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CONTROL AND STABILITY ANALYSIS OF
SPATIALLY DEPENDENT NUCLEAR REACTOR SYSTEMS

by

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CONTROL AND STABILITY ANALYSIS OF SPATIALLY DEPENDENT NUCLEAR REACTOR SYSTEMS

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ABSTRACT

A fundamental approach to the study of the dynamics of spatially dependent nuclear reactor systems is developed. This approach treats reactor dynamics problems from the system's partial-differential equations without finding the solution or engaging any approximation to a lumped-parameter model. A dynamical model of a spatially dependent process in nuclear reactors is formulated. The model is general and may include the spatial effects of deterministic processes such as control devices and temperature.

The extended method of Lyapunov, for distributed parameter systems, is then employed to analyze the stability of the system formulated above. A method of using the well-known spectral theory of Von Neumann is also introduced to analyze certain classes of linear and/or nonlinear spatially dependent dynamical systems.

The application of the maximum principle to the optimal control system design for spatially dependent reactor systems is demonstrated for several classes of optimization problems. The practicality of optimal control for nonlinear systems is discussed, and a practical technique for suboptimal control design for nonlinear reactor systems using Lyapunov functionals is introduced.

CHAPTER I

INTRODUCTION

Recent technical advances in research reactors and the successful long-time operation of the relatively small experimental power reactors have successfully laid the foundations for the future of nuclear power generation. However, to be attractive to the consumers, nuclear power must be economically competitive to power generated using fossil fuel. Many analyses on nuclear power costs have been conducted both by government

agencies¹ and by commercial companies.^{2,3} The results from these analyses indicated that, in most cases, the power cost is inversely proportional to the capacity of the power plant. Therefore, the future in nuclear power generation is in large power reactors, and consequently the study of the dynamical behavior of large reactors becomes essential (e.g., stability analysis and optimal control of reactors).

Most reactors built in the past often used the classical point-model reactor-kinetics equation, which approximates the spatially-dependent reactor system as a space-independent system by employing certain spatial averages of the system variables.⁴ When the reactor in question is small, the point model is usually adequate for the control and stability analysis since the assumption that the system responses are space-time separable is valid.⁵ However, for the much larger reactors (many times the neutron migration length),⁶ the neutron flux in the various regions of an inhomogeneous reactor core can become loosely coupled and any non-uniform changes in the properties of the reactor cause local distortions of the power distribution.⁷ In this case, the space-time separable assumption is no longer valid, and the use of the point model in the study of spatially-dependent reactor systems is open to question. Yasinsky, Henry, and Clarke⁸ illustrated the importance of considering the spatial effect in the control and stability analysis.

In Ref. 8, the authors analyzed the spatial flux distribution of different size reactors (one-dimensional, 240 and 60 cm in core length). The numerical solutions using the time-dependent diffusion equation were obtained and compared with the point-model approximation. The results showed that the errors were far more extreme than expected. A perturbation (a step change in reactivity) was assumed to occur at one side of the core. The result shows that for the 240-cm core, the point model can have an error as much as five orders of magnitude larger in the fast transient (less than 100 msec) and four times as large for the slow transient (within 1 sec). As for the 60-cm core, the error is only four times and 20% for the fast and slow transients, respectively. This example shows that the error involved by using the point-model approximation increases with the size of the core. It also leads to the conclusion that, if the point model were used to analyze the stability of the large reactor system, the space-independent parameters in the point-model equation would not be reliable, since they are obtained from the spatial distribution of the unperturbed reactor.⁹ Consequently, the result of the stability analysis is not reliable. Similarly, if an optimal control system were designed to have uniform burnup using the point model, the system may indeed be optimal with respect to its model, but it may not be optimal in the true sense because no matter how good the design is, it is only as good as the mathematical model. The above discussion indicates that it is not only desirable but necessary to study the control and stability problem of the large reactor systems by taking into account the spatial distribution of the system responses.

The problem of spatially-dependent reactor dynamics was first triggered by the well-known phenomenon of xenon spatial oscillation in the early study of thermal reactors. Henry and Germann¹⁰ studied the oscillations in the spatial distribution of power due to xenon buildup for a reactor of sufficient size and operating at high power density. The reactor under consideration was divided into three distinct regions (a region of pure moderator was assumed to be between two fuel regions), and a simple mathematical model was formulated. The spatial transient behavior of the flux was analyzed, and possible spatial oscillation was discussed. Many papers have followed, both in theoretical analysis and experimental verification of these studies. The various techniques employed for these studies were discussed by Henry¹¹ at the 1960 Idaho Conference on Reactor Kinetics. At this conference, he cited over a hundred papers on this topic. Among the techniques for the study of spatial reactor dynamics, two of them are essential to the study of the stability and control of reactor systems, namely, the "modal" and the "nodal" methods.

In the modal method, Garabedian and Leffert¹² proposed a technique that permitted the investigation of the change in flux shape when there is a local reactivity variation in an inhomogeneous reactor. This method employed the modal expansion, which does not use the usual assumption of separability of the flux into a single product of space and time alone, but a series of products of such functions. Kaplan¹³ further extended this work in analyzing the problem by various types of modal expansion. The first to apply the modal approach to control systems were Mochizuki and Takeda.¹⁴ They considered both the temperature effect and xenon buildup in a large power reactor. A set of linearized modal transfer functions was formulated, and a control system was designed to compensate for the dynamic effects caused by spatial variation in the core. Gyorey¹⁵ further investigated, in detail, the effect of modal interaction in the xenon spatial-instability problem. The work in modal expansion by Henry¹⁶ was on the so-called "inhour modes" expansion, which generates a cluster of eigenvalues from the inhour equation for each mode.

The nodal method divides an inhomogeneous or a homogeneous reactor into regions or zones and ascribes to each section a separate kinetics behavior determined partly by its own characteristics and partly by a coupling through neutron leakage to neighboring regions. The best-known analysis using this approach is attributed to Avery.^{17,18} For a reactor having one part of the core fast and the other part thermal, Avery treated the system in terms of lumped parameters, which explicitly characterized the individual regions and the couplings between them. He then formulated the general mathematical description of a system containing many such regions. Baldwin's¹⁹ approach on the study of the Argonaut Reactor can be considered as an application of Avery's theory. Henry and Curlee^{20,21} also applied this method to calculate the shape function of a reactor with spatial temperature effects.

The only significant publications on spatially-dependent reactor optimal control were by Kliger²² and Wiberg,²³ both of whom used the modal expansion. However, the success of this approach depends on whether a noninteracting modal expansion can be obtained for the system. In most cases, this is difficult, even for inhomogeneous linear systems. As for nonlinear systems, it is not always possible to justify whether a modal expansion exists unless the asymptotic stability of the nonlinear system is assured. This is true since the responses of the system may not be bounded everywhere in the reactor, and the usual modal expansion may not even converge to the solution.

In the stability analysis of spatially-dependent reactor systems, most of the work has been either to study the solution of the reactor equations or to employ modal or nodal expansions to reduce the original system to a spatially-independent set of equations. In the latter case, the stability of the system is then analyzed using the well-developed methods for lumped-parameter systems. Both steps are not only difficult, but they also encounter mathematical problems as to the validity of approximations used, such as the space-time separability in the nodal method and the truncation in the modal method.

The objective of this investigation is to develop a foundation for studying the dynamics of spatially-dependent reactor systems, without taking advantage of either the nodal or modal techniques. The desire here is to develop a method of stability analysis which deals directly with the general spatially-dependent reactor system that is governed by a set of nonlinear partial differential equations. The analysis has been concentrated in the following two areas:

1. The stability analysis of linear and nonlinear spatially-dependent reactor systems.
2. The control and intrinsic characteristics of spatially-dependent reactor systems.

As a first step (Chapter II), a mathematical model of a power reactor system is formulated which takes into account the group diffusion properties of the neutron flux, the heat transfer, and hydrodynamics. The system parameters are treated generally as functions of space and system variables. The variation of the control devices (e.g., control rods) is also accounted for by assuming that certain system parameters are functions of these variations. The system stability is then analyzed (Chapter III) via the well-known methods of Lyapunov. The methods are generalized to study the stability of systems of partial differential equations. Since the stability of a nonlinear system is directly related to the stability of the linearized system about the neighborhood of

its equilibrium state, the first section of Chapter III is devoted to analyzing linear systems. A sufficient condition for the stability (asymptotic stability) of the general linear reactor system is obtained directly from the system equations using both the conventional method (i.e., analyzing the spectrum of the linear system) and Lyapunov's methods. The advantages and disadvantages of the various methods are also discussed. Then, a technique is developed for the construction of Lyapunov functionals for the class of nonlinear reactor systems discussed in Chapter II. In Chapter IV, the optimal control of spatially-dependent reactor systems is analyzed using the maximum principle. Finally, a brief discussion is given on the use of digital computers to implement the optimal control obtained from the above-mentioned method.

CHAPTER II

MATHEMATICAL MODEL OF THE REACTOR SYSTEM

Before undertaking the stability and control analysis of reactor systems, we must formulate a mathematical model of the system. The reactor system in this investigation is a power reactor in which the fuel is stationary and the heat generated is removed from the reactor by the coolant traveling through the reactor core. Equations for the more commonly encountered processes in the reactor are formulated in this chapter. This will allow the techniques developed herein to be applied to various reactor systems.

A. The Neutron Kinetics Equations

In the general case, the neutron balance in a reactor system is described by the Boltzman or transport equation. The neutron flux depends, not only on space and time, but also on the energy level of the neutrons and their direction of motion. Since in a power reactor system there are no sharp gradients in the flux distribution, the neutron dynamics can be adequately approximated by multigroup diffusion equations which depend only on space and time. The M-group diffusion equations with I-group delayed neutron precursors are given in

$$\left[\nabla \cdot D \nabla - \Sigma + (1 - \beta) \vec{\chi}_p \Sigma_f^T \right] \vec{\phi}(\vec{r}, t) + \sum_{i=1}^I \lambda_i C_i(\vec{r}, t) \vec{\chi}_i = v^{-1} \frac{\partial}{\partial t} \vec{\phi}(\vec{r}, t),$$

$$\beta_i \Sigma_f^T \vec{\phi}(\vec{r}, t) - \lambda_i C_i(\vec{r}, t) = \frac{\partial}{\partial t} C_i(\vec{r}, t), \quad i = 1, 2, \dots, I \quad (1)$$

where

D = a diagonal matrix of the group diffusion coefficients,

Σ = matrix of the group total removal cross sections,

$\vec{\chi}_p$ = a column vector of group emission spectrum for prompt neutrons,

$\vec{\chi}_i$ = a column vector of group emission spectrum for i th group delayed neutrons,

Σ_f = a column vector of group neutron fission rate ($v \Sigma_f^n$),

$\vec{\phi}(\vec{r}, t)$ = a column vector of group fluxes,

λ_i = the decay constant of precursors of type i ,

$C_i(\vec{r}, t)$ = a concentration of delayed neutron precursors of type i ,

v^{-1} = a diagonal matrix of the group velocities,

β = total delayed group fraction,

and

β_i = delayed group fraction for group i .

Among the parameters in Eq. (1), D , Σ , and Σ_f are spatially dependent in general and in a power reactor may be functions of the system variables as well as time; $\bar{\chi}_p$, $\bar{\chi}_i$, ν , λ_i , and v^{-1} are in general assumed to be constants.

The first vector equation in Eq. (1) indicates that the rate of change of neutron densities for each energy group is equal to the rate of production less the rate of removal of neutrons in the same energy group. On the left-hand side of the vector equation, the off-diagonal elements of $\Sigma\vec{\phi}$,

$(1 - \beta) \bar{\chi}_p \Sigma_f^T \vec{\phi}$, and $\sum_{i=1}^I \lambda_i C_i(\vec{r}, t) \bar{\chi}_i$ represent the rate of neutron production

from scattering, prompt fission, and delayed-group neutrons, respectively. The terms $\nabla \cdot D \nabla \vec{\phi}$ and the diagonal elements of $\Sigma\vec{\phi}$ represent the removal of neutrons from leakage, absorption, and scattering, respectively. The second set of equations indicates that the rate of change of the delay neutron precursors is equal to the fractions of fission that produces these precursors, less the decay rate of the precursors.

The allowable boundary conditions for the system (1) are:

(a) The neutron flux vanishes at the extrapolated boundaries for all energy groups; and (b) a certain linear combination of the current (gradient of the flux) and the flux vanishes at the boundary for all energy groups. The dependence of the reactor parameters to the system variables will be discussed later in this chapter.

The derivation of system (1) from Boltzmann's equation can be easily found (e.g., Ref. 9).

B. Temperature Distribution

In a power reactor, the large amounts of heat energy generated within the fuel must be efficiently removed from the reactor core to prevent high-temperature buildup in the fuel. For this reason, the reactor core is usually composed of many tiny fuel pins or thin fuel plates separated by a constantly moving coolant. This heterogeneous property can result in sharp temperature gradient between the fuel and the coolant. To study this property in detail is desirable, but analyzing an extremely large number of equations simultaneously is laborious and impractical.

To simplify the analysis without completely ignoring the heterogeneous property of the reactor core, a procedure is used to formulate the temperature distributions for the fuel and coolant in the reactor. In this procedure, the reactor is considered to be composed of many identical unit cells, each consisting of a fuel pin (or plate) and its surrounding equivalent coolant channel. The height of the unit cell is the same as that of the reactor core. An example of the unit cell for a fuel pin is shown in Figs. 1 and 2.

(a) FUEL PIN ARRANGEMENT

(b) UNIT CELL CROSS SECTION

Fig. 1. Unit Cell Configuration

Fig. 2. Reactor Core Configuration

Figure 1a is the top view of a reactor core; the shaded areas are the fuel pins, and the blank area is the coolant. The unit cell is the dotted circle. Figure 1b is the side view of a unit cell. Figure 2 shows a cross section of a reactor core composed of many such unit cells.

For a typical unit cell, the temperature distributions in the fuel and coolant can be calculated from the basic heat-transfer equations and an internal heat source due to nuclear fission. From these temperature distributions, the spatial averages (in the r -direction) can then be obtained. These averaged quantities can be used to represent the fuel and coolant temperatures at the point where the unit cell is located. The overall temperature distributions are then described by these locally averaged temperatures.

The derivation given below assumes that the reactor core is composed of fuel pins. The following basic assumptions are used for the analysis:

a. The heat source (flux) in the fuel pin of a unit cell is uniformly distributed in the r-direction. This is justifiable since the fuel pin is much smaller than the reactor core.

b. There is no heat conduction in the axial (z) direction²⁴ since most of the heat is transferred horizontally to the coolant.

c. The temperature profile is uniform in the coolant in the direction perpendicular to the coolant flow. Since the flow is turbulent flow in normal condition, this is also justified.²⁵

By using the above assumptions, we can derive the temperature distributions as follows:

Two equations describing the energy balance for the temperatures in the unit cell are

$$\rho_f C_{pf} \frac{\partial T_f}{\partial t} = k \nabla_r^2 T_f + q_1'''(R, z, t), \quad (2)$$

and

$$\rho_c C_{pc} \frac{\partial T_c}{\partial t} = - \frac{w C_{pc}}{\pi (r_1^2 - r_0^2)} \frac{\partial T_c}{\partial z} + \frac{h}{2\pi r_0} [T_f(r_0, z, t) - T_c(z, t)] + q_2'''(R, z, t), \quad (3)$$

where

k = the thermal conductivity of the fuel,

ρ_f = the density of the fuel,

C_{pf} = the heat capacity of the fuel,

q_1''' = the volumetric heat source (and/or sinks) in the fuel,

w = the mass flow rate of the coolant,

h = the heat-transfer coefficient,

ρ_c = the density of the coolant,

C_{pc} = the heat capacity of the coolant,

q_2''' = the volumetric heat source (and/or sinks) in the coolant,

R, z = space variable with respect to the reference coordinates of the reactor,

and

r = space variable with respect to the unit cell.

Equation (2) indicates that the rate of heat energy change in the fuel is equal to the heat generation within the fuel, less the heat transferred to the coolant from the fuel. Equation (3) indicates that the rate of change of the heat energy in the coolant is equal to the heat generation in the coolant, plus the heat transferred to the coolant from the fuel, less the heat removed from the system.

The boundary conditions for the two equations are

$$k \frac{\partial T_f}{\partial r} \Big|_{r=r_0} = h[T_f(r_0, z, t) - T_c(z, t)]; \quad \frac{\partial T_f}{\partial r} \Big|_{r=0} = 0,$$

$$T_f(R, z, 0) = T_{f0},$$

$$T_c(0, t) = T_i, \quad T_c(z, 0) = T_{c0}. \quad (4)$$

If the equilibrium equations of Eqs. (2) and (3) are solved for in the fuel pin of a unit cell with respect to a stationary heat source (heat generation due to neutron fission in the fuel and heat taken out due to heat transfer in the coolant), the average fuel temperature can be found by integrating the temperature profile over the cross-sectional area within the fuel region. The space-time dependent fuel-temperature equation can be formulated by considering the heat energy balance condition in the entire reactor in terms of the average fuel temperature so obtained. Since the heat source is dependent on R , the fuel temperature would also be dependent on R . Assuming that the fuel surface temperature is proportional to the average fuel temperature, the final result can be written as

$$\frac{\partial \bar{T}_f}{\partial t} = A_1 \Sigma_f \phi + A_2 T_c - A_3 \bar{T}_f. \quad (5)$$

The equation for the coolant temperature would in turn be written as

$$\frac{\partial T_c}{\partial t} = -B_1 \frac{\partial T_c}{\partial z} + B_2 (\bar{T}_f - T_c) + B_3 \phi. \quad (6)$$

The boundary condition for the system (6) is that $T_c(R, 0, t) = T_{c0}$, the coolant inlet temperature, where the A 's and B 's depend on the heat-transfer parameters.

Equation (5) can also be obtained from Eq. (2) by assuming that $T_f = T_r(r) \bar{T}_f(R, z, t)$, and that the fuel surface temperature is proportional

to the integral over r of the fuel temperature. Let us integrate Eq. (2) over the fuel region, and write the heat balance equation,

$$\int_0^{r_0} k \nabla_r^2 T_r(r) \bar{T}_f(R, z, t) dr + \int_0^{r_0} q'''(R, z, t) dr = \rho_f C_{pf} \int_0^{r_0} T_r(r) dr \frac{\partial}{\partial t} \bar{T}_f(R, z, t), \quad (7)$$

where

$$\int_0^{r_0} q'''(R, z, t) dr = A_1' \Sigma_f \phi - A_2' (K \bar{T}_f - T_c),$$

K is the proportionality constant, which relates the fuel surface temperature to the average fuel temperature, and $T_r(r)$ may be the normalized solution (or shape function) calculated from the equilibrium equations of Eqs. (2) and (3).

By rearranging the constants in Eq. (6) and the constants obtained after integration in Eq. (7), we can obtain two equations of the form of Eq. (5).

In this derivation, the final heat-transfer equations, Eqs. (5) and (6), were derived assuming that the fuel and coolant temperature follows the power distribution in the reactor. This assumption implies that the radial conduction in the core has to be neglected. The reasons for neglecting the conduction (generally accepted in the design of reactor systems) are as follows:

1. In a reactor system, the heat-conduction loss is generally much smaller compared with the heat removed by convection at normal operating condition. To illustrate this point, a sample calculation for the EBR-II reactor was made. For simplicity, the reactor core is assumed to be a 1-ft cube. The general steady-state solution of a three-dimensional conduction and axial convection problem is obtained. The general solution shows that the coolant temperature is inversely proportional to the flow rate. Using the design parameters from Ref. 26, we found that when the maximum coolant temperature reaches the boiling point (2000°F for sodium at 100 psi) of the coolant (maximum conduction loss case), the heat-conduction loss only amounts to 0.21% of the total heat generation.

2. Neglecting of the conduction term yields the worst possible²⁷ temperature peaking in the reactor. Thus, if a design is based on the

worst case, the safety margin of the reactor system is increased. Example 2 (Chapter III) shows that from the stability analysis, the inclusion of the conduction terms in the heat-transfer equation yields a more stable system than the one without these terms.

C. Xenon and Other Processes

In a thermal reactor system, the xenon process affects the absorption cross section on the system. The thermal-absorption cross section of Xe^{135} is many orders greater than that of the other types of material in the reactor. Thus, after a thermal reactor has been operated for a period of time, the xenon effect on the system parameters has to be taken into consideration.

The contribution of Xe^{135} to the thermal-absorption cross section can be found by adding the buildup of the xenon atoms to the existing materials. If the thermal-absorption cross section without Xe^{135} is designated Σ_{a0} , the concentration of xenon atoms N_x , and the microscopic cross section of Xe^{135} , σ_{ax} , then the total thermal-absorption cross section of the reactor can be written as

$$\Sigma_a = \Sigma_{a0}(\vec{r}) + \sigma_{ax}N_x(\vec{r},t). \quad (8)$$

For breeder or converter systems, the buildup of Pu^{239} from the absorption of neutrons by U^{238} may be of some interest. The governing equations for this process can be written as

$$\frac{\partial}{\partial t} N_{pu}(\vec{r},t) = \lambda_{np}N_{np}(\vec{r},t) - \sum_m^M \sigma_{cm}^{pu}N_{pu}\phi_m, \quad (9)$$

and,

$$\frac{\partial}{\partial t} N_{np}(\vec{r},t) = \sum_m^M \gamma_m \Sigma_{cm}^{38} \phi_m - \lambda_{np}N_{np}, \quad (10)$$

where

ϕ_m = the m th energy-group neutron flux,

λ_{np} = decay constant of neptunium,

$N_{pu}(\vec{r},t)$ = number density of plutonium,

$N_{np}(\vec{r},t)$ = number density of neptunium,

σ_{cm}^{pu} = total microscopic capture cross section of plutonium,

γ_m = fraction that produces neptunium,

and

$$\Sigma_{cm}^{38} = \text{total macroscopic capture cross section of } U^{238}.$$

Equation (9) indicates that the rate of change of plutonium concentration is equal to the rate of plutonium production from neptunium decay, less the rate of loss of plutonium due to absorption and fission. Equation (10) shows that the rate of neptunium production from U^{238} neutron absorption less the rate of neptunium decay.

In this case, the total fission cross section is affected and can be written as

$$\Sigma_f = \Sigma_{fo} + \sigma_f N_{pu}(\vec{r}, t).$$

Similar equations governing the xenon effect can be derived from the balance equations of xenon and iodine.

D. The Control-rod Effect

In the formulation of the reactor equations, the parameters of the system are determined from the material composition of the reactor. If a control device moves in or out of the reactor core, the material composition of the reactor system changes in time. This effect can be taken into account in the analysis by either of two approaches--namely, a multi-region moving-boundary system, or a space-time varying-parameters system.

In the former method, the reactor is divided into sectors with some suitable boundary condition between adjacent regions. The motion of the control devices is then considered as a moving of the boundaries of each region. The composition of each region and the boundary condition between regions remain unchanged. This method is difficult to handle analytically since the geometry of each region may be too complex for analysis.

The latter method allows us to treat the reactor system as one set of equations with fixed boundary conditions. However, the system parameters will be considered depending upon both space and time. As a typical example, the absorption cross section (Σ_a) can be expressed as

$$\Sigma_a = \frac{\Sigma_{a1} + \Sigma_{a2}}{2} + \left(\frac{\Sigma_{a1} - \Sigma_{a2}}{2} \text{Sgn}\{z - f[x, y, z_1(t), \dots, z_s(t)]\} \right), \quad (11)^\dagger$$

$^\dagger \text{Sgn}[\cdot]$ is defined as

$$\begin{aligned} \text{Sgn}[x] &= +1, & x > 0; \\ &= 0, & x = 0; \\ &= -1, & x < 0. \end{aligned}$$

where

$$f[x, y, z_1(t), \dots, z_s(t)] = z_k(t) \text{ when } \begin{cases} x_k \leq x \leq x'_k \\ y_k \leq y \leq y'_k \end{cases},$$

$$= 0 \text{ otherwise,}$$

$$k = 1, 2, \dots, s,$$

s = the total number of control rods in the reactor,

$z_k(t)$ = the time motion of the k th control rod,

and

$$\begin{Bmatrix} x_k, x'_k \\ y_k, y'_k \end{Bmatrix} = \text{the space coordinates where the } k\text{th control rod is located.}$$

Other parameters can be expressed similarly.

For small variations of the control-rod positions, the expression in Eq. (11) can be further simplified. The idea is to approximate the implicit function form (space and control-rod motion) of the system parameters by a product of a spatially-dependent function and the control-rod motion. If Eq. (11) is expanded into a Taylor's series about a stationary (initial) position of the control rods ($\vec{z}_0$), Eq. (11) can be written as

$$\Sigma a[\vec{z}_0 + \Delta \vec{z}(t)] = \Sigma a(\vec{z}_0) + \frac{\partial}{\partial \vec{z}_0} [\Sigma a(\vec{z}_0)] \cdot \Delta \vec{z}(t) + \dots^\dagger$$

If the higher-order variations are neglected from the above expression, the control-rod effect becomes the second term in the equation and can be considered as the time-dependent feedback.

E. Effects of System States of Parameters

In a reactor system, the nuclear parameters discussed in Section II-A are determined from the chemical and physical properties of the material used in the reactor. When the system variables change their states or a control device moves, the properties of the material, as well as the material in the system, will change. Therefore, the nuclear parameters of the system should be considered in general, not only as functions of space and time, but also as functions of the system variables. In this study, only the fission and absorption cross sections in Eq. (1) are assumed to be functions of the fuel and coolant temperatures and the control-rod motion. Since the emphasis in this work is on fast breeder reactor systems, the above assumption should be justifiable. For a typical example, the system macroscopic cross section is written as

[†] The derivative of $\text{Sgn}[\cdot]$ is defined as a Dirac delta function.

$$\Sigma = \Sigma[r, t, T_f, T_c, z_1(t), z_2(t), \dots, z_S(t)].$$

The exact functional form of the above expression can only be defined for the specific reactor system. A typical example for the absorption cross section of a sodium-cooled fast breeder reactor²⁶ is

$$\Sigma_a(T_f, T_c) = \Sigma_a(T_{fo}, T_{co})[1 + \delta\sigma_a(T_f - T_{fo}) - \alpha_c(T_c - T_{co}) + \alpha_f(T_f - T_{fo})],$$

where

T_f, T_c = fuel and coolant temperatures,

T_{fo}, T_{co} = steady-state fuel and coolant temperatures,

α_c, α_f = respective temperature coefficients,

and

$\delta\sigma_a(T_f - T_{fo})$ = Doppler coefficient of the fuel.

F. Equations for Coupled Neutronic Systems

The multigroup diffusion Eq. (1) does not explicitly involve either the temperature distribution or other processes discussed in Section II-D above. However, the parameters in the equation depend on these system variables (Section II-E). Therefore, the multigroup diffusion equations (Section II-A) are coupled with the other equations governing the temperature and other processes in the reactor system; this coupling is through the nuclear parameters. The total reactor system can be represented by the combination of Eqs. (1), (5), and (6).

For simplicity, a two-dimension, two-group diffusion equation with one delayed-neutron-group equation, and fuel-temperature and coolant-temperature equations will be used for the stability analysis. These equations are

$$\left. \begin{aligned} & [\nabla \cdot D_1 \nabla - \Sigma_1 + \nu_1(1 - \beta) \chi_{p1} \Sigma_{f1}] \phi_1 + \nu_2(1 - \beta) \chi_{p1} \Sigma_{f2} \phi_2 \\ & + \lambda C \chi_{i1} = \frac{1}{v_1} \frac{\partial}{\partial t} \phi_1, \\ & [\nabla \cdot D_2 \nabla - \Sigma_2 + \nu_2(1 - \beta) \chi_{p2} \Sigma_{f2}] \phi_2 + \nu_2(1 - \beta) \chi_{p2} \Sigma_{f1} \phi_1 \\ & + \lambda C \chi_{i2} = \frac{1}{v_2} \frac{\partial}{\partial t} \phi_2, \\ & \beta(\nu_1 \Sigma_{f1} \phi_1 + \nu_2 \Sigma_{f2} \phi_2) - \lambda C = \frac{\partial}{\partial t} C, \end{aligned} \right\} \quad \begin{array}{l} (12) \\ \text{Contd.} \end{array}$$

$$\left. \begin{aligned}
 &A_1(\Sigma_{f1}\phi_1 + \Sigma_{f2}\phi_2) + A_2T_c - A_3T_f = \frac{\partial}{\partial t} T_f, \\
 \text{and} \\
 &B_3(\phi_1 + \phi_2) + B_2(T_f - T_c) - B_1 \frac{\partial}{\partial z} T_c = \frac{\partial}{\partial t} T_c.
 \end{aligned} \right\} \text{Contd. (12)}$$

with the boundary condition that

$$\phi_1(\text{at extrapolated boundary}) = 0,$$

$$\phi_2(\text{at extrapolated boundary}) = 0,$$

$$T_c \Big|_{z=0} = 0.$$

Since in stability analysis we only look at solution surface, the initial conditions are not specified. In these equations, D_1 and D_2 depend only on space; Σ_1 and Σ_2 may depend on both space and coolant temperature, T_c ; Σ_{f1} and Σ_{f2} may depend on space and the fuel temperature, T_f . Furthermore, we assume that the control device is made of natural or depleted uranium. Thus, Σ_{f1} and Σ_{f2} may also be functions of time, depending upon the type of control rod used.

CHAPTER III

STABILITY ANALYSIS OF REACTOR SYSTEMS

Lyapunov's methods have proven useful in the stability analysis of systems involving linear or nonlinear ordinary differential equations. The theories have also been extended to systems governed by partial differential equations.^{29,30} Bellman³¹ and Narashimhan³² have tried approaches of a similar nature to systems of parabolic equations. The advantages of this line of approach are: (1) They do not require the solving of the system equations, and (2) they provide information about the effect of system-parameter variations on the system stability. The major disadvantage is that the condition for system stability is, in general, only sufficient. Thus, the results may be too restrictive. However, with some ingenuity and experience, the results can be improved.

In this chapter, therefore, the general reactor system stability will be analyzed by using both the conventional methods and Lyapunov's methods. An outline of this chapter is as follows:

1. The essential definitions and notations used in functional analysis.
2. A presentation of Lyapunov's stability theorems and the technique of applying them to analyze the stability of reactor systems.
3. A comparison of Lyapunov's methods with conventional techniques in the analysis of linear and nonlinear reactor system stability.

A. Definitions and Notations³³

Consider a distributed-parameter system,

$$\frac{\partial}{\partial t} \vec{u}(\vec{r}, t) = \vec{h}[\vec{u}(\vec{r}, t), \vec{f}(\vec{r}, t), \vec{r}, t], \quad (13)$$

defined on a fixed spatial domain denoted by R (an open, connected subset of an M -dimensional Euclidean space, E_M). The boundary of R will be denoted by Ω , the closure of R by $\bar{R}$ ($\bar{R} = R \cup \Omega$), and the spatial coordinate vector by $\vec{r}$ [$\vec{r} = \text{col}(r_1 \dots r_M)$].

The state of such a system at any fixed time, t , can be generally specified by a state vector function, $\vec{u}(\vec{r}, t)$, of a set of real-valued functions, $u_i(\vec{r}, t)$, $i = 1, 2, \dots, N$, defined for all $\vec{r} \in R$. The set of all possible functions of $\vec{r}$ that $u_i(\vec{r}, t)$ can take on at any time, t , will be called the state component function space, $\Gamma_i(\bar{R})$, and the product space,

$$\Gamma(\bar{R}) = \prod_{i=1}^N \Gamma_i(\bar{R}),$$

will be called the state function space. The state of the control function denoted by the vector function, $\vec{f}(\vec{r}, t)$, consists of a set of real-valued functions, $f_i(\vec{r}, t)$, $i = 1, 2, \dots, K$, defined for all, or certain, subsets of $\vec{r} \in \bar{R}$. The set of all possible functions of $\vec{r}$ that $\vec{f}(\vec{r}, t)$ can take on at any fixed time, t , will be denoted by $\Gamma_c(\bar{R})$.

In system (13), $\vec{h}$ is a column vector with specified spatial differential operators whose parameters may depend on $\vec{r}$ and t .

Starting from a given initial state $\vec{u}(\vec{r}, 0)$, the differential equation (13) generally permits the construction of many possible solutions. To choose the solution that is appropriate to the physical situation, we introduce additional constraints or boundary conditions. The constraint used in this analysis may be represented by the vector equation,

$$B[\vec{u}(\vec{r}', t)] = \vec{0}, \quad \vec{r}' \in \Omega, \quad (14)$$

where B is matrix of specified spatial differential operators whose parameters may depend upon $\vec{r}'$ and t .

For a variable spatial-domain system, the entire spatial domain undergoes deformation in the course of time. To study the variable spatial-domain system usually involves mathematical difficulties. However, in many cases, it is possible to find some appropriate transformations which map the variable spatial domain onto a fixed domain. The resulting transformed system generally reduces to an equivalent fixed spatial-domain distributed parameter system and can be treated accordingly.

If the partial differential equation (13) with boundary conditions (14) is to represent a physical system, the following basic requirements must be met:

1. Its solution must exist and should be uniquely determined.
2. Its solution should depend continuously on the initial state.

The first requirement essentially excludes any ambiguous and contradictory properties in the physical situation. The second requirement implies that an arbitrary small variation of the given state can only lead to a small change in the corresponding solution. If both requirements are satisfied, then the problem of determining the dynamical behavior of the system is said to be well posed (in the sense of Hadamard). Some of the mathematical implications of the above fundamental idea are described in Ref. 33. For simplicity, the definition of a "dynamical system" by Zubov³⁴ will be used to define the properties of the physical system described by Eq. 13.

Definition. The system (13) is a dynamical system in the state function space $\Gamma(\bar{R})$ ³⁴ if the solution of the system satisfies the following conditions in the state function space $\Gamma(\bar{R})$ formed by a set of continuous and square integrable functions $\vec{\phi}(\vec{r})$ defined for all $\vec{r} \in \bar{R}$:

- a. $\lim_{t \rightarrow 0} \vec{u}[\vec{r}, t, \vec{\phi}(\vec{r})] \rightarrow \vec{\phi}(\vec{r})$.
- b. $\vec{u}[\vec{r}, t, \vec{\phi}(\vec{r})]$ is uniquely determined for any given $\vec{\phi}(\vec{r}) \in \Gamma(\bar{R})$.
- c. $\vec{u}[\vec{r}, t, \vec{\phi}(\vec{r})]$ is continuous and differentiable with respect to all its arguments.
- d. $\vec{u}[\vec{u}(\vec{r}, \vec{\phi}, t_1), \vec{r}, t_2] = \vec{u}[\vec{\phi}(\vec{r}), \vec{r}, t_1 + t_2]$.

Since in most nonlinear cases it is difficult to determine whether the system in question satisfies the above definition, it is assumed, from physical arguments, that all the physical systems treated here are "dynamical systems."

Additional definitions relevant to the ensuing analysis are:

Definition. A fixed spatial-domain distributed parameter dynamical system described by Eqs. (13) and (14) is said to be autonomous (free or enforced) if $\vec{f}(\vec{r}, t) = \vec{0}$ for all $\vec{r} \in \bar{R}$.

Definition. An equilibrium state $\vec{u}_{eq}(\vec{r})$ of a dynamical system (13) is defined to be the solution of the system with the left-hand side equal to a null vector and for $\vec{f}(\vec{r}, t) = \vec{0}$ for all $t \geq 0$, $\vec{r} \in \bar{R}$.

Definition. An equilibrium state $\vec{u}_{eq}(\vec{r})$ is called stable in the sense of Lyapunov if for any number $\epsilon > 0$ it is possible to find a number $\delta > 0$ such that $\xi[\vec{\phi}, \vec{u}_{eq}(\vec{r})]^\dagger < \delta$, $\xi[\vec{u}(\vec{r}, t, \vec{\phi}), \vec{u}_{eq}] < \epsilon$ for all $t > 0$. Furthermore, if $\xi[\vec{u}(\vec{r}, t, \vec{\phi}), \vec{u}_{eq}] \rightarrow 0$ as $t \rightarrow \infty$, then the equilibrium state $\vec{u}_{eq}(\vec{r})$ is called asymptotically stable.

Theorem 1 (Lyapunov's Stability Theorem). For an equilibrium state $\vec{u}_{eq}$ to be stable in the sense of Lyapunov, it is necessary and sufficient that in a sufficiently small neighborhood of the equilibrium state there exists a specified functional $V(\vec{u}, t)$ with the following properties:

- a. V is positive definite with respect to $\xi(\vec{u}, \vec{u}_{eq})$; i.e., for any sufficiently small $C_1 > 0$, there exists a $C_2 > 0$ depending on C_1 such that for $\xi > C_1$, $V > C_2$ for all $t \geq 0$, and $V \rightarrow 0$ as $\xi(\vec{u}, \vec{u}_{eq}) \rightarrow 0$.

[†] $\xi[\vec{u}_1(\vec{r}, t), \vec{u}_2(\vec{r}, t)]$ is the defined metric in $\Gamma(\bar{R})$, i.e., the distance between the states $\vec{u}_1$ and $\vec{u}_2$.

- b. V is continuous with respect to $\zeta(\vec{u}, \vec{u}_{eq})$; i.e., for any real $\epsilon > 0$, there exists a real $\delta > 0$, such that $V < \epsilon$ for all $\vec{u} \in \Gamma(\bar{R})$, $0 < t < \infty$, whenever $\zeta(\vec{\phi}, \vec{u}_{eq}) < \delta$.
- c. V evaluated along any system trajectory (the solution of the system) is nonincreasing for all $t > 0$ provided $\zeta[\vec{\phi}(\vec{r}), \vec{u}_{eq}(\vec{r})] < \delta_0$, where δ_0 is a sufficiently small positive number. Furthermore, if $V \rightarrow 0$ as $t \rightarrow \infty$, the system is said to be asymptotically stable.

This is the extension of the Lyapunov stability theorem for systems of ordinary differential equations to the distributed parameter dynamical system by Zubov.³⁵ Since Zubov only stated the theorem without proof, this theorem is proved here.

Necessity. If we assume that the equilibrium state $\vec{u}_{eq}$ is stable in the sense of Lyapunov, then by the definition of stability of an equilibrium state, when

$$\zeta(\vec{\phi}, \vec{u}_{eq}) < \delta,$$

it follows that

$$\zeta(\vec{u}, \vec{u}_{eq}) < \epsilon$$

(δ depends on ϵ , an arbitrary positive constant) for all $t \geq 0$. Let a V -functional be chosen as

$$V = \sup_{t \geq 0} \zeta(\vec{u}, \vec{u}_{eq})$$

for $\zeta(\vec{\phi}, \vec{u}_{eq}) < \delta$. Then $V(\vec{u}) < \epsilon$ for all $\zeta(\vec{\phi}, \vec{u}_{eq}) < \delta$. This proves condition b in the theorem. Also, since

$$V[\vec{u}(\vec{\phi}, 0)] = V(\vec{\phi}) = \zeta(\vec{\phi}, \vec{u}_{eq}),$$

thus $V(\vec{\phi}) \rightarrow 0$ as $\zeta(\vec{\phi}, \vec{u}_{eq}) \rightarrow 0$, and since

$$V = \sup_{t \geq 0} \zeta(\vec{u}, \vec{u}_{eq}) > C_1 > 0, \text{ thus } V(\vec{u}) > C_2 = C_1$$

for all $t \geq 0$. Hence V is positive definite and condition a is proved.

For condition c, let $\vec{\phi}$ be in the neighborhood of the equilibrium state $\vec{u}_{eq}$ since the system is assumed to be a dynamical system. Thus,

$$V(\vec{u}) = \sup_{t \geq t_1} \zeta\{\vec{u}[\vec{u}(\vec{\phi}, t_1), t] \vec{u}_{eq}\}$$

for some $0 \leq t_1 < t$. It follows that

$$\begin{aligned} V(\vec{u}) &= \sup_{t \geq t_1} \zeta[\vec{u}(\vec{\phi}, t_1 + t), \vec{u}_{eq}] \\ &\leq \sup_{t \geq 0} \zeta[\vec{u}(\vec{\phi}, t), \vec{u}_{eq}] \\ &\leq \sup_{t \geq 0} \zeta(\vec{\phi}, \vec{u}_{eq}) = V(\vec{\phi}). \end{aligned}$$

Consequently,

$$V(\vec{u}) \leq V(\vec{\phi}),$$

since t_1 is arbitrary.

Sufficiency. If there exists a V -functional that satisfies the above conditions, if we choose an arbitrary ϵ ,

$$\delta > \epsilon > 0,$$

and sets a constant λ ,

$$\lambda = \inf_{\vec{\phi} \in \Gamma(R)} V(\vec{\phi})$$

for $\epsilon = \zeta(\vec{\phi}, \vec{u}_{eq})$, then, since V is positive definite,

$$\lambda > 0.$$

Now, from condition b,

$$V(\vec{u}) < \lambda$$

for $\zeta(\vec{\phi}, \vec{u}_{eq}) < \delta$. To show that when

$$\zeta(\vec{\phi}, \vec{u}_{eq}) < \delta,$$

$$\zeta(\vec{u}, \vec{u}_{eq}) < \epsilon$$

for all $t \geq 0$, assume that the opposite is true; i.e., for some $t_1 > 0$,

$$\zeta(\vec{u}, \vec{u}_{eq}) \geq \epsilon.$$

Then, it must be that

$$V[\vec{u}(\vec{\phi}, t_1)] \geq \lambda,$$

which contradicts condition c since

$$\lambda \geq V(\vec{\phi}) \geq V(\vec{u})$$

for all $t \geq 0$. This proves that

$$\zeta(\vec{u}, \vec{u}_{eq}) < \epsilon$$

when

$$\zeta(\vec{\phi}, \vec{u}_{eq}) < \delta.$$

Q.E.D.

To apply this theorem to distributed parameter systems, the objective is to find a Lyapunov functional or V-functional which satisfies the above properties; i.e., V is positive definite and $\dot{V}$ negative definite (semidefinite). For ordinary differential equations, the V-functional is replaced by a V-function which has the same properties.

B. Stability of Linear Systems

Linear systems are of importance in the stability analysis of nonlinear systems since the dynamical behavior of many nonlinear physical systems in the neighborhood of certain prescribed motions can be approximated by that of a linear system. Furthermore, for certain classes of nonlinear systems, the conditions for stability of the nonlinear system rely on the relative stability of the approximated linear system. Therefore, it may be helpful to study the stability of linear systems before attacking the nonlinear stability problems.

Many linear distributed-parameter systems defined on a fixed spatial domain, $\bar{R}$, can be expressed in the following form:

$$\frac{\partial \vec{u}}{\partial t} = L(\vec{r}, t) \vec{u}(\vec{r}, t) + D(\vec{r}, t) \vec{f}(\vec{r}, t) \quad (15)$$

with the possible boundary conditions

$$B[\vec{u}(\vec{r}, t)] = \vec{0}, \quad \vec{r} \in \Omega, \quad (16)$$

where

L, B = matrices of linear spatial differential operators whose parameters may depend on $\vec{r}$ and t ,

$D(\vec{r}, t)$ = matrix whose elements are specified functions of $\vec{r}$ and t ,
and

$\vec{f}(\vec{r}, t)$ = input or forcing vector function.

1. Stability of Time-invariant[†] Autonomous Linear Systems

For a linear, time-invariant, distributed-parameter system, the stability of the system can be studied by many methods other than the Lyapunov methods. However, all these methods are centered about the linear spatial operator, L , whose properties give the condition for system stability. The main theorem is as follows:

Theorem 2. If the linear time-invariant, L , in (15) is an infinitesimal generator of a semigroup,^{††} and its spectrum, $\sigma(L)$, satisfies the condition,

$$\operatorname{Re}[\sigma(L)] \leq -\gamma, \quad \gamma = \text{constant} > 0, \quad (17)$$

then the solution[‡] of Eq. (15) is asymptotically stable in the large around its equilibrium state. Moreover, the solutions of Eq. (15) satisfy an estimate of the form

$$||\vec{u}(\vec{r}, t, \vec{u}(\vec{r}, 0))|| \leq \exp(-\gamma t) \cdot ||\vec{u}(\vec{r}, 0)|| \quad (18)$$

for $t \geq 0$. In general, $||\vec{u}||$ is defined as the norm of the vector function, $\vec{u}$, and in particular, it is defined here as

$$||\vec{u}|| = \left(\int_R \vec{u}^T \vec{u} d\vec{r} \right)^{1/2} \quad (19)$$

To apply the above theorem to a practical problem, we must verify that L is an infinitesimal generator of a semigroup. Secondly, we must establish the upper bound of the real part of the spectrum associated with L . These tasks may be difficult. It is not unreasonable to assume that for the dynamical reactor system, L is in general an infinitesimal generator of a semigroup.^{‡‡} The upper bound of the real part of the spectrum of L can be closely approximated by one of the following methods:

[†] Time-invariant system refers to a system where L does not depend upon t .

^{††} The properties of a semigroup and the conditions for L being an infinitesimal generator are given in Appendix A.

[‡] In most cases, for the sake of convenience, the equilibrium state is chosen to be the null state; i.e., $\vec{u} = \vec{0}$.

^{‡‡} See Appendix A.

1. System (15) can be reduced to that of a set of spatially-independent differential-difference equations³⁶ (in space), which can be written in matrix form as

$$\frac{d\vec{v}}{dt} = A\vec{v}, \quad (20)$$

where

$\vec{v}$ = vector function of the spatial nodes,

and

A = square matrix whose elements are constant (if L depends on t , then A could be time-dependent).

The largest real part of the eigenvalues of A approximates that of the spectrum of L .

2. System (16) can be reduced to a set of ordinary differential equations (in time) by modal expansion.³⁷ The system would in general be a denumerably infinite system. If this system is truncated to a finite system, the eigenvalues of the truncated system approximate the spectrum of L .

3. Variational methods such as those of Ritz or Galenkin³⁸ can be used instead of the conventional modal expansion.

The application of Lyapunov's second method to the general linear system is discussed in this section.

For the system (13) with boundary condition (14), a general V-functional,[†]

$$V = \frac{1}{2} \langle \vec{u}, P(\vec{r}) \vec{u} \rangle,$$

can be chosen for the analysis, where $P(\vec{r})$ is a spatially-dependent, positive definite matrix operator. The time derivative of the V-functional evaluated along the system trajectory would be

$$\begin{aligned} \dot{V} &= \frac{dV}{dt} = \langle \frac{1}{2} \dot{\vec{u}}, P(\vec{r}) \vec{u} \rangle + \frac{1}{2} \langle \vec{u}, P(\vec{r}) \dot{\vec{u}} \rangle \\ &= \frac{1}{2} \langle \dot{\vec{u}}, [L^+ P(\vec{r}) + P(\vec{r}) L] \vec{u} \rangle \\ &= \frac{1}{2} \langle \dot{\vec{u}}, M \vec{u} \rangle, \end{aligned}$$

[†] $\langle \vec{u}, \vec{v} \rangle$ is the inner product and is defined as

$$\langle \vec{u}, \vec{v} \rangle = \int_R \vec{u}^T \vec{v} d\vec{r}.$$

where $M = L^+P + PL$, a self-adjoint operator. Then, the system is stable (asymptotically stable) if $\dot{V}$ is negative semidefinite (definite). From the stability theorem (Theorem 1), if the linear system is asymptotically stable, there exists a $P(\vec{r})$. However, in most cases, finding $P(\vec{r})$ is rather difficult; hence, we usually assume a positive definite $P(\vec{r})$, then investigate the properties of V .

The generalized reactor system equation can be written in matrix form as

$$\sum_{ij}^M \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} \vec{u} + \sum_i^M B_i \frac{\partial}{\partial x_i} \vec{u} + C\vec{u} = \frac{\partial \vec{u}}{\partial t}, \quad (21)$$

where A_{ij} , B_i , and C are square matrices whose elements may depend on the spatial variable, $\vec{r}$, as well as time, t . The system may also have a boundary condition

$$B(\vec{u}) = \vec{0}, \quad (22)$$

where B may be a spatially-dependent, linear differential operator.

For system (21) with boundary condition (22), a sufficient condition† for the asymptotic stability can be obtained by first choosing a V -functional of the form

$$V = \frac{1}{2} \|\vec{u}\|^2 = \frac{1}{2} \langle \vec{u}, \vec{u} \rangle; \quad (23)$$

i.e., $P(r) = I$ the identity matrix. The time derivative is then

$$\begin{aligned} \dot{V} &= \frac{dV}{dt} = \frac{1}{2} (\langle \vec{u}, L\vec{u} \rangle + \langle \vec{u}, L^+\vec{u} \rangle) \\ &= \frac{1}{2} \langle \vec{u}, (L + L^+) \vec{u} \rangle. \end{aligned}$$

In system (21),

$$\begin{aligned} \langle \vec{u}, (L + L^+) \vec{u} \rangle &= \left\langle \vec{u}, \left[\sum_{ij}^M \frac{\partial}{\partial x_i} (A_{ij} + A_{ij}^T) \frac{\partial}{\partial x_j} \right. \right. \\ &\quad \left. \left. + \sum_i^M \left(B_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} B_i^T \right) + (C + C^T) \right] \vec{u} \right\rangle \end{aligned} \quad (24)$$

†Although Lyapunov's stability theorem is necessary and sufficient, the assumed V -function may not be the ultimate one; thus, by assuming a positive definite V -function, we can obtain only a sufficient condition.

The V-functional in Eq. (23) is positive definite. Thus, if $\dot{V}$ in Eq. (24) can be restricted to be negative definite (semidefinite), a sufficient condition for the system asymptotic stability (stability) is obtained. To make $\dot{V}$ negative definite, the $[(M+1)n]$ th-order matrix,

$$G = \begin{bmatrix} +A & -B \\ -B^T & -\left[C + C^T - \sum_i^M \frac{1}{2} \frac{\partial}{\partial x_i} (B_i + B_i^T) \right] \end{bmatrix}, \quad (25)$$

must be positive definite, where the elements of A are

$$A_{ml} = \left[\frac{1}{2} (A_{ij} + A_{ij}^T) \right]_{ml}, \quad m, l = 1, 2, \dots, M$$

and

$$B = \text{col.} \left[\frac{1}{2} (B_i - B_i^T) \right].$$

Condition (25) may sometime be too restrictive since it is only a sufficient condition. However, it can be improved for specific systems by choosing a different V-functional. Furthermore, for a reactor system represented by the multienergy group diffusion equation, condition (25) can be further simplified since the system equation is

$$\sum_i^M \frac{\partial}{\partial x_i} A \frac{\partial}{\partial x_i} \vec{u} + \sum_i^M B_i \frac{\partial}{\partial x_i} \vec{u} + C\vec{u} = \frac{\partial \vec{u}}{\partial t}, \quad (26)$$

where A is a diagonal matrix, and B_i is symmetrical. If the same V-functional in Eq. (23) is used, condition (25) is reduced to the matrix,

$$G = \left(-\Lambda - \frac{1}{2}(C + C^T) - \sum_i^M \frac{\partial}{\partial x_i} B_i \right), \quad (27)$$

where Λ is the matrix containing the largest eigenvalues of the system,

$$\sum_i^M \frac{\partial}{\partial x_i} A_i \frac{\partial}{\partial x_i} u_j = \lambda u_j.$$

[†]The subscript m and l refer to the matrix element related to the term $(\partial \vec{u} / \partial x_m)(\partial \vec{u} / \partial x_l)$.

The conclusion obtained from the above analysis for linear distributed parameter reactor systems may be stated as follows: The application of Lyapunov's second method to the stability analysis of a linear system does not necessarily provide the best results since

1. It only gives a sufficient condition for system stability, thus inducing unduly restrictive conditions to the system.

2. To obtain a better result, we must obtain the solution of a simpler system; however, it may not be analytically solvable. Therefore, it may not be possible to avoid the use of numerical techniques. However, the method does provide a sufficient condition for the system stability (no matter how restrictive it may be) without solving the system equations.

2. Stability of General Linear Systems

For the general linear systems, the parameters of the system may depend explicitly on time or its boundary may change with time. The stability theorem (Theorem 2) may or may not be generalized to time-variant systems. For time-variant, lumped-parameter systems, the system stability can be analyzed by conventional methods.^{39,40} The second method of Lyapunov is readily valid for the general linear system of this type since the spectrum of the system does not need to be examined. The only modification to that of condition (25) is that the matrix G would be positive definite for all $t \geq 0$.

3. Examples

Two examples are given to illustrate the techniques discussed above:

Example 1: Homogeneous Zero-power Reactor.⁹ This example deals with a simple linear reactor model. Three different methods illustrate the discussion given in Section III-B-1.

The mathematical model of the reactor is assumed to have only one energy group and one delayed-neutron group. The geometry of the reactor is assumed to be an infinite slab (one dimension in space) with a width ℓ . The reactor is assumed to be homogeneous. The system equation is thus given as

$$\left. \begin{aligned} v \left\{ D \frac{\partial^2}{\partial x^2} \phi - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \phi + \lambda C \right\} &= \frac{\partial \phi}{\partial t}, \\ \beta \nu \Sigma_f \phi - \lambda C &= \frac{\partial C}{\partial t}, \end{aligned} \right\} \quad (28)$$

and has the boundary condition that

$$\phi(0,t) = \phi(l,t) = 0.$$

$\phi(x,0)$ and $C(x,0)$ are arbitrary, where v , D , Σ_a , Σ_f , β , and λ are positive constants and have the same meaning as given in Section II-A.

Assume that there exists a nontrivial, steady-state solution, ϕ_0 , C_0 (flux and delayed-neutron flux); i.e., the system

$$\left. \begin{aligned} v \left[D \frac{\partial^2}{\partial x^2} \phi_0 - (\Sigma_a - \nu \Sigma_f) \phi_0 \right] &= 0, \\ \beta \nu \Sigma_f \phi_0 - \lambda C_0 &= 0, \end{aligned} \right\} \quad (29)$$

has a nontrivial solution.

Let[†] the variation of ϕ from ϕ_0 be ψ , where

$$\psi = \phi - \phi_0; \quad c = C - C_0.$$

If Eqs. (28) and (29) are combined, the equation of the variations becomes

$$\left. \begin{aligned} v \left\{ D \frac{\partial^2}{\partial x^2} \psi - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \psi + \lambda c \right\} &= \frac{\partial \psi}{\partial t}, \\ \beta \nu \Sigma_f \psi - \lambda c &= \frac{\partial c}{\partial t}. \end{aligned} \right\} \quad (30)$$

Method a. Spectral Analysis. Solve for the eigenvalue problem (30); i.e.,

$$D \frac{\partial^2}{\partial x^2} \psi - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \psi + \lambda c = \frac{\mu}{v} \psi,$$

$$\beta \nu \Sigma_f \psi - \lambda c = \mu c.$$

By substituting the second equation into the first, we obtain

$$D \frac{\partial^2}{\partial x^2} \psi - \left[\Sigma_a - (1 - \beta) \nu \Sigma_f - \frac{\lambda \beta \nu \Sigma_f}{\lambda + \mu} + \frac{\mu}{v} \right] \psi = 0.$$

Since the reactor is homogeneous, all the parameters are constants. Therefore, the eigenvalue (μ) of the system can be determined from the second-order equation,

[†]This change of variables moves the equilibrium state of the system from (ϕ_0, C_0) to $(0, 0)$.

$$\frac{\mu}{v} - \frac{\lambda \beta v \Sigma_f}{\lambda + \mu} - (1 - \beta) v \Sigma_f + \Sigma_a + \frac{n^2 \pi^2}{\ell^2} D = 0. \quad (31)$$

Furthermore, since Eq. (29) has a nontrivial solution, the condition

$$D \frac{\pi^2}{\ell^2} + \Sigma_a - v \Sigma_f = 0 \quad (32)$$

must be satisfied. Hence, the largest eigenvalue, $\mu_{\max}$, from Eq. (31) is

$$\mu_{\max} = 0,$$

and the spectrum of this system satisfies the inequality,

$$\operatorname{Re}[\sigma(L)] \leq 0.$$

Thus the system is only stable but not asymptotically stable.

Method b. Lyapunov's Method; Trial Functional I. If we choose a positive definite V-functional of the form,

$$V = \frac{1}{2} \int_0^\ell (\psi^2 + c^2) dx,$$

then the time derivative of V along the system trajectory [Eq. (30)] is

$$\begin{aligned} \dot{V} \leq - \int_0^\ell \left\{ D v \frac{\pi^2}{\ell^2} \psi^2 + v [\Sigma_a - (1 - \beta) v \Sigma_f] \psi^2 \right. \\ \left. - (v \lambda + \beta v \Sigma_f) \psi c + \lambda c^2 \right\} dx. \end{aligned}$$

For $\dot{V}$ to be negative definite, the following conditions must be satisfied:

$$\left. \begin{aligned} D \frac{\pi^2}{\ell^2} + \Sigma_a - (1 - \beta) v \Sigma_f &> 0, \\ \lambda &> 0, \end{aligned} \right\} \quad (33)$$

and

$$4 \lambda v \left\{ D \frac{\pi^2}{\ell^2} + [\Sigma_a - (1 - \beta) v \Sigma_f] \right\} - (v \lambda + \beta v \Sigma_f)^2 > 0. \quad (34)$$

Conditions in Eq. (33) are automatically satisfied from Eq. (32), but Eq. (34) is not satisfied. Therefore, this approach indicated that for this V-functional, no definite conclusion can be obtained. If we restrict $\dot{V}$ to be negative definite, then the condition (32) will be violated; i.e., the system does not have a nontrivial solution.

Method c. Lyapunov's Method; Trial Functional II. Instead of the above V-functional, choose another V-functional of the form,

$$V = \int_0^{\ell} \begin{bmatrix} \psi \\ c \end{bmatrix}^T \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{v}{\beta \nu \Sigma_f} \end{bmatrix} \begin{bmatrix} \psi \\ c \end{bmatrix} dx.$$

Since $\lambda, v, \beta, \nu \Sigma_f > 0$, hence, V is positive definite. The time derivative of V is

$$\dot{V} = \int_0^{\ell} \begin{bmatrix} \psi \\ c \end{bmatrix}^T \begin{bmatrix} 2 \frac{v}{\lambda} \left\{ -D \frac{\pi^2}{\ell^2} - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \right\} & 2v \\ 2v & -\frac{2\lambda v}{\beta \nu \Sigma_f} \end{bmatrix} \begin{bmatrix} \psi \\ c \end{bmatrix} dx.$$

If the condition (32) is used, we find that $\dot{V} = 0$ for any (ψ, c) satisfies the condition $(\beta \nu \Sigma_f \psi - \lambda c = 0)$ and we obtain the same conclusion as in Method a.

Example 2: Homogeneous Power Reactor. A linearized model of a simple power reactor system is used in this example. The system in terms of the variations from the steady state is

$$\left. \begin{aligned} v \left\{ D \frac{\partial^2 \psi}{\partial x^2} - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \psi - b\tau + \lambda c \right\} &= \frac{\partial \psi}{\partial t}, \\ \beta \nu \Sigma_f \psi - \lambda c &= \frac{\partial c}{\partial t}, \\ a_1 \psi - a_2 \tau &= \frac{\partial \tau}{\partial t}, \end{aligned} \right\} \quad (35)$$

where ψ , c , and τ are the deviation from the steady state (ϕ_0, C_0, T_0) .

If we assume that a steady-state solution (ϕ_0, C_0, T_0) exists, then the condition in Eq. (32) is still satisfied. Also, the same boundary conditions and homogeneity are assumed as in Example 1.

Method a. Spectral Analysis. Solve for the eigenvalue problem (35); i.e.,

$$D \frac{\partial^2}{\partial x^2} \psi - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \psi - b\tau + \lambda c = \frac{\mu}{v} \psi,$$

$$\beta \nu \Sigma_f \psi - \lambda c = \mu c,$$

$$a_1 \psi - a_2 \tau = \mu \tau.$$

Combining the above equations, we see that the largest eigenvalue can be obtained from the equation

$$\frac{\mu}{v} + \frac{D\pi^2}{l^2} + \Sigma_a - \nu \Sigma_f + \beta \nu \Sigma_f + \frac{a_1 b}{a_2 + \mu} - \frac{\lambda \beta \nu \Sigma_f}{\lambda + \mu} = 0.$$

Imposing the condition (32), we can further reduce the above equation to a third-order equation in μ (the eigenvalue) as

$$\mu^3 + (a_2 + \lambda + v\beta\nu\Sigma_f) \mu^2 + (a_2\lambda + a_2v\beta\nu\Sigma_f + a_1bv) \mu + a_1b\lambda v = 0. \quad (36)$$

The stability of this linear system can be examined by using Routh's criterion (this is only valid for homogeneous systems).⁴¹ For the above system, if a_1 , a_2 , and b are positive constants, the system is asymptotically stable.

If the parameter values of the EBR-II⁴² are used in this example, the roots of the above equation are

$$\mu = -2.93 \times 10^{-2}, -9.27 \times 10^{-1}, \text{ and } -9.27 \times 10^5;$$

hence,

$$\mu_{\max} = -2.93 \times 10^{-2}.$$

Method b. Lyapunov's Method. For the system (35), the V-functional is of the form,

$$V = \int_0^l \begin{bmatrix} \psi \\ c \\ \tau \end{bmatrix}^T \begin{bmatrix} \frac{1}{b\lambda} & 0 & 0 \\ 0 & \frac{v}{b\beta\nu\Sigma_f} & 0 \\ 0 & 0 & \frac{v}{a_1\lambda} \end{bmatrix} \begin{bmatrix} \psi \\ c \\ \tau \end{bmatrix} dx.$$

The time derivative of V evaluated along the system trajectory is then [again, by applying condition (32)]

$$\dot{V} = - \int_0^{\ell} \left[\frac{v}{b\beta\nu\Sigma_f\lambda} (\beta\nu\Sigma_f\lambda - \lambda c)^2 + \frac{a_2 v}{a_1 \lambda} \tau^2 \right] dx \leq 0.$$

The equal sign of the above inequality holds only when

$$\beta\nu\Sigma_f\lambda - \lambda c = 0 \text{ and } \tau = 0,$$

which does not belong to the system trajectory. Therefore, the system is asymptotically stable if a_2 , a_1 , and b are positive.

In this example, the temperature equation does not contain the conduction terms. To justify the assumption made in Chapter II concerning the neglecting of the conduction in the coolant, we add another term in this example. The system equation then becomes

$$\left. \begin{aligned} v \left\{ D \frac{\partial^2 \psi}{\partial x^2} - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \psi - b\tau + \lambda c \right\} &= \frac{\partial \psi}{\partial t}, \\ \beta \nu \Sigma_f \psi - \lambda c &= \frac{\partial c}{\partial t}, \\ a_1 \psi - a_2 \tau - a_3 \frac{\partial^2 \tau}{\partial x^2} &= \frac{\partial \tau}{\partial t}, \end{aligned} \right\} \quad (37)$$

with the additional boundary condition, $\tau(0, t) = \tau(\ell, t) = 0$, where a_3 is the heat-conduction coefficient of the reactor core composition.

If the same Lyapunov functional is used, the time derivative of V (denoted by $\dot{V}_1$) evaluated along the system trajectory is then found, by applying condition (32), to be

$$\dot{V}_1 = - \int_0^1 \left[\frac{v}{b\beta\nu\Sigma_f\lambda} (\beta\nu\Sigma_f\psi - \lambda c)^2 + \frac{a_2 v}{a_1 \lambda} \right] dx + \int_0^1 \tau \frac{va_3}{a_1 \lambda} \frac{\partial^2 \tau}{\partial x^2} dx,$$

since the second term in $\dot{V}_1$ is less than or equal to zero if $a_3 > 0$; therefore,

$$\dot{V}_1 \leq - \int_0^1 \left[\frac{v}{b\beta\nu\Sigma_f\lambda} (\beta\nu\Sigma_f\psi - \lambda c)^2 + \frac{a_2 v}{a_1 \lambda} \tau^2 \right] dx \leq \dot{V}.$$

Since $\dot{V}_1$ is more negative, the stability analysis made without the heat-conduction term provides a better safety margin.

C. Stability of Nonlinear Systems

In the study of nonlinear dynamical systems, although we assume that the solution of the system exists and that it can be uniquely determined, it is not always possible to analyze the stability of the system using conventional methods. The difficulties involved in the conventional methods can be summarized as follows:

1. The use of spectral theory for linear systems such as Theorem 2 may not be defined for nonlinear systems.
2. The usual technique in reducing the distributed-parameter system to a lumped-parameter system is no longer mathematically justifiable.
3. The finite differential-difference method discussed in Eq. (35) involves a far greater number of equations for the analysis.

Therefore, the methods of Lyapunov become essential for the stability analysis of nonlinear systems.

Assume a nonlinear system has the form

$$\frac{\partial}{\partial t} \vec{u} = L(\vec{r})\vec{u} + \vec{g}(\vec{f}, \vec{u}, \vec{r}, t), \quad (38)$$

with the possible boundary condition

$$B[\vec{u}(\vec{r}^1, t)] = \vec{0}, \quad \vec{r}^1 \in \Omega, \quad (39)$$

where, as in the past,

$L(\vec{r})$ = spatially-dependent linear matrix operator,

and

$\vec{g}(\vec{f}, \vec{u}, \vec{r}, t)$ = general nonlinear vector functional.

Note that a general nonlinear system (13) can always be reduced to the form of Eq. (38) by expanding the right-hand side about its equilibrium state. The vector function, $\vec{g}$, would then contain all the higher-order nonlinear terms in the expansion.

1. Stability Analysis of Time-invariant Nonlinear Systems

Before analyzing the stability of system (38) with boundary condition (39), we make the following basic assumptions with respect to the linear operator, L , and the nonlinear vector functional, $\vec{g}(\vec{f}, \vec{u}, \vec{r}, t)$:

a. The linear system,

$$\frac{\partial}{\partial t} \vec{u} = L\vec{u},$$

with boundary condition (39) is asymptotically stable.

b. L is an infinitesimal generator of a continuous semigroup, and its spectrum $\sigma(L)$ satisfies the condition,

$$\operatorname{Re}[\sigma(L)] \leq -\gamma, \gamma = \text{constant} > 0.$$

c. The vector functional $\vec{g}(\vec{f}, \vec{u}, \vec{r}, t)$ has the property that

$$\vec{g}(\vec{f}, \vec{u}, \vec{r}, t) = \vec{0}$$

for $\vec{u} = \vec{0}$ and satisfies a Lipschitz condition,

$$\|\vec{g}(\vec{f}, \vec{u}, \vec{r}, t)\| \leq \rho(t) \|\vec{u}\|,$$

where $\rho(t)$ is a continuous, nonnegative scalar function of t defined for $t \geq 0$.

The first assumption is necessary since if the system is not asymptotically stable about its equilibrium state, the nonlinear system certainly would not be asymptotically stable (in the sense of Lyapunov).

Condition c actually provides an upper bound for the nonlinearities in the system, as will be discussed below.

Theorem 3. For the system (38) with boundary condition (39) satisfying assumptions a-c, the equilibrium state of (38) will be asymptotically stable if

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \rho(t') dt' < \gamma, \quad (40)$$

where γ has the same meaning as in Theorem 2, i.e., the largest real part of the spectrum of L .

Proof. Since L is an infinitesimal generator of continuous semigroup operator $[\Phi(t, t')]$,[†] where $\Phi(t, t')$ satisfies

$$\|\Phi(t, t')\| \leq \exp[-\gamma(t - t')], \quad (41)$$

[†]See Ref. 43. For the system discussed here, $[\Phi(t, t')]$ is the generalized Green's matrix operator; i.e.,

$$\Phi(t, t') \vec{u}(\vec{r}', t) = \int_R G(\vec{r}, \vec{r}', t, t') \vec{u}(\vec{r}', t) d\vec{r}.$$

the solution of system (38) satisfies the integral equation,

$$\vec{u}(\vec{r}, t) = \Phi(t, 0) \vec{u}(\vec{r}, 0) + \int_0^t \Phi(t, t') \vec{g}(\vec{f}, \vec{u}, \vec{r}, t') dt' \quad (42)$$

Thus,

$$\begin{aligned} \|\vec{u}(\vec{r}, t)\| &\leq \|\Phi(t, 0)\| \|\vec{u}(\vec{r}, 0)\| + \int_0^t \|\Phi(t, t')\| \|\vec{g}(\vec{f}, \vec{u}, \vec{r}, t')\| dt' \\ &\leq \|\vec{u}(\vec{r}, 0)\| \exp(-\gamma t) + \int_0^t \exp[-\gamma(t - t')] \|\vec{g}(\vec{f}, \vec{u}, \vec{r}, t')\| dt'. \end{aligned}$$

By the use of the Bellman-Gronwall Inequality,⁴⁴ we can show that

$$\|\vec{u}(\vec{r}, t)\| \leq \|\vec{u}(\vec{r}, 0)\| \exp(-\gamma t) \cdot \exp\left[\int_0^t \rho(t') dt'\right].$$

In order that $\|\vec{u}(\vec{r}, t)\| \rightarrow 0$ as $t \rightarrow \infty$, a sufficient condition is that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \rho(t') dt' < \gamma.$$

Q.E.D.

Theorem 3 gives a sufficient condition for asymptotic stability of system (38), and condition c dictates the restriction on the nature of the nonlinearities in the system. The inequality (40) indicates that for a non-linear system with an asymptotically linear part, the stability of the system depends strongly on the relative stability of the linear part of the system.

A similar result can be found by the use of Lyapunov's second method. The following theorem illustrates the method:

Theorem 4. For the system (38) with boundary condition (39) satisfying assumption a, it is asymptotically stable in the sense of Lyapunov if the nonlinear vector functional, $\vec{g}(\vec{u}, \vec{r}, t)$, is within the sector,

$$\|\vec{g}(\vec{u}, \vec{r}, t)\| < \gamma \|\vec{u}\|, \gamma > 0. \quad (43)$$

This theorem can be proved by choosing a V-functional of the form,

$$V = \frac{1}{2} \langle \vec{u}, P(\vec{r}) \vec{u} \rangle,$$

where P is a positive definite operator. The time derivative of the V -functional along the system trajectory is

$$\begin{aligned}\dot{V} &= \frac{1}{2} \langle \dot{\vec{u}}, P\vec{u} \rangle + \frac{1}{2} \langle \vec{u}, P(\vec{r})\dot{\vec{u}} \rangle \\ &= \frac{1}{2} \langle \vec{u}, (L^+P + PL)\vec{u} \rangle + \langle \vec{g}, P\vec{u} \rangle \\ &\leq -k \|\vec{u}\|^2 + \|\vec{g}\| \|\vec{u}\|, \quad k > 0, \\ &\leq -k \|\vec{u}\|^2 + \|\vec{g}\| \|P\| \|\vec{u}\|.\end{aligned}$$

For $\dot{V}$ to be negative definite, it must be that

$$\|\vec{g}\| < \frac{k}{\|P\|} \|\vec{u}\| = \gamma \|\vec{u}\|,$$

where

$$\gamma = \frac{k}{\|P\|}^\dagger$$

and k is the smallest eigenvalue of the system,^{††}

$$(L^+P + PL)\vec{u} = -\mu\vec{u}.$$

Corollary. If the system (38) with boundary condition (39) satisfies assumption a of Theorem 3 and

$$\langle \vec{u}, \vec{g} \rangle > 0, \tag{44}$$

then the system is asymptotically stable in the large. The proof is straightforward by choosing a V -functional,

$$V = \frac{1}{2} \|\vec{u}\|^2,$$

and $\dot{V}$ is negative definite if the condition (44) is satisfied.

In the above discussion, the system nonlinearity is quite general; thus the condition may be much too difficult to verify. However, for special types of nonlinearity, simpler conditions can be obtained. Two special types of nonlinearity have been investigated.

[†] $\|P\| = \sup(\|\vec{P}\vec{u}\| : \|\vec{u}\| = 1).$

^{††} See Appendix B.

Case I. If system (38) can be written as

$$\left. \begin{aligned} L\vec{u} + \vec{b}f(\sigma) &= \frac{\partial}{\partial t} \vec{u} \\ \sigma &= \vec{c}^T \vec{u}, \end{aligned} \right\} \quad (45)$$

and $f(\sigma)$ satisfies

$$\left. \begin{aligned} \text{(i)} \quad f(\sigma) &= 0, \quad \sigma = 0, \\ \text{(ii)} \quad \sigma f(\sigma) &> 0, \quad \sigma \neq 0, \end{aligned} \right\} \quad \text{for all } t \geq 0; \quad (46)$$

then the system is asymptotically stable within the sector

$$f^2(\sigma) < \frac{1}{\beta} \sigma f(\sigma), \quad \beta > 0,$$

where $\beta_{\min}$ corresponding to the largest sector is given by

$$\beta_{\min} = \frac{1}{4k} \left| \frac{\vec{c}}{|\vec{b}|} \right| \left| \vec{b} + \frac{|\vec{b}|}{|\vec{c}|} \vec{c} \right|^2, \quad (47)$$

and $\vec{b}$ and $\vec{c}$ are constant column vectors.

Case II. If the system (38) can be written as

$$\left. \begin{aligned} L\vec{u} + B\vec{f}(\sigma) &= \frac{\partial}{\partial t} \vec{u}, \\ \vec{\sigma} &= C\vec{u}, \end{aligned} \right\} \quad (48)$$

where $f_i(\sigma_i) = \text{col. } [f_1(\sigma_1), f_2(\sigma_2), \dots, f_n(\sigma_n)]$ and $f_i(\sigma_i)$ satisfies

$$\left. \begin{aligned} f_i(\sigma_i) &= 0, \quad \sigma_i = 0, \\ \sigma_i f_i(\sigma_i) &> 0, \quad \sigma_i \neq 0, \end{aligned} \right\} \quad (49)$$

then, system (48) with condition (49) is asymptotically stable within the sector,

$$0 < f_i^2(\sigma_i) < \frac{1}{\beta_i} \sigma_i f_i(\sigma_i),$$

$$\beta_i > 0, \quad i = 1, 2, \dots, n,$$

where $\beta_{i_{\min}}$ is given by the set S ,

$$S = \left\{ \beta_{\min} : \underset{\vec{\alpha} > 0}{\text{Min}} [G] \text{ positive definite} \right\}, \quad (50)$$

where

$$G = \begin{bmatrix} K & -\frac{1}{2}(C\vec{\alpha}^T \vec{1} + B) \\ -\frac{1}{2}(C\vec{\alpha}^T \vec{1} + B) & D_{\alpha\beta} \end{bmatrix},$$

$\vec{\alpha}, \vec{\beta} = \text{constant vectors},$

$D_{\alpha\beta} = \text{scalar matrix of } (\vec{\alpha}, \vec{\beta}),$

and

$\vec{1} = \text{column vector of unity.}$

The conditions (47) and (50) can be easily derived by choosing V-functionals,

$$V = \frac{1}{2} \|\vec{u}\|^2 + \exp(-at) \int_0^t \alpha \langle \sigma - \beta g(\sigma), g(\sigma) \rangle dt,$$

and

$$V = \frac{1}{2} \|\vec{u}\|^2 + \exp(-at) \int_0^t \langle \vec{\alpha}^T [\vec{\sigma} \vec{1}^T - \vec{\beta} g^T(\sigma)], \vec{g} \rangle dt,$$

respectively.

The above discussion indicates that there is no way to obtain a universal V-function for general nonlinear systems. The reason is that the second method of Lyapunov is merely an approach rather than a systematic method. The success of its application depends, to a large extent, on the ingenuity and experience of the user.

The following examples apply the method to a reactor system discussed in Chapter II.

2. Examples

Example 3. The reactor model used in this example is the same as in Example 2 except that a nonlinear term (which depends on the temperature variation, τ) is added to the first equation in Eq. (35). The system equation is

$$\left. \begin{aligned} v \left\{ D \frac{\partial^2 \psi}{\partial x^2} - [\Sigma_a - (1 - \beta) \nu \Sigma_f] \psi - b\tau - g(\tau) + \lambda c \right\} &= \frac{\partial \psi}{\partial t}, \\ \beta \nu \Sigma_f \psi - \lambda c &= \frac{\partial c}{\partial t}, \\ a_1 \psi - a_2 \tau &= \frac{\partial \tau}{\partial t}. \end{aligned} \right\} \quad (51)$$

The assumption used in Example 2 will be used here. Furthermore,

$$g(\tau) = 0, \tau = 0, \text{ for all } \vec{r} \in \bar{R},$$

is assumed to hold.

a. Spectral Analysis. If the maximum eigenvalue obtained from Example 2 (EBR-II), i.e.,

$$\mu_{\max} = -2.93 \times 10^{-2},$$

is used to determine the stability of the nonlinear system (51), then, if

$$\|g(\tau)\|^2 < \mu_{\max} \|\tau\|^2, \quad (52)$$

the system will be asymptotically stable in the large. However, if condition (52) only holds for certain values of τ , then a region of stability can be obtained from the condition.

Fig. 3. Region of Stability for Example 3 (Spectral Analysis)

The condition (52) can be interpreted geometrically as shown in Fig. 3. The shaded area indicates the region of asymptotic stability. The value of $\mu_{\max}$ seems rather small. However, the effect of temperature change to the absorption cross section is usually of the order of 10^{-4} ; i.e., $(\delta \Sigma_a / \Sigma_a) / ^\circ\text{C} \simeq 10^{-4}$.

b. Lyapunov's Method. Write the system (51) in operator form as in Eq. (45); i.e.,

$$\left. \begin{aligned} L\vec{u} + \vec{b}g(\sigma) &= \frac{\partial \vec{u}}{\partial t}, \\ \sigma &= \vec{c}^T \vec{u}, \end{aligned} \right\} \quad (53)$$

where

$$L = \begin{bmatrix} vD \frac{\partial^2}{\partial x^2} - v[\Sigma_a - (1 - \beta) \nu \Sigma_f] & v\lambda & -vb \\ \beta \nu \Sigma_f & -\lambda & 0 \\ a_1 & 0 & -a_2 \end{bmatrix},$$

$$u = \begin{bmatrix} \psi \\ c \\ \tau \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and

$$f(\sigma) = f(\tau).$$

If we use the condition (46), the system will be asymptotically stable within the sector,

$$f^2(\sigma) < \frac{1}{\beta_{\min}} \sigma f(\sigma) \text{ for all } \vec{r} \in \bar{R}, \quad (54)$$

where $\beta_{\min}$ can be determined from Eq. (47), and in this case

$$\beta_{\min} = \frac{1}{k},$$

Fig. 4. Region of Stability for Example 3 (Lyapunov's Method)

where k is obtained from the method discussed in Theorem 4. If the P in Example 2 is used, numerically

$$k = -5.1 \times 10^{-3}.$$

Condition (54) can also be interpreted geometrically as shown in Fig. 4. The two results obtained for this example indicate that a better result can be obtained from the spectral analysis. However, as mentioned in Section III-C, the result obtained from Lyapunov's method is only a sufficient condition. From

experience, the result can be improved considerably. Example 4 deals with a slightly more complex system than Example 3. However, by selecting a different V-function, we obtain a much better result.

Example 4. Consider a system described by

$$\left. \begin{aligned} \nabla \cdot \{ \nabla \cdot D \nabla \phi - [\Sigma_a(T) - (1 - \beta) \nu \Sigma_f] \phi + \lambda C \} &= \frac{\partial \phi}{\partial t}, \\ \beta \nu \Sigma_f \phi - \lambda C &= \frac{\partial}{\partial t} C, \\ a_1 \phi - a_2 T - a_3 \frac{\partial T}{\partial z} &= \frac{\partial T}{\partial t}, \end{aligned} \right\} \quad (55)$$

where ϕ , C , and T have the same meaning as in Example 3 (all parameters > 0).

By assuming that a steady-state solution of the system exists, and performing a transformation as in Example 2, we can write system (55) in terms of the variation as

$$\begin{aligned} \nabla \cdot \{ \nabla \cdot D \nabla \psi - [\Sigma_a(T_0) - (1 - \beta) \nu \Sigma_f] \psi + \lambda c - (\phi_0 + \psi) g(T) \} &= \frac{\partial \psi}{\partial t}, \\ \beta \nu \Sigma_f \psi - \lambda c &= \frac{\partial c}{\partial t}, \\ a_1 \psi - a_2 T - a_3 \frac{\partial T}{\partial z} &= \frac{\partial T}{\partial t}. \end{aligned}$$

The system is assumed to have a boundary condition such that

$$\begin{aligned} \psi(\vec{r}, t) \Big|_{\vec{r} \in \Omega} &= 0, \\ T(\vec{r}, t) \Big|_{z=0} &= 0. \end{aligned}$$

If a V-functional of the form

$$V = \int_R \int_0^T g(T) dT + \int_0^\psi a_1 \frac{\psi}{\phi_0 + \psi} d\psi + \int_0^c a_1 \frac{c}{C_0 + c} dc d\vec{r}$$

is chosen, then the time-derivative along the system trajectory is

$$\begin{aligned}
\dot{V} &= \int_R \left(a_1 \psi - a_2 \tau - a_3 \frac{\partial \tau}{\partial y} \right) g(\tau) d\vec{r} \\
&+ \int_R \frac{a_1 \psi}{\phi_0 + \psi} \left\{ v \nabla \cdot D \nabla \psi - v [\Sigma_a(T_0) - (1 - \beta) v \Sigma_f] \psi \right. \\
&+ \left. \lambda v c - (\phi_0 + \psi) g(\tau) \right\} d\vec{r} \\
&+ \int_R \frac{a_1 c}{C_0 + c} (\beta v \Sigma_f \psi - \lambda c) d\vec{r} \\
&\leq - \int_R \left(a_2 \tau + a_3 \frac{\partial \tau}{\partial z} \right) g(\tau) d\vec{r} \\
&- \int_R a_1 v \frac{(\beta v \Sigma_f \psi - \lambda)^2}{\lambda (C_0 + c) (\phi_0 + \psi)} d\vec{r}.
\end{aligned}$$

For $\dot{V}$ to be negative definite, $g(\tau)$ must satisfy the condition that

$$\left. \begin{aligned} g(\tau) &= 0, & \tau &= 0, \\ \tau g(\tau) &> 0, & \tau &\neq 0. \end{aligned} \right\} \quad (56)$$

The proof is straightforward since the first term can be written as

$$\begin{aligned}
- \int_R \left\{ [a_2 \tau g(\tau)] + a_3 \frac{\partial}{\partial z} [\tau g(\tau)] \right\} d\vec{r} &= - \int_R a_2 \tau g(\tau) d\vec{r} - a_3 \tau g(\tau) \Big|_{z=\ell z} \\
&> 0 \text{ for } \tau \neq 0 \\
&= 0 \text{ for } \tau = 0
\end{aligned}$$

[from condition (56)].

The second term is negative semidefinite. However, the term is equal to zero when

$$\beta v \Sigma_f \psi = \lambda c,$$

which is not the system trajectory; thus, the system is asymptotically stable. The geometrical interpretation of the result is shown in Fig. 5.

Fig. 5. Region of Stability for Example 4

The result in Example 1 shows that, in a linear system, the spectral analysis yields the necessary and sufficient condition for system stability. For the zero-power reactor considered in the example, the criticality condition [Eq. (32)] can thus be obtained. On the other hand, Lyapunov's method can, in general, provide only a sufficient condition for stability. However, with experience, the necessary and sufficient condition can be obtained by selecting the V-functional as shown in Method c (p. 36).

In Example 2, an asymptotically stable system is analyzed. Both methods yield the same conclusion; i.e., the system is indeed asymptotically stable. However, the spectral analysis provides a little more information as to the relative stability of the system. This can be done by examining how close the maximum real part of the eigenvalue is to zero. On the other hand, Lyapunov's method eliminates the calculation of eigenvalues of the system.

As for the two examples of nonlinear system analyses (Examples 3 and 4), bounds or regions of stability are obtained. However, these bounds on system variables or regions of stability are valid only if the system model is valid. If the system model is known to be valid in certain parts of the regions of stability (e.g., the heat-transfer equation developed in Chapter II where the model is only valid when the coolant temperature is below the boiling point of the coolant), additional bounds on system variables should be imposed in the analysis.

CHAPTER IV

OPTIMAL CONTROL OF NUCLEAR REACTOR SYSTEMS

The classical design techniques for control systems (e.g., the Nyquist diagram, Bode plots, and Root-locus method of the conventional frequency-domain techniques and the modification of the eigenvalues of the system equations in the time-domain techniques) are more or less empirical. These techniques usually provide the designer with qualitative information regarding the performance of the controlled system. However, they fail to yield a unique control system design, and the designer is left the task of evaluating the quality of the final design. The modern control theory, on the other hand, provides the designer a means to find the control system that would yield the desired performance for a given system subjected to the given control constraints. In the past few years, the application of the optimal control theory to general systems governed by ordinary differential equations has been under intensive investigation, and progress has also been made in the design of control systems for nuclear reactors. However, little has been done in the area of optimal control for distributed parameter systems.

In the design of control systems for large reactors, the reactor, in general, cannot be satisfactorily represented by a set of ordinary differential equations. However, we can approximate the distributed parameter reactor system by various methods, but the difficulties involved in such an approximation (see Chapter III) cannot always be avoided. Therefore, the objective in this chapter is to study the application of the optimal control theory to the spatially-dependent reactor systems.

A. Definitions and Formulation of the Problem³³

Before introducing the optimal control theory for the distributed parameter systems, we give several definitions and notations. Then, we formulate the problem of optimal control for reactor systems.

1. Definitions^{45,46}

a. Distributed Control. The distributed control function denoted by $\vec{f}(\vec{r}, t)$ is a set of input functions distributed over all, or a certain subset, of $\bar{R}$. The set of all possible functions of $\vec{f}(\vec{r}, t)$ defined for all $\vec{r} \in \bar{R}$, $0 \leq t \leq T$, will be called the admissible controls and denoted by $I_c(\bar{R} \times T)$.

b. Output Transformation. An output transformation is a transformation that transforms the state function $\vec{u}(\vec{r}, t)$ into a set of directly measurable variables (output) $\vec{w}(\vec{r}, t)$ of the system. The transformation will be denoted by $M(\vec{r}, t)$.

c. Controllability. Controllability is defined as the ability of the given admissible control functions to steer the system variables from one given state to another in a finite or infinite amount of time. A state $\vec{u}(\vec{r}, 0) \in \Gamma(\vec{R})$ is said to be null-controllable if an admissible control function exists which will transfer $\vec{u}(\vec{r}, 0)$ to the null-state in a finite time interval, T . If the condition $\vec{u}(\vec{r}, T) = \vec{0}$ is placed by $\rho\{\vec{u}[\vec{r}, T, \vec{u}(\vec{r}, 0)], \vec{0}\} < \delta$, where δ is a positive number, then the system is said to be null- δ -controllable. Furthermore, if the system is controllable for all $\vec{u}(\vec{r}, 0) \in \Gamma(\vec{R})$, then the system is said to be completely null- or null- δ -controllable.

d. Observability. Observability is defined as the ability to determine the system state at any time t by observing the output $\vec{w}(\vec{r}, t)$ over a finite time interval ($0 \leq t \leq T$). If the system state at $t = t_0$ can be determined by observing $\vec{w}(\vec{r}, t)$ over a finite time interval, the system is said to be completely observable at $t = t_0$. If the system is completely observable for any t , the system is said to be completely observable.

2. Problem Formulation

In the optimal control of lumped-parameter systems, the performance of the system is evaluated by means of a functional of the state of the system and time. Such a functional (commonly referred to as performance index, cost function, or value function) assigns a value (number) to the system under given inputs and/or initial conditions. The task is to extremize (minimize) the functional with respect to the control function for the given system and the admissible controls. This concept can be readily extended to the distributed parameter systems by a functional of the system state, time, and space. Two simple examples of such functionals are the integrals;

$$J = \int_0^{\infty} (\|\vec{u}\| + \|\vec{f}\|) dt,$$

and

$$J = \int_0^{\infty} (\|\vec{u}\|^2 + \|\vec{f}\|^2) dt.$$

The physical interpretation of the above performance indices is that the variation of the system state and the control effort is minimized over the defined region and time. The choice of the form of the performance index is not obvious (in relation to the system performance); however, we can improve our choice from experience.

From the above discussion, the problem can be formulated as follows:

A distributed parameter system is specified by a set of partial differential equations,

$$\frac{\partial \vec{u}}{\partial t} = \vec{h}(\vec{u}, \vec{r}, \vec{f}), \text{ and a boundary condition } B(\vec{u}, \vec{r}) = \vec{0} \quad (57)$$

where $\vec{u}$ has the same meaning as in Section III-A, with the assumption that the system is both completely null-controllable and observable.

A performance index is given as

$$J = \int_0^{t_1} \int_R P(\vec{u}, \vec{f}, \vec{r}) dt d\vec{r}, \quad (58)$$

and

$$\vec{f}(\vec{r}, t) \in I_C(\bar{R} \times t_1).$$

The optimal control problem becomes the minimizing (or maximizing) of the functional (58) with respect to the admissible control function $\vec{f}(\vec{r}, t) \in I_C(\bar{R} \times t_1)$ under the condition that the system state $\vec{u}$ satisfies Eq. (57). The above statement indicates that the above problem falls into the study of the variational problem of functionals. The next section of this chapter will discuss the application of the maximum principle to the distributed parameter systems.

B. The maximum Principle of Pontryagin[†]

One of the more outstanding, recent contributions to the optimal control theory is the maximum principle of Pontryagin.⁴⁷ The essence of the maximum principle is that it enables us to handle variational problems with constraints of boundedness in the control function where the classical variational techniques fail. Since the theorem for ordinary differential equations is only a special case of the general theorem, only the extension of the theorem to systems of partial differential equations will be presented here.

1. Theorem

Let $\vec{f}(\vec{r}, t)$, $t \in [0, t_1]$ be an admissible control such that the corresponding trajectory $\vec{u}(\vec{r}, t)$ which begins at $t = 0$ passes through a point $\vec{u}(\vec{r}, t_1)$ at time t_1 . For $\vec{f}(\vec{r}, t)$ and $\vec{u}(\vec{r}, t)$ to be optimal [the functional in Eq. (58) is minimized], it is necessary that there exists a nonzero continuous vector function $\vec{v}(\vec{r}, t)$ (the co-state, auxiliary, or adjoint functions) defined in $\Gamma(\bar{R})$ corresponding to $\vec{f}(\vec{r}, t)$ and $\vec{u}(\vec{r}, t)$ such that

[†]For the lumped-parameter system, see Ref. 47.

a. For every $t \in [0, t_1]$, the functional

$$G = \int_R H(\vec{u}, \vec{v}, \vec{f}, \vec{r}) d\vec{r} \quad (59)$$

of the variable $\vec{f} \in I_c$ attains its maximum at the point,

$$\vec{f}(\vec{r}, t) = \vec{f}^*(\vec{r}, t),$$

where $H(\vec{u}, \vec{v}, \vec{f}, \vec{r})$ is defined as

$$H = -P(\vec{u}, \vec{f}, \vec{r}) + [\vec{v}, \vec{h}]^\dagger.$$

Furthermore, if we denote a functional,

$$G^* = \max_{\vec{f} \in I_c} \int_R H(\vec{u}, \vec{v}, \vec{f}, \vec{r}) d\vec{r},$$

the optimal trajectory satisfies

$$\left. \begin{aligned} \frac{\partial u_i}{\partial t} &= \left(\frac{\delta H^*}{\delta v_i} \right)^T, \\ \frac{\partial v_i}{\partial t} &= - \left(\frac{\delta H^*}{\delta u_i} \right)^T, \end{aligned} \right\} \quad (60)$$

$$i = 1, 2, \dots, n.$$

b. At the terminal time, t_1 , the relation,

$$G^* = \text{const at } t = t_1, \quad (61)$$

is satisfied. Furthermore, if $\vec{u}, \vec{v}$, and $\vec{f}$ satisfy Eq. (60) and condition a, the functional, G^* , is a constant. Thus, Eq. (61) is true at any time $t \in [0, t_1]$ and not just at t_1 .

Proof. Define two new variables $u_{n+1}(\vec{r}, t)$ and $h_{n+1}(\vec{u}, \vec{f}, \vec{r})$, where

$$u_{n+1} = \int_0^t P(\vec{u}, \vec{f}, \vec{r}) dt',$$

$^\dagger [\vec{u}, \vec{v}]$ is defined as the vector inner product of the vectors $\vec{u}$ and $\vec{v}$.

and

$$h_{n+1} = P(\vec{u}, \vec{f}, \vec{r}).$$

Then, we have an equation such that

$$\frac{\partial u_{n+1}}{\partial t} = h_{n+1}(\vec{u}, \vec{f}, \vec{r}). \quad (62)$$

Combining Eqs. (57) and (62), we obtain a set of $(n + 1)$ equations; i.e.,

$$\frac{\partial \vec{u}}{\partial t} = \vec{h}(\vec{u}, \vec{f}, \vec{r}), \quad (63)$$

where $\vec{u}$ represents an $(n + 1)$ vector function which includes the n state equation plus the performance index integral.

The H function of the system (63) is thus given by

$$H = [\vec{v}, \vec{h}] = \sum_{i=1}^{n+1} v_i(\vec{r}, t) h_i(\vec{u}, \vec{f}, \vec{r}). \quad (64)$$

From this procedure, the problem of minimizing the functional J in Eq. (58) becomes that of minimizing the $(n + 1)$ st state function at the terminal time, t_1 . This problem can be considered as a special case of the more general problem of minimizing a functional of the form,

$$J' = \langle \vec{b}(\vec{r}), \vec{u}(\vec{r}, t_1) \rangle = \int_R \sum_{i=1}^{n+1} b_i(\vec{r}) \vec{u}_i(\vec{r}, t_1) d\vec{r}, \quad (65)^\dagger$$

with respect to variations in $\vec{u}$ and $\vec{f}$.

Since Eq. (64) depends only on the final state of the system variables, it can be written as

$$J' = - \int_0^{t_1} \frac{\partial}{\partial t} \langle \vec{v}(\vec{r}, t), \vec{u}(\vec{r}, t) \rangle dt, \quad (66)$$

providing that

$$\langle \vec{v}(\vec{r}, 0), \vec{u}(\vec{r}, 0) \rangle = 0,$$

and

$$\langle \vec{v}(\vec{r}, t_1), \vec{u}(\vec{r}, t_1) \rangle = \langle -\vec{b}(\vec{r}), \vec{u}(\vec{r}, t_1) \rangle. \quad (67)$$

[†]For this case, $b_i = 0$; $i = 1, 2, \dots, n$; $b_{n+1} = 1$.

If the functional (66) is expanded, it becomes

$$\begin{aligned} J' &= - \int_0^{t_1} \left(\left\langle \frac{\partial}{\partial t} \vec{v}, \vec{u} \right\rangle + \left\langle \vec{v}, \frac{\partial}{\partial t} \vec{u} \right\rangle \right) dt \\ &= - \int_0^{t_1} \left\langle \frac{\partial}{\partial t} \vec{v}, \vec{u} \right\rangle dt - \int_0^{t_1} G(\vec{u}, \vec{v}, \vec{f}) dt. \end{aligned}$$

If the variation of the functional J' is to vanish with respect to the variation of the control function $\vec{f}$, i.e.,

$$\left(\frac{\delta J'}{\delta \vec{f}} \right)^T = \vec{0},$$

it must be that

$$\left(\frac{\delta G}{\delta \vec{f}} \right)^T = \vec{0}. \quad (68)$$

Equation (68) is one of the necessary conditions for J' to have an extreme. Furthermore, it is also the necessary condition that G attains an extreme with respect to $\vec{f}$. If an admissible control, $\vec{f}^*$, is obtained from Eq. (68), then J' will be at an extreme if, in addition,

$$\left(\frac{\delta J'}{\delta \vec{u}} \right)^T = \vec{0},$$

and

$$\left(\frac{\delta J'}{\delta \vec{v}} \right)^T = \vec{0}.$$

This leads directly to the canonical equations, Eq. (60).

To prove that Eq. (68) yields a necessary condition for J' to be minimum whereas G attains a maximum, it is necessary to show that $\delta J'$ will be positive if the control function deviates from $\vec{f}^*$.

The variation in J' can be written as

$$\delta J' = J' - J'^*,$$

where the starred symbol designates the quantities corresponding to the optimal case. Therefore,

$$J' = - \int_0^{t_1} \int_R \left(\frac{\partial}{\partial t} \vec{v} \right)^T \vec{u} \, d\vec{r} \, dt - \int_0^{t_1} G(\vec{u}, \vec{v}, \vec{f}) \, dt$$

Assume that $\vec{u}, \vec{v}, \vec{f}$ deviates only slightly from the optimal value $\vec{u}^*, \vec{v}^*,$ and $\vec{f}^*$. Denote $\vec{u} = \vec{u}^* + \delta\vec{u}$, $\vec{v} = \vec{v}^* + \delta\vec{v}$, and $\vec{f} = \vec{f}^* + \delta\vec{f}$. Then $\delta J'$ can be written as

$$\begin{aligned} \delta J' = & - \int_0^{t_1} [G(\vec{u}^*, \vec{v}^*, \vec{f}) - G^*] \, dt - \int_0^{t_1} \int_R \left(\frac{\partial}{\partial t} \vec{v}^* \right)^T \delta\vec{u} \, d\vec{r} \, dt \\ & - \int_0^{t_1} \int_R \left(\frac{\partial}{\partial t} \delta\vec{v} \right)^T \vec{u}^* \, d\vec{r} \, dt \\ & - \int_0^{t_1} \int_R \vec{v}^{*T} \frac{\partial h(\vec{u}^*, \vec{f})}{\partial \vec{u}^*} \delta\vec{u} \, d\vec{r} \, dt \\ & - \int_0^{t_1} \int_R \delta\vec{v}^T h(\vec{u}^*, \vec{f}) \, d\vec{r} \, dt + 0(\epsilon). \end{aligned}$$

The last term contains the terms of at least the second order of the variations. If we use the canonical equations, $\delta J'$ can be simplified as

$$\begin{aligned} \delta J' = & - \int_0^{t_1} [G(\vec{u}^*, \vec{v}^*, \vec{f}) - G^*] \, dt \\ & - \int_0^{t_1} \int_R \vec{v}^{*T} \frac{\partial}{\partial \vec{u}} \left(\frac{\partial h}{\partial \vec{f}} \delta\vec{f} \right) \delta\vec{u} \, d\vec{r} \, dt + 0(\epsilon). \end{aligned}$$

The second term in the above expression contains the second-order variation. Thus,

$$\delta J' = - \int_0^{t_1} [G(\vec{u}^*, \vec{v}^*, \vec{f}) - G^*] \, dt + 0(\epsilon).$$

Since $\vec{u}$, $\vec{v}$, and $\vec{h}$ are all continuous functions, it is always possible to find small variations such that the first term in $\delta J'$ is dominant in the vicinity

of optimality. Therefore, for $\delta J'$ to be positive, it is necessary that G^* be a maximum with respect to the control function $\vec{f}$. This proves condition a in the theorem.

To prove condition b of the theorem, the definition of the derivative of a functional is used. If the system is optimal,

$$\frac{d}{dt} G^* = \frac{\partial}{\partial t} G^* + \frac{\delta G^*}{\delta \vec{u}} \frac{\partial \vec{u}}{\partial t} + \frac{\delta G^*}{\delta \vec{v}} \frac{\partial \vec{v}}{\partial t} + \frac{\delta G^*}{\delta \vec{f}} \frac{\partial \vec{f}}{\partial t} = \frac{\partial}{\partial t} G^*.$$

Furthermore, G^* is independent of t . Therefore,

$$\frac{dG^*}{dt} = 0,$$

and G^* is a constant for $t \in [0, t_1]$. This proves condition b of the theorem.

The application of the maximum principle to the optimal control problem can be outlined as follows:

2. Statement of Problem

Given a dynamical system,

$$\frac{\partial}{\partial t} \vec{u}(\vec{r}, t) = \vec{h}(\vec{u}, \vec{f}, \vec{r}),$$

with its boundary conditions and a performance index,

$$J = \int_R \int_0^{t_1} P(\vec{u}, \vec{f}, \vec{r}) d\vec{r} dt,$$

subject to the constraints on the control function, $\vec{f}$ (if there are constraints), find an admissible control function, $\vec{f}^*(\vec{r}, t) \in I_c$, such that J is minimized.

3. Procedure

a. Form the G-functional from Eq. (64):

$$G = \langle \vec{v}, \vec{h} \rangle - \int_R P(\vec{u}, \vec{f}, \vec{r}) d\vec{r}.$$

b. Find the optimal admissible control function $\vec{f}^*(\vec{r}, t)$ by maximizing G with respect to $\vec{f}$; i.e., solve for $\vec{f}^*$ from

$$\left(\frac{\delta G}{\delta f} \right)^T = \vec{0}. \quad (69)$$

c. Solve the set of canonical equations,

$$\left. \begin{aligned} \frac{\partial \vec{u}}{\partial t} &= \left(\frac{\delta H}{\delta \vec{v}} \right)^T \\ \frac{\partial \vec{v}}{\partial t} &= - \left(\frac{\delta H}{\delta \vec{u}} \right)^T \end{aligned} \right\} \quad (70)$$

with the given boundary conditions on $\vec{u}$ and the corresponding boundary conditions on $\vec{v}$ from Eq. (67).

In summary, the procedure is to solve the two-point, boundary-value problem of Eq. (70).

C. Optimal Control of Linear Reactor Systems

In this section, the application of the maximum principle to two classes of linear optimization problems will be discussed. These problems usually are encountered in the optimal control system design for reactor systems.

1. Regulator Problem

In the regulator problem, the system is considered to have a bounded disturbance at time $t = 0$, and the optimal control system will be the design that would return the system state to its equilibrium state satisfying a given performance.

Consider a linear reactor system given by the equation,

$$\frac{\partial}{\partial t} \vec{u} = L\vec{u} + M\vec{f},$$

given

$$\vec{u}(\vec{r}, 0); \quad \text{and} \quad \vec{u}(\vec{r}, T) = \vec{0},$$

$$C\vec{u}(\vec{r}_b, t) = \vec{0}, \quad \vec{r}_b \in \Omega, \quad (71)$$

and a performance index,

$$J = \int_0^T (\langle \vec{u}, A\vec{u} \rangle + \langle \vec{f}, B\vec{f} \rangle) dt.$$

Denote the auxiliary vector function $\vec{v}(\vec{r}, t)$ and form the G-functional,

$$G = \langle \vec{u}, L\vec{u} + M\vec{f} \rangle - \langle \vec{u}, A\vec{u} \rangle - \langle \vec{f}, B\vec{f} \rangle.$$

Maximizing G with respect to $\vec{f}$ results in

$$\delta G \text{ due to } \delta \vec{f} = \langle \vec{u}, M\delta \vec{f} \rangle - 2\langle \delta \vec{f}, B\vec{f} \rangle = 0.$$

Thus,

$$\vec{f}^* = \frac{1}{2}B^{-1}M^T\vec{v},$$

and

$$G^* = \left\langle \vec{v}, L\vec{u} + \frac{1}{2}MB^{-1}M^T\vec{v} \right\rangle - \left\langle -\frac{1}{2}B^{-1}M^T\vec{v}, \frac{1}{2}M^T\vec{v} \right\rangle - \langle \vec{u}, A\vec{u} \rangle.$$

Then the canonical equations are

$$\left. \begin{aligned} \frac{\partial}{\partial t} \vec{u} &= L\vec{u} + \frac{1}{2}MB^{-1}M^T\vec{v}, \\ \frac{\partial}{\partial t} \vec{v} &= -L^+\vec{v} + 2A\vec{u}, \\ \vec{u}(\vec{r}, T) &= \vec{0}, \vec{u}(\vec{r}, 0) \text{ is bounded}, \\ \vec{v}(\vec{r}, 0) &= \vec{0}, \\ C^+\vec{v}(\vec{r}_b, t) &= \vec{0}, \vec{r}_b \in \Omega. \end{aligned} \right\} \quad (72)$$

Solving for $\vec{v}(\vec{r}, t)$, we obtain

$$\vec{v}(\vec{r}, t) = 2 \int_0^t \Phi^+(t, t') A\vec{u}(t') dt',$$

where $\Phi^+(t, t')$ is the state transition operator, L^+ , and hence

$$\vec{f}^* = B^{-1}M^T \int_0^t \Phi^+(t, t') A\vec{u}(t') dt'. \quad (73)$$

Equation (73) gives the optimal control function in terms of the system state at any time, t .

From Eq. (73) the solution of the system trajectory can be found as

$$\begin{aligned} \vec{u}(\vec{r}, t) = & \Phi(t, 0) \vec{u}(\vec{r}, 0) - \int_0^t \Phi(t, t') MB^{-1}M^T \\ & \left[\int_0^{t'} \Phi^+(t', t'') A\vec{u}(t'') dt'' \right] dt'. \end{aligned} \quad (74)$$

From the discussion on controllability in Section IV-B, for the above system to be completely null-controllable, the operator

$$\Psi = \int_0^T \Phi(T, t) MB^{-1}M^T \int_0^t \Phi^+(t, t') A(\cdot) dt' dt$$

must have an inverse. The proof is straightforward since Eq. (74) can be written as

$$\Psi \vec{u} = \Phi(T, 0) \vec{u}(\vec{r}, 0),$$

and to satisfy this condition, Ψ must have an inverse.

2. Time Optimal Problem

This problem treats the time optimal control of the linear reactor system (71) with the same spatial boundary condition; $\vec{u}(\vec{r}, 0)$ is bounded in the minimal time interval, and $\vec{u}(\vec{r}, T) = \vec{0}$. The performance index in this case would be

$$J = \int_0^T \int_R d\vec{r} dt.$$

Furthermore, there are constraints on the control function, $\vec{f}(\vec{r}, t)$, such as

$$||\vec{f}(\vec{r}, t)|| \leq 1.$$

Therefore,

$$H = [\vec{v}, L\vec{u} + M\vec{f}] - 1.$$

For G to be at a maximum, it is necessary that

$$\langle \vec{v}, M\vec{f} \rangle$$

always attains a maximum with the given constraint; i.e.,

$$\vec{f}(\vec{r}, t) = M^T \vec{v} \cdot ||M^T \vec{v}||^{-1},$$

where $\vec{v}$ is the solution from the equation,

$$\frac{\partial \vec{v}}{\partial t} = -L^+ \vec{v},$$

with the boundary condition,

$$B^+ \vec{v} = \vec{0},$$

and the terminal condition for $\vec{v}$ at $t = 0$ can be obtained from the relationship that

$$\begin{aligned} & \int_0^T \Phi(T, t') M M^T \Phi^+(t', 0) \vec{v}(\vec{r}, 0) ||M^T \Phi^+(t', 0) \vec{v}(\vec{r}, 0)||^{-1} dt' \\ & = \Phi(T, 0) \vec{u}(\vec{r}, 0). \end{aligned}$$

3. Examples

A linear reactor model was chosen for the following two examples. The reactor is assumed to be an infinite slab reactor (one spatial dimension) with a width ℓ . Two infinitesimally thin control rods are located at x_1 and x_2 , respectively. The control rods are assumed to be made of absorption material, and the absorption cross section can change with time $f_1(t)$ and $f_2(t)$. The configuration of the system is shown in Fig. 6.

Fig. 6

Infinite Slab Reactor and Control Rods
Configuration for Example 5

Furthermore, it is assumed that the system can be described by

$$\frac{1}{v'} \frac{\partial}{\partial t} \phi = D \frac{\partial^2}{\partial x^2} \phi - \Sigma_a \phi + \nu \Sigma_f \phi \\ + \Sigma'_a [\delta(x - x_1) f_1(t) + \delta(x - x_2) f_2(t)],$$

with a boundary condition that $\phi(0,t) = \phi(\ell,t) = 0$.

a. Example 5: Optimal Regulator. Let the performance index in this case be

$$J = \int_0^\infty \int_0^\ell (\phi - \phi_0)^2 dx dt + \int_0^\infty (f_1^2 + f_2^2) dt,$$

where ϕ_0 is the desired state of the system. Following the procedures given in Section IV-B and from the G-functional,

$$G = - \int_0^\ell (\phi - \phi_0)^2 dx - f_1^2 - f_2^2 + \int_0^\ell v' v \left\{ D \frac{\partial^2}{\partial x^2} \phi - \Sigma_a \phi + \nu \Sigma_f \phi \right. \\ \left. + \Sigma'_a [\delta(x - x_1) f_1 + \delta(x - x_2) f_2] \right\} dx.$$

Then by maximizing G with respect to the control functions, f_1 and f_2 , we obtain the optimal control functions,

$$f_1 = v' \frac{\Sigma'_a}{2} v(x_1, t),$$

and

$$f_2 = v' \frac{\Sigma'_a}{2} v(x_2, t),$$

where v is the co-state function.

The function v can be obtained from solving the equation,

$$\frac{1}{v'} \frac{\partial v}{\partial t} = -D \frac{\partial^2}{\partial x^2} v + \Sigma_a v - \nu \Sigma_f v + 2(\phi - \phi_0) \frac{1}{v'}.$$

The boundary condition is that

$$v(0,t) = v(\ell,t) = 0,$$

and from the condition $V(x, \infty) = 0$, it must be that $v(x, 0) = 0$. The solution is, then

$$\begin{aligned} {}^{\dagger}v(x, t) = & \int_0^t \int_0^{\ell} 2 \sum_n \sin \frac{n\pi}{\ell} x' \sin \frac{n\pi}{\ell} x \exp[-\lambda_n(t-t')] \\ & \cdot [\phi(x', t') - \phi_0(x')] dx' dt' \end{aligned}$$

and the control functions,

$$\begin{aligned} f_1(t) = & +\Sigma_a' v' \sum_n \sin \frac{n\pi}{\ell} x_1 \int_0^t \int_0^{\ell} \sin \frac{n\pi}{\ell} x' [\phi(x', t') - \phi_0(x')] \\ & \cdot \exp[-\lambda_n(t-t')] dx' dt', \\ f_2(t) = & +\Sigma_a' v' \sum_n \sin \frac{n\pi}{\ell} x_2 \int_0^t \int_0^{\ell} \sin \frac{n\pi}{\ell} x' [\phi(x', t') - \phi_0(x')] \\ & \cdot \exp[-\lambda_n(t-t')] dx' dt', \end{aligned}$$

where

$$\lambda_n = \left[-D \frac{n^2 \pi^2}{\ell^2} - \Sigma_a + \nu \Sigma_f \right] v'.$$

The control functions obtained here are functionals of the system state, $\phi(x, t)$, which, in the case of distributed parameter system, can be considered as control laws; i.e., the control is a feedback of the system state.

b. Example 6: Time Optimal Control. For the time optimal control problem, the system for Example 5 is used here except that the control function is distributed throughout the reactor. This is a hypothetical case; however, it does demonstrate the procedure for treating the general time optimal control problems.

The system equation is given as

$$\frac{1}{v'} \frac{\partial}{\partial t} \phi = D \frac{\partial^2}{\partial x^2} \phi - \Sigma_a \phi + \nu \Sigma_f \phi + f(x, t), \quad (75)$$

with a boundary condition that $\phi(0, t) = \phi(\ell, t) = 0$.

[†]The semigroup operator $\Phi^{\dagger}(t, t') = \int_0^{\ell} \sum_n \sin \frac{n\pi}{\ell} x' \sin \frac{n\pi}{\ell} x \exp[-\lambda_n(t-t')] (\cdot) dx'$.

The performance index is of the form,

$$J = \int_0^{t_1} dt, \quad (76)$$

and a control constraint that

$$||f(x,t)|| \leq 1. \quad (77)$$

The G-functional is given by

$$G = \int_0^{\ell} v' v \left[D \frac{\partial^2}{\partial x^2} \phi - \Sigma_a \phi + \nu \Sigma_f \phi + f(x,t) \right] dx - 1.$$

For G to be maximized with respect to f, which satisfies the constraint, it must be that

$$f(x,t) = v(x,t) ||v(x,t)||^{-1}, \quad (78)$$

where $v(x,t)$ and $\phi(x,t)$ satisfy the canonical equations,

$$-\frac{1}{v'} \frac{\partial v}{\partial t} = D \frac{\partial^2}{\partial x^2} v - \Sigma_a v + \nu \Sigma_f v \quad (79)$$

and

$$\frac{1}{v'} \frac{\partial \phi}{\partial t} = D \frac{\partial^2}{\partial x^2} \phi - \Sigma_a \phi + \nu \Sigma_f \phi + v(x,t) ||v(x,t)||^{-1}. \quad (80)$$

The boundary condition for $v(x,t)$ is that

$$v(0,t) = v(\ell,t) = 0,$$

and the initial (or terminal) condition can be found through the given conditions in ϕ . Since $\phi(x,0)$ is given, $\phi(x,t_1) = 0$ and the solution of $\phi(x,t)$ is

$$\phi(x,t) = \Phi(t,0) \phi(x,0) + \int_0^t \Phi(t,t') v(x,t') ||v(x,t')||^{-1} dt',$$

at $t = t_1$

$$\phi(x,t_1) = \Phi(t_1,0) \phi(x,0) + \int_0^{t_1} \Phi(t_1,t') v(x,t') ||v(x,t')||^{-1} dt' = 0,$$

and

$$v(x,t) = \Phi^+(t,0) v(x,0).$$

Hence,

$$\int_0^{t_1} \Phi(t_1, t') \Phi^+(t', 0) v(x, 0) \|\Phi^+(t', 0) v(x, 0)\|^{-1} dt' = \Phi(t_1, 0) \phi(x, 0), \quad (81)$$

where

$$\Phi(t, t') = \int_0^\ell \sum_n^\infty \sin \frac{n\pi}{\ell} x \sin \frac{n\pi}{\ell} x' \exp[\lambda_n(t - t')] (\cdot) dx'$$

$$\Phi^+(t, t') = \int_0^\ell \sum_n^\infty \sin \frac{n\pi}{\ell} x \sin \frac{n\pi}{\ell} x' \exp[-\lambda_n(t - t')] (\cdot) dx'$$

If $v(x, 0)$ is obtained from Eq. (81), then the control function $f(x, t)$ is uniquely determined through Eq. (78).

D. Optimal Control of Nonlinear Reactor Systems

For the linear optimization problem, if the performance index is quadratic, the canonical equations involved will be linear and are readily solvable. However, for a nonlinear system, this is no longer true. Therefore, an optimal design cannot always be obtained for the control system. Fortunately, since the maximum principle holds for any system, the performance of the system can be improved from it. Sometimes, the improvement of performance is called suboptimization. This section treats certain classes of nonlinear optimization problems by using the maximum principle first, followed by the use of Lyapunov's method for suboptimal systems.

1. Optimal Regulating Problem

Let the system be of the form of

$$\frac{\partial \vec{u}}{\partial t} = L\vec{u} + M(\vec{u}) \vec{f} \quad (82)$$

and a quadratic performance index,

$$J = \int_0^T (\langle \vec{u}, A\vec{u} \rangle + \langle \vec{f}, B\vec{f} \rangle) dt,$$

where L is a linear asymptotically stable operator, and $M(\vec{u})$ a nonlinear operator which depend on the system state as well as spatial variables.

For this system, the G-functional is

$$G = \langle \vec{v}, L\vec{u} + M(\vec{u}) \vec{f} \rangle - \langle \vec{u}, A\vec{u} \rangle - \langle \vec{f}, B\vec{f} \rangle.$$

Following the same procedure as in Section IV-B, the optimal control function and the auxiliary vector function must satisfy the following equations:

$$2B\vec{f} - M^T(\vec{u}) \vec{v} = \vec{0}; \quad (83)$$

$$\frac{\partial \vec{v}}{\partial t} = -L^+ \vec{v} + 2A\vec{u} - \left[\frac{\partial M(\vec{u}) \vec{f}}{\partial \vec{u}} \right]^T \vec{v}. \quad (84)$$

Equation (83) can be used to solve the control function; i.e.,

$$\vec{f} = +\frac{1}{2} B^{-1} M^T(\vec{u}) \vec{v},$$

and the canonical equations are

$$\left. \begin{aligned} \frac{\partial \vec{u}}{\partial t} &= L\vec{u} + \frac{1}{2} M B^{-1} M^T \vec{v} \\ \text{and} \\ \frac{\partial \vec{v}}{\partial t} &= -L^+ \vec{v} + 2A\vec{u} - \frac{1}{2} \left[\frac{\partial}{\partial \vec{u}} (M B^{-1} M^T \vec{v}) \right]^T \vec{v} \end{aligned} \right\} \quad (85)$$

This is a set of $2n$ nonlinear equations. In general, there is no way to find the solution, and therefore no optimal system can be obtained through these procedures. However, if it can be assumed that $\vec{u}(\vec{r}, t)$ can be closely approximated from measurement at finite spatial points, the auxiliary equation can be written as an integral equation,

$$\begin{aligned} \vec{v}(\vec{r}, t) &= + \int_0^t 2\Phi^+(t, t') A\vec{u}(\vec{r}, t') dt' \\ &\quad - \int_0^t \frac{1}{2} \Phi^+(t, t') \left[\frac{\partial}{\partial \vec{u}} (M B^{-1} M^T \vec{v}) \right]^T \vec{v} dt' \end{aligned}$$

where $\vec{v}(\vec{r}, t)$ can be approximately generated at discrete time intervals. Furthermore, if the system response is much slower than the generation of $\vec{v}(\vec{r}, t)$, it can be used for the control-function calculation. In such cases, the control function is

$$\vec{f}(\vec{r}, t) = + \frac{1}{2} B^{-1} M^T \left\{ \int_0^t 2 \Phi^+(t, t') A \vec{u}(t') dt' - \int_0^t \frac{1}{2} \Phi^+(t, t') \left[\frac{\partial}{\partial \vec{u}} (M B^{-1} M^T \vec{v}) \right]^T \vec{v} dt' \right\}.$$

The control function so obtained depends on both the system state, $\vec{u}$, and auxiliary functions, $\vec{v}$, whereas in the linear case, it is only a function of $\vec{u}$. This indicates that a control law (control as a function of system states) can be obtained for the linear system, but only a control function can be obtained for the nonlinear system.

2. Minimal Time Problem

In this case, as in Section IV-B, the performance index becomes

$$J = \int_R \int_0^T d\vec{r} dt,$$

and the control constraint becomes

$$|| \vec{f}(\vec{r}, t) || \leq p.$$

The G-functional thus becomes

$$G = \langle \vec{v}, L\vec{u} + M(\vec{u}) \vec{f} \rangle - 1.$$

For this problem, the control function is the same as in the linear case; i.e.,

$$\vec{f}(\vec{r}, t) = p M^T(\vec{u}) \vec{v} || M^T(\vec{u}) \vec{v} ||^{-1},$$

where $\vec{v}$ is the solution from the equation,

$$\frac{\partial \vec{v}}{\partial t} = -L^+ \vec{v} + p \left\{ \frac{\partial}{\partial \vec{u}} \left[M M^T(\vec{u}) \vec{v} || M^T(\vec{u}) \vec{v} ||^{-1} \right] \right\}^T \vec{v},$$

which again becomes a nonlinear equation even if $\vec{u}(\vec{r}, t)$ is known.

3. Suboptimal Control Using Lyapunov Functionals

The discussion in the last two sections indicated that it is difficult to obtain the optimal control function for nonlinear systems. Therefore, a method of improving the system performance using Lyapunov functionals is introduced.

The basic idea of this method is to choose a V-functional for the system without control such that the system is asymptotically stable (this condition is necessary for reactor control because of the requirement of fail-safe operation). Then the time derivative of the V-functional is made more negative with an added control function. The physical interpretation is that the system state would converge faster when V is made more negative. However, the exact performance of the system cannot be evaluated as to the nature of the improvement. Furthermore, since the V-functional is not unique, the improvement thus depends on the choice of the V-functional. The procedure of the method is as follows:

- a. Find a V-functional for the system without control such that the system is asymptotically stable.
- b. Add a control function to the system, and evaluate the time derivative of the V-functional.
- c. Obtain a minimum for V with respect to the control function.

The application of this method can be illustrated using the problems in Section IV-D.

For system (82), choose a V-functional,

$$V = \frac{1}{2} \|\vec{u}\|^2,$$

$$\dot{V} = \frac{1}{2} \langle \vec{u}, (L + L^+) \vec{u} \rangle + \langle \vec{f}, M(\vec{u})^T \vec{u} \rangle \leq -k \|\vec{u}\|^2 + \langle \vec{f}, M(\vec{u})^T \vec{u} \rangle.$$

For V to be more negative, the control function may be chosen from the condition that the functional,

$$\langle \vec{f}, M(\vec{u})^T \vec{u} \rangle,$$

be a minimum; i.e.,

$$\vec{f}(\vec{r}, t) = -KM^T(\vec{u}) \vec{u}.$$

If there is no constraint on $\vec{f}(\vec{r}, t)$, K can be made as large as possible, and $\dot{V} \rightarrow -\infty$ as $K \rightarrow +\infty$. This is not physically possible; thus a constraint has to be added to the control function. If the constraint is chosen as

$$\|\vec{f}(\vec{r},t)\| \leq p,$$

then

$$\vec{f}(\vec{r},t) = -pM^T(\vec{u}) \vec{u} \|M^T(\vec{u}) \vec{u}\|^{-1},$$

which is very similar to the time-optimal problem as given in Section IV-C-2. However, since the control function depends on $\vec{u}$ rather than $\vec{v}$, the control function can be obtained directly from the system states.

E. Practical Aspects of the Optimal Control for Distributed-parameter Systems

From the procedures discussed in Section IV-B, the solution of the optimal-control problem is to solve the two-point-boundary-value problem of the canonical equations. In general, the solution of these equations in closed form cannot be obtained except for a few exceptionally simple cases. Unlike the lumped-parameter systems, in most cases, even if a closed-form control law can be obtained analytically (e.g., the control function in Example 5), the control function involves the spatial integral of the system states. Therefore, high-speed digital computers, or in general, hybrid computers become essential in the practical application of optimal control for distributed-parameter systems. In the area of optimal control of reactor systems, it is always desirable to have a large digital computer if the control objectives are complex. However, for some simple objectives (such as uniform burnup), a compact special digital or hybrid computer may be used.

In conjunction with the use of digital or hybrid computers, even with the large, high-speed computers available, the problem of the practical application of the optimal control theory is only partially solved. More efficient computational techniques and effective approximation procedures in numerical methods must be developed. Since the nature of the control function depends, to a certain extent, on the form of the performance index, the selection of the performance index deserves further attention.

CHAPTER V

CONCLUSION AND SUGGESTED FUTURE WORK

A: Conclusion

The problem of the stability analysis and the optimal control system design for spatially-dependent reactor systems has been attacked in a very general manner. Although the emphasis was placed on nuclear reactor systems, the application of the approaches used here is not restricted to nuclear reactors. The generalization to related engineering fields would be relatively simple to someone familiar with the special characteristics of the system in question.

The formulation of the spatially-dependent reactor system (Chapter II) is not completely novel, but it permits the inclusion of the spatial effects of temperatures, control-rod motion, and other processes within a reactor. Also, the effects on the nuclear parameters from the variations of the system state are included directly in the formulation.

The stability analysis in Chapter III clearly indicates that it is not only possible but also desirable to analyze a reactor system stability directly from the governing set of partial differential equations. The advantages of these methods are threefold:

1. They do not require the finding of the system solution. Hence the laborious work in computation is eliminated.
2. They do not need to engage any further approximation beyond those in the derivation of the system equations. Hence the result of the analysis is more reliable.
3. It is possible to find the relationship between the system stability and the system parameters.

The methods introduced in this work are not the best methods for all reactor stability problems. The success of the application depends on the nature of the problem and the experience of the user.

Based on past experience, a preference table (Table I) illustrates the correlation of the type of method to use for various types of systems.

Note that the Lyapunov functions used for the lumped-parameter reactor system can sometimes be modified for the distributed-parameter systems.

TABLE I. Summary--Method for Stability Analysis of Various Types of Systems.

Type of Systems		Methods and Conditions		
		Lyapunov	Spectral Analysis	Conventional ^a Approximation
Linear	Self-adjoint	SC ^b but NASC ^{c,d} possible	NASC ^e	SC
	Nonself-adjoint	SC ^d only	NASC ^e	SC if it converges from above
Nonlinear with linear part which is a generator of a semigroup	Self-adjoint	SC ^e	SC	f
	Nonself-adjoint	SC ^d	SC ^e	f
General nonlinear system		SC ^d	Sometimes valid SC	f

^aReferring to the approximation method discussed in Section III-B.

^bSC = sufficient condition.

^cNASC = necessary and sufficient condition.

^dPreferable for the preliminary design of reactors since it provides the relationship of stability and the system parameters.

^ePreferable for final analysis of system stability.

^fThese methods are not generally feasible, but may be valid for special cases.

The extension of the maximum principle of Pontryagin (Chapter IV) to the distributed-parameter system may be a valuable tool for the design of optimal control systems for reactors. In particular, it permits the design of optimal control systems with bounded control constraints; such design cannot be made using the classical techniques from the calculus of variation. Even though Chapter IV discusses only two classes of optimization problems, other classes of problems may be relatively simple to derive from the given theorem. Furthermore, the examples in Chapter IV indicate that designing optimal control systems for the spatially-dependent reactor system is mathematically no more complicated than the lumped-parameter model. For the control devices used in the example (control rods), the results are similar to those for the optimal control design using the modal or nodal method. The application of the theory to nonlinear systems is sound in theory; however, since the finding of the optimal control function depends upon the solution of a nonlinear, two-point, boundary-value problem (the canonical equations), the method is not practically useful. The introduction of the suboptimal method using the Lyapunov functional may shed some light in the control of nonlinear systems.

B. Suggestions for Future Work

The results of this investigation indicated that there are still areas left to be explored. Concerning the investigation made here, some basic foundations were laid for the study of spatially-dependent reactor systems. Note that this analysis is not solely restricted to reactor systems. The same techniques can be applied to other physical systems such as aero-elastic, heat-transfer problems, etc.

Further improvements may be made in the stability analysis, and these are listed below.

1. A generalized Lyapunov functional for nonself-adjoint systems should be developed such that the necessary and sufficient condition for the general coupled neutronic system can be obtained. This work will relax the conditions on the stability of linear and nonlinear reactor systems as well as of systems in other engineering fields.

2. New numerical or analytical techniques for the calculation of eigenvalues of general linear reactor systems should be developed. Such an investigation could yield fruitful results in determining the stability condition for certain class of nonlinear reactor systems.

3. Further study in analyzing the stability of the variable domain system should be advanced so that the stability of reactors with moving control devices can be treated directly.

4. The analytical study of spatially-dependent reactor dynamics should be followed by simulation of the reactor models on analog or digital computers. Experiments in measuring the dynamic response of system variables subject to various types of excitations should then be developed for the simulated reactor model. The objective of this study should be aimed at developing new concepts such as spatially-dependent transfer function (space-frequency) or impulse response function (space-time). The results of these studies should improve the understanding of dynamical characteristics of spatially-dependent systems and consequently lead to the development of techniques to determine nonlinear system stability from experiments.

As for optimal control of spatially-dependent reactors, the application of the maximum principle to linear systems is well defined and should be simple to implement. However, in nonlinear optimal control, the implementation of the theoretical results is still relatively unexplored (even in the lumped-parameter case). Therefore, future work for improvement should be in the following two areas:

1. Although the analytical solution of the optimal control function for continuous control can be obtained, it has to be evaluated by digital computers. Since the computer speed is finite, it requires finite time for computation. Thus, new approaches to the optimal control for nonlinear spatially-dependent system have to be investigated. The most promising approach seems to be in the area of digital sampled-data control of distributed-parameter systems. Also, in relation to this approach, the intrinsic properties of sampled-data system should be included in the study.

2. The suboptimal control using Lyapunov's method is quite useful since simple control laws can be obtained with little effort. Therefore, it is desirable to study the relationship between the system performance and the Lyapunov functional chosen for the design of optimal control. From this result, we can improve our basic understanding of this technique.

APPENDIX A

Properties of Semigroup Operators

To establish the explicit forms for the operator $[\Phi(t, t')]$ in terms of L and the condition on L such that L has the properties of being the generator of a continuous semigroup operator $[\Phi(t, t')]$, the solution of the linear system.

$$\frac{\partial}{\partial t} \vec{u} = L\vec{u} + M\vec{f} \quad (A-1)$$

is examined without imposing specific boundary conditions. The solution to the above system can be formally represented by

$$\vec{u} = \Phi(t, 0) \vec{u}(\vec{r}, 0) + \int_0^t \Phi(t, t') M(t') \vec{f}(\vec{r}, t') dt'.$$

The problem of establishing the form of $[\Phi(t, t')]$ for a bounded operator, L {i.e., there exists a constant, K , such that $\|L\vec{u}\| \leq K\|\vec{u}\|$ for all $\vec{u} \in [\Gamma(R)]$ } is rather simple. The semigroup operator $[\Phi(t, t')]$ corresponds to the fundamental (transition) matrix of an ordinary differential system. If L is independent of t , then

$$\Phi(t, t') = \exp[(t - t') L]. \quad (A-2)$$

As for the unbounded operator, L , the problem is no longer simple. A precise definition for the operator-valued function in Eq. (A-2) has been given by Hille and Yosida⁴³ in the framework of semigroup theory. The main result due to them can be stated in the following theorem:

Theorem

The necessary and sufficient conditions that the operator L be the infinitesimal generator of a continuous semigroup operator $[\Phi(t, t')]$ are that

1. The domain of L [denoted by $D(L)$] is dense in $\Gamma_B(\Omega)$ (a Banach space).
2. All the real numbers $\lambda > k$ are in the resolvent set of L , and

$$\|R(\lambda, L)\| \leq (\lambda - k)^{-1} \text{ for } \lambda > k,$$

where $R(\lambda, L) = (\lambda I - L)^{-1}$ is the resolvent of L , and I is the identity matrix operator.

For each L satisfying the above conditions, there exists a unique operator-valued function, $\exp(tL)$, defined for $t \geq 0$ with the following properties:

1. $\exp(tL)$ is bounded, and

$$\|\exp(tL)\| \leq \exp(kt).$$

2. $\exp(tL)$ is strongly continuous in t with $\exp(tL) = I$ when $t = 0$.

3. $\exp[(t+t')L] = \exp(tL) \cdot \exp(t'L)$.

4. $\exp(tL)$ maps $D(L)$ into itself, and for each $L \in D(L)$, $\exp(tL)$ is strongly differentiable in time, with

$$\frac{d}{dt} [\exp(tL) u] = \exp(tL) \cdot L u.$$

5. $\exp(tL)$ is permutable with the resolvent $(\lambda I - L)^{-1}$.

From this theorem, Theorem 2 can be easily proved.

Proof of Theorem 2

Since L is assumed to be an infinitesimal generator of a semigroup, it follows directly from condition 2 and property 1 of this theorem that

$$\|\exp(tL)\| \leq \exp(-\gamma t), \text{ for all } t \geq 0,$$

where $-\gamma$ corresponds to the constant k in this theorem. Furthermore, since

$$\vec{u}(\vec{r}, t, \vec{u}_0) = \exp(tL) \vec{u}_0,$$

it follows immediately that

$$\|\vec{u}\| \leq \exp(-\gamma t) \|\vec{u}_0\|.$$

This completes the proof.

APPENDIX B

A Functional Inequality

In this appendix, we prove that the inequality

$$\langle \vec{u}, L\vec{u} \rangle \leq \lambda_{\max} \|\vec{u}\|^2$$

holds for any $\vec{u} \in \Gamma(\bar{R})$. The equal sign holds only when $\vec{u}$ is the eigenvector corresponding to the largest eigenvalue of the equation,

$$L\vec{u} = \lambda\vec{u},$$

where L is a self-adjoint, time-independent linear operator, and the system satisfies a homogeneous boundary condition,

$$B\vec{u} = \vec{0},$$

where B may be a linear differential operator.

Proof

From the well-known spectral theory of Von Neumann, since L is a self-adjoint linear operator, the following is true:

1. The domain of L , i.e., $D(L)$, is dense in H , and abstract linear Hilbert space.
2. The eigenelements of L , i.e., $\vec{\phi}_i$, form a complete orthonormal set for H .
3. The eigenvalues of L , the spectrum of L , are all real.

Therefore, for any $\vec{u} \in \Gamma(\bar{R})$, let $\vec{u}$ be represented by a linear combination of the eigenelements of L ; i.e.,

$$\vec{u} = \sum_i^{\infty} a_i \vec{\phi}_i.$$

By using the orthonormal properties of the eigenelements, we obtain

$$\langle \vec{u}, L\vec{u} \rangle = \sum_i^{\infty} a_i^2 \lambda_i.$$

Since

$$||\vec{u}||^2 = \sum_i^{\infty} a_i^2,$$

if we let the largest eigenvalue be $\lambda_{\max}$, then it follows directly that

$$\langle \vec{u}, L\vec{u} \rangle \leq \lambda_{\max} \sum_i^{\infty} a_i^2 = \lambda_{\max} ||\vec{u}||^2.$$

This completes the proof.

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