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BEAM TRANSPORT CALCULATIONS USING GRAPHS

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BEAM TRANSPORT CALCULATIONS USING GRAPHS

Abstract

The spread of a beam of particles can be represented by an ellipse in phase space, which represents the spread of the beam in both angle and distance. It is shown that the changes in this ellipse can be followed simply by graphical means, as the beam passes through an optical system consisting of thick and thin lenses and drift spaces. Such calculations are of value when an injected beam must be matched to an accelerator, or when a beam of particles from an accelerator must be transported to an experiment or through an analysing system. The graphical calculation is quicker than a computer calculation, and should be of sufficient accuracy in most cases. More usefully, the graphical calculation gives more insight into what changes are necessary in a beam transport system to give it the desired properties.

TRANSFER MATRICES AND PHASE ELLIPSES

If x and $\dot{x}$ are the displacement of a particle in distance and angle from the central trajectory of a beam either horizontally or vertically, then the values at position 2 can be calculated from the values at position 1, when the transfer matrix is known.

$$\begin{pmatrix} x_2 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \end{pmatrix} \quad \text{where } ad - bc = 1 \quad (1)$$

An ellipse in the $x_1 \dot{x}_1$ space is given by

$$f_1 x^2 + 2h_1 x_1 \dot{x}_1 + g_1 \dot{x}_1^2 = w \quad \text{where } f_1 g_1 - h_1^2 = 1$$

and similarly at the position 2 for subscript 2.

The relation between the two ellipses is given by

$$\begin{pmatrix} f_2 \\ h_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} d^2 & -2cd & c^2 \\ -bd & ad+bc & -ac \\ b^2 & -2ab & a^2 \end{pmatrix} \begin{pmatrix} f_1 \\ h_1 \\ g_1 \end{pmatrix} \quad \text{and } f_2 g_2 - h_2^2 = 1 \quad (2)$$

and the value of w remains unaltered.

The matrix for a drift distance D is $\begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix}$

The matrix for a thin lens of strength L , where positive value for L is a defocussing lens of focal length $1/L$ is $\begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix}$

The matrix for a thick lens is of the general form above and this can be decomposed into an equivalent set of thin lenses and drift distances in two ways as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a-1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d-1}{c} \\ 0 & 1 \end{pmatrix} \quad (3)$$

that is, a drift distance $\frac{d-1}{c}$ followed by a lens of strength c and again by a drift distance $\frac{a-1}{c}$. This decomposition gives the positions of the principal planes of the thick lens.

Or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{d-1}{b} & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{a-1}{b} & 1 \end{pmatrix} \quad (4)$$

that is a thin lens of strength $\frac{a-1}{b}$ followed by a drift distance b and followed again by a thin lens of strength $\frac{d-1}{b}$.

UNITS OF MEASUREMENT

It is frequently convenient to choose appropriate units for the beam spatial spread x and for the beam angular spread $\dot{x}$ or θ . The unit of path length along the beam then becomes equal to the unit of x divided by the unit of $\dot{x}$, and the unit of lens strength the reciprocal of this. Thus, for the Cambridge Electron Accelerator beam, half-widths are suitably measured in tenths of inches, the semi-angular spread in milliradians; the drift distances are suitably measured in units of 100 inches, and a lens of unit strength has a focal length of 100 inches.

If we change from units where the quantities of interest have values $x, \dot{x}, a, b, c, d, f, h, g, w$, to other units where these have values $X, \dot{X}, A, B, C, D, F, H, G, W$, the relations are $x=mX, a=A, b=\frac{m}{n}B, c=\frac{n}{m}C, d=D,$

$$\dot{x}=n\dot{X}, f=\frac{n}{m}F, h=H, g=\frac{m}{n}G, w=mnW. \quad (6)$$

ELLIPSE COORDINATES

The quantities f, h, g , determine the shape of the ellipse and the value of w determines the area of the ellipse. Since w is not altered as the beam passes through an optical system, it is necessary only to follow the shape of the ellipse and for this purpose we can set $w=1$. It is then more meaningful to define the ellipse shape with a set of four parameters: y, θ, D , and L . We take $y=+\sqrt{g}$, $\theta=+\sqrt{f}$ if h is negative and $\theta=-\sqrt{f}$ if h is positive; y is the maximum value of x on the ellipse and

$\pm\theta$ is the greatest value of $\dot{x}$, and thus y and θ measure the half-width and half-divergence of the beam and the sign of θ gives the lean of the ellipse. It is an advantage to have two extra coordinates, where $D = \frac{-h}{f}$, and $L = \frac{-h}{g}$; a drift distance of $-D$, or a thin lens of strength $-L$ would put the beam at a crossover where the h value equals zero. Thus the values y , θ , D and L are natural parameters of the phase ellipse and give directly the beam diameter, beam divergence and the departure of the ellipse from being on axis, that is, the distance of the beam from a crossover. For other values of w , the beam semi-diameter is $y\sqrt{w}$ and its semi-divergence is $\theta\sqrt{w}$, and D and L remain the same. Fig. 1 shows representative ellipses with a series of values of f , h , g , y , θ , L and D .

ELLIPSE TRANSFORMATIONS

Either θ and D , or y and L fully define the ellipse.

Given θ and D , we can find y and L from

$$y = +\sqrt{\frac{1}{\theta^2} (1+D^2\theta^4)}, \quad L = D\left(\frac{\theta^4}{1+D^2\theta^4}\right) \quad (7)$$

or given y and L we can find θ and D :

$$\theta = \pm \sqrt{\frac{1}{y^2} (1+L^2y^4)}, \quad D = L\left(\frac{y^4}{1+L^2y^4}\right) \quad (8)$$

A drift distance A sends the ellipse defined by θ and D into the ellipse defined by θ and $D+A$, and the new values of y and L can be determined from Fig. 2, or from Eq.(7). A lens of strength M sends the ellipse defined by y and L into the ellipse defined by y and $L+M$, and the corresponding values of θ and D can be found from Fig. 3 or Eq. (8). The ellipse transformations involving a drift distance are performed graphically using Fig. 2 and those involving thin lenses are done using Fig. 3.

The main diagonal in Fig. 2 and 3 refers to on-axis ellipses where the beam is at a waist or crossover. The top right hand half refers to forward leaning ellipses with negative h and positive values of θ , D , and L . The lower left hand half

refers to backward leaning ellipses with positive h and negative values of θ , D and L . Note that the directions of the y and θ axes change direction by 90 degrees in passing across the main diagonal, this arrangement is necessary to plot both halves on the same sheet of graph paper. In Fig. 2, curves of constant D are plotted with y and θ log-log coordinates, and in Fig. 3, similarly are plotted curves of constant L . All these curves are congruent and can be drawn using a single template.

EXAMPLE OF BEAM TRANSPORT

The injection linac of the Cambridge Electron Accelerator produces an electron beam with a semi-diameter of .085 inches and a semi-divergence of .85 milliradian in the horizontal. The output beam is at a crossover at the linac output, and its coordinates are thus $f = 1.0$, $h = 0$, $g = 1.0$, $w = .72$; $y = 1.0$, $\theta = 1.0$, $D = 0$, $L = 0$, where the beam width is measured in tenth of inch units, and the angular spread is measured in milliradian units. The unit of beam drift distance is then 100 inches, and a unit strength lens has a focal length of 100 inches.

In the same units, the horizontal acceptance ellipse of the accelerator at the inflector is given by $f = 1.83$, $h = -2.073$, $g = 2.89$; $y = 1.70$, $\theta = 1.35$, $D = 1.13$, $L = 0.72$.

The transformation of the linac ellipse to the acceptance ellipse by the injection optics is shown in Fig. 4. Using Fig. 2 we transform the ellipse through a drift distance of 140 inches, 1.4 units of 100 inches each to the point B, then a defocussing thin lens of strength .125 takes it to C by using Fig. 3, then a drift of 1.83 units takes it to D using Fig. 2, then a focussing lens of strength -.62 takes it through a crossover at E to the point F, using Fig. 3, then a drift of 1.70 units takes it to G using Fig. 2. The fringing field transfer matrix is decomposed into a drift distance of 0.9 units, a thin lens of strength 1.5 units (defocussing) and a drift distance of 0.7 units taking the

ellipse to H, K, and finally to the acceptance ellipse coordinates at L in Fig. 4. Alternately by taking the acceptance ellipse from L back to F and the linac emittance forward from A to B, this graphical method could be used to determine the required values of the thin lenses in the injection run to give the required matching.

A MEASURE OF MATCHING ERROR

Two ellipses, f_1, h_1, g_1 , and f_2, h_2, g_2 , are matched when $f_1 = f_2, g_1 = g_2, h_1 = h_2$. A measure of the mismatch is given by $2M = f_1 g_2 + f_2 g_1 - 2h_1 h_2$ which equals 2 for matched ellipses and is otherwise greater than 2. For example, if the suffix 1 refers to the acceptance ellipse in a machine and the suffix 2 to the actual injected ellipse, the betatron oscillation amplitude will exceed the value for a matched beam by a factor $= (M + \sqrt{M^2 - 1})^{\frac{1}{2}}$.

In Fig. 5 are plotted a number of curves the points of which represent all those ellipses of equal mismatch $M = 1.125$ from the ellipse represented at the center of these curves. This value of M corresponds to betatron oscillation increase by a factor of 1.28 over that for $M = 1.0$. These curves in Fig. 5 give some idea of the permissible errors when doing injection matching or other beam transport calculations.

FIG. 1 PHASE SPACE ELLIPSES

FIG. 2 Drift Distance Transformation

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FIG. 3 Thin Lens Transformations

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FIG.4 Linac to Inflector Beam Matching

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FIG. 5 Curves of Equal Mismatch

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