 - Y_2) (\Lambda)_{1-2}^n + 2 \frac{(X_7 - X_3)(X_1 - X_2)(Y_1 + Y_2)}{A_2 (Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{1-2}^n \right. \\
& \left. + \left(\frac{Y_2 - Y_8}{A_3} \right) (Y_3 - Y_4) (\Lambda)_{3-4}^n + 2 \frac{(X_2 - X_8)(X_3 - X_4)(Y_3 + Y_4)}{A_3 (Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{3-4}^n \right]
\end{aligned}$$

$$\begin{aligned}
& + T_{\textcircled{3}}^{\textcircled{n}} \left[\left(\frac{Y_6 - Y_7}{A_2} \right) (Y_1 - Y_2) (\Lambda)_{1-2}^{\textcircled{n}} + \frac{2(X_6 - X_7)(X_1 - X_2)(Y_1 + Y_2)}{A_1(Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{1-2}^{\textcircled{n}} \right] \\
& + T_{\textcircled{3}}^{\textcircled{n}} \left[\left(\frac{Y_8 - Y_9}{A_3} \right) (Y_3 - Y_4) (\Lambda)_{3-4}^{\textcircled{n}} + 2 \frac{(X_8 - X_9)(X_3 - X_4)(Y_3 + Y_4)}{A_3(Y_1 + Y_2 + Y_3 + Y_4)} (\Lambda)_{3-4}^{\textcircled{n}} \right] \Bigg\} \\
& + W_{\textcircled{3}}^{\textcircled{n}},
\end{aligned}$$

and

$$W_{\textcircled{3}}^{\textcircled{n}} = + \frac{1}{2} \left\{ - \left[p^{\textcircled{n}} + q^{\textcircled{n}+1/2} + \left(\frac{\partial E}{\partial V} \right)_T^{\textcircled{n}} \right] \left[v^{\textcircled{n}+1} - v^{\textcircled{n}} \right] + v_{\textcircled{3}}^{\textcircled{n}+1/2} \left(\frac{J^2}{C} \right)_{\textcircled{3}}^{\textcircled{n}} \Delta r^{\textcircled{n}+1/2} + dZ \right\}.$$

(See Ref. 1. for the calculation of dZ .)

To calculate Ohmic heating we expand J^2/C as follows:

$$\left(\frac{J^2}{C} \right)_{\textcircled{3}} = \left[\frac{1}{2} \left(\frac{J^2}{C} \right)_{\textcircled{3}}^k + \left(\frac{J^2}{C} \right)_{\textcircled{3}}^j \right] \quad (\text{B-3})$$

$$\begin{aligned}
\left(\frac{J^2}{C} \right)_{\textcircled{3}}^k = \frac{C_{\textcircled{3}}}{2 \cdot 16\pi^2} & \left[\left(\frac{a_1^2 + a_2^2}{2} + \frac{b_1^2 + b_2^2}{2} \right) \left(\frac{1}{C_{1-2}} \right)^2 \right. \\
& \left. + \left(\frac{a_3^2 + a_4^2}{2} + \frac{b_3^2 + b_4^2}{2} \right) \left(\frac{1}{C_{3-4}} \right)^2 \right]
\end{aligned}$$

$$\begin{aligned}
\left(\frac{J^2}{C} \right)_{\textcircled{3}}^j = \frac{C_{\textcircled{3}}}{2 \cdot 16\pi^2} & \left[\left(\frac{a_2^2 + a_3^2}{2} + \frac{b_2^2 + b_3^2}{2} \right) \left(\frac{1}{C_{2-3}} \right)^2 \right. \\
& \left. + \left(\frac{a_4^2 + a_1^2}{2} + \frac{b_4^2 + b_1^2}{2} \right) \left(\frac{1}{C_{4-1}} \right)^2 \right]
\end{aligned}$$

The superscripts k and j refer to the k and j sweeps, where:

$$\begin{aligned}
a_1^k = & \frac{1}{2A_1} \left[H_{\textcircled{3}}^{\textcircled{n}+1} (X_2 - X_4) + H_{\textcircled{3}}^{\textcircled{n}} (X_4 - X_{1-1}) + H_{\textcircled{3}}^{\textcircled{n}} (X_{1-2} - X_5) + H_{\textcircled{3}}^{\textcircled{n}+1} (X_5 - X_2) \right. \\
& \left. + \frac{(HA)_{\textcircled{3}}^{\textcircled{n}+1} + (HA)_{\textcircled{3}}^{\textcircled{n}} + (HA)_{\textcircled{3}}^{\textcircled{n}} + (HA)_{\textcircled{3}}^{\textcircled{n}+1}}{2\bar{Y}_1} \right]
\end{aligned}$$

$$\begin{aligned}
a_2^k = & \frac{1}{2A_2} \left[H_{\textcircled{3}}^{\textcircled{n}} (X_7 - X_3) + H_{\textcircled{3}}^{\textcircled{n}+1} (X_3 - X_1) + H_{\textcircled{3}}^{\textcircled{n}+1} (X_1 - X_6) + H_{\textcircled{3}}^{\textcircled{n}} (X_6 - X_7) \right. \\
& \left. + \frac{(HA)_{\textcircled{3}}^{\textcircled{n}} + (HA)_{\textcircled{3}}^{\textcircled{n}+1} + (HA)_{\textcircled{3}}^{\textcircled{n}+1} + (HA)_{\textcircled{3}}^{\textcircled{n}}}{2\bar{Y}_2} \right]
\end{aligned}$$

$$a_3^k = + \frac{1}{2A_3} \left[H_{\odot}^n (X_8 - X_9) + H_{\odot}^{n+1} (X_8 - X_4) + H_{\odot}^{n+1} (X_4 - X_2) + H_{\odot}^n (X_2 - X_8) \right. \\ \left. + \frac{(HA)_{\odot}^n + (HA)_{\odot}^{n+1} + (HA)_{\odot}^{n+1} + (HA)_{\odot}^n}{2\bar{Y}_3} \right]$$

$$a_4^k = + \frac{1}{2A_4} \left[H_{\odot}^{n+1} (X_3 - X_{10}) + H_{\odot}^n (X_{10} - X_{11}) + H_{\odot}^n (X_{11} - X_1) + H_{\odot}^{n+1} (X_1 - X_3) \right. \\ \left. + \frac{(HA)_{\odot}^{n+1} + (HA)_{\odot}^n + (HA)_{\odot}^n + (HA)_{\odot}^{n+1}}{2\bar{Y}_4} \right]$$

$$b_1^k = \frac{-1}{2A_1^{n+1/2}} \left[H_{\odot}^{n+1} (Y_2 - Y_4) + H_{\odot}^n (Y_4 - Y_{12}) + H_{\odot}^n (Y_{12} - Y_5) + H_{\odot}^{n+1} (Y_5 - Y_2) \right]$$

$$b_2^k = \frac{-1}{2A_1^{n+1/2}} \left[H_{\odot}^n (Y_7 - Y_3) + H_{\odot}^{n+1} (Y_3 - Y_1) + H_{\odot}^{n+1} (Y_1 - Y_6) + H_{\odot}^n (Y_6 - Y_7) \right]$$

$$b_3^k = \frac{-1}{2A_3^{n+1/2}} \left[H_{\odot}^n (Y_8 - Y_9) + H_{\odot}^{n+1} (Y_9 - Y_4) + H_{\odot}^{n+1} (Y_4 - Y_2) + H_{\odot}^n (Y_2 - Y_8) \right]$$

$$b_4^k = \frac{-1}{2A_4^{n+1/2}} \left[H_{\odot}^{n+1} (Y_3 - Y_{10}) + H_{\odot}^n (Y_{10} - Y_{11}) + H_{\odot}^n (Y_{11} - Y_1) + H_{\odot}^{n+1} (Y_1 - Y_3) \right]$$

$$a_1^j = + \frac{1}{2A_1^{n+1/2}} \left[H_{\odot}^{n+1} (X_2 - X_4) + H_{\odot}^{n+1} (X_4 - X_{12}) + H_{\odot}^n (X_{12} - X_5) + H_{\odot}^n (X_5 - X_2) \right. \\ \left. + \frac{(HA)_{\odot}^{n+1} + (HA)_{\odot}^{n+1} + (HA)_{\odot}^n + (HA)_{\odot}^n}{2\bar{Y}_1} \right]$$

$$a_2^j = + \frac{1}{2A_2^{n+1/2}} \left[H_{\odot}^{n+1} (X_7 - X_3) + H_{\odot}^{n+1} (X_3 - X_1) + H_{\odot}^n (X_1 - X_6) + H_{\odot}^n (X_6 - X_7) \right. \\ \left. + \frac{(HA)_{\odot}^{n+1} + (HA)_{\odot}^{n+1} + (HA)_{\odot}^n + (HA)_{\odot}^n}{2\bar{Y}_2} \right]$$

$$a_3^j = + \frac{1}{2A_3^{n+1/2}} \left[H_{\odot}^n (X_8 - X_9) + H_{\odot}^n (X_9 - X_4) + H_{\odot}^{n+1} (X_4 - X_2) + H_{\odot}^{n+1} (X_2 - X_8) \right. \\ \left. + \frac{(HA)_{\odot}^n + (HA)_{\odot}^n + (HA)_{\odot}^{n+1} + (HA)_{\odot}^{n+1}}{2\bar{Y}_3} \right]$$

$$a_4^j = + \frac{1}{2A_4^{n+1/2}} \left[H_{\odot}^n (X_3 - X_{10}) + H_{\odot}^n (X_{10} - X_{11}) + H_{\odot}^{n+1} (X_{11} - X_1) + H_{\odot}^{n+1} (X_1 - X_3) \right. \\ \left. + \frac{(HA)_{\odot}^n + (HA)_{\odot}^n + (HA)_{\odot}^{n+1} + (HA)_{\odot}^{n+1}}{2\bar{Y}_4} \right]$$

$$b_1^j = - \frac{1}{2A_1^{n+1/2}} \left[H_{\odot}^{n+1} (Y_2 - Y_4) + H_{\odot}^{n+1} (Y_4 - Y_{12}) + H_{\odot}^n (Y_{12} - Y_5) + H_{\odot}^n (Y_5 - Y_2) \right]$$

$$b_2^j = - \frac{1}{2A_2^{n+1/2}} \left[H_{\odot}^{n+1} (Y_7 - Y_3) + H_{\odot}^{n+1} (Y_3 - Y_1) + H_{\odot}^n (Y_1 - Y_6) + H_{\odot}^n (Y_6 - Y_7) \right]$$

$$b_3^j = - \frac{1}{2A_3^{n+1/2}} \left[H_{\odot}^n (Y_8 - Y_9) + H_{\odot}^n (Y_9 - Y_4) + H_{\odot}^{n+1} (Y_4 - Y_2) + H_{\odot}^{n+1} (Y_2 - Y_8) \right]$$

$$b_4^j = - \frac{1}{2A_4^{n+1/2}} \left[H_{\odot}^n (Y_3 - Y_{10}) + H_{\odot}^n (Y_{10} - Y_{11}) + H_{\odot}^{n+1} (Y_{11} - Y_1) + H_{\odot}^{n+1} (Y_1 - Y_3) \right]$$

Calculation Scheme for j-Sweep.

Fig. B-4. Calculational grid for j-sweep.

Calculation of ∇T to be used for the thermal flux change in the j-direction proceeds as follows. See Fig. B-4 for time and position number scheme.

$$\nabla T = \frac{\partial T}{\partial X} \vec{i} + \frac{\partial T}{\partial Y} \vec{j}$$

$$a. \quad \left. \frac{\partial T}{\partial Y} \right|_1^j = + \frac{1}{2A_1^{n+1/2}} \left[T_{\text{⑤}}^{n+1} (X_2 - X_4) + T_{\text{⑥}}^{n+1} (X_4 - X_{12}) + T_{\text{⑩}}^n (X_{12} - X_5) + T_{\text{②}}^n (X_5 - X_2) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_1^j = - \frac{1}{2A_1^{n+1/2}} \left[T_{\text{⑤}}^{n+1} (Y_2 - Y_4) + T_{\text{⑥}}^{n+1} (Y_4 - Y_{12}) + T_{\text{⑩}}^n (Y_{12} - Y_5) + T_{\text{②}}^n (Y_5 - Y_2) \right]$$

$$\rho_1^j = \left. \frac{\partial T}{\partial X} \right|_1^j ; m_1^j = \left. \frac{\partial T}{\partial Y} \right|_1^j$$

$$b. \quad \left. \frac{\partial T}{\partial Y} \right|_1^j = - \frac{1}{2A_2^{n+1/2}} \left[T_{\text{⑤}}^{n+1} (X_7 - X_3) + T_{\text{⑥}}^{n+1} (X_3 - X_1) + T_{\text{⑩}}^n (X_1 - X_6) + T_{\text{②}}^n (X_6 - X_7) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_2^j = - \frac{1}{2A_2^{n+1/2}} \left[T_{\text{⑤}}^{n+1} (Y_7 - Y_3) + T_{\text{⑥}}^{n+1} (Y_3 - Y_1) + T_{\text{⑩}}^n (Y_1 - Y_6) + T_{\text{②}}^n (Y_6 - Y_7) \right]$$

$$\rho_2^j = \left. \frac{\partial T}{\partial X} \right|_2^j ; m_2^j = \left. \frac{\partial T}{\partial Y} \right|_2^j$$

$$c. \quad \left. \frac{\partial T}{\partial Y} \right|_3^j = - \frac{1}{2A_3^{n+1/2}} \left[T_{\text{⑤}}^n (X_8 - X_9) + T_{\text{⑥}}^n (X_9 - X_4) + T_{\text{⑩}}^{n+1} (X_4 - X_2) + T_{\text{②}}^{n+1} (X_2 - X_8) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_3^j = - \frac{1}{2A_3^{n+1/2}} \left[T_{\text{⑤}}^n (Y_8 - Y_9) + T_{\text{⑥}}^n (Y_9 - Y_4) + T_{\text{⑩}}^{n+1} (Y_4 - Y_2) + T_{\text{②}}^{n+1} (Y_2 - Y_8) \right]$$

$$\rho_3^j = \left. \frac{\partial T}{\partial X} \right|_3^j ; m_3^j = \left. \frac{\partial T}{\partial Y} \right|_3^j$$

$$d. \quad \left. \frac{\partial T}{\partial Y} \right|_4^j = + \frac{1}{2A_4^{n+1/2}} \left[T_{\text{⑤}}^n (X_3 - X_{10}) + T_{\text{⑦}}^n (X_{10} - X_{11}) + T_{\text{⑥}}^{n+1} (X_{11} - X_1) + T_{\text{⑧}}^{n+1} (X_1 - X_3) \right]$$

$$\left. \frac{\partial T}{\partial X} \right|_4^j = \frac{-1}{2A_4^{n+1/2}} \left[T_{\text{⑥}}^n (Y_3 - Y_{10}) + T_{\text{⑦}}^n (Y_{10} - Y_{11}) + T_{\text{⑥}}^{n+1} (Y_{11} - Y_1) + T_{\text{⑧}}^{n+1} (Y_1 - Y_3) \right]$$

$$\ell_4^j = \left. \frac{\partial T}{\partial X} \right|_4^j ; \quad m_4^j = \left. \frac{\partial T}{\partial Y} \right|_4^j$$

With ∇T from the above we can calculate the change in internal energy from fluxes in the j-direction (i.e. across 2-3 and 4-1 in Fig. B-5).

Fig. B-5. Computational grid.

$$dT \left(\left. \frac{\partial E}{\partial T} \right|_V \right) = \nabla \nabla \cdot (\Lambda \nabla T) dt + W \quad (\text{B-4})$$

$$W = \frac{1}{2} \left[- \left(P + q + \left. \frac{\partial E}{\partial V} \right|_T \right) dV + V \frac{J^2}{C} dt + dZ \right]$$

But

$$\nabla T = \ell \vec{i} + m \vec{j}$$

and

*NOTE: The factor (1/2) is present because the total energy change by thermal diffusion will be calculated by adding the flux changes from two directions. Hence the W terms will appear twice.

$$\nabla \cdot (\Lambda \nabla T) = \Lambda \left(\frac{\partial \ell}{\partial X} + \frac{1}{Y} \frac{\partial Y m}{\partial Y} \right)$$

where

$$\ell = \frac{\partial T}{\partial X}$$

$$m = \frac{\partial T}{\partial Y}$$

$$\Lambda = K + \left(\frac{4}{3} a_R c \right) T^3 \lambda.$$

Therefore

$$\Lambda \frac{\partial \ell}{\partial X} = - \frac{1}{2A_{\oplus}} \left[(\Lambda)_{2-3}^n (\ell_2^i + \ell_3^i) (Y_2 - Y_3) + (\Lambda)_{4-1}^n (\ell_4^i + \ell_1^i) (Y_4 - Y_1) \right]$$

and

$$\Lambda \frac{1}{Y} \frac{\partial Y m}{\partial Y} = \frac{1}{A_{\oplus} (Y_1 + Y_2 + Y_3 + Y_4)} \left\{ (\Lambda)_{2-3}^n (m_2^i + m_3^i) (X_2 - X_3) (Y_2 + Y_3) \right. \\ \left. + (\Lambda)_{4-1}^n (m_4^i + m_1^i) (X_4 - X_1) (Y_4 + Y_1) \right\}.$$

Rewriting Eq. (B-4)

$$\begin{aligned} & (T^{n+1} - T^n)_{\oplus} \left(\frac{\partial E}{\partial T} \right)_V \Big|_{\oplus}^n \\ &= \frac{\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} V_{\oplus}^{n+1/2} \left\{ T_{\oplus}^{n+1} \left[\left(\frac{Y_3 - Y_1}{A_2} + \frac{Y_4 - Y_2}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n \right. \right. \\ & \quad \left. \left. + \left(\frac{Y_1 - Y_3}{A_4} + \frac{Y_2 - Y_4}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \right. \\ & \quad \left. \left. + 2 \left(\frac{X_3 - X_1}{A_2} + \frac{X_4 - X_2}{A_3} \right) (X_2 - X_3) \left(\frac{Y_1 + Y_2 + Y_3 + Y_4}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right. \right. \\ & \quad \left. \left. + 2 \left(\frac{X_1 - X_3}{A_4} + \frac{X_2 - X_4}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 - Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + T_{\oplus}^{n+1} \left[\left(\frac{Y_2 - Y_3}{A_2} + \frac{Y_2 - Y_6}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n \right. \\
& + 2 \left(\frac{X_7 - X_3}{A_2} + \frac{X_2 - X_6}{A_3} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \Big] \\
& + T_{\oplus}^{n+1} \left[\left(\frac{Y_{11} - Y_1}{A_4} + \frac{Y_4 - Y_{12}}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \\
& + 2 \left(\frac{X_{11} - X_1}{A_4} + \frac{X_4 - X_{12}}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \\
& + T_{\oplus}^n \left[\left(\frac{Y_1 - Y_6}{A_2} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n + \left(\frac{Y_5 - Y_2}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \\
& + 2 \left(\frac{X_1 - X_6}{A_2} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \\
& + 2 \left(\frac{X_5 - X_2}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \Big] \\
& + T_{\oplus}^n \left[\left(\frac{Y_9 - Y_4}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n + \left(\frac{Y_3 - Y_{10}}{A_4} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \\
& + 2 \left(\frac{X_9 - X_4}{A_3} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n + 2 \left(\frac{X_3 - X_{10}}{A_4} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \Big] \\
& + T_{\oplus}^n \left[\left(\frac{Y_{12} - Y_5}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n + 2 \left(\frac{X_{12} - X_5}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_6 - Y_7}{A_2} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n + 2 \left(\frac{X_6 - X_7}{A_2} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_8 - Y_9}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n + 2 \left(\frac{X_8 - X_9}{A_3} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right]
\end{aligned}$$

$$+ T_{\textcircled{2}}^n \left[\left(\frac{Y_{10} - Y_{11}}{A_4} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n + 2 \left(\frac{X_{10} - X_{11}}{A_4} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \right] \\ + W_{\textcircled{3}}^n, \quad (\text{B-5})$$

where:

$$W_{\textcircled{3}}^n = \frac{1}{2} \left\{ - \left[p^n + q^{n+1/2} + \left(\frac{\partial E}{\partial V} \right)_T^n \right]_{\textcircled{3}} (V^{n+1} - V^n)_{\textcircled{3}} \right. \\ \left. + V_{\textcircled{3}}^{n+1/2} \left(\frac{J^2}{C} \right)_{\textcircled{3}} \Delta t^{n+1/2} \right\}.$$

See Eq. (B-3) for J^2/C .

For a given k-line, as shown in Fig. B-6:

Fig. B-6. Grid for calculating temperature flux changes in zone ⓐ from contributions in the j-direction.

$$T_{i+1} \equiv T_{\textcircled{6}}^{n+1} = T_{j+3/2, k+1/2}^{n+1}$$

$$T_i \equiv T_{\textcircled{5}}^{n+1} = T_{j+1/2, k+1/2}^{n+1}$$

$$T_{i-1} \equiv T_{\textcircled{4}}^{n+1} = T_{j-1/2, k+1/2}^{n+1}$$

Rewriting again Eq. (B-5):

$$A_i T_{i+1} + B_i T_i + C_i T_{i-1} = D_i,$$

where

$$A_1 = -\frac{\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} v_{\oplus}^{n+1/2} \left[\left(\frac{Y_7 - Y_3}{A_2} + \frac{Y_2 - Y_8}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n \right. \\ \left. + 2 \left(\frac{X_7 - X_3}{A_2} + \frac{X_2 - X_8}{A_3} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right],$$

$$B_1 = \left(\frac{\partial E}{\partial T} \right)_V \Big|_{\oplus}^n - \frac{\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} v_{\oplus}^{n+1/2} \left[\left(\frac{Y_3 - Y_1}{A_2} + \frac{Y_4 - Y_2}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n \right. \\ \left. + \left(\frac{Y_1 - Y_3}{A_4} + \frac{Y_2 - Y_4}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \\ \left. + 2 \left(\frac{X_3 - X_1}{A_2} + \frac{X_4 - X_2}{A_3} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right. \\ \left. + 2 \left(\frac{X_1 - X_3}{A_4} + \frac{X_2 - X_4}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \right]$$

$$C_1 = -\frac{\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} v_{\oplus}^{n+1/2} \left[\left(\frac{Y_{11} - Y_1}{A_4} + \frac{Y_4 - Y_{12}}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \\ \left. + 2 \left(\frac{X_{11} - X_1}{A_4} + \frac{X_4 - X_{12}}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \right]$$

and

$$D_1 = T_{\oplus}^0 \left(\frac{\partial E}{\partial T} \right)_V \Big|_{\oplus}^n + \frac{\Delta t^{n+1/2}}{4A_{\oplus}^{n+1/2}} v_{\oplus}^{n+1/2} \left[T_{\oplus}^0 \left[\left(\frac{Y_1 - Y_6}{A_2} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n \right. \right. \\ \left. \left. + \left(\frac{Y_5 - Y_2}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \right. \\ \left. \left. + 2 \left(\frac{X_1 - X_6}{A_2} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right. \right. \\ \left. \left. + 2 \left(\frac{X_5 - X_2}{A_1} \right) (X_4 - Y_1) (\Lambda)_{4-1}^n \right] \right]$$

$$\begin{aligned}
& + 2 \left(\frac{X_5 - X_2}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \Big] \\
& + T_{\oplus}^n \left[\left(\frac{Y_9 - Y_4}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n + \left(\frac{Y_3 - Y_{10}}{A_4} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n \right. \\
& + 2 \left(\frac{X_2 - Y_4}{A_3} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n + 2 \left(\frac{X_3 - Y_{10}}{A_4} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \Big] \\
& + T_{\oplus}^n \left[\left(\frac{Y_{12} - Y_5}{A_1} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n + 2 \left(\frac{X_{12} - X_5}{A_1} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_6 - Y_7}{A_2} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n + 2 \left(\frac{X_6 - X_7}{A_2} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_3 - Y_9}{A_3} \right) (Y_2 - Y_3) (\Lambda)_{2-3}^n + 2 \left(\frac{X_8 - X_9}{A_3} \right) (X_2 - X_3) \left(\frac{Y_2 + Y_3}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{2-3}^n \right] \\
& + T_{\oplus}^n \left[\left(\frac{Y_{10} - Y_{11}}{A_4} \right) (Y_4 - Y_1) (\Lambda)_{4-1}^n + 2 \left(\frac{X_{10} - X_{11}}{A_4} \right) (X_4 - X_1) \left(\frac{Y_4 + Y_1}{Y_1 + Y_2 + Y_3 + Y_4} \right) (\Lambda)_{4-1}^n \right] \Big\} \\
& + W_{\oplus}^n .
\end{aligned}$$

APPENDIX C

BACKWARD SUBSTITUTION METHOD FOR SOLVING A SYSTEM OF LINEAR EQUATIONS OF THE FORM: $A_i H_{i+1} + B_i H_i + C_i H_{i-1} = D_i$

Given:

$$A_i H_{i+1} + B_i H_i + C_i H_{i-1} = D_i ; \quad (C-1)$$

$H_{i \max}$ and $H_{i \min}$ are known quantities.

Define:

$$E_{i+1} H_i + F_{i+1} = H_{i+1} . \quad (C-2)$$

Substitute (C-2) into (C-1):

$$A_i (E_{i+1} H_i + F_{i+1}) + B_i H_i + C_i H_{i-1} = D_i ,$$

or

$$H_i = \frac{D_i - A_i F_{i+1} - C_i H_{i-1}}{A_i E_{i+1} + B_i} . \quad (C-3)$$

Rewrite Eq. (C-2) as

$$H_i = E_i H_{i-1} + F_i . \quad (C-4)$$

Equate coefficients of (C-3) and (C-4):

$$E_i = \frac{-C_i}{A_i E_{i+1} + B_i} ; \quad F_i = \frac{D_i - A_i F_{i+1}}{A_i E_{i+1} + B_i} \quad (C-5)$$

for:

$$i = i_{\max-1} \text{ to } i = i_{\min+1}$$

where

$$E_{i \max} = 0 \text{ and } F_{i \max} = H_{i \max} .$$

Store E_i and F_i , then calculate H_i for $i = i_{\min+1}$ to $i = i_{\max-1}$.

$$H_i = E_i H_{i-1} + F_i .$$

Example:

$$\left. \begin{aligned} A_3 H_4 + B_3 H_3 + C_3 H_2 + 0 + 0 &= D_3 \\ A_2 H_3 + B_2 H_2 + C_2 H_1 + 0 &= D_2 \\ A_1 H_2 + B_1 H_1 + C_1 H_0 &= D_1 \end{aligned} \right\} \quad (C-6)$$

Given: H_4, H_0 and A_i, B_i, C_i for $i = 1 \rightarrow i = 3$.

To calculate H_3, H_2 , and H_1 , define the following recursive equations:

$$\left. \begin{aligned} E_3 H_2 + F_3 &= H_3, \\ E_2 H_1 + F_2 &= H_2, \\ E_1 H_0 + F_1 &= H_1. \end{aligned} \right\} \quad (C-7)$$

Solving Eq. (C-6) in the form of Eq. (C-7) provides the following identifications:

$$\begin{aligned} E_3 &= -\frac{C_3}{B_3} & ; & \quad F_3 = \frac{D_3 - A_3 H_4}{B_3} \\ E_2 &= -\frac{C_2}{A_2 E_3 + B_2} & ; & \quad F_2 = \frac{D_2 - A_2 F_3}{A_2 E_3 + B_2} \\ E_1 &= -\frac{C_1}{A_1 E_2 + B_1} & ; & \quad F_1 = \frac{D_1 - A_1 F_2}{A_1 E_2 + B_1} \end{aligned}$$

or:

$$E_i = -\frac{C_i}{A_i E_{i+1} + B_i} \quad \text{and} \quad F_i = \frac{D_i - A_i F_{i+1}}{A_i E_{i+1} + B_i}.$$

Substitute: E_i and F_i into Eq. (C-7).