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HOSE INSTABILITY AT ARBITRARY CONDUCTIVITY

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HOSE INSTABILITY AT ARBITRARY CONDUCTIVITY*

ABSTRACT

A model is developed for studying the dynamics of a low-current, highly relativistic beam propagating in a conducting medium. Here the conductivity (σ) is of arbitrary magnitude – the usual assumption being that the scale beam radius (a) is small compared with the magnetic skin length ($4\pi\sigma a^2/c$).

A dispersion formula for the hose instability is derived for the case of uniform σ and Bennett current profile $J_b(r) \propto (a^2 + r^2)^{-2}$. The peak growth rate at fixed laboratory position, maximized with respect to σ as well as driver frequency, is approximately $0.465 c/a$. This growth rate is realized when $4\pi\sigma a/c = \sqrt{12/5}$.

INTRODUCTION

The study of resistive beam instability has been previously carried out under the assumption that conductivity is large,¹ e.g.,

$$4\pi\sigma a/c \gg 1,$$

where σ is the conductivity and a is the scale beam radius. Since the conductivity is usually generated by the beam in its passage through the background medium, this condition is violated near the pulse head. A more general analysis is thus required. The problem is of considerable importance because the growth rates derived under the assumption of high conductivity increase as σ^{-1} ; when growth is most rapid the analysis is not valid. It is expected that growth rates actually vanish as $\sigma \rightarrow 0$, since in vacuum the magnetic and electric forces of the beam cancel.

In Sections I and II below, the field equations are derived using the low-current and large- γ approximations. The force law for the beam is given in Section III. Section IV contains an analytic derivation of the hose dispersion relation assuming uniform σ and Bennett current profile.

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I. FIELD EQUATIONS

It is necessary to include both the magnetic and electric fields in the force equation for the beam particles. These fields satisfy Maxwell's equations with beam and plasma currents as a source:

$$\nabla \times \underline{\underline{B}} = \frac{4\pi}{c} (\underline{\underline{J}}_b + \sigma \underline{\underline{E}}) + \frac{1}{c} \frac{\partial \underline{\underline{E}}}{\partial t}, \quad (1)$$

$$\nabla \times \underline{\underline{E}} = -\frac{1}{c} \frac{\partial \underline{\underline{B}}}{\partial t}. \quad (2)$$

The conductivity is a scalar function of space and time, but the physics governing its formation is not considered here.

The fields $\underline{\underline{E}}$ and $\underline{\underline{B}}$ are derived from potentials $\underline{\underline{A}}$ and ϕ :

$$\underline{\underline{B}} = \nabla \times \underline{\underline{A}}, \quad (3)$$

$$\underline{\underline{E}} = -\frac{1}{c} \frac{\partial \underline{\underline{A}}}{\partial t} - \nabla \phi, \quad (4)$$

so Eq. (2) is satisfied automatically. Substituting these expressions into Eq. (1) yields

$$\nabla \cdot \underline{\underline{A}} - \nabla^2 \underline{\underline{A}} = \frac{4\pi}{c} \left[\underline{\underline{J}}_b - \sigma \left(\frac{1}{c} \frac{\partial \underline{\underline{A}}}{\partial t} + \nabla \phi \right) \right] - \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \underline{\underline{A}}}{\partial t} + \nabla \phi \right). \quad (5)$$

We are free to impose a scalar gauge condition on the potentials; a very convenient one is²

$$\nabla \cdot \underline{\underline{A}} + \frac{1}{c} \frac{\partial \phi}{\partial t} + \frac{4\pi\sigma\phi}{c} = 0. \quad (6)$$

Then Eq. (5) becomes

$$\nabla^2 \underline{\underline{A}} - \frac{1}{c^2} \frac{\partial^2 \underline{\underline{A}}}{\partial t^2} - \frac{4\pi\sigma}{c^2} \frac{\partial \underline{\underline{A}}}{\partial t} = -\frac{4\pi}{c} (\underline{\underline{J}}_b + \sigma \nabla \phi). \quad (7)$$

The charge density can be determined from the equation of continuity,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\underline{\underline{J}}_b + \sigma \underline{\underline{E}}) = -\nabla \cdot \underline{\underline{J}}_b + \nabla \cdot \sigma \left(\frac{1}{c} \frac{\partial \underline{\underline{A}}}{\partial t} + \nabla \phi \right), \quad (8)$$

and the scalar potential satisfies

$$4\pi\rho = \nabla \cdot \underline{E} = -\nabla^2\phi + \frac{1}{c^2} \frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t} \frac{4\pi\sigma\phi}{c^2}. \quad (9)$$

Equations (8) and (9) are redundant with Eqs. (6) and (7).

The potentials are not completely specified by the condition (6); we may still make a gauge transformation

$$\begin{aligned} \underline{A} &= \underline{A}' + \nabla \chi, \\ \phi &= \phi' - \frac{1}{c} \frac{\partial \chi}{\partial t}, \end{aligned} \quad (10)$$

where χ is any scalar solution of

$$\nabla^2\chi - \frac{1}{c^2} \frac{\partial^2\chi}{\partial t^2} - \frac{4\pi\sigma}{c^2} \frac{\partial\chi}{\partial t} = 0. \quad (11)$$

This property is used in Section II to find the most convenient set of boundary conditions.

The beam current is low compared with the Alfvén limit³:

$$\frac{I_b}{(\beta\gamma mc^3/q)} \equiv \epsilon^2 \ll 1. \quad (12)$$

Thus the transverse beam particle velocity ($\underline{v}_\perp$) is small of order ϵ compared with v_z . We also order the transverse beam current small:

$$\underline{J}_{b\perp}/J_{bz} = O(\epsilon), \quad (13)$$

but the transverse plasma current is of order unity. The fields and derivatives in Eqs. (6)-(11) are treated as quantities of order unity, so $\underline{J}_{b\perp}$ is set equal to zero (as a source).

The beam considered is also highly relativistic ($\gamma \gg 1$), so v_z is set equal to c and we introduce the Doppler-shifted time coordinate

$$\xi = ct - z. \quad (14)$$

ξ and z are used as the independent variables instead of t and z . For any function $f(z,t)$,

$$\left(\frac{1}{c} \frac{\partial f}{\partial t}\right)_z = \left(\frac{\partial f}{\partial \xi}\right)_z, \quad (15)$$

$$\left(\frac{\partial f}{\partial z}\right)_t = \left(\frac{\partial f}{\partial z}\right)_\xi - \left(\frac{\partial f}{\partial \xi}\right)_z. \quad (16)$$

The first term on the right in Eq. (16) is assumed to be $0(\epsilon)$ because it is the rate of change of f experienced by a moving particle. Hence we set

$$\left(\frac{\partial f}{\partial z}\right)_t = -\left(\frac{\partial f}{\partial \xi}\right)_z. \quad (17)$$

This very important approximation is consistent and depends on the fact that the beam current is the basic source for σ and the fields.

Substituting from Eqs. (15) and (17) into Eqs. (6) and (7) gives

$$\nabla_{\perp} \cdot \underline{A}_{\perp} - \frac{\partial A_z}{\partial \xi} + \frac{\partial \phi}{\partial \xi} + \frac{4\pi\sigma\phi}{c} = 0, \quad (18)$$

$$\nabla_{\perp}^2 \underline{A}_{\perp} - \frac{4\pi\sigma}{c} \frac{\partial \underline{A}_{\perp}}{\partial \xi} = -\frac{4\pi}{c} \phi \nabla_{\perp} \sigma, \quad (19)$$

$$\nabla_{\perp}^2 A_z - \frac{4\pi\sigma}{c} \frac{\partial A_z}{\partial \xi} = -\frac{4\pi}{c} \left(J_{bz} - \phi \frac{\partial \sigma}{\partial \xi} \right). \quad (20)$$

Equations (18)-(20), along with boundary conditions, determine ϕ , $\underline{A}_{\perp}$, and A_z . Note that the choice of gauge has decoupled $\underline{A}_{\perp}$ from A_z when σ is constant.

II. MULTIPOLE EXPANSION AND BOUNDARY CONDITIONS

We employ cylindrical coordinates (r, θ, z) . Then Eqs. (18)-(20) become

$$\frac{1}{r} \frac{\partial}{\partial r} r A_r + \frac{1}{r} \frac{\partial A_{\theta}}{\partial \theta} - \frac{\partial A_z}{\partial \xi} + \frac{\partial \phi}{\partial \xi} + \frac{4\pi\sigma\phi}{c} = 0, \quad (21)$$

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} r A_r + \frac{1}{r} \frac{\partial A_{\theta}}{\partial \theta} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial}{\partial r} r A_{\theta} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right) - \frac{4\pi\sigma}{c} \frac{\partial A_r}{\partial \xi} = -\frac{4\pi}{c} \phi \frac{\partial \sigma}{\partial r}, \quad (22)$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial}{\partial r} r A_r + \frac{1}{r} \frac{\partial A_{\theta}}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} r A_{\theta} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right) - \frac{4\pi\sigma}{c} \frac{\partial A_{\theta}}{\partial \xi} = -\frac{4\pi}{c} \phi \frac{1}{r} \frac{\partial \sigma}{\partial \theta}, \quad (23)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial A_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} - \frac{4\pi\sigma}{c} \frac{\partial A_z}{\partial \xi} = -\frac{4\pi}{c} J_{bz} + \frac{4\pi}{c} \phi \frac{\partial \sigma}{\partial \xi}. \quad (24)$$

It is useful to expand the beam current and each component of potential as a Fourier series in θ , e.g.,

$$A_z = A_z^0 + \sum_{n=1}^{\infty} \left[A_{zs}^n \sin(n\theta) + A_{zc}^n \cos(n\theta) \right]. \quad (25)$$

The conductivity is similarly expanded; thus the field equations are not in general decoupled for differing n .

We consider here the equations for the study of the hose in the linear limit, in which the monopole ($n = 0$) terms are large, dipole ($n = 1$) terms are small, and terms of higher n are negligible. The monopole equations, which do decouple in this case, are

$$\frac{1}{r} \frac{\partial}{\partial r} r A_r^0 - \frac{\partial A_z^0}{\partial \xi} + \frac{\partial \phi^0}{\partial \xi} + \frac{4\pi\sigma^0 \phi^0}{c} = 0, \quad (26)$$

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r A_r^0 - \frac{4\pi\sigma^0}{c} \frac{\partial A_r^0}{\partial \xi} = -\frac{4\pi}{c} \phi^0 \frac{\partial \sigma^0}{\partial r}, \quad (27)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial A_z^0}{\partial r} - \frac{4\pi\sigma^0}{c} \frac{\partial A_z^0}{\partial \xi} = -\frac{4\pi}{c} \left(J_{bz}^0 - \phi^0 \frac{\partial \sigma^0}{\partial \xi} \right), \quad (28)$$

and $A_\theta^0 = 0$.

For the dipole components it is assumed, without loss of generality, that only the $\sin(\theta)$ terms of A_r , A_z , ϕ , and σ , and the $\cos(\theta)$ term of A_θ , are present. Equations (21)-(24) yield

$$\frac{1}{r} \frac{\partial}{\partial r} r A_r^1 - \frac{A_\theta^1}{r} - \frac{\partial A_z^1}{\partial \xi} + \frac{\partial \phi^1}{\partial \xi} + \frac{4\pi\sigma^0}{c} \phi^1 + \frac{4\pi\sigma^1}{c} \phi^0 = 0, \quad (29)$$

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r A_r^1 - \frac{A_r^1}{r^2} + \frac{2A_\theta^1}{r^2} - \frac{4\pi\sigma^0}{c} \frac{\partial A_r^1}{\partial \xi} - \frac{4\pi\sigma^1}{c} \frac{\partial A_r^0}{\partial \xi} = -\frac{4\pi}{c} \left(\phi^1 \frac{\partial \sigma^0}{\partial r} + \phi^0 \frac{\partial \sigma^1}{\partial r} \right), \quad (30)$$

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r A_\theta^1 - \frac{A_\theta^1}{r^2} + \frac{2A_r^1}{r^2} - \frac{4\pi\sigma^0}{c} \frac{\partial A_\theta^1}{\partial \xi} = -\frac{4\pi}{c} \phi^0 \frac{\sigma^1}{r}, \quad (31)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial A_z^1}{\partial r} - \frac{A_z^1}{r^2} - \frac{4\pi\sigma^0}{c} \frac{\partial A_z^1}{\partial \xi} - \frac{4\pi\sigma^1}{c} \frac{\partial A_z^0}{\partial \xi} = -\frac{4\pi}{c} J_{bz}^1 + \frac{4\pi}{c} \left(\phi^1 \frac{\partial \sigma^0}{\partial \xi} + \phi^0 \frac{\partial \sigma^1}{\partial \xi} \right). \quad (32)$$

All components of the potential are taken to vanish ahead of the beam (which is traveling at the speed of light). The boundary conditions at $r = 0$ result from the requirement that the fields be finite and single valued:

$$A_r^0(0) = \frac{\partial \phi^0}{\partial r}(0) = \frac{\partial A_z^0}{\partial r}(0) = 0, \quad (33)$$

$$\frac{\partial A_r^1}{\partial r}(0) = \frac{\partial A_\theta^1}{\partial r}(0) = A_z^1(0) = \phi^1(0) = 0.$$

Boundary conditions are also applied as $r \rightarrow \infty$. Conductivity vanishes at large enough r , so we must match the potentials to their vacuum forms. The gauge (10) is selected to be such that

$$A_z^0(R) = \phi^0(R) = 0 \quad (34)$$

on some surface $r = R$ in the vacuum region. Beyond this surface

$$A_r^0 \propto r^{-1}, \quad (35)$$

$$\phi^0 = A_z^0 \propto \log(r/R).$$

Note that in vacuum

$$E_z^0 = 0, \quad E_r^0 = B_\theta^0 \propto r^{-1}. \quad (36)$$

The dipole potentials, in the appropriate gauge, vary in the external region as

$$A_z^1 = \phi^1 \sim r^{-1}, \quad (37)$$

$$A_r^1 = A_\theta^1 \sim r^{-2}.$$

The corresponding $\underline{E}$ and $\underline{B}$ fields are of the form

$$E_z^1 = B_z^1 = 0, \quad (38)$$

$$E_r^1 = B_\theta^1 \propto r^{-2}.$$

III. FORCE LAW

The equation of motion for a beam particle of charge q and mass m is

$$\frac{d\underline{P}}{dt} = q(\underline{E} + \frac{\underline{v}}{c} \times \underline{B}), \quad (39)$$

with momentum

$$\underline{P} = \gamma m \underline{v} \quad (40)$$

and relativistic factors

$$\beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}, \quad (41)$$

$$\gamma = (1 - \beta^2)^{-1/2} = \left(1 + \frac{P^2}{m^2 c^2}\right)^{1/2}.$$

Applying the approximations based on high γ and low current, we have for the transverse motion:

$$\frac{d\underline{P}_\perp}{dt} = q(\underline{E}_\perp + \hat{e}_z \times \underline{B}_\perp), \quad (42)$$

$$\frac{d\underline{r}_\perp}{dt} = \frac{\underline{P}_\perp}{\gamma m}. \quad (43)$$

γ will be regarded as a known function here.

The potentials are inserted in the transverse force to get

$$\begin{aligned} q(\underline{E}_\perp + \hat{e}_z \times \underline{B}_\perp) &= q \left[-\left(\frac{1}{c} \frac{\partial \underline{A}_\perp}{\partial t} + \underline{\nabla}_\perp \phi\right)_z + \hat{e}_y \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)_t - \hat{e}_x \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)_t \right] \\ &= q \underline{\nabla}_\perp (A_z - \phi); \end{aligned} \quad (44)$$

terms involving $\underline{A}_\perp$ are seen to cancel out. This expression is identical with that used in the high- σ hose calculations if A_z is substituted for $(A_z - \phi)$.

The total derivative d/dt follows the particle motion and may be written

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_\xi = c \left(\frac{\partial}{\partial z}\right)_\xi. \quad (45)$$

Substituting Eqs. (44) and (45) into Eqs. (42) and (43) we have

$$c \left(\frac{\partial \underline{P}_\perp}{\partial z}\right)_\xi = q \underline{\nabla}_\perp (A_z - \phi), \quad (46)$$

$$c \left(\frac{\partial \underline{r}_\perp}{\partial z}\right)_\xi = \frac{\underline{P}_\perp}{\gamma m}. \quad (47)$$

In a simulation of the beam Eqs. (46) and (47) are employed directly to push particles.

In the "rigid beam model" a beam segment is assumed to displace sideways with no internal distortions; taking the displacement to be $\underline{Y} = Y(\xi, z)\hat{e}_y$, the beam current is

$$J_{bz}(\underline{r}) = J_b^0(\underline{r} - \underline{Y}) \cong J_b^0 - \underline{Y} \cdot \nabla J_b^0 = J_b^0 - Y \frac{\partial J_b^0}{\partial r} \sin(\theta) = J_b^0 + J_b^1 \sin(\theta). \quad (48)$$

The force (46) is averaged over the displaced segment. The result may be obtained from previous work⁴ by substituting $(A_z - \phi)$ for A_z :

$$\frac{1}{\gamma} \frac{\partial}{\partial z} \left(\gamma \frac{\partial Y}{\partial z} \right) = \frac{\pi q}{\gamma m c^2 I_b} \int_0^\infty dr r \left[-Y \frac{\partial J_b^0}{\partial r} \frac{\partial}{\partial r} (A_z^0 - \phi^0) + J_b^0 \frac{1}{r} \frac{\partial}{\partial r} r (A_z^1 - \phi^1) \right]. \quad (49)$$

IV. THE HOSE INSTABILITY

A dispersion formula for the hose instability of a continuous beam with the Bennett profile is derived using the rigid beam model. For all ξ , z , and r , we take

$$\sigma = \sigma^0 = \text{constant},$$

$$J_b^0 = \frac{I_b}{\pi a^2} \left(1 + \frac{r^2}{a^2} \right)^{-2}. \quad (50)$$

The monopole Eqs. (26)-(28) yield

$$A_r^0 = \phi^0 = 0,$$

$$B_\theta^0 = -\frac{\partial A_z^0}{\partial r} = \frac{2I_b}{ac} \frac{r/a}{1 + r^2/a^2}. \quad (51)$$

The effective mean betatron frequency, which appears in Eq. (49), is evaluated to be

$$k_s^2 = \frac{\pi q}{\gamma m c^2 I_b} \int_0^\infty dr r \frac{\partial J_b^0}{\partial r} \frac{\partial A_z^0}{\partial r} = \frac{1}{3} \left(\frac{2q I_b}{\gamma m c^3 a^2} \right) \equiv \frac{1}{3} k_m^2. \quad (52)$$

The frequency $k_m = \sqrt{3} k_s$ is the actual betatron frequency of a particle with orbit close to the beam axis.

Dipole potentials A_r^1 and A_θ^1 vanish by Eqs. (30) and (31) because $d\sigma/dr = 0$. Non-vanishing dipole quantities are assumed to depend on ξ and z as

$$\phi^1, A_z^1, Y \propto e^{P\xi - i\Omega z}. \quad (53)$$

Equation (29) gives immediately

$$\phi^1 = \frac{PA_z^1}{P + 4\pi\sigma/c}. \quad (54)$$

Defining the charge relaxation length

$$\lambda_c \equiv \frac{c}{4\pi\sigma}, \quad (55)$$

we have

$$(A_z^1 - \phi^1) = \frac{A_z^1}{1 + P\lambda_c}. \quad (56)$$

A_z^1 is determined from Eq. (32), which becomes

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r A_z^1 - \frac{4\pi\sigma P}{c} A_z^1 = -\frac{4\pi}{c} J_b^1. \quad (57)$$

Note that this is just the equation satisfied by A_z^1 in the high-conductivity limit.⁴ Using Eqs. (48) and (50), it may be written

$$\frac{\partial}{\partial x} \frac{1}{x} \frac{\partial}{\partial x} x A_z^1 - P\lambda_s A_z^1 = \left(\frac{4I_b Y}{ac} \right) \frac{d}{dx} \frac{1}{(1+x^2)^2}, \quad (58)$$

with

$$x = \frac{r}{a}, \quad \lambda_s = \frac{4\pi\sigma a^2}{c}. \quad (59)$$

Equation (58) is readily solved using the method of Bessel transforms⁵ to obtain

$$A_z^1 = \left(\frac{2I_b Y}{ac} \right) \int_0^\infty dy y^3 J_1(xy) \frac{K_1(y)}{y^2 + P\lambda_s}. \quad (60)$$

Substituting from Eqs. (52), (53), and (56), the force law (49) becomes

$$-\Omega^2 Y = -k_s^2 Y + \frac{\pi q}{\gamma m c^2 I_b} \int_0^\infty dr r J_b^0 \frac{1}{r} \frac{\partial}{\partial r} r \left(\frac{A_z^1}{1 + P\lambda_c} \right). \quad (61)$$

The second term on the right is evaluated using Eq. (60) for A_z^1 :

$$\begin{aligned} \frac{\pi q}{\gamma m c^2 I_b} \int_0^\infty dr r J_b^0 \frac{1}{r} \frac{\partial}{\partial r} r \left(\frac{A_z^1}{1 + P\lambda_c} \right) &= -\frac{\pi q}{\gamma m c^2 I_b} \int_0^\infty dr r \frac{A_z^1}{1 + P\lambda_c} \frac{\partial}{\partial r} J_b^0 \\ &= -3k_s^2 \frac{Y}{1 + P\lambda_c} \int_0^\infty dx x \left[\frac{\partial}{\partial x} \frac{1}{(1 + x^2)^2} \right] \int_0^\infty dy y^3 \frac{J_1(xy) K_1(y)}{y^2 + P\lambda_s} \\ &= 3k_s^2 \frac{Y}{1 + P\lambda_c} \int_0^\infty dy \frac{y^3 K_1(y)}{y^2 + P\lambda_s} \int_0^\infty dx J_1(xy) \frac{4x^2}{(1 + x^2)^3} \\ &= k_s^2 \frac{Y}{1 + P\lambda_c} \frac{3}{2} \int_0^\infty dy \frac{y^5 K_1^2(y)}{y^2 + P\lambda_s}. \end{aligned} \quad (62)$$

All terms of Eq. (61) are proportional to Y , so for a nontrivial solution P and Ω satisfy the dispersion formula

$$-\Omega^2 = -k_s^2 + \frac{k_s^2}{1 + P\lambda_c} \frac{3}{2} \int_0^\infty dy \frac{y^5 K_1^2(y)}{y^2 + P\lambda_s}. \quad (63)$$

Accurate evaluation of the growth rate must be done numerically. However, a simple analytical analysis may be made using the "one-pole approximation"; that is, we take

$$\frac{3}{2} \int_0^\infty dy \frac{y^5 K_1^2(y)}{y^2 + P\lambda_s} \cong \left(1 + \frac{5}{12} P\lambda_s \right)^{-1}. \quad (64)$$

The form of the function on the right agrees with the simple models of the hose instability,⁶ while the coefficients are selected to give agreement for $P = 0$ and $P \rightarrow \infty$. The integral cannot be expanded as a power series in P ; this is a consequence of the infinite channel radius.

Define the effective magnetic decay length by

$$\lambda_m \equiv \frac{5}{12} \lambda_s = \frac{5}{12} \frac{4 \pi \sigma a^2}{c}. \quad (65)$$

Then the dispersion relation, in its approximate form, is

$$(1 + P \lambda_c)(1 + P \lambda_m) = \frac{k_s^2}{k_s^2 - \Omega^2}. \quad (66)$$

In the limit of high conductivity ($\lambda_c \rightarrow 0$) we recover the well-known result⁵

$$P \lambda_m = \frac{\Omega^2}{k_s^2 - \Omega^2}. \quad (67)$$

To solve Eq. (66) for arbitrary σ it is useful to introduce dimensionless variables. Noting that the product

$$\lambda_m \lambda_c = \frac{5}{12} a^2 \quad (68)$$

is independent of σ , we define

$$\bar{P} = \sqrt{\lambda_m \lambda_c} P = \sqrt{\frac{5}{12}} P a, \quad (69)$$

$$\psi = \frac{1}{2} \left(\sqrt{\frac{\lambda_c}{\lambda_m}} + \sqrt{\frac{\lambda_m}{\lambda_c}} \right). \quad (70)$$

Equation (66) becomes

$$\bar{P}^2 + 2\psi\bar{P} - \frac{\Omega^2}{k_s^2 - \Omega^2} = 0. \quad (71)$$

For given real Ω there is instability in the range $\Omega^2 < k_s^2$ with growth rate

$$\bar{P} = -\psi + \sqrt{\psi^2 + \frac{\Omega^2}{k_s^2 - \Omega^2}}. \quad (72)$$

The growth rate contains a square-root singularity, but it is eliminated with spread betatron frequency as described below.

It has been previously demonstrated⁷ that a beam of the Bennett profile is adequately modeled with the distribution of (squared) betatron response frequency

$$f(k^2) = 6 \frac{k^2}{k_m^4} \left(1 - \frac{k^2}{k_m^2} \right). \quad (73)$$

The function f is normalized to unity and is related to k_s^2 by

$$\frac{1}{k_s^2} = \int_0^{k_m^2} dk^2 \frac{f(k^2)}{k^2} = \frac{3}{k_m^2}. \quad (74)$$

In the dispersion relation (71) it is appropriate to make the substitution

$$\frac{\Omega^2}{k_s^2 - \Omega^2} \rightarrow \int_0^{k_m^2} dk^2 f(k^2) \frac{\Omega^2}{k^2 - \Omega^2} = 6S \left[\frac{1}{2} - S + (S - S^2) \left(\pi i + \log \frac{1-S}{S} \right) \right], \quad (75)$$

where $S = \Omega^2/k_m^2$.

The maximum real part of this expression is approximately 0.69, occurring at $S = 0.275$. Hence we expect maximum growth rate, for given σ , to be crudely given by

$$\bar{P}_0 \approx -\psi + \sqrt{\psi^2 + 0.69}. \quad (76)$$

The growth rate increases monotonically as ψ decreases. However, ψ is bounded below by unity, so the absolute maximum growth rate is

$$\bar{P}_{00} \approx -1 + \sqrt{1.69} = 0.3,$$

or in dimensional terms

$$P_{00} \approx 0.3 \sqrt{\frac{12}{5}} a^{-1} = 0.465 a^{-1}. \quad (77)$$

It is realized with $\lambda_m = \lambda_c = \sqrt{5/12} a$, i.e.,

$$\frac{4 \pi \sigma a}{c} = \sqrt{\frac{12}{5}} \approx 1.55. \quad (78)$$

Finally it must be emphasized that two ad-hoc substitutions have been made in Eqs. (64) and (76). The actual growth rate must be determined by finding the roots of the complete dispersion relation including spread k_s^2 :

$$(1 + P\lambda_c) \left[\frac{3}{2} \int_0^\infty dy \frac{y^5 K_1^2(y)}{y^2 + P\lambda_s} \right]^{-1} - 1 = \int_0^{k_m^2} dk^2 f(k^2) \frac{\Omega^2}{k^2 - \Omega^2}. \quad (79)$$

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NOTICE

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