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A Soluble Theoretical Model of the Probable
Amount of Material in an Inventory

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Abstract

The concept of Markov chains, borrowed from probability theory, can be used to study various aspects of special nuclear material inventory control that are amenable to probabilistic treatment.

As an illustration, a model of the operation of a storage vault or warehouse is set up which is soluble analytically, and yet is reasonably realistic.

The inventory is assumed to have additions (receipts) and withdrawals (shipments) on a regular basis, and to have a periodic random selection of some of the items in inventory made for assay or testing. As a result of testing, it is assumed that knowledge of the content of individual items has been improved, narrowing the probability of deviation from the nominal value of such tested items. The inventory is thus composed of two populations with distinctly different levels of assurance associated with each.

Using the model and the Markov theory, the time evolution of the probable content and assurance level of the inventory can be followed.

A Soluble Theoretical Model of the Probable Amount of Material in an Inventory

Anybody at all familiar with the operation of a nuclear fuel fabrication plant knows that it is quite presumptuous to set up a mathematical model to represent the complex workings of that plant, even for such a limited purpose as a safeguard's study. Yet, it is the very fact of that complexity which makes attractive some sort of an operations research approach.

In discussions concerning setting up some computer-based operations research calculations, it occurred to us that many of the probabilistically determined processes which one would have to mock up on the computer for an operations research simulation study, are amenable to treatment by what is called in probability theory a Markov chain. To the extent this is the case, it then followed that it would be possible to set up analytic models of processes which would otherwise have to be simulated by Monte Carlo techniques. Even if the process had to be drastically simplified to fit the format of a Markov chain, it might still be worthwhile to do so, in that the analytical Markov calculation would furnish a standard with which to compare the more accurate, but much more complex and troublesome, Monte Carlo calculation.

In the present paper we would like to briefly mention the relevant theory of Markov chains and use it in a simple, not too unrealistic, calculation. The particular operation studied will be that of a nuclear materials warehouse. This will be a trivial application; it is hoped that it will serve to introduce the topic and a more important problem in the following companion paper by Epel, et al.

A reasonable model of a warehouse holding special nuclear material might be as follows (Figure 1): A vault has a number of nominally identical items in inventory; these are indicated by square boxes. There are shipments to and from the vault, and a periodic sampling of the contents of the vault in which an assay or test is performed. The items as received from the supplier are supposed to be known to a lesser confidence level than that which can be established by the assay; these are denoted by the open boxes, whereas

Figure 1

MATERIAL BALANCE AREA BEING STUDIED

samples which have undergone test are marked or tagged by crosses. Suppose now that this process of shipment, receipt, and assay or testing of a sample of the inventory, has been proceeding for some length of time; the question raised is what confidence limit can be placed on the total amount of material thought to be present. It will be supposed that our interest is in the total weight of material present, and that in this case the assay performed on samples drawn from the inventory is simply a more accurate weighing than whatever weighing procedure was heretofore performed on the sample.

In order to answer the question, it is first necessary to more precisely define the particular inventory cycle to be studied and its parameters. This choice can be fairly arbitrary; there are several alternate cycles which would also fit within the mathematical framework which will be used. For definiteness however the following particular cycle will be considered:

- (1) The inventory will start out each cycle with N nominally identical items
- (2) Of these, at some time it is assumed A have been assayed or tested and labeled or tagged as such; these measurements have an uncertainty of σ_A . The $N-A = B$ remaining items are just as received from a vendor, have not been assayed, and thus have an uncertainty σ_B , where $\sigma_B > \sigma_A$, associated with each.
- (3) We suppose the cycle begins with the shipment of some n units taken at random from the inventory. The shipment therefore contains in general a mixture of (a) assayed or tagged and (b) unassayed and untagged items, where $n = a + b$. The probability of making such a particular selection of (a) assayed and (b) unassayed items from such an inventory of n is from combinatorial analysis:

$$h(a, n) = \frac{\binom{A}{a} \binom{B}{b}}{\binom{N}{n}} = \frac{\binom{A}{a} \binom{N-A}{n-a}}{\binom{N}{n}}, \text{ where } \binom{A}{a} = \frac{A!}{(A-a)!a!} \text{ etc.}$$

(This is just the so-called hypergeometric probability which pertains in the case of sampling without replacement.)

- (4) There is then a receipt of n items into the inventory from the vendor. These being untested have an uncertainty σ_B associated with them. The total number of items in inventory is now up to the constant total of N items.
- (5) A random selection of v items is taken from the inventory for accurate assay. It will in general be a mixture of α already assayed and β previously untested items. Regardless of previous history all are tested, tagged and returned to stock. Samples previously assayed (weighed in this example) are tested again because to assume the previous result to still hold would give a would-be diverter the opportunity to take material from such previously assayed samples without it being discovered. The probability of a given selection $v = \alpha + \beta$ being made is given, similarly as before, by

$$h(\alpha, v) = \frac{\binom{A-a}{\alpha} \binom{B+a}{\beta}}{\binom{N}{v}} = \frac{\binom{A-a}{\alpha} \binom{B+a}{v-\alpha}}{\binom{N}{v}}$$

The sample, consisting now of v assayed items, is returned to stock. This action will be considered as completing the typical inventory cycle.

The various changes in stock during the cycle can be summarized as in Table I below:

Table I

Changes in Composition in the Stock
During a Typical Inventory Cycle

	<u>Start of Cycle</u>	<u>Just After Shipment</u>	<u>After Receipt of New Material</u>	<u>(After Assay)</u>
Tested	A	A-a	A-a	A-a+ β
Untested	B	B-b	B-b+n	B+a+ α -v
TOTAL	N	N-n	N	N

$$N = A=B, n = a+b, v = \alpha+\beta$$

The probability of the particular sequence above occurring during one cycle is just given by the product of $h(a,n)$ by $h(\alpha,v)$. To get the total probability that the inventory contains a given number $A-a+\beta$ of assayed items at the end of the cycle it is necessary to sum over all such probabilities $h(a,n) h(\alpha,v)$ consistent with starting with A tagged items and ending with $A-a+\beta$ tagged items:

$$P_{A,A+\delta} = \sum_{a,\alpha} h(a,n) h(\alpha,v), \text{ where } \delta = v - (a+\alpha) = -a+\beta.$$

such that $\delta = v - (a+\alpha)$

This probability $P_{A,A+\delta}$ can be interpreted as the probability of a change or "transition" in the composition or "state" of the inventory during one inventory cycle, from a situation in which there were A tested items (and $N-A = B$ untested) to one in which there were $A+\delta$ tested items; δ can in general be positive or negative, corresponding to states with a greater or lesser amount of assayed items in the stockroom.

The probabilities $P_{A,A'}$ for all possible values of A,A' can be considered as forming a square array or matrix $\underline{P}$ which appropriately is termed the transition probability matrix.

The method of Markov chains can now be used to study the time evolution of this warehouse or vault.

Between any two consecutive inventory cycles, the composition of the system in general is only known probabilistically, i.e., there is a probability p_A that there are exactly A tagged (and $B = N-A$ untagged) items residing in inventory.

The p_A are a set of numbers which can be considered as forming a linear array or vector $\underline{p}$. Since each represents the probability of there being exactly A assayed items in the inventory, the sum of all the p_A must be unity. The collection of probabilities, $\underline{p}$, define the probability of the inventory having a certain composition or state, where this state is defined in our example by the number (A) of tested items present; it is appropriately termed therefore the probability or state vector.

The utility of the Markov chain method rests upon the fact that if the probability of a given composition is known

at one point in time, then the probable composition after one cycle of operation can be found just by multiplying the probability vector corresponding to the initial state $\underline{p}_i$ by the transition probability matrix, $\underline{P}$, yielding a new probability vector representing the final state $\underline{p}_f$ after the one cycle of operation. That this is so follows from the fact that the standard methods for compounding probabilities in this situation give the same algebraic results as the rules for matrix multiplication. Thus, the probable composition of the inventory as it changes after each inventory cycle can be followed by a process of matrix multiplication. This process can be repeated indefinitely, producing in effect the temporal evolution of the probability as a result of the operation of the system. Symbolically, letting $\underline{p}^{(0)}$ represent an initial value of $\underline{p}$, and $\underline{p}^{(k)}$, etc., represent the probability vector after k cycles,

$$\underline{p}^{(0)} \underline{P} = \underline{p}^{(1)}, \underline{p}^{(1)} \underline{P} = \underline{p}^{(2)}, \dots, \underline{p}^{(k-1)} \underline{P} = \underline{p}^{(k)}$$

Equivalently, the result after k cycles is given by the k^{th} power of the matrix $\underline{P}$ multiplying the initial probability $\underline{p}^{(0)}$:

$$\underline{p}^{(k)} = \underline{p}^{(0)} \underline{P}^k, \text{ where } \underline{P}^k = \underbrace{\underline{P} \cdot \underline{P} \dots \underline{P}}_{k \text{ factors}}$$

Before proceeding it might be appropriate to illustrate this process. Assume the following parameters, the numbers kept small to be manageable:

Total inventory	$N = 10$,
Number shipped and received, per cycle	$n = 1$,
Number sampled, tested, and returned, per cycle	$v = 1$.

For convenience, the state of the system, A , will be determined by the composition at the end of the cycle. Then the value of A must be at least 1; it can't be greater than 10; thus $\underline{P}$ is a 10×10 array. Calculating the transition probabilities as indicated, one finds most of the elements $P_A, A+\delta$ are zero:

$$\underline{P} = \begin{bmatrix} .19 & .81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .02 & .34 & .64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .06 & .45 & .49 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .12 & .52 & .36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .20 & .55 & .25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .30 & .54 & .16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .42 & .49 & .09 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .56 & .40 & .04 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .72 & .27 & .01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .90 & .10 \end{bmatrix}$$

The values comprising the probability vector are determined by considerations like the following: Suppose that at a given time the inventory is inspected and it is determined that there are three tested items present out of the ten. Then the initial probability vector would be:

$$\underline{p}^{(0)} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$$

where the 1 in the third position signifies that there is a certainty that three assayed items are present. On the other hand, suppose there is a complete lack of information as to the state of the inventory. This would correspond to any state (i.e., any number of assayed items) being equally likely, in which case:

$$\underline{p}^{(0)} = (.1, .1, .1, .1, .1, .1, .1, .1, .1, .1).$$

Once an initial probability vector is assumed, the probability at any given time (i.e., any given number of inventory cycles) after is determined by matrix multiplication as indicated previously.

Of particular interest is what the probability vector becomes if the inventory cycling is allowed to proceed indefinitely. Very often a limiting form for the probability will be reached irregardless of what the initial state was. In the present instance the equilibrium probability distribution (customarily signified by $\underline{\pi} \equiv \underline{p}^{(\infty)}$) is given by:

$$\underline{\pi} = (.0001, .0044, .0466, .1907, .3436, .2867, .1094, .0176, .0010, .0000) ,$$

which was obtained by repeated operation on the vector $\underline{p}^{(0)}$ representing a uniform distribution with the matrix $\underline{P}$ a number of times until some convergence criteria were satisfied. Alternatively, simply multiplying $\underline{P}$ by itself a sufficient number of times will yield the equilibrium vector π , as the theory indicates that if an equilibrium can be reached, then the rows of $\underline{P}^k$, $k \rightarrow \infty$ are all identical, and in fact each represents the equilibrium probability vector.

It is not possible to go into the question of the rapidity with which equilibrium is approached without going into more theory than is desirable here. Let it suffice to say that practically speaking this approach is fairly rapid (geometrically), and will of course depend on how close the initial state is to the steady state. Table II below summarizes the progress towards equilibrium for the assumed uniform probability.

Table II

Example of Markov Chain Calculation, II

Starting with a uniform probability (i.e., A equally likely to have a value from 1 to 10), then

$$\underline{p}^{(0)} = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$$

After five complete inventory cycles

$$\underline{p}^{(5)} = (.007, .0145, .0821, .1905, .2502, .2426, .1608, .0525, .0060, .0001)$$

After twenty complete inventory cycles

$$\underline{p}^{(20)} = (.0001, .0044, .0465, .1903, .3433, .2870, .1097, .0177, .0010, .0000)$$

(Convergence after 24 cycles, using $< 0.1\%$ change in components of the vector as a criteria)

The equilibrium distribution at equilibrium in the present case is shown in Figure 2. Also pictured there is the initial

Figure 2

state the system was assumed to be in, and what the equilibrium probability vector would be if another sampling plan had been assumed, namely if $v=2$ samples were chosen to be assayed per cycle instead of only one.

Having the probability $p^{(k)}$ at any time that there are A tested and B untested items, the evaluation of the confidence with which the amount of material in the inventory is known can proceed. As mentioned earlier it will be assumed that the assay process narrows the uncertainty in a sample, so that $\sigma_A < \sigma_B$. Assume the weights of the individual items are gaussian distributed about their nominal values and that there are no systematic errors. Then the probability of the inventory having a total weight w assuming that these are A samples which have been tested or assayed is

$$\Pr(w|A) = \frac{1}{\sqrt{2\pi}} \frac{1}{(A\sigma_A^2 + B\sigma_B^2)^{1/2}} \exp \left[- (w - \bar{w})^2 / 2 (A\sigma_A^2 + B\sigma_B^2) \right]$$

where σ_A^2 and σ_B^2 are the variances associated with tested and untested items respectively, $\bar{w}$ is the mean value of w, and the notation $\Pr(w|A)$ specifically calls attention to this being a conditional probability, i.e., it is the probability of a weight w, given that there were A tagged items in the inventory.

For a given number A of assayed items, the probable weight after k inventory cycles is given by $\Pr(w|A)$ multiplied by the probability that there are A assayed items at that time, $p^{(k)}(A)$. Thus the probability of a total weight w regardless of what number of assayed samples are present is given by the sum of this product over all possible values of A:

$$\Pr^{(k)}(w) = \sum_{A=1}^N \Pr(w|A) p^{(k)}(A)$$

where $p^{(k)}(A)$ are the components of the probability state vector $\underline{p}^{(k)}$ as computed above, representing the probability of the presence of A assayed items after the kth cycle.

The $(1-\alpha)$ confidence limits by definition are the limits t_1 and t_2 such that

$$\int_{-\infty}^{t_1} \Pr(w) dw = \alpha/2, \quad \int_{t_2}^{\infty} \Pr(w) dw = \alpha/2.$$

Figure 3

These integrals can be computed numerically from the values $\text{Pr}^{(k)}(w)$ computed from the above expression. The meaning of these limits is that with a confidence of $1-\alpha$, the weight is given by $\bar{w} \pm t_1 \frac{\sigma}{\sqrt{v}}$.

To continue the above numerical example: Assume that each item is a (nominally) 5 kg can of special nuclear material of some sort, that $\sigma_A = 0.1$ kg, and $\sigma_B = 1$ kg, so that the assay of each sample (weighing under the present assumption) reduces the uncertainty in its value by a factor of 10. The total nominal weight of the inventory is thus $\bar{w} = 50$ kg. Suppose the only interest is in the long term equilibrium situation. Then the steady state vector $p^{(\infty)}(A) = \pi$ is used, resulting in the curves of probability density of the total weight of the inventory versus the total weight, for the two sampling plans, $v=1$ and $v=2$ (Figure 3). The integration of the $\text{Pr}(w)$ as discussed above gives a confidence limit at 2σ of ± 3.9 kg, for $v=1$. As can be seen, taking two samples per cycle ($v=2$) results in the smaller uncertainty of ± 3.3 kg.

Sampling a greater number each cycle would increase the confidence with which the total amount in the inventory is known. Naturally this increased confidence comes at an increased economic cost. Carrying out the same type of calculation but for $v = 3, 4, \dots$ yields the following results; a steady state condition is assumed in all cases:

<u>Sample size</u> <u>v</u>	<u>Inventory and uncertainty</u> <u>(kg)</u>	<u>Average number of assayed items</u> <u>(A)</u>
1	50 $\pm$ 3.9	5.3
2	$\pm$ 3.3	7.1
4	$\pm$ 2.5	8.7
7	$\pm$ 1.7	9.6
8	$\pm$ 1.4	9.8
9	$\pm$ 0.9	9.9
10	$\pm$ 0.6	10.0

These results are plotted in Figure 4. Note how rapidly the average number of assayed items present in inventory increases with the sample size.

Figure 4

UNCERTAINTY IN
THE INVENTORY AS
A FUNCTION OF SAMPLING RATE

These data can be interpreted as the cost-benefit relation between the amount of sampling done per cycle and the resultant degree of assurance. The question of what the optimum sampling plan might be could now in principle be discussed for this system, if the economic and other ground rules were made clear.

The steady state probability vector depends only upon the $\underline{P}$. For example, corresponding to $\underline{P}$ ($N=10, n=1, v=2$) there is the steady state probability $\underline{\pi}(A)$ given by

$$\underline{\pi}(A) = \underline{p}^{(\infty)}(A) = (0.0, .900001, .00014067, .00444424, .04775055, .20545699, .37468991, .28343766, .07857781, .00550117).$$

The steady state $\underline{\pi}$ can also be gotten by solving the matrix equation:

$$\underline{\pi} = \underline{\pi} \underline{P}, \text{ or } \underline{\pi}(\underline{I} - \underline{P}) = 0,$$

where $\underline{I}$ is the matrix with ones on the main diagonal and zeros elsewhere, though it is not clear whether, for practical purposes this is better than the previous. The resulting homogeneous equations can be solved for the components of $\underline{\pi}$ if the determinant $|\underline{I} - \underline{P}|$ vanishes. A unique solution is determined by the condition that the components of $\underline{\pi}$ have to add up to unity.

The general time dependence of the transition probability matrix $\underline{P}$ can be obtained from the following calculation. First the eigenvalues λ_i and eigenvectors $\underline{q}_i$ are determined by solving the matrix equations

$$\underline{P} \underline{q}_i = \lambda_i \underline{q}_i$$

Then the matrix $\underline{Q}$ is formed by making its columns up out of the column vectors $\underline{q}_i$. It follows from matrix theory that $\underline{P}$ can be written as

$$\underline{P} = \underline{Q}^{-1} \underline{\Lambda} \underline{Q}$$

where $\underline{Q}^{-1}$ is the inverse matrix to $\underline{Q}$, i.e.,

$\underline{Q}^{-1} \underline{Q} = \underline{Q} \underline{Q}^{-1} = \underline{I}$, and $\underline{\Lambda}$ is a matrix whose diagonal elements are the λ_i , with zeros elsewhere. Then

$$\begin{aligned} \underline{P} \cdot \underline{P} &= \underline{P}^2 = \underline{Q}^{-1} \underline{\Lambda} \underline{Q} \underline{Q}^{-1} \underline{\Lambda} \underline{Q} \\ &= \underline{Q}^{-1} \underline{\Lambda}^2 \underline{Q}, \end{aligned}$$

and similarly, in general,

$$\underline{P}^k = \underline{Q}^{-1} \underline{\Lambda}^k \underline{Q}$$

Since $\underline{\lambda}$ is a diagonal matrix, $\underline{\lambda}^k$ is very simple to find, being a diagonal matrix where each element is the corresponding element of the matrix raised to the k^{th} power. Thus the result of the k^{th} cycle can be represented in terms of the constant matrix $\underline{Q}$ and its inverse $\underline{Q}^{-1}$, and the simple diagonal matrix $\underline{\lambda}^k$.

If there is a steady state solution, then there will be an eigenvalue exactly equal to unity. The eigenvector corresponding to this eigenvalue is the same as that obtained either by repeated application of $\underline{P}$ to some initial state, or to itself, as mentioned in the text, or by solving the equation $\underline{\pi}(\underline{I} - \underline{P}) = 0$ as outlined above in the appendix.

Since the eigenvalues that are less than unity will tend to zero if raised to a power which is allowed to increase indefinitely, the matrix $\underline{\lambda}^k$ will approach as a limiting value a matrix which is zero everywhere except in one location on the main diagonal, which element would be equal to unity.

To illustrate some of these points, for the $\underline{P}$ used in the text, i.e., $\underline{P}$ ($N = 10$, $n = 1$, $v = 1$), the eigenvalues with their corresponding eigenvectors listed beneath them as column vectors are as shown below (there is no significance to the order; this just happens to be the order in which these vectors and eigenvalues were found by the computer code which calculated them).

Eigenvalues:

.81000000	1.00000000	.64000000	.49000000	.25000000	.36000000	.01000000	.16000000	.09000000	.04000000

Eigenvectors:

-.00057711	.00010825	.00169820	-.00367567	-.00889944	.00633681	-.00227243	.01048799	-.00936946	.00735001
-.01789049	.00438416	.03820948	-.05513505	-.02669831	.05386289	.02045185	-.01573198	.04684730	-.05512506
-.13235112	.04676438	.16812171	-.08821608	.16018983	-.06759264	-.08180740	-.09439189	-.06870937	.17640021
-.30163744	.19095456	.06240882	.26464825	-.12459209	-.23657425	.19088393	.31201762	-.04372414	-.30870036
-.11311404	.34371820	-.37445220	.17643217	-.22426576	.35486137	-.28632589	-.33037160	.26234486	.30870036
.26393276	.28643183	-.18722645	-.35286434	.37377627	.05914356	.28632589	.05506193	-.34979115	-.15435018
.23700085	.10911689	.17831091	-.06301149	-.12459209	-.23657425	-.19088393	.14683182	.21862072	.00000000
.06431970	.01753664	.10125512	.10081838	-.05339661	.03802086	.08180740	-.10225788	-.05621676	.04410005
.00455278	.00097426	.01146284	.02047873	.02669831	.02745951	-.02045185	.01573198	-.00312315	-.02205001
.00006612	.00001083	.00021227	.00052510	.00177989	.00105614	.00227243	.00262200	.00312315	.00367509

The eigenvectors taken together as a matrix, constitute Q.

Note that the eigenvector corresponding to $\lambda = 1$ is that found in the text by repeated multiplication (except for small differences due to round-off errors, etc.), and that since it is also a probability vector, all its terms are ≥ 0 and sum to unity. The sum over components of each of the other eigenvectors is identically zero.

The inverse matrix, Q⁻¹ is as shown below:

Inverse of Matrix of Eigenvectors:

1.11418534	1.00000000	.64594444	-.19584683	-.32740875	.35560719	.00000000	-.66675456	.55977138	-.66213190
-1.45181726	1.00000000	1.42107777	-.63976632	.98222625	-.53341079	.82457128	-.66675456	-.41049902	-.44142127
-.16881596	1.00000000	-.43062963	.17408607	-.18709071	-.25400514	.19632650	-.31750217	.21324624	.31530091
2.39718664	1.00000000	4.65079999	7.12882475	7.85781003	4.26728630	-4.94742770	-8.00105469	-.89563422	8.60771475
.47268469	1.00000000	-.25837778	-.41780658	.37418143	.05080102	-.11779590	.38100260	-.34119399	.06306018
-2.09331791	1.00000000	3.44503703	-4.26510882	-1.74618001	-.94828584	-2.74857095	1.77801215	2.98544739	3.75208079
1.75568599	1.00000000	2.28233703	1.94976403	-.87309000	-1.54096450	.54971419	1.77801215	-.89563422	.66213190
-2.73481857	1.00000000	6.20106665	-11.51579383	-23.57343008	-25.60371777	-14.84228311	-8.00105469	-24.18212382	17.87756141
3.03868730	1.00000000	7.75133332	16.45113404	47.14686016	64.00929443	74.21141557	80.01054692	80.60707941	29.79593569
-.81031661	1.00000000	.12918889	.47003240	-.18709071	.43180873	-.35338769	.38100260	-.06397387	-.37836109

It has the peculiarity that the second column, which is the one corresponding to the eigenvector $\pi = \underline{p}^{(\infty)}$, is composed of all ones.

Additional theory on Markov chains may be found in "The Theory of Stochastic Processes" by D. R. Cox and H. D. Miller, Methuen and Co. Ltd., 1965.