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**AN ANALYSIS OF THE KELVIN-HELMHOLTZ INSTABILITY
AT THE GAS-LIQUID INTERFACE OF
LIQUID-SUBMERGED SUBSONIC AND SONIC GAS JETS
RELATIVE TO FISSION-GAS JET IMPINGEMENT
IN LMFBR SUBASSEMBLIES**

by

T. C. Chawla

BASE TECHNOLOGY

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ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS

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Reactor Analysis and Safety Division

March 1974

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NOMENCLATURE

A_1, A_2, A_3 = constants defined, respectively,
by Eqs. (34) through (36)

a = radius of orifice or breach

C = constant defined by Eq. (39)

C_g = constant defined following
Eq. (2)

I_n = modified Bessel function of the
first kind of order n

$J_n(t)$ = Bessel function of the first
kind of order n

$K_n(z)$ = modified Bessel function of
second kind of order n

ℓ = parameter defined by Eq. (26)

M = Mach number

P = pressure

P_o = pressure at orifice or breach

r = radial coordinate

t = time

U_g = velocity of gas jet at orifice
or breach

u = velocity at a point in liquid
phase in axial direction

v = velocity at a point in liquid
phase in radial direction

x = axial coordinate

Greek Letters

α = $\alpha_r - i\alpha_i$

α_r = amplification factor

α_i = angular frequency

$\Gamma_\mu = (\rho_g/\rho) (\mu U_g/\sigma)^{5/2}$

$\tilde{\Gamma}_\mu = (\rho_g/\rho)^{1/2} (\mu U_g)/[\sigma(1 - M^2)^{1/4}]$

$\Gamma_\omega = \sigma/(\mu U_g)$

γ = ratio of specific heats

η = radial displacement of gas-liquid
interface from mean radius of jet

η_o = initial amplitude of displacement

κ = wave number

κ_c = parameter defined by Eq. (25)

λ = wavelength

μ = dynamic viscosity of liquid

σ = surface tension

ν = kinematic viscosity of liquid

ρ = liquid density

ϕ = velocity potential

ψ = stream function

$\omega^* = |\alpha| \lambda / (2\pi U_g)$

Subscripts

g = gas jet

m = maximum instability

Superscripts

$*, ^\wedge$ = nondimensional quantities per-
taining to sonic gas jet

$\sim$ = nondimensional quantities per-
taining to subsonic gas jet

ABSTRACT

The analysis of the stability of the gas-liquid interface of subsonic and sonic gas jets submerged in a liquid is of primary importance for the determination of the rate of liquid entrainment and the size of the droplets formed. The determination of these quantities is required in formulating a correlation for the heat-transfer coefficient in the spray-cooled region of the fission-gas jet impingement in LMFBR subassemblies. The motion of gas in subsonic and sonic gas jets is described by linearized compressible potential flow theory with negligible shear at the gas-liquid interface; the conditions of the applicability of the linearized equations of motion are also given. The motion of the surrounding liquid is assumed to be described by linearized incompressible-flow equations. It is reasoned that this assumption is compatible with the conditions and configuration pertaining to fission-gas release in LMFBR subassemblies from a nonvented fuel pin at its end of life.

The analysis shows that for the case of subsonic gas jet the pressure perturbation exerted by gas phase on the liquid layer at the gas-liquid interface is in phase with the wave amplitude and transfers energy to the liquid layer predominantly through a "lift" component or by sucking in at wave crests and pushing out at wave troughs. In case of sonic gas jet, the pressure perturbation is always out of phase with the wave amplitude; the phase angle is shown to be a function of the flow and instability parameters. Consequently, depending upon the value of phase angle, the energy to the liquid layer may be transferred both through wave-drag and "lift" components of the pressure perturbation. The analysis shows that in case of sonic gas jet, pressure perturbation becomes in phase with wave amplitude

at and above the cutoff wave number which only exists for very low viscosity liquids due to stabilizing effect of surface tension; for liquids having finite or very high viscosity, the pressure perturbation is always out of phase with the amplitude, and the liquid layer is always unstable. For a subsonic gas jet, there always exists one cutoff wave number below which the disturbance is unstable; in contrast to a sonic gas jet, the cutoff wave number is not altered by the presence of viscosity.

The analysis further shows that maxima exist in the distributions of the amplification factor of the disturbances at the gas-liquid interface as a function of wave number for both subsonic and sonic gas jets. For low viscosity liquids, these maxima are functions of the density ratio of gas to liquid. For liquids of large viscosity, however, the viscosity is the governing parameter.

I. INTRODUCTION

Experiments [1-3] performed to assess the failure potential of fission-gas release from a nonvented ruptured fuel pin near the end of life in the form of a persistent gas jet impinging on an adjacent pin (see Fig. 1) have shown that the gas in the jet entrains coolant droplets on its passage through the coolant, thus producing a spray in the impingement area. The experimental results with water as a coolant [2,3] show that, as a result of spray cooling, a significant increase in heat transfer from the heated surface in the impingement area is obtained over that which would be obtained from a single-component gaseous cooling fluid. However, recent data taken with liquid-metal sodium as coolant [4] show similar increases in heat-transfer rates compared to those due to impingement of a gas jet without the entrainment of the coolant; these rates are, however, still significantly

less than those due to liquid-metal coolant alone during normal operation. Therefore, to obtain a realistic assessment of the failure potential of a fission-gas jet impinging on an opposite pin and overheating that pin, it is essential to obtain an estimate of the heat-transfer coefficient which includes the effect of spray.

The mechanism of liquid-drop entrainment at the surface of the gas jet surrounded by coolant is governed by the Kelvin-Helmholtz instability of the gas-liquid interface. This instability is dominantly governed by energy transfer to the liquid layer at the gas-liquid interface by a pressure perturbation exerted by the gas phase on the interface. The amount of energy transfer to the liquid layer depends on the magnitude and phase of the pressure with respect to the disturbance wave at the gas-liquid interface. This phase depends on whether the gas jet is subsonic or sonic. The pressure perturbation exerted by subsonic gas flow in the jet on the liquid layer is in phase with the wave amplitude, and thus acts on the interface by pushing out at the wave troughs and sucking at the wave crests, thereby feeding energy to the disturbance in the liquid layer dominantly through "lift" component of pressure perturbation. For a sonic gas jet, on the other hand, the pressure perturbation is out of phase with the wave slope by an angle that varies with flow and instability parameters; depending upon the value of phase angle, the energy to the liquid layer may be transferred due to both "lift" and drag components of the pressure perturbation.

Several investigators have examined analytically the Kelvin-Helmholtz instability of the gas-liquid interface. Chang and Russell [5] analyzed the case of a plane liquid layer exposed to subsonic and supersonic gas streams. Nachtsheim [6] has considered the three-dimensional disturbance of a shear flow of a thin liquid film adjacent to a supersonic gas stream

with wave fronts oblique to the external stream. A nonlinear analysis of the Kelvin-Helmholtz instability of a liquid film adjacent to a compressible gas and under the influence of a body force directed either outward from or toward the liquid has been presented by Nayfeh and Saric [7]. Drazin [8] analyzed the nonlinear Kelvin-Helmholtz instability of two parallel horizontal streams of inviscid incompressible fluids. Craik [9] examined both experimentally and analytically the stability of thin liquid films in the presence of an incompressible air stream.

The objective of the present investigation is to study the instability of the gas-liquid interface of liquid-submerged axisymmetric sonic and subsonic gas jets under the action of pressure perturbations, liquid viscosity, and surface tension. It may be noted that the results of instability analysis are essential to obtaining the rate of liquid entrainment and the size of droplets resulting from the breakup of liquid being torn off at the unstable gas-liquid interface by gas flows in subsonic and sonic flow regimes. These quantities are, in turn, utilized together with experimental data on the heat-transfer coefficient to obtain a correlation for the heat-transfer coefficient in the spray-cooled region of the gas-jet impingement.

II. FORMULATION

A. General Considerations

The physical problem considered is shown in Figs. 1 and 2. The fission-gas jet issues from a pin-hole failure of radius a in the cladding of a nonvented fuel pin with a gas-plenum pressure and coolant velocity such that gas jet (whether sonic or subsonic) travels across the pin spacing and impinges on the adjacent pin, with negligible deflection of its centerline. Furthermore, only those rates of fission-gas release are considered that

lead to a persistent [10,11] gas jet of sufficient duration that the basic flow in the gas jet can be considered quasi-stationary. These rates of gas release depend on size of breach, the axial location, and the impedance to the flow of gas internally in the pin during its passage to the breach. The fluid surrounding the jet is a liquid coolant. The effect of coolant velocity transverse to the gas jet is neglected. This assumption is justified by the fact that the initial transverse (to the jet) coolant velocity becomes negligibly small due to the presence of the gas in the affected subchannels (see Fig. 1), i.e., the coolant surrounding the jet is almost "stagnant"; the experimental results [4] support this assumption.

1. Governing Equations for Gas Jet

The flow at the breach is assumed to be uniform and has a velocity U_g ; for sonic gas jet U_g is the sonic velocity at the throat conditions. The expansion of the mean jet boundary is not considered in anticipation of the short-wave approximation to be employed in the subsequent analysis, which renders the dispersion equation to become independent of mean jet radius. Therefore, it is assumed that gas jet has a constant mean radius equal to the breach radius a . Furthermore, we assume a continuous, frictionless (also neglecting the thin shear layer at the jet boundary) nonheat-conducting gas flow, free from shock waves of finite strength (this assumption is only relevant to sonic gas jet) and body forces. The basis for neglecting shear layer at the jet boundary is that the pressure at the liquid surface is not affected by the presence of shear layer [5]. In consequence of the above discussion, we can assume the flow in both subsonic and sonic gas jets can be described by potential flow theory. Furthermore, we assume that flow in the jet is described by linearized potential flow theory, which may be noted is known to hold well for subsonic gas jet.

For a sonic gas jet the limitations imposed by linearization are more stringent than for a subsonic gas jet. The linearized theory becomes valid only if the unsteadiness introduced into flow by oscillation of the disturbed gas-liquid interface is sufficiently large. In Ref. 12, the condition for the applicability of the linearized theory to unsteady flow in the sonic gas jet when the unsteadiness is caused by an oscillating gas-liquid interface has been obtained. It was shown that for short waves (i.e., $\lambda \ll a$, where λ is the wavelength of the disturbance at the gas-liquid interface) this condition is $\omega^* \gg \lambda/a$, where ω^* is the nondimensional wave velocity of the disturbance. It may be noted that this condition provides a measure of the unsteadiness.

In view of the above assumptions and discussion, the equations of motion for both subsonic and sonic two-dimensional gas jets in the axisymmetric coordinate system shown in Fig. 2 are [12]

$$\frac{\partial^2 \phi_g}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_g}{\partial r} + (1 - M^2) \frac{\partial^2 \phi_g}{\partial x^2} - 2 \frac{M^2}{U_g} \frac{\partial^2 \phi_g}{\partial x \partial t} - \frac{M^2}{U_g^2} \frac{\partial^2 \phi_g}{\partial t^2} = 0 ; \quad (1)$$

$$P_g = C_g - \rho_g \left(\frac{\partial \phi_g}{\partial t} + U_g \frac{\partial \phi_g}{\partial x} \right) , \quad (2)$$

where M is the Mach number at the breach, ϕ_g the velocity potential related to velocity vector $q(r,x,t)$ through the relation $q = \nabla \phi_g$, ρ_g the density of gas at throat conditions, P_g the pressure at any point in the gas jet, and C_g a constant defined by

$$C_g = P_o + \rho_g U_g^2 \quad \text{for } M < 1,$$

$$C_g = \frac{\gamma + 1}{\gamma} \rho_g U_g^2 \quad \text{for } M = 1.$$

Here P_0 is the pressure at the throat of breach.

2. Governing Equations for Liquid Phase

The transient motion of the surrounding coolant with density ρ and kinematic viscosity ν is represented by the following linearized equations:

$$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (rv) = 0 ; \quad (3)$$

$$\frac{\partial u}{\partial t} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right] ; \quad (4)$$

$$\frac{\partial v}{\partial t} = - \frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left\{ \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (rv) \right] \right\} , \quad (5)$$

where u and v are axial and radial components of the induced motion of the liquid, respectively, and P is the pressure at any point in the liquid.

Equations (4) and (5) can be simplified by use of a stream function $\psi(r, x, t)$ defined as

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r} ; \quad v = - \frac{1}{r} \frac{\partial \psi}{\partial x} ,$$

which follows directly from the continuity equation [Eq. (3)]. The substitution of the above expressions into Eqs. (4) and (5) yields

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial t} \right) = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial}{\partial r} \left[\frac{\partial^2 \psi}{\partial x^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right] ; \quad (6)$$

$$- \frac{1}{r} \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial t} \right) = - \frac{1}{\rho} \frac{\partial P}{\partial r} - \nu \frac{\partial}{\partial x} \left[\frac{\partial^2 \psi}{\partial x^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right] . \quad (7)$$

Elimination of the pressure P between Eqs. (6) and (7) gives

$$\left(D - \frac{1}{v} \frac{\partial}{\partial t}\right) D\psi = 0, \quad (8)$$

where

$$D \equiv \frac{\partial^2}{\partial x^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right).$$

Since the operators $(D - \partial/v\partial t)$ and D commute, the function ψ can be separated into two parts, of which ψ_1 satisfies

$$D\psi_1 = 0 \quad (9)$$

and ψ_2 satisfies

$$D\psi_2 - \frac{1}{v} \frac{\partial \psi_2}{\partial t} = 0. \quad (10)$$

The solutions of Eqs. (9) and (10) can then be combined to give the general solution of Eq. (8) as

$$\psi = \psi_1 + \psi_2. \quad (11)$$

Substituting Eq. (11) into Eqs. (6) and (7), and using Eqs. (9) and (10), we obtain

$$\frac{1}{\rho} \frac{\partial P}{\partial x} = - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial \psi_1}{\partial t} \right); \quad (12)$$

$$\frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{1}{r} \frac{\partial}{\partial x} \left(\frac{\partial \psi_1}{\partial t} \right). \quad (13)$$

3. Boundary Conditions at the Gas-liquid Interface

The normal components of the liquid and the gas velocities are each equal to the normal velocity of the interface; the linearized forms of

this interfacial kinematical boundary condition for the liquid and the gas phases are given as, respectively,

$$v = - \frac{1}{r} \frac{\partial \psi}{\partial x} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad r = a ; \quad (14)$$

$$\frac{\partial \phi_g}{\partial r} = \frac{\partial \eta}{\partial t} + U_g \frac{\partial \eta}{\partial x} \quad \text{at} \quad r = a , \quad (15)$$

where η is the displacement of the gas-liquid interface from the mean jet radius. The balance of normal forces on the interface gives

$$-P + 2\mu \frac{\partial v}{\partial r} + P_g = \sigma \left(\frac{1}{a} - \frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} \right) ; \quad (16)$$

where μ is the dynamic viscosity of liquid and σ is the surface tension.

The balance of tangential forces at the interface gives

$$\mu \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x} \right) = 0 , \quad (17)$$

III. DERIVATION OF THE DISPERSION EQUATION

As usual in linear stability analysis, we assume that the gas-liquid interface is set into motion according to expression

$$\eta(x,t) = \eta_0 e^{i\kappa x + \alpha t} , \quad (18)$$

and seek solutions of Eqs. (1), (9), and (10) in the following form, respectively:

$$\phi_g(r,x,t) = U_g x + \phi_g(r) e^{i\kappa x + \alpha t} ; \quad (19)$$

$$\psi_l(r,x,t) = \psi_l(r) e^{i\kappa x + \alpha t} ; \quad (20)$$

$$\psi_2(r, x, t) = \psi_2(r) e^{i\kappa x + \alpha t} . \quad (21)$$

Here κ is the wave number and η_0 is the initial amplitude. The substitution of the above solutions into the appropriate equations yields

$$\frac{d^2 \phi_g(r)}{dr^2} + \frac{1}{r} \frac{d\phi_g(r)}{dr} + \kappa_c^2 \phi_g(r) = 0 ; \quad (22)$$

$$\frac{d^2 \psi_1(r)}{dr^2} - \frac{1}{r} \frac{d\psi_1(r)}{dr} - \kappa^2 \psi_1(r) = 0 ; \quad (23)$$

$$\frac{d^2 \psi_2}{dr^2} - \frac{1}{r} \frac{d\psi_2(r)}{dr} - \ell^2 \psi_2(r) = 0 , \quad (24)$$

where

$$\kappa_c^2 = -\kappa^2 \left\{ (1 - M^2) + M^2 \left[\frac{2i\alpha}{\kappa U_g} + \left(\frac{\alpha}{\kappa U_g} \right)^2 \right] \right\} \quad (25)$$

and

$$\ell^2 = \kappa^2 + \frac{\alpha}{v} . \quad (26)$$

By use of the solution of Eq. (22) which vanishes at jet axis and the solutions of Eqs. (23) and (24), either of which vanishes at infinity, we obtain for Eqs. (19) through (21)

$$\phi_g(r, x, t) = U_g x + A_1 J_0(\kappa_c r) e^{i\kappa x + \alpha t} ; \quad (27)$$

$$\psi_1(r, x, t) = A_2 r K_1(\kappa r) e^{i\kappa x + \alpha t} ; \quad (28)$$

$$\psi_2(r, x, t) = A_3 r K_1(\ell r) e^{i\kappa x + \alpha t} . \quad (29)$$

Combination of Eqs. (28) and (29) gives

$$\psi = \psi_1 + \psi_2 = \left[A_2 K_1(\kappa r) + A_3 K_1(\ell r) \right] \text{re}^{i\kappa x + \alpha t} . \quad (30)$$

The substitution of Eqs. (18) and (27) into Eq. (15) gives

$$\eta_o(\alpha + i\kappa U_g) = -\kappa_c A_1 J_1(\kappa_c a) \quad \text{at} \quad r = a; \quad (31)$$

similarly, from the substitution of Eqs. (18) and (30) into Eq. (14) we obtain

$$\alpha \eta_o = -i\kappa \left[A_2 K_1(\kappa a) + A_3 K_1(\ell a) \right] \quad \text{at} \quad r = a. \quad (32)$$

Substituting Eq. (30) into Eq. (17) yields

$$2A_2 \kappa^2 K_1(\kappa a) + A_3 (\kappa^2 + \ell^2) K_1(\ell a) = 0 \quad \text{at} \quad r = a. \quad (33)$$

The solutions of Eqs. (31) through (33) give

$$A_1 = -\frac{i\kappa U_g + \alpha}{\kappa_c J_1(\kappa_c a)} \eta_o , \quad (34)$$

$$A_2 = -\frac{i\alpha}{\kappa} \frac{\kappa^2 + \ell^2}{\kappa^2 - \ell^2} \frac{\eta_o}{K_1(\kappa a)} , \quad (35)$$

and

$$A_3 = \frac{2i\kappa\alpha}{\kappa^2 - \ell^2} \frac{\eta_o}{K_1(\ell a)} \quad (36)$$

The substitution of Eq. (27) with coefficient A_1 given by Eq. (34) into Eq. (2) yields

$$P_g = C_g - \rho_g U_g^2 + \rho_g \left[\alpha + i\kappa U_g \right]^2 \frac{J_0(\kappa_c r)}{\kappa_c J_1(\kappa_c a)} \eta_o e^{i\kappa x + \alpha t} \quad (37)$$

The use of Eq. (28) with coefficient A_2 given by Eq. (35) into Eq. (12) or Eq. (13) and subsequent integration give

$$P = C - \rho \alpha^2 \frac{\kappa^2 + \ell^2}{\kappa^2 - \ell^2} \frac{K_o(\kappa r)}{\kappa K_1(\kappa a)} \eta_o e^{i\kappa x + \alpha t} . \quad (38)$$

The constant C in Eq. (38) is evaluated by applying Eq. (16) at $x = 0$ (i.e., at the location where disturbance originates, and as such at $x = 0$ the deviation η of the gas-liquid interface is zero). Thus the substitution of Eqs. (30) [with coefficients A_2 and A_3 given by Eqs. (35) and (36), respectively], (37), and (38) into Eq. (16), and then dropping the terms that involve η , give

$$C = C_g - \rho_g U_g^2 - \frac{\sigma}{a} . \quad (39)$$

Finally, with constant C given by Eq. (39), and by substitution of Eqs. (18), (37), (38), and (30) with constants A_2 and A_3 given by Eqs. (35) and (36), respectively, into Eq. (16), we obtain

$$\begin{aligned} & \rho \alpha^2 \frac{\kappa^2 + \ell^2}{\kappa^2 - \ell^2} \frac{K_o(\kappa a)}{\kappa K_1(\kappa a)} + 2\mu \alpha \left\{ \left(\frac{\kappa^2 + \ell^2}{\kappa^2 - \ell^2} \right) \left[\frac{1}{a} + \frac{\kappa K_o(\kappa a)}{K_1(\kappa a)} \right] - \frac{2\kappa^2}{\kappa^2 - \ell^2} \right. \\ & \times \left. \left[\frac{1}{a} + \frac{\ell K_o(\ell a)}{K_1(\ell a)} \right] \right\} + \rho_g (i\kappa U_g + \alpha)^2 \frac{J_0(\kappa_c a)}{\kappa_c J_1(\kappa_c a)} = \left(\frac{\kappa^2 a^2 - 1}{a^2} \right) \sigma . \quad (40) \end{aligned}$$

IV. SHORT WAVE APPROXIMATION

The dispersion equation (40) is highly complex and cannot be solved analytically for α . However, by employing asymptotic expansions of the

Bessel functions for large and small arguments, Eq. (40) can be simplified considerably. The use of large arguments, i.e., $\kappa a \gg 1$, $|\ell|a \gg 1$, and $|\kappa_c|a \gg 1$, implies that we consider only the contribution of short waves; the use of small argument implies contribution of only long waves.

Physically, the short-wave approximation is relevant to liquid breakup by a high-speed gas jet at the gas-liquid interface. The long-wavelength approximation is relevant to other jet phenomena involving scale lengths of the order of the jet diameter, e.g., breakup of a liquid jet at low flow velocity [13] or to the stability of thin liquid films in presence of high-speed gas streams. Thus, for large arguments, we employ the following asymptotic expansions [14]:

$$J_n(z) \sim \frac{1}{\sqrt{2\pi z}} \left\{ \exp \left[i \left(z - \frac{2n+1}{4} \pi \right) \right] + \exp \left[-i \left(z - \frac{2n+1}{4} \pi \right) \right] \right\} ; \quad (41)$$

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad I_n(z) = i^{-n} J_n(iz) \sim \frac{1}{\sqrt{2\pi z}} e^z . \quad (42)$$

The neglected terms in these asymptotic expansions are of the order of $1/|z|$.

The asymptotic expression for $J_n(z)$ can be simplified further: with α expressed as

$$\alpha = \alpha_r - i\alpha_i \quad (43)$$

(where α_r is the time amplification factor and α_i is the angular frequency of the disturbance), we obtain from Eq. (25) for $M = 1$

$$\kappa_c = \kappa_{cr} - i\kappa_{ci} , \quad (44a)$$

where

$$\kappa_{cr} = \sqrt{\frac{\kappa}{U_g}} \left[\sqrt{\left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g \kappa} \right)^2 + \left(\alpha_r - \frac{\alpha_i \alpha_r}{U_g \kappa} \right)^2} - \left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g \kappa} \right) \right]^{1/2} ; \quad (44b)$$

$$\kappa_{ci} = \sqrt{\frac{\kappa}{U_g}} \left[\sqrt{\left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g \kappa} \right)^2 + \left(\alpha_r - \frac{\alpha_i \alpha_r}{U_g \kappa} \right)^2} + \left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g \kappa} \right) \right]^{1/2} , \quad (44c)$$

and for $M < 1$

$$\kappa_c = i\kappa(1 - M^2)^{1/2} , \quad (45)$$

where in obtaining Eq. (45), we have made use of the inequality

$$\frac{|\alpha|}{\kappa U_g} \ll \left(\frac{1 - M^2}{M^2} \right) . \quad (46)$$

Satisfaction of the above condition implies that flow in the subsonic gas jet can be represented by quasi-steady flow. In the analysis to follow, it will be demonstrated that almost all gas-liquid systems of physical interest satisfy condition (46). Clearly, for $M = 1$, we have from Eq. (44)

$$- \operatorname{Im}(\kappa_c) = \kappa_{ci} > R(\kappa_c) = \kappa_{cr} > 0 .$$

Further, if $a|\kappa_c| \gg 1$, then

$$a\kappa_{ci} \gg 1 . \quad (47)$$

Substituting Eqs. (44) into Eq. (41) and making use of condition (47), we obtain the following simpler asymptotic formula for $J_n(\kappa_c a)$:

$$J_n(\kappa_c a) \sim \frac{1}{(2\pi\kappa_c a)^{1/2}} \exp \left[i \left(\kappa_c a - \frac{2n+1}{4} \pi \right) \right]. \quad (48)$$

Through the substitution of Eqs. (42) and (48) into Eq. (40), and employing the short-wave approximation, we obtain for $M = 1$

$$\frac{\kappa^2 + \ell^2}{\kappa^2 - \ell^2} \alpha^2 + 2\nu\kappa^2 \alpha \frac{(\kappa - \ell)^2}{\kappa^2 - \ell^2} + i \frac{\rho_g \kappa}{\rho \kappa_c} (i\kappa U_g + \alpha)^2 = \frac{\sigma \kappa^3}{\rho}. \quad (49)$$

With the use of Eqs. (42), (45), and the assumption that the wave velocity is much smaller than the gas velocity, i.e.,

$$\frac{|\alpha|}{\kappa U_g} \ll 1, \quad (50)$$

in Eq. (40), as well as employing the short-wave approximation, we obtain for $M < 1$,

$$\frac{\kappa^2 + \ell^2}{\kappa^2 - \ell^2} \alpha^2 + 2\nu\kappa^2 \alpha \frac{(\kappa - \ell)^2}{(\kappa^2 - \ell^2)} + \frac{\rho_g U_g^2 \kappa^2}{\rho (1 - M^2)^{1/2}} = \frac{\sigma \kappa^3}{\rho}. \quad (51)$$

From the conditions (46) and (50) it is clear that so long as gas flow is subsonic (not including $M \rightarrow 1$), condition (50) embraces condition (46); therefore, we consider (50) to be the necessary condition that gas-jet-liquid systems with gas jets in a subsonic flow regime must satisfy in order that Eq. (51) be applicable.

By using Eqs. (43), (44a), and (48) for $M = 1$, and Eqs. (42), (45), and condition (50) for $M < 1$, with Eq. (37), we obtain the following expressions

for the pressure perturbation exerted by gas phase on the liquid at gas-liquid interface: for $M = 1$,

$$\begin{aligned} \Delta P_g &= P_g - \frac{1}{\gamma} \rho_g U_g^2 = \operatorname{Re} \left\{ - \frac{\rho_g \kappa U_g^2}{|\kappa_c|^2} \left[(w_{rcr} \kappa_{ci} + w_{ici} \kappa_{cr}) + i(w_{rci} \kappa_{ci} - w_{icr} \kappa_{cr}) \right] \frac{\partial \eta}{\partial x} \right\} \\ &= \frac{\rho_g U_g^2 \kappa^2}{|\kappa_c|^2} \sqrt{(w_{rci} \kappa_{ci} - w_{icr} \kappa_{cr})^2 + (w_{rcr} \kappa_{ci} + w_{ici} \kappa_{cr})^2} \eta_o \sin(\kappa x - \alpha_i t + \theta) e^{\alpha_r t}; \end{aligned} \quad (52)$$

for $M < 1$,

$$\begin{aligned} \Delta P_g &= P_g - P_o = \operatorname{Re} \left[\frac{\rho_g U_g^2 \kappa}{(1 - M^2)^{1/2}} \eta_o e^{i\kappa x + \alpha t} \right] \\ &= \frac{\rho_g U_g^2 \kappa}{(1 - M^2)^{1/2}} \eta_o \cos(\kappa x - \alpha_i t) e^{\alpha_r t}, \end{aligned} \quad (53)$$

where

$$w_r = 1 - \frac{2\alpha_i}{\kappa U_g} + \left(\frac{\alpha_i}{\kappa U_g} \right)^2 - \left(\frac{\alpha_r}{\kappa U_g} \right)^2; \quad (54a)$$

$$w_i = 2 \left[\frac{\alpha_r}{\kappa U_g} - \frac{\alpha_r \alpha_i}{(\kappa U_g)^2} \right]; \quad (54b)$$

$$|\kappa_c|^2 = \kappa_{cr}^2 + \kappa_{ci}^2. \quad (55)$$

Equation (52) for the sonic gas jet shows that the pressure perturbation ΔP_g is out of phase with the wave amplitude by an angle $(\pi/2) - \theta$ since $\sin(\kappa x - \alpha_i t + \theta) = \cos[\kappa x - \alpha_i t - (\pi/2 - \theta)]$, where θ is the angle given by

$$\theta = \tan^{-1} [(w_{r\kappa ci} - w_{i\kappa cr}) / (w_{r\kappa cr} + w_{i\kappa ci})] . \quad (56)$$

It may be noted that θ is a measure of the relative effectiveness of wave-drag component as compared with the "lift" component of the pressure perturbation. For example, at $\theta = 90^\circ$ the pressure perturbation is in phase with the wave amplitude; as a result, energy is transferred to the liquid layer at the gas-liquid interface predominantly through the lift component, as with a subsonic gas jet [see Eq. (53)]. With $\theta = 0^\circ$, the pressure perturbation is in antiphase with the wave slope; as a result, the dominant mode of energy transfer is through wave drag, as in supersonic gas flow [7]. It may also be pointed out that for a plane layer with a subsonic gas stream flowing over it (the case treated in Refs. 5 and 7), the pressure perturbation is in antiphase with the wave amplitude and transfers energy to the liquid layer by sucking at the wave crests and pushing at the wave troughs; for a submerged subsonic gas jet, however, the pressure perturbation is in phase with wave amplitude and transfers energy to the liquid by sucking in at the narrowest part and pushing out at the widest part of the disturbed jet.

V. SOLUTION OF THE DISPERSION EQUATIONS

For ease in obtaining roots, Eqs. (49) and (51) are separated into real and imaginary parts by substituting for ℓ , α , and κ_c from Eqs. (26), (43), and (44), respectively. The resultant equations in the dimensionless form are: for $M = 1$,

$$\begin{aligned} & -4\Gamma_\mu \kappa^{*2} \alpha_r^* - \alpha_r^{*2} + \alpha_i^{*2} - 4\Gamma_\mu^2 \kappa^{*4} + \sqrt{8} \Gamma_\mu^2 \kappa^{*4} \ell_r^* \\ & + \frac{\Gamma_\mu \kappa^{*3}}{|\kappa_c^*|^2} (w_{r\kappa ci}^* - w_{i\kappa cr}^*) = \Gamma_\mu \kappa^{*3} ; \end{aligned} \quad (57a)$$

$$4\Gamma_{\mu}\kappa^{*2}\alpha_i^* + 2\alpha_r^*\alpha_i^* - \sqrt{8}\Gamma_{\mu}^2\kappa^{*4}\ell_i^* - \frac{\Gamma_{\mu}\kappa^{*3}}{|\kappa_c^*|^2} (w_{r\kappa}^* + w_{i\kappa}^*) = 0, \quad (57b)$$

where

$$\kappa^* = \frac{\kappa\sigma^{3/2}}{\sqrt{\mu}\rho_g U_g^{5/2}}; \quad \alpha_r^* = \frac{\alpha_r\sqrt{\mu\sigma}}{\rho_g U_g^{5/2}}; \quad (58)$$

$$\alpha_i^* = \frac{\alpha_i\sqrt{\mu\sigma}}{\rho_g U_g^{5/2}}; \quad \Gamma_{\mu} = \frac{\rho_g}{\rho} \left(\frac{\mu U_g}{\sigma} \right)^{5/2};$$

$$\ell_r^* = \left[\sqrt{\left(1 + \frac{\alpha_r^*}{\Gamma_{\mu}\kappa^{*2}}\right)^2 + \left(\frac{\alpha_i^*}{\Gamma_{\mu}\kappa^{*2}}\right)^2} + \left(1 + \frac{\alpha_r^*}{\Gamma_{\mu}\kappa^{*2}}\right) \right]^{1/2}; \quad (59a)$$

$$\ell_i^* = \left[\sqrt{\left(1 + \frac{\alpha_r^*}{\Gamma_{\mu}\kappa^{*2}}\right)^2 + \left(\frac{\alpha_i^*}{\Gamma_{\mu}\kappa^{*2}}\right)^2} - \left(1 + \frac{\alpha_r^*}{\Gamma_{\mu}\kappa^{*2}}\right) \right]^{1/2}; \quad (59b)$$

$$w_r^* = 1 - 2\Gamma_{\omega} \frac{\alpha_i^*}{\kappa^*} + \left(\frac{\Gamma_{\omega}\alpha_i^*}{\kappa^*} \right)^2 - \left(\frac{\Gamma_{\omega}\alpha_r^*}{\kappa^*} \right)^2; \quad (60a)$$

$$w_i^* = 2 \frac{\Gamma_{\omega}\alpha_r^*}{\kappa^*} \left(1 - \frac{\Gamma_{\omega}\alpha_i^*}{\kappa^*} \right); \quad (60b)$$

$$\kappa_{cr}^* = \sqrt{\kappa^*} \left[\sqrt{\left(\alpha_i^* + \Gamma_{\omega} \frac{\alpha_r^{*2} - \alpha_i^{*2}}{2\kappa^*} \right)^2 + \left(\alpha_r^* - \Gamma_{\omega} \frac{\alpha_r^*\alpha_i^*}{\kappa^*} \right)^2} - \left(\alpha_i^* + \Gamma_{\omega} \frac{\alpha_r^{*2} - \alpha_i^{*2}}{2\kappa^*} \right) \right]^{1/2}; \quad (61a)$$

$$\kappa_{ci}^* = \sqrt{\kappa^*} \left[\sqrt{\left(\alpha_i^* + \Gamma_\omega \frac{\alpha_r^{*2} - \alpha_i^{*2}}{2\kappa^*} \right)^2 + \left(\alpha_r^* - \Gamma_\omega \frac{\alpha_r^* \alpha_i^*}{\kappa^*} \right)^2} + \left(\alpha_i^* + \Gamma_\omega \frac{\alpha_r^{*2} - \alpha_i^{*2}}{2\kappa^*} \right) \right]^{1/2} ; \quad (61b)$$

$$|\kappa_c^*|^2 = \kappa_{cr}^{*2} + \kappa_{ci}^{*2} ; \quad \Gamma_\omega = \frac{\sigma}{\mu U_g} = \left(\frac{\rho_g}{\rho \Gamma_\mu} \right)^{2/5} . \quad (62)$$

For $M < 1$,

$$- 4\tilde{\Gamma}_\mu \tilde{\alpha}_r \tilde{\kappa}^2 - \tilde{\alpha}_r^2 + \tilde{\alpha}_i^2 - 4\tilde{\Gamma}_\mu^2 \tilde{\kappa}^4 + \sqrt{8} \tilde{\Gamma}_\mu^2 \tilde{\kappa}^4 \tilde{\ell}_r + \tilde{\kappa}^2 = \tilde{\kappa}^3 , \quad (63a)$$

$$\left(2\tilde{\Gamma}_\mu \tilde{\kappa}^2 + \tilde{\alpha}_r - \frac{\sqrt{2} \tilde{\Gamma}_\mu \tilde{\kappa}^2}{\tilde{\ell}_r} \right) \tilde{\alpha}_i = 0 , \quad (63b)$$

where

$$\tilde{\kappa} = \frac{\kappa \sigma (1 - M^2)^{1/2}}{\rho_g U_g^2} ; \quad \tilde{\alpha}_r = \left(\frac{\rho}{\rho_g} \right)^{1/2} \frac{\alpha_r \sigma (1 - M^2)^{3/4}}{\rho_g U_g^3} \quad (64)$$

$$\tilde{\alpha}_i = \left(\frac{\rho}{\rho_g} \right)^{1/2} \frac{\alpha_i \sigma (1 - M^2)^{3/4}}{\rho_g U_g^3} ; \quad \tilde{\Gamma}_\mu = \left(\frac{\rho_g}{\rho} \right)^{1/2} \frac{\mu U_g}{\sigma (1 - M^2)^{1/4}} ;$$

$$\tilde{\ell}_r = \left[\sqrt{\left(1 + \frac{\tilde{\alpha}_r}{\tilde{\Gamma}_\mu \tilde{\kappa}^2} \right)^2 + \left(\frac{\tilde{\alpha}_i}{\tilde{\Gamma}_\mu \tilde{\kappa}^2} \right)^2} + \left(1 + \frac{\tilde{\alpha}_r}{\tilde{\Gamma}_\mu \tilde{\kappa}^2} \right) \right]^{1/2} . \quad (65)$$

The reason for the choice for the combination of flow parameters for the purpose of nondimensionalizing α_r , α_i , and κ as indicated by expression (58)

for $M = 1$ and expression (64) for $M < 1$ will be made clear in the subsequent analysis.

The solution of Eqs. (57) and (63) consists of obtaining, respectively

$$\alpha_r^* = \alpha_r^*(\kappa^*, \Gamma_\mu, \Gamma_\omega), \quad \alpha_i^* = \alpha_i^*(\kappa^*, \Gamma_\mu, \Gamma_\omega), \quad (66)$$

$$\tilde{\alpha}_r = \tilde{\alpha}_r(\tilde{\kappa}, \tilde{\Gamma}_\mu), \quad \text{and} \quad \tilde{\alpha}_i = \tilde{\alpha}_i(\tilde{\kappa}, \tilde{\Gamma}_\mu), \quad (67)$$

where, for a given value of Γ_μ in Eqs. (66), the second of Eq. (62) shows that Γ_ω is only a function of density ratio ρ_g/ρ . Thus, in the solution of Eqs. (57) for α_r^* and α_i^* as a function κ^* , Γ_μ and ρ_g/ρ are the parameters.

Prior to obtaining solutions as indicated by their functional forms (66) and (67), it will be very instructive to obtain asymptotic behavior of the solutions. These may be solutions of Eqs. (57) and (63) for low and high values of liquid viscosity, and we search for maxima of α_r (if they exist) with respect to κ for various values of the parameters in Eqs. (66) and (67). We would also seek solutions for high and low values of the wave velocity (only relevant to the case of sonic gas jet).

It may be pointed out that some features of the following analysis for a subsonic gas jet are similar in some details to analyses presented in Refs. 5 and 7. These features include, in particular, some aspects of low- and high-liquid-viscosity approximations. However, the method of nondimensionalizing α_r , α_i , and κ deduced from the low-liquid-viscosity approximation, which reduces the number of independent parameters to a single nondimensional parameter $\tilde{\Gamma}_\mu$, is introduced here for the first time. As a consequence of the method of nondimensionalization, we are able to obtain a general solution to the full dispersion equation which was not considered in Refs. 5 or 7.

The maximum value of α_r (or nondimensional α_r) with respect to κ (or nondimensional κ) for a given value of the parameters is obtained by differentiating Eqs. (57) and (63), setting $d\alpha_r^*/d\kappa^*$ and $d\tilde{\alpha}_r/d\tilde{\kappa}$ equal to zero, and eliminating $d\alpha_r^*/d\kappa^*$ and $d\tilde{\alpha}_r/d\tilde{\kappa}$ between the resulting equations: for $M = 1$,

$$\left\{ 8\alpha_r^* + 16\Gamma_\mu \kappa^{*2} - 3\sqrt{8} \Gamma_\mu \kappa^{*2} \ell_r^* - 2\sqrt{8} \Gamma_\mu \kappa^{*2} \frac{\ell_r^*}{|\ell^*|^2} - \frac{5\kappa^{*1/2}}{2|\kappa_c^*|^2} (w_{r-ci}^* \kappa^* - w_{i-cr}^*) - \frac{\kappa^{*3/2}}{|\kappa_c^*|^2} \left[\frac{2\kappa_{ci}^* \Gamma_\omega}{\kappa^{*2}} \left(\alpha_i^* - \Gamma_\omega \frac{\alpha_i^{*2}}{\kappa^*} + \Gamma_\omega \frac{\alpha_r^{*2}}{\kappa^*} \right) + w_{r-ri}^* C_{ri} \right. \right. \\ \left. \left. + \frac{2\Gamma_\omega \kappa_{cr}^* \alpha_r^*}{\kappa^{*2}} \left(1 - \frac{2\Gamma_\omega \alpha_i^*}{\kappa^*} \right) - w_{i-rr}^* C_{rr} \right] - \kappa^{*3/2} (w_{r-ci}^* \kappa^* - w_{i-cr}^*) C_{rc} + 3\kappa^* \right\} /$$

$$\left\{ 2\alpha_i^* + \frac{\sqrt{8}\alpha_i^*}{|\ell^*|^2 \ell_r^*} + \frac{\Gamma_\mu \kappa^{*5/2}}{|\kappa_c^*|^2} \left[w_{r-ii}^* C_{ii} - \frac{2\Gamma_\omega \kappa_{ci}^*}{\kappa^*} \left(1 - \frac{\Gamma_\omega \alpha_i^*}{\kappa^*} \right) - w_{i-ir}^* C_{ir} \right. \right. \\ \left. \left. + \frac{2\kappa_{cr}^* \alpha_r^* (\Gamma_\omega / \kappa^*)^2}{\kappa^*} \right] + \Gamma_\mu \kappa^{*5/2} (w_{r-ci}^* \kappa^* - w_{i-cr}^*) C_{ic} \right\} = \left\{ -8\alpha_i^* + 3\sqrt{8} \Gamma_\mu \kappa^{*2} \ell_i^* \right. \\ \left. - 2\sqrt{8} \Gamma_\mu \kappa^{*2} \frac{\ell_i^*}{|\ell^*|^2} + \frac{5\kappa^{*1/2}}{2|\kappa_c^*|^2} (w_{r-cr}^* \kappa^* + w_{i-ci}^*) + \frac{\kappa^{*3/2}}{|\kappa_c^*|^2} \left[w_{r-rr}^* C_{rr} + \frac{2\kappa_{cr}^* \Gamma_\omega}{\kappa^{*2}} \right. \right. \\ \left. \left. \times \left(\alpha_i^* - \frac{\Gamma_\omega \alpha_i^{*2}}{\kappa^*} + \frac{\Gamma_\omega \alpha_r^{*2}}{\kappa^*} \right) + w_{i-ri}^* C_{ri} - \frac{2\Gamma_\omega \alpha_{r-ci}^*}{\kappa^{*2}} \left(1 - \frac{2\Gamma_\omega \alpha_i^*}{\kappa^*} \right) \right] + \kappa^{*3/2} C_{rc} \right. \\ \left. \times (w_{r-cr}^* \kappa^* + w_{i-ci}^*) \right\} / \left\{ 4\Gamma_\mu \kappa^{*2} + 2\alpha_r^* - \frac{\sqrt{8}\alpha_i^*}{|\ell^*|^2 \ell_i^*} - \frac{\Gamma_\mu \kappa^{*5/2}}{|\kappa_c^*|^2} \right.$$

$$\begin{aligned}
& \times \left[w_{\mathbf{r}}^* C_{\mathbf{i}\mathbf{r}} - 2 \frac{\Gamma_{\omega}}{\kappa^*} \left(1 - \frac{\Gamma_{\omega} \alpha_{\mathbf{i}}^*}{\kappa^*} \right) \frac{\kappa_{\mathbf{c}\mathbf{r}}^*}{\kappa^*} + w_{\mathbf{i}}^* C_{\mathbf{i}\mathbf{i}} - 2 \alpha_{\mathbf{r}}^* \frac{\kappa_{\mathbf{c}\mathbf{i}}^*}{\kappa^*} (\Gamma_{\omega}/\kappa^*)^2 \right] - \Gamma_{\mu} \kappa^{*5/2} C_{\mathbf{i}\mathbf{c}} \\
& \times (w_{\mathbf{i}}^* \frac{\kappa_{\mathbf{c}\mathbf{r}}^*}{\kappa^*} + w_{\mathbf{i}}^* \frac{\kappa_{\mathbf{c}\mathbf{i}}^*}{\kappa^*}) \Bigg\} , \tag{68}
\end{aligned}$$

where

$$|\underline{\kappa}_{\mathbf{c}}^*|^2 = |\kappa_{\mathbf{c}}^*|^2 / \kappa^* ; \quad \underline{\kappa}_{\mathbf{c}\mathbf{i}}^* = \kappa_{\mathbf{c}\mathbf{i}}^* / \sqrt{\kappa^*} ; \quad \underline{\kappa}_{\mathbf{c}\mathbf{r}}^* = \kappa_{\mathbf{c}\mathbf{r}}^* / \sqrt{\kappa^*} ;$$

$$|\underline{\ell}^*|^2 = \ell_{\mathbf{r}}^{*2} + \ell_{\mathbf{i}}^{*2} ;$$

$$C_{\mathbf{r}\mathbf{i}} = - \frac{C_{\mathbf{r}}}{\underline{\kappa}_{\mathbf{c}\mathbf{i}}^* |\underline{\kappa}_{\mathbf{c}}^*|^2} - \frac{\Gamma_{\omega} (\alpha_{\mathbf{r}}^{*2} - \alpha_{\mathbf{i}}^{*2})}{4 \kappa^{*2} \underline{\kappa}_{\mathbf{c}\mathbf{i}}^*} ;$$

$$C_{\mathbf{r}\mathbf{r}} = - \frac{C_{\mathbf{r}}}{\underline{\kappa}_{\mathbf{c}\mathbf{r}}^* |\underline{\kappa}_{\mathbf{c}}^*|^2} + \frac{\Gamma_{\omega} (\alpha_{\mathbf{r}}^{*2} - \alpha_{\mathbf{i}}^{*2})}{4 \kappa^{*2} \underline{\kappa}_{\mathbf{c}\mathbf{r}}^*} ; \quad C_{\mathbf{r}\mathbf{c}} = \frac{4 C_{\mathbf{r}}}{(|\underline{\kappa}_{\mathbf{c}}^*|^2)^3} ;$$

$$C_{\mathbf{r}} = \frac{\Gamma_{\omega} (\alpha_{\mathbf{r}}^{*2} - \alpha_{\mathbf{i}}^{*2})}{2 \kappa^{*2}} \left(\alpha_{\mathbf{i}}^* + \Gamma_{\omega} \frac{\alpha_{\mathbf{r}}^{*2} - \alpha_{\mathbf{i}}^{*2}}{2 \kappa^*} \right) - \frac{\Gamma_{\omega} \alpha_{\mathbf{r}}^{*2} \alpha_{\mathbf{i}}^*}{\kappa^{*2}} \left(1 - \Gamma_{\omega} \frac{\alpha_{\mathbf{i}}^*}{\kappa^*} \right) ;$$

$$C_{\mathbf{i}\mathbf{c}} = - \frac{4 C_{\mathbf{i}}}{(|\underline{\kappa}_{\mathbf{c}}^*|^2)^3} ; \quad C_{\mathbf{i}\mathbf{r}} = \frac{C_{\mathbf{i}}}{\underline{\kappa}_{\mathbf{c}\mathbf{r}}^* |\underline{\kappa}_{\mathbf{c}}^*|^2} - \frac{(1 - \Gamma_{\omega} \alpha_{\mathbf{i}}^* / \kappa^*)}{2 \kappa_{\mathbf{c}\mathbf{r}}^*} ;$$

$$C_{\mathbf{i}\mathbf{i}} = \frac{C_{\mathbf{i}}}{\underline{\kappa}_{\mathbf{c}\mathbf{i}}^* |\underline{\kappa}_{\mathbf{c}}^*|^2} + \frac{(1 - \Gamma_{\omega} \alpha_{\mathbf{i}}^* / \kappa^*)}{2 \kappa_{\mathbf{c}\mathbf{i}}^*} ;$$

$$C_{\mathbf{i}} = \left(\alpha_{\mathbf{i}}^* + \Gamma_{\omega} \frac{\alpha_{\mathbf{r}}^{*2} - \alpha_{\mathbf{i}}^{*2}}{2 \kappa^*} \right) \left(1 - \frac{\Gamma_{\omega} \alpha_{\mathbf{i}}^*}{\kappa^*} \right) - \left(1 - \frac{\Gamma_{\omega} \alpha_{\mathbf{i}}^*}{\kappa^*} \right) \frac{\Gamma_{\omega} \alpha_{\mathbf{r}}^{*2}}{\kappa^*} .$$

For $M < 1$,

$$-8\tilde{\Gamma}_{\mu}\tilde{\alpha}_r - 16\tilde{\Gamma}_{\mu}^2\tilde{\kappa}^2 \left(1 - \sqrt{1 + \frac{\tilde{\alpha}_r}{\tilde{\Gamma}_{\mu}\tilde{\kappa}^2}}\right) - \frac{4\tilde{\Gamma}_{\mu}\tilde{\alpha}_r}{\left(1 + \frac{\tilde{\alpha}_r}{\tilde{\Gamma}_{\mu}\tilde{\kappa}^2}\right)^{1/2}} + 2 = 3\tilde{\kappa} , \quad (69)$$

where in obtaining Eq. (69), we have utilized $\tilde{\alpha}_i = 0$ as one of roots of Eqs. (63b). As will be shown specifically for low- and high-viscosity liquids, the second root of Eq. (63b), viz.,

$$\tilde{\alpha}_r + \sqrt{2}\tilde{\Gamma}_{\mu}\tilde{\kappa}^2(\sqrt{2} - 1/\tilde{\ell}_r) = 0 , \quad (70)$$

leads to stable mode of disturbance.

A. Low-viscosity-liquid Approximation

The low-viscosity liquid is defined as one for which

$$\frac{\nu\kappa^2}{|\alpha|} = \frac{\Gamma_{\mu}\kappa^{*2}}{|\alpha|} = \frac{\tilde{\Gamma}_{\mu}\tilde{\kappa}^2}{|\tilde{\alpha}|} \ll 1 . \quad (71)$$

1. Low-wave Velocity

The low-wave velocity is characterized by the condition

$$\omega^* = \frac{|\alpha|}{\kappa U_g} = \frac{\Gamma_{\omega}|\alpha^*|}{\kappa^*} = \frac{(\rho_g/\rho)^{1/2}}{(1 - M^2)^{1/4}} \frac{|\tilde{\alpha}|}{\tilde{\kappa}} \ll 1 . \quad (72)$$

As discussed previously for the case of a sonic gas jet, the following condition necessary for the linearized equations of motion to apply:

$$\omega^* = \frac{\Gamma_{\omega}|\alpha^*|}{\kappa^*} \gg (\lambda/a) , \quad (73a)$$

must also be satisfied. For the case of subsonic gas jet, the corresponding condition of linearization, obtained in Ref. 12, is

$$\frac{\lambda}{a} \ll |1 - M^{-2}|. \quad (73b)$$

With the use of inequalities (71) and (72), Eqs. (57) simplify to

$$\alpha_i^{*2} - \alpha_r^{*2} + \frac{\Gamma_\mu \kappa^{*5/2}}{2} \left(\frac{\sqrt{\alpha_i^{*2} + \alpha_r^{*2} + \alpha_i^{*2}}}{\alpha_i^{*2} + \alpha_r^{*2}} \right)^{1/2} = \Gamma_\mu \kappa^{*3}; \quad (74)$$

$$2\alpha_r^* \alpha_i^* - \frac{\Gamma_\mu \kappa^{*5/2}}{2} \left(\frac{\sqrt{\alpha_i^{*2} + \alpha_r^{*2} - \alpha_i^{*2}}}{\alpha_i^{*2} + \alpha_r^{*2}} \right)^{1/2} = 0; \quad (75)$$

with the use of condition (71), Eqs. (63) simplify to

$$\tilde{\alpha}_i^2 - \tilde{\alpha}_r^2 + \tilde{\kappa}^2 = \tilde{\kappa}^3, \quad (76)$$

$$\tilde{\alpha}_r \tilde{\alpha}_i = 0. \quad (77)$$

From Eqs. (74) and (75), we obtain for the stable gas-liquid interface for $M = 1$

$$\alpha_r^* = 0; \quad (78)$$

$$\alpha_i^{*2} + \frac{\Gamma_\mu \kappa^{*5/2}}{2} \left(\frac{2}{\alpha_i^*} \right)^{1/2} = \Gamma_\mu \kappa^{*3}.$$

The unstable gas-liquid interface for $M = 1$ is governed by Eq. (74) and the second root of Eq. (75), viz.,

$$2\alpha_i^* - \frac{\Gamma_\mu \kappa^{*5/2}}{2 \left[(\alpha_i^{*2} + \alpha_r^{*2}) \left(\sqrt{\alpha_i^{*2} + \alpha_r^{*2} + \alpha_r^*} \right)^{1/2} \right]} = 0. \quad (79)$$

For $M < 1$, we obtain from Eqs. (76) and (77) for the stable gas-liquid interface

$$\tilde{\alpha}_r = 0; \quad (80a)$$

$$\tilde{\alpha}_i = \sqrt{\tilde{\kappa}^3 - \tilde{\kappa}^2} \quad \text{or} \quad \alpha_i = \left[\frac{\kappa^3 \sigma}{\rho} - \frac{\rho_g}{\rho} \frac{U^2 \kappa^2}{(1 - M^2)^{1/2}} \right]^{1/2}, \quad (80b)$$

and for the unstable gas-liquid interface,

$$\tilde{\alpha}_i = 0; \quad (81a)$$

$$\tilde{\alpha}_r = \sqrt{\tilde{\kappa}^2 - \tilde{\kappa}^3} \quad \text{or} \quad \alpha_r = \left[\frac{\rho_g}{\rho} \frac{U^2 \kappa^2}{(1 - M^2)^{1/2}} - \frac{\kappa^3 \sigma}{\rho} \right]^{1/2}. \quad (81b)$$

Equations (81b) show that the compressibility effect, indicated by factor $(1 - M^2)^{-1/2}$, is destabilizing, i.e., increasing values of M tend to increase the rate of amplification; in addition, Eq. (81b) shows that in the unstable mode the rate of amplification of the disturbance decreases if the density ratio ρ_g/ρ is decreased.

From Eqs. (74) and (79), the cutoff wave number κ_0 and the corresponding frequency for $M = 1$ are given in dimensional form by

$$\kappa_0 = \frac{5}{4} \left(\frac{\rho}{\rho_g} \right)^{0.2} \frac{\rho_g U^2}{\sigma} \quad (82)$$

and

$$\alpha_i = \frac{5}{8} \left(\frac{\rho_g}{\rho} \right)^{0.2} \frac{\rho_g U_g^3}{\sigma} . \quad (83)$$

For $M < 1$, the cutoff wave number and the corresponding frequency are obtained from Eqs. (81) as

$$\tilde{\kappa}_0 = 1 \quad \text{or} \quad \kappa_0 = \frac{\rho_g U_g^2}{\sigma (1 - M^2)^{1/2}} , \quad (84)$$

$$\alpha_i = 0 .$$

For a subsonic gas jet, it is clear from Eqs. (81) that, for $\tilde{\kappa} \geq 1$, the gas-liquid interface is stable due to the stabilizing effect of surface tension, and Eq. (80) shows that above the cutoff wave number the disturbance consists of pure oscillatory motion with a constant amplitude. For a sonic gas jet, it will be shown in the analyses to follow that similarly all disturbances having wave numbers above the cutoff value decay due to stabilizing effect of the surface tension. As can be seen from Eq. (56), at and above the cutoff number or for the stable mode of disturbance, angle θ becomes 90° , i.e., the pressure perturbation (Eq. (52)) exerted by a sonic gas-flow regime becomes in phase with the wave amplitude and the energy to the liquid layer is transferred dominantly through lift component as in the case of subsonic gas jet.

The solution of Eqs. (74) and (79) for $M = 1$ as function of wave number will be presented as a special case of a more general solution to be given subsequently. However, the solution corresponding to the mode of maximum instability is examined below. The equation corresponding to Eq. (68) for low-viscosity liquids and low wave velocity is obtained more simply from Eqs. (74) and (79) as

$$\frac{3\alpha_i^*(\alpha_i^{*2} + \alpha_r^{*2})^{1/2} + 2\alpha_i^{*2} + \alpha_r^{*2}}{4\alpha_i^{*2} + 2\alpha_r^{*2} + \alpha_i^* \sqrt{\alpha_i^{*2} + \alpha_r^{*2}}} = \frac{3\kappa^{*3} \left(\sqrt{\alpha_i^{*2} + \alpha_r^{*2} + \alpha_i^*} \right)^{1/2}}{\frac{5}{2} \kappa^{*5/2}} . \quad (85)$$

The use of Eqs. (74) and (79) into Eq. (85) yields, after some algebraic manipulations,

$$25(\alpha_i^*/\alpha_r^*)^8 + 575(\alpha_i^*/\alpha_r^*)^6 - 205(\alpha_i^*/\alpha_r^*)^4 - 215(\alpha_i^*/\alpha_r^*)^2 - 36.0 = 0. \quad (86)$$

This polynomial equation is solved numerically, and the resultant solution that also satisfies Eqs. (74) and (85) simultaneously is

$$\alpha_i^* \approx 0.92245661\alpha_r^* . \quad (87)$$

Substituting Eq. (87) into Eqs. (74) and (85), and solving for κ^* , α_r^* , and α_i^* , we obtain

$$\kappa_m^* \approx 0.80327205 \Gamma_\mu^{-0.2} ; \quad (88a)$$

$$\alpha_{rm}^* \approx 0.35717695 \Gamma_\mu^{0.2} ; \quad (88b)$$

$$\alpha_{im}^* \approx 0.32948022 \Gamma_\mu^{0.2} . \quad (88c)$$

In dimensional form, these expressions become, respectively,

$$\kappa_m \approx 0.803 \left(\frac{\rho}{\rho_g} \right)^{0.2} \frac{\rho_g U^2}{\sigma} ; \quad (89a)$$

$$\alpha_{rm} \approx 0.357 \left(\frac{\rho}{\rho_g} \right)^{0.2} \frac{\rho_g U^3}{\sigma} ; \quad (89b)$$

$$\alpha_{im} \approx 0.329 \left(\frac{\rho}{\rho_g} \right)^{0.2} \frac{\rho_g U^3}{\sigma} . \quad (89c)$$

In the above set of equations, the subscript m has been added to denote the values of κ and α_i at the maximum value α_{rm} of the amplification factor of the waves. Equations (89) show clearly that the parameters of instability, viz., κ , α_r , and α_i for low-viscosity liquids characterized by condition (71), become independent of viscosity. These equations also show that the surface tension acts to reduce the frequency and the rate of amplification at the maximum instability according to the inverse law for σ . By comparison of Eq. (89a) with Eq. (82), we find that the maximum instability occurs at a wave number below the cutoff number; consequently, the disturbance with wave numbers above cutoff are stable.

Analogously, for a subsonic gas jet the solution for the mode of maximum instability is obtained below from Eqs. (76) and (77) as

$$\tilde{\kappa}_m = \frac{2}{3} \quad \text{or} \quad \kappa_m = \frac{2}{3} \frac{\rho_g U_g^2}{\sigma(1 - M^2)^{1/2}}, \quad (90a)$$

$$\tilde{\alpha}_{rm} = \frac{2}{3\sqrt{3}} \quad \text{or} \quad \alpha_{rm} = \frac{2}{3\sqrt{3}} \left(\frac{\rho_g}{\rho} \right)^{1/2} \frac{U_g^3}{\sigma(1 - M^2)^{3/4}}, \quad (90b)$$

$$\tilde{\alpha}_{im} = 0 \quad \text{or} \quad \alpha_{im} = 0. \quad (90c)$$

It is the above set of equations that suggested the nondimensionalization employed previously [see Eqs. (64)]. By comparing Eqs. (90) for the subsonic gas jet with Eqs. (89) for the sonic gas jet, the similarities and dissimilarities in the manner in which the various instability parameters (viz., κ , α_r , and α_i) at maximum instability depend on various flow parameter are readily seen.

With the solution of Eqs. (74) and (79) given by Eqs. (89) at the maximum instability for a sonic gas jet, conditions (71) and (72) become

$$\Gamma_{\mu}^{0.4} \ll 1, \quad (91)$$

$$\left(\frac{\rho_g}{\rho}\right)^{0.4} \ll 1, \quad (92)$$

and

$$a \gg 13 \left(\frac{\rho}{\rho_g}\right)^{0.2} \frac{\sigma}{\rho_g U_g^2} \quad \text{or} \quad \frac{2a\rho_g U_g^2}{\sigma} = We \gg 26(\rho/\rho_g)^{0.2}, \quad (93)$$

where We is the Weber number based on the gas-jet diameter. In addition, the conditions necessary for employing short-wave approximation:

$$\kappa a \gg 1, \quad |\ell|a \gg 1, \quad \text{and} \quad |\kappa_c|a \gg 1, \quad (94)$$

must also be satisfied. From consideration of Eqs. (25), (26), and condition (72) it is clear that only the third of the above inequalities needs to be satisfied. From Eqs. (44) and (89) and condition (72), the third of inequalities (94) becomes

$$a \gg \frac{\sigma}{\rho_g U_g^2}. \quad (95)$$

From comparison of condition (93) with condition (95), it is evident that the (93) is more stringent than (95); therefore, condition (95) becomes redundant. Analogously for the case of subsonic gas jet, the use of Eqs. (90) with conditions (71), (72), (73b), and the third of (94) yields

$$\tilde{\Gamma}_{\mu} \ll 1, \quad (96)$$

$$\frac{(\rho_g/\rho)^{1/2}}{(1 - M^2)^{1/4}} \ll 1, \quad (97)$$

$$a \gg \frac{3\pi\sigma M^2}{\rho_g U_g^2 (1 - M^2)^{1/2}} \quad \text{or} \quad We \gg \frac{6\pi M^2}{(1 - M^2)^{1/2}}, \quad (98)$$

and

$$a \gg \frac{3\sigma}{2\rho_g U_g^2} \quad \text{or} \quad We \gg 1. \quad (99)$$

From comparison of condition (98) with condition (99), it is clear that for the subsonic gas jet (but not including $M \rightarrow 0$ and $M \rightarrow 1$) the two conditions are equivalent. From condition (97) it is evident that in the case of subsonic gas jet most gas-liquid systems will satisfy this condition.

2. Wave Velocity of the Same Order as Gas Velocity

When the wave velocity becomes of the same order as the gas velocity (only relevant for a sonic gas jet), condition (72) or its equivalent (92) for maximum instability can no longer be satisfied (i.e., when the gas stagnation pressure becomes high). It follows, then, that all the terms in the expressions (60) and (61) for w_r^* , w_i^* and κ_{cr}^* , and κ_{ci}^* , respectively, are of the same order, and Eqs. (57) become on application of condition (71) only:

$$\alpha_i^{*2} - \alpha_r^{*2} + \frac{\Gamma_\mu \kappa_c^{*3}}{|\kappa_c^*|^2} (w_{r\,ci}^* \kappa_{ci}^* - w_{i\,cr}^* \kappa_{cr}^*) = \Gamma_\mu \kappa_c^{*3}; \quad (100)$$

$$2\alpha_r^* \alpha_i^* - \frac{\Gamma_\mu \kappa_c^{*3}}{|\kappa_c^*|^2} (w_{r\,cr}^* \kappa_{cr}^* + w_{i\,ci}^* \kappa_{ci}^*) = 0. \quad (101)$$

If the nondimensionalization of α_r , α_i , and κ is carried out on the basis of Eqs. (89) instead of Eqs. (58), then

$$\hat{\alpha}_r = \frac{\alpha_r^\sigma}{\rho_g U_g^3} \left(\frac{\rho}{\rho_g} \right)^{0.2} = \alpha_r^* \Gamma_\mu^{-0.2} ; \quad \hat{\alpha}_i = \frac{\alpha_i^\sigma}{\rho_g U_g^3} \left(\frac{\rho}{\rho_g} \right)^{0.2} = \alpha_i^* \Gamma_\mu^{-0.2} ;$$

$$\hat{\kappa} = \frac{\kappa \sigma}{\rho_g U_g^2} \left(\frac{\rho}{\rho_g} \right)^{0.2} = \kappa^* \Gamma_\mu^{0.2} . \quad (102)$$

The use of the above nondimensional variables in Eqs. (100) and (101) yields for an unstable gas-liquid interface

$$\hat{\alpha}_i^2 - \hat{\alpha}_r^2 + \frac{\hat{\kappa}^3}{|\hat{\kappa}_c|^2} (\hat{w}_r \hat{\kappa}_{ci} - \hat{w}_i \hat{\kappa}_{cr}) = \hat{\kappa}^3 ; \quad (103a)$$

$$2\hat{\alpha}_i - \frac{\hat{\kappa}^3}{|\hat{\kappa}_c|^2} \left\{ \left[\hat{\kappa} - \hat{\alpha}_i \left(\frac{\rho_g}{\rho} \right)^{0.4} \right] \frac{\hat{w}_r}{\hat{\kappa}_{ci}} + 2 \left(\frac{\rho_g}{\rho} \right)^{0.4} \left[1 - \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_i}{\hat{\kappa}} \right] \frac{\hat{\kappa}_{ci}}{\hat{\kappa}} \right\} = 0 ,$$

where

$$\hat{w}_r = 1 - 2 \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_i}{\hat{\kappa}} + \left(\frac{\rho_g}{\rho} \right)^{0.8} \left(\frac{\hat{\alpha}_r}{\hat{\kappa}} \right)^2 - \left(\frac{\rho_g}{\rho} \right)^{0.8} \left(\frac{\hat{\alpha}_i}{\hat{\kappa}} \right)^2 , \quad (104a)$$

$$\hat{w}_i = 2 \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_r}{\hat{\kappa}} \left[1 - \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_i}{\hat{\kappa}} \right] , \quad (104b)$$

$$\hat{\kappa}_{cr} = \sqrt{\hat{\kappa}} \left\{ \sqrt{\left[\hat{\alpha}_i + \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}} \right]^2 + \left[\hat{\alpha}_r - \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_r \hat{\alpha}_i}{\hat{\kappa}} \right]^2} - \left[\hat{\alpha}_i + \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}} \right] \right\}^{1/2} , \quad (105a)$$

$$\hat{\kappa}_{ci} = \sqrt{\hat{\kappa}} \left\{ \sqrt{\left[\hat{\alpha}_i + \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}} \right]^2 + \left[\hat{\alpha}_r - \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_r \hat{\alpha}_i}{\hat{\kappa}} \right]^2} + \left[\hat{\alpha}_i + \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}} \right] \right\}^{1/2}, \quad (105b)$$

and

$$|\hat{\kappa}_c|^2 = \hat{\kappa}_{cr}^2 + \hat{\kappa}_{ci}^2.$$

For the stable gas-liquid interface, we obtain from Eqs. (100) and (101) on use of nondimensional variables given by Eqs. (102)

$$\hat{\alpha}_r = 0$$

and

$$\hat{\alpha}_i^2 + \frac{\hat{\kappa}^3 \left[1 - 2 \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_i}{\hat{\kappa}} + \left(\frac{\rho_g}{\rho} \right)^{0.8} \left(\frac{\hat{\alpha}_i}{\hat{\kappa}} \right)^2 \right]}{\left\{ 2\hat{\kappa}\hat{\alpha}_i \left[1 - \left(\frac{\rho_g}{\rho} \right)^{0.4} \frac{\hat{\alpha}_i}{2\hat{\kappa}} \right] \right\}^{1/2}} = \hat{\kappa}^3. \quad (106)$$

Equations (103) through (105) clearly show that the solution of Eqs. (103) for $\hat{\alpha}_r$ and $\hat{\alpha}_i$ as a function of $\hat{\kappa}$ depends only on the parameter ρ_g/ρ , i.e.,

$$\hat{\alpha}_r = \alpha_{r\mu}^* \Gamma_\mu^{-0.2} = \hat{\alpha}_r(\hat{\kappa}, \rho_g/\rho)$$

and

$$\hat{\alpha}_i = \alpha_{i\mu}^* \Gamma_\mu^{-0.2} = \hat{\alpha}_i(\hat{\kappa}, \rho_g/\rho).$$

(107)

To obtain the above solution, Eqs. (103) are solved numerically varying $\rho_g/\rho = 0$ to 1.0 to cover a very wide range of stagnation pressures (range of ρ_g/ρ of interest to LMFBR is between 0.01 to 0.2) or gas-liquid systems. It may be noted that Eqs. (74) and (75) are a special case of Eqs. (103) for $\rho_g/\rho \rightarrow 0$. Figure 3 displays such a solution of Eqs. (103). It can be seen that in the inviscid case there exists one cutoff wave number which varies with the density ratio or wave velocity. At or near the cutoff wave number, the rate of amplification decays very rapidly due to a very rapid decrease in wave drag (i.e., θ approaching 90° ; see Fig. 4), and the disturbance above the cutoff wave number consists of pure oscillatory motion with constant amplitude. From Fig. 4, it may be deduced that at low density ratios and for values of the wave number below the values at maximum instability, the energy is transferred to the liquid layer at the interface by both wave drag and "lift"; however, with increasing density ratio, the wave-drag component of the pressure perturbation increases in comparison to the "lift" component and becomes most effective (i.e., $\theta = 0$) at $\rho_g/\rho \approx 0.75$. However, at the maximum instability the wave-drag component of pressure perturbation monotonically increases in comparison to the "lift" component with increasing density ratio, as can be seen in Fig. 5. The manner in which cutoff wave number varies with density ratio is displayed in Fig. 5 which also shows a similar plot of wave number at which the amplification factor displays maximum (see Fig. 3). It can be seen that the cutoff wave number and the wave number of the maximum instability decrease with increasing density ratio, i.e., the modes of stability and maximum instability shift toward lower values of wave numbers with increasing density ratios. The cutoff wave number always lie above the wave number of maximum instability over the range of density ratios displayed in Fig. 5. Figure 5 also displays the plots of the maximum values (with respect to wave

number) of the amplification factor, and the corresponding values of the frequency and the wave velocity; it can be seen that the amplification factor and the frequency at the maximum instability display maxima with respect to density ratio. From the comparison of plots of wave velocity and wave number at maximum instability in Fig. 5, it can be seen that for a given gas-liquid system and stagnation temperature (i.e., for a fixed value of σ , ρ , and U_g), the rate of increase of wave velocity with respect to gas density is much greater than the rate of decrease of wave number or rate of increase of wavelength with the gas density (see also condition (93)). This, in turn, implies that any gas-jet-liquid system that satisfies condition (73a) at low values of gas density would satisfy this condition with greater margin as the gas density increases, i.e., the linearization of equations of motion for the sonic gas jet becomes more valid as the gas density increases.

B. High-viscosity-liquid Approximation

A highly viscous liquid is characterized by the following condition:

$$\frac{|\alpha|}{\nu \kappa^2} = \frac{|\alpha^*|}{\Gamma_\mu \kappa^{*2}} = \frac{|\tilde{\alpha}|}{\tilde{\Gamma}_\mu \tilde{\kappa}^2} \ll 1. \quad (108)$$

1. Low Wave Velocity

Using conditions (108) and (72) with Eqs. (57), we obtain for

$$M = 1,$$

$$-2\alpha_r^* + \frac{\kappa^{*1/2}}{2} \left(\frac{\sqrt{\alpha_i^{*2} + \alpha_r^{*2} + \alpha_i^{*2}}}{\alpha_i^{*2} + \alpha_r^{*2}} \right)^{1/2} = \kappa^* ; \quad (109a)$$

$$2\alpha_i^* - \frac{\kappa^* 1/2}{2} \left(\frac{\sqrt{\alpha_i^{*2} + \alpha_r^{*2}} - \alpha_i^*}{\alpha_i^{*2} + \alpha_r^{*2}} \right)^{1/2} = 0 . \quad (109b)$$

The solution of Eqs. (109) for α_r^* and α_i^* as a function of κ^* will be presented subsequently as a special case of the more general solution.

The equation corresponding to Eq. (68) for maximum instability is obtained more simply from Eqs. (109) as

$$\frac{\frac{\sqrt{\alpha_i^{*2} + \alpha_r^{*2}} - \alpha_i^*}{4\kappa^* 1/2 (\alpha_i^{*2} + \alpha_r^{*2})^{1/2}}}{2 + \frac{\kappa^* 1/2}{4} \left[\frac{\sqrt{\alpha_i^{*2} + \alpha_r^{*2}} + 2\alpha_i^*}{(\alpha_i^{*2} + \alpha_r^{*2})^{3/2}} \right] \left(\sqrt{\alpha_i^{*2} + \alpha_r^{*2}} - \alpha_i^* \right)^{1/2}} = \frac{\sqrt{\alpha_i^{*2} + \alpha_r^{*2}}}{\frac{2}{\alpha_r^*} \left(2\alpha_i^{*2} + \alpha_r^{*2} + 2\alpha_i^* \sqrt{\alpha_i^{*2} + \alpha_r^{*2}} \right)} . \quad (110)$$

The substitution of Eqs. (109) into Eq. (110) yields, after some algebraic manipulations,

$$-27(\alpha_i^*/\alpha_r^*)^6 - 3(\alpha_i^*/\alpha_r^*)^4 + 12(\alpha_i^*/\alpha_r^*)^2 + 4 = 0 . \quad (111)$$

The solution of this polynomial equation that also satisfies Eqs. (109) simultaneously is

$$\alpha_i^* \approx 0.86483545 \alpha_r^* . \quad (112)$$

The substitution of the above into Eqs. (109) yields in dimensional form

$$\kappa_m \approx 0.35191129 \sqrt{\frac{\mu}{\sigma}} \frac{\rho_g U}{\sigma} \frac{5/2}{\sigma} \approx 0.352 \frac{\rho_g U^2}{\sigma \sqrt{\Gamma_\omega}} ; \quad (113a)$$

$$\alpha_{rm} \approx 0.19740649 \frac{\rho_g U^{5/2}}{\sqrt{\mu\sigma}} - 0.197 \frac{\rho_g U^3}{\sigma} \sqrt{\Gamma_\omega} ; \quad (113b)$$

$$\alpha_{im} \approx 0.17072413 \frac{\rho_g U^{5/2}}{\sqrt{\mu\sigma}} - 0.171 \frac{\rho_g U^3}{\sigma} \sqrt{\Gamma_\omega} . \quad (113c)$$

This set of equations provided the motivation for employing the above combination of flow parameters for the purpose of nondimensionalizing α_r , α_i , and κ in the manner as indicated by expressions (58). The above solution shows that the viscosity and surface tension act to reduce the frequency and the rate of amplification at the maximum instability according to an inverse-square-root law for $\mu\sigma$. By comparison of Eqs. (113) with Eqs. (89), it can be seen that the parameter Γ_ω plays the same part (except for the power) as does the density ratio ρ_g/ρ in case of low-viscosity liquids.

By use of the solution given by Eqs. (113) for α_r , α_i , and κ at maximum instability, the conditions (108), (72), and (73a) become, respectively,

$$\Gamma_\mu \gg 1 ; \quad (114)$$

$$\Gamma_\omega \ll 1 \quad \text{or} \quad (\rho_g/\rho)^{0.4} \ll \Gamma_\mu^{0.4} ; \quad (115)$$

$$\Gamma_\omega \gg (\lambda/a) \quad \text{or} \quad a \gg 18\sigma/(\sqrt{\Gamma_\omega} \rho_g U_g^2) \quad \text{or} \quad We \gg 36/\sqrt{\Gamma_\omega} . \quad (116)$$

For the solution given by Eqs. (113), once again it can easily be shown that the third of the conditions (94) is less stringent than condition (116); therefore, for high-viscosity liquids, condition (94) is no longer a necessary condition.

For a subsonic gas jet, we obtain from the use of condition (108) in Eqs. (63),

$$\frac{3}{2} \tilde{\alpha}_i^2 - \frac{3}{2} \tilde{\alpha}_r^2 - 2\tilde{\Gamma}_\mu \tilde{\alpha}_r \tilde{\kappa}^2 + \tilde{\kappa}^2 = \tilde{\kappa}^3 ; \quad (117a)$$

$$\left(\tilde{\Gamma}_\mu \tilde{\kappa}^2 + \frac{3}{2} \tilde{\alpha}_r \right) \tilde{\alpha}_i = 0 , \quad (117b)$$

where in the above equations terms of $O[(|\tilde{\alpha}|/\tilde{\Gamma}_\mu \tilde{\kappa}^2)^3]$ and higher have been neglected. The unstable gas-liquid interface corresponds to the following roots of Eqs. (117):

$$\tilde{\alpha}_i = 0 ; \quad (118a)$$

$$\tilde{\alpha}_r = -\frac{2}{3} \tilde{\Gamma}_\mu \tilde{\kappa}^2 + \frac{2}{3} \tilde{\Gamma}_\mu \tilde{\kappa}^2 \sqrt{1 + \frac{3}{2\tilde{\Gamma}_\mu^2 \tilde{\kappa}^2} (1 - \tilde{\kappa})} . \quad (118b)$$

From Eq. (118b), the cutoff wave number once again [see Eqs. (84)] is given by $\tilde{\kappa}_0 = 1$, implying that the cutoff wave number for a subsonic gas jet is not altered by the viscosity of the liquid. Since α_r (or $\tilde{\alpha}_r$) must always be a real number, this in turn implies that, for the unstable mode of disturbance, the expression under the radical sign in Eq. (118b) must always be greater than zero; this, however, will certainly be assured provided $\tilde{\kappa} < 1$. However, if $\tilde{\kappa} > 1$ and the expression under the radical sign is still greater than zero, the gas-liquid interface becomes stable. Furthermore, if

$$\frac{|1 - \tilde{\kappa}|}{\tilde{\Gamma}_\mu^2 \tilde{\kappa}^2} \ll 1 , \quad (119)$$

then Eq. (118b) simplifies to

$$\tilde{\alpha}_r = \frac{1 - \tilde{\kappa}}{2\tilde{\Gamma}_\mu} \quad (120a)$$

From the above expression we obtain

$$\frac{|\tilde{\alpha}_r|}{\tilde{\Gamma}_\mu \tilde{\kappa}^2} = \frac{|1 - \tilde{\kappa}|}{2\tilde{\Gamma}_\mu^2 \tilde{\kappa}^2} . \quad (120b)$$

It follows that condition (119) is consistent with condition (108), which is the condition necessary for high-viscosity-liquid approximation to apply. In dimensional form, Eq. (120a) is

$$\alpha_r = \frac{1}{2\nu} \left[\frac{\rho_g}{\rho} \frac{U^2}{(1 - M^2)^{1/2}} - \frac{\kappa\sigma}{\rho} \right] ; \quad (120c)$$

it is clear that the viscosity tends to reduce the growth of the amplitude according to an inverse law for ν .

The equation corresponding to Eq. (69) for maximum instability for high-viscosity liquids is obtained more simply from Eq. (118b) as

$$\tilde{\alpha}_r = \frac{2 - 3\tilde{\kappa}}{4\tilde{\Gamma}_\mu} . \quad (121a)$$

Substitution of Eq. (121a) into Eq. (118b) yields

$$16\tilde{\Gamma}_\mu^2 \tilde{\kappa}^3 - 27\tilde{\kappa}^2 + 36\tilde{\kappa} - 12 = 0 , \quad (121b)$$

which for $\tilde{\Gamma}_\mu \gg 1$ [see condition (123) below] simplifies to

$$16\tilde{\Gamma}_\mu^2 \tilde{\kappa}^3 - 12 = 0$$

or

$$\tilde{\kappa}_m = \left(\frac{3}{4\tilde{\Gamma}_\mu^2} \right)^{1/3} \quad \text{or} \quad \kappa_m = \left[\frac{3}{4} \frac{\rho}{\rho_g} \frac{\sigma^2 (1 - M)^{1/2}}{(\mu U_g)^2} \right]^{1/3} . \quad (122a)$$

With the use of the above expression, Eq. (121a) becomes

$$\tilde{\alpha}_{rm} = \frac{1}{2\tilde{\Gamma}_{\mu}} - \left(\frac{3}{4}\right)^{4/3} \frac{1}{\tilde{\Gamma}_{\mu}^{5/3}} \approx \frac{1}{2\tilde{\Gamma}_{\mu}} \quad \text{or} \quad \alpha_{rm} = \frac{\rho_g U_g^2}{2\mu(1 - M^2)^{1/2}} . \quad (122b)$$

By consideration of the solution given by Eqs. (118a) and (122) at maximum instability, the conditions (108), (72), (94), and (73b) become, respectively,

$$\tilde{\Gamma}_{\mu}^{2/3} \gg 1 ; \quad (123)$$

$$\frac{(\rho_g/\rho)^{1/2}}{(1 - M^2)^{1/4}} \frac{1}{\tilde{\Gamma}_{\mu}^{1/3}} = \frac{1}{(1 - M^2)^{1/6}} \left(\frac{\rho_g}{\rho} \frac{\sigma}{\mu U_g} \right)^{1/3} \ll 1 ; \quad (124)$$

$$a \gg \left(\frac{\rho_g}{\rho} \right)^{1/3} \left[\frac{\mu U_g}{\sigma(1 - M^2)} \right]^{2/3} ; \quad (125)$$

$$a \gg \frac{2\pi M^2}{(1 - M^2)^{7/6}} \left(\frac{\rho_g}{\rho} \right)^{1/3} \left(\frac{\mu U_g}{\sigma} \right)^{2/3} . \quad (126)$$

It may be noted that, as for low-viscosity liquids, conditions (125) and (126) are also equivalent for high-viscosity liquids except when $M \rightarrow 0$.

The alternative root of Eq. (117b), viz.,

$$\tilde{\alpha}_r = -\frac{2}{3} \tilde{\Gamma}_{\mu}^{2\tilde{\kappa}^2} , \quad (127a)$$

when used in Eq. (117a) gives

$$\tilde{\alpha}_i = -\frac{2}{3} \tilde{\Gamma}_{\mu}^{2\tilde{\kappa}^2} \sqrt{\frac{3}{2} \frac{1}{\tilde{\Gamma}_{\mu}^{2\tilde{\kappa}^2}} (\tilde{\kappa} - 1) - 1} . \quad (127b)$$

Expression (127a) shows that the disturbance decreases with time. The choice of negative sign in Eq. (127b) implies that the disturbance propagates upstream; this, however, is not relevant to a liquid-submerged gas jet in which the disturbance originates at the orifice and travels downstream; in our previous discussion we found that the frequency is always zero.

2. Wave Velocity of the Same Order as Gas Velocity

For a sonic gas jet with a wave velocity of the same order as the gas velocity, the only conditions that need to be satisfied are conditions (108) and (73a). The application of condition (108) to Eq. (57) yields

$$-2\alpha_r^* + \frac{\kappa^*}{|\kappa_c^*|^2} (w_{r\,ci}^* \kappa^* - w_{i\,cr}^* \kappa^*) = \kappa^* \quad (128a)$$

and

$$-2\alpha_i^* - \frac{\kappa^*}{|\kappa_c^*|^2} (w_{i\,cr}^* \kappa^* + w_{i\,ci}^* \kappa^*) = 0 \quad (128b)$$

The above equations are solved numerically for $\alpha_r^* = \alpha_r^*(\kappa^*, \Gamma_\omega)$ and $\alpha_i^* = \alpha_i^*(\kappa^*, \Gamma_\omega)$ with Γ_ω as a parameter, and their solution is displayed in Figs. 6 and 7. Figure 6 shows that in highly viscous liquids, unlike low-viscosity liquids, there does not exist any cutoff wave number, and the physical condition of the liquid layer at the interface is always one of instability due to destabilizing effect of "lift" and wave drag components of the pressure perturbation on the liquid layer. As can be seen from Fig. 8, which displays angle θ as a function of wave number κ^* with Γ_ω as a parameter, θ approaches 90° (i.e., pressure perturbation becoming in phase wave amplitude) only asymptotically at large values of κ^* , unlike the case for low-viscosity liquids. In contrast to a sonic gas jet,

the cutoff wave number for a subsonic gas jet is, as noted earlier, unaltered by the presence of viscosity. Distributions of the amplification factor and the frequency as a function of wave number shown in Fig. 6 possess maxima, but these maxima occur at different wave numbers. To show the effect of parameter Γ_ω on the instability parameters at maximum instability, Fig. 7 displays these parameters as a function of Γ_ω . By comparing the plot of amplification factor given in Fig. 7 with that given in Fig. 5 for low-viscosity liquids, it can be seen that the effect of Γ_ω on the amplification factor at maximum instability is somewhat analogous to the effect of density ratio in case of low-viscosity liquids; analogous behavior of the frequency, wave numbers, and wave velocity also can be seen, except that at large values of Γ_ω for the case of highly viscous liquids, in practice it is not possible to realize these values for any physical system involving high-viscosity liquids. Further, it may be seen from Fig. 7 that wave velocity increases very slowly with increasing value of Γ_ω for $\Gamma_\omega < 0.1$. In practice, for most physical systems values of Γ_ω lie well below $\Gamma_\omega = 0.1$; this, in turn, implies that the low-wave-velocity approximation may be valid at or near maximum instability for the case of most of the highly viscous liquids.

C. General Solution

1. The Case of Sonic Gas Jet

The general solution of Eq. (57) obtained numerically in the form indicated by functions (66) is presented in Figs. 9 and 10. The values of Γ_μ were varied over a wide range; however, for the sake of clarity in the graphical presentation, plots are shown only for three values of Γ_μ corresponding, respectively, to very high, very low, and intermediate values of Γ_μ . As discussed previously, for a given value of Γ_μ , the parameter Γ_ω

depends only on the density ratio ρ_g/ρ ; therefore, in these plots the density ratio ρ_g/ρ instead of Γ_ω is shown as a parameter. For very high values of Γ_μ , e.g., $\Gamma_\mu = 10^4$, the distributions of the amplification factor and frequency as a function of κ^* given in Figs. 9 and 10, respectively, are very similar to the corresponding distributions shown in Fig. 6 for high-viscosity liquids. The reason for this similarity can be seen from the display of the plot of $\Gamma_\mu \kappa^{*2}/|\alpha^*|$ as a function of κ^* in Fig. 11; this figure shows that for the value of Γ_μ given above, $\Gamma_\mu \kappa^{*2}/|\alpha^*|$ has a finite value which increases rapidly as κ^* increases and the condition of high-viscosity-liquid approximation is satisfied with an increasing margin. Furthermore, it can be seen from Fig. 9 that, like the case for a high value of Γ_μ , there does not exist any cutoff wave number even at intermediate values of Γ_μ , e.g., $\Gamma_\mu = 1.0$, due to the destabilizing action of the "lift" and wave drag components of pressure perturbation. As can be seen from Fig. 12, which shows angle θ as a function of κ^* with Γ_μ and density ratio as parameters, θ approaches 90° asymptotically at large values of κ^* for high and moderate values of Γ_μ . Figures 9 and 10 further show that for high values of the wave number, the behavior of both the frequency and amplification factor for intermediate values of Γ_μ approaches that for high values of Γ_μ ; the explanation for this follows readily by reference to Fig. 11. For low values of Γ_μ , e.g., $\Gamma_\mu = 5 \times 10^{-6}$, the distributions of α_r^* and α_i^* behave very similarly to those at very low liquid viscosities so long as the condition $\Gamma_\mu \kappa^{*2}/|\alpha^*| \ll 1$ (see Fig. 11) is satisfied. This condition is satisfied, however, with a sufficiently wide margin for values of the wave number below those which correspond to maximum instability, and the margin decreases very rapidly as the wave number increases further, i.e., the effect of the viscosity of the liquid becomes more pronounced with increasing wave number. Consequently, the distribution of α_r^* does not show

a well-defined cutoff wave number as in the low-viscosity-liquid approximation, although the rapid fall off in the distribution for α_r^* and rapid increase in angle θ , i.e., rapid decrease in the wave-drag component of pressure perturbation (see Figs. 9 and 12) beyond the maximum instability as seen previously in Figs. 3 and 4 for low-viscosity liquids due to stabilizing effect of surface tension, are also evident at very low values of Γ_μ .

The merging or collapse of profiles of both α_r^* and α_i^* for different values of the density ratio at a very high value of Γ_μ as the wave number increases is due to rapid decrease in the wave velocity with wave number, as can be seen in Fig. 13. On the other hand, the wide scatter between profiles at intermediate and low values of Γ_μ , near the maximum instability, is due to the fact that the wave velocities at low and intermediate values of liquid viscosity are no longer small but of the same order as gas velocity, and they increase with increasing density ratio (see Fig. 13).

The evaluation of the condition of linearization for the sonic gas jet for various gas-liquid systems (covering wide range of parameters including that of interest to LMFBR safety) corresponding to general solution at maximum instability presented here is given in Ref. 12, where it is demonstrated that almost all gas-liquid systems of physical interest satisfy the condition of linearization.

2. The Case of Subsonic Gas Jet

The general solution of Eq. (63a) with $\tilde{\alpha}_i = 0$ as a root of Eq. (63b), is obtained numerically in the form $\tilde{\alpha}_r = \tilde{\alpha}_r(\tilde{\kappa}, \tilde{\Gamma}_\mu)$ as a function of $\tilde{\kappa}$, with $\tilde{\Gamma}_\mu$ as a parameter. The solution is displayed in Fig. 14, where the parameter $\tilde{\Gamma}_\mu$ is shown to vary from a very low value of 5×10^{-6} to a high value of 7.0. As can be seen from Fig. 15, which shows the plot of $\tilde{\Gamma}_\mu \tilde{\kappa}^2 / |\tilde{\alpha}|$

as a function of wave number $\tilde{\kappa}$, with $\tilde{\Gamma}_\mu$ as a parameter, at very low values of $\tilde{\Gamma}_\mu$, e.g., $\tilde{\Gamma}_\mu = 5 \times 10^{-5}$, the low-viscosity-liquid condition, namely, $\tilde{\Gamma}_\mu \tilde{\kappa}^2 / |\tilde{\alpha}| \ll 1$, is satisfied over all values of the wave number lying below the cutoff value (i.e., $\tilde{\kappa} = 1$); consequently, at very low values of $\tilde{\Gamma}_\mu$ the solution of the dispersion equation shown in Fig. 14 should correspond approximately to that given by Eqs. (81). Furthermore, Fig. 14 shows that the rate of amplification in the vicinity of wave number at maximum instability decreases rapidly as the value of $\tilde{\Gamma}_\mu$ or as liquid viscosity increases, i.e., liquid viscosity has a stabilizing effect. Figure 15 shows that at large values of $\tilde{\Gamma}_\mu$, e.g., $\tilde{\Gamma}_\mu = 7.0$, as the wave number increases beyond the value at the maximum instability, the high-viscosity-liquid condition, i.e., $\tilde{\Gamma}_\mu \tilde{\kappa}^2 / |\alpha| \gg 1$, is satisfied with increasing margin and the corresponding solution of the dispersion equation should approximately correspond to that given by Eqs. (118). For the sake of clarity of presentation, we have not shown the high- and low-viscosity-liquid approximations in Fig. 14, but one can easily verify how well the solutions shown in Fig. 14 for high and low values of $\tilde{\Gamma}_\mu$ correspond to the approximate solutions. It can be seen from Fig. 14 that the cutoff value of the wave number is always equal to unity, i.e., the cutoff value is independent of $\tilde{\Gamma}_\mu$ or liquid viscosity.

The discussion of the condition of linearization for a subconic gas jet is presented in Ref. 12, where it is noted that for short waves ($\lambda/a \ll 1$), this condition is satisfied by all gas-liquid systems of physical interest.

VI. CONCLUDING REMARKS ON THE INSTABILITY ANALYSIS

The analysis shows that the pressure perturbation exerted by the gas phase on the liquid layer at the gas-liquid interface in the case of sonic

gas jets is always out of phase with the disturbance waves, and the value of the phase angle depends upon the flow and instability parameters. On the other hand, for a subsonic gas jet, the pressure perturbation is always in phase with the waves. In the case of a sonic gas jet, the energy to the liquid layer at the gas-liquid interface is transferred both through the wave-drag and "lift" components of the pressure perturbation; however, at low liquid viscosities and near the cutoff wave number, the dominant mode of energy transfer is through the "lift" component of the pressure perturbation. In the case of subsonic gas jet, the transfer of energy to the liquid layer is predominantly through the "lift" component of the pressure perturbation. For the sonic gas jet, except at low liquid viscosities, the liquid layer at the gas-liquid interface is always unstable in spite of stabilizing effect of viscosity and surface tension; however, for a low liquid viscosity, there exists a cutoff wave number which is a function of density ratio ρ_g/ρ or wave velocity, and above which the disturbance at the gas-liquid interface is always stable, consisting only of the oscillatory motion with constant amplitude. In contrast, for a subsonic gas jet there always exists a cutoff wave number which is unaltered by the presence of viscosity.

The analysis further shows that, in both subsonic and sonic gas jets, the distributions of the amplification factor as a function of wave number show maxima which are functions of the density ratio at low liquid viscosities; at high liquid viscosity, however, the liquid viscosity rather than the density ratio becomes the governing parameter. At high liquid viscosities for both subsonic and sonic gas jets, the rate of amplification at maximum instability decreases with increasing viscosity. For both subsonic and sonic gas jets, the maximum of the instability at high liquid viscosities shifts towards lower values of wave number with increasing liquid viscosity.

In a subsequent paper, we will utilize the results of the present instability analysis to obtain the rate of liquid entrainment and the droplet size resulting from tearing off the liquid from the unstable liquid layer at the gas-liquid interface by a gas jet.

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Fig. 1

Schematic of the Fission-gas Jet Impingement in LMFBR Subassemblies.
ANL Neg. No. 900-2934 Rev. 2.

Fig. 2. Description of the Coordinate System.
ANL Neg. No. 900-1310 Rev. 1.

Fig. 3. The Dependence of the Nondimensional Amplification Factor and Frequency on the Nondimensional Wave Number and on the Density-ratio Parameter ρ_g/ρ in the Case of Low-viscosity Liquids ($M = 1$). ANL Neg. No. 900-2488.

Fig. 4

Angle θ as a Function of Nondimensional Wave Number with Density Ratio ρ_g/ρ as a Parameter in Case of Low-viscosity Liquids ($M = 1$). ANL Neg. No. 900-2495.

Fig. 5. The Dependence of the Nondimensional Amplification Factor, Frequency, Wave Number, Wave Velocity, Angle θ , All at Maximum Instability and Nondimensional Cutoff Wave Number, on the Density Ratio for Low-viscosity Liquids ($M = 1$). ANL Neg. No. 900-2484 Rev. 1.

Fig. 6. The Variation of the Nondimensional Amplification Factor and Frequency as a Function of Nondimensional Wave Number with Γ_ω as a Parameter in Case of High-viscosity Liquids ($M = 1$). ANL Neg. No. 900-2497.

Fig. 7. The Dependence of the Nondimensional Amplification Factor, Frequency, Wave Number, Wave Velocity, and Angle θ , All at Maximum Instability, on Density Ratio ρ_g/ρ in Case of High-viscosity Liquids ($M = 1$). ANL Neg. No. 900-2483 Rev. 1.

Fig. 8

The Variation of Angle θ as a Function of the Nondimensional Wave Number with Γ_ω as Parameter in Case of High-viscosity Liquids ($M = 1$). ANL Neg. No. 900-2494.

Fig. 9. The Variation of the Nondimensional Amplification Factor as a Function of the Nondimensional Wave Number with Γ_μ and density ratio ρ_g/ρ as Parameters, Corresponding to General Solution of the Dispersion Equation ($M = 1$). ANL Neg. No. 900-2491.

Fig. 10

The Variation of the Nondimensional Frequency as a Function of the Nondimensional Wave Number with Γ_μ and Density Ratio ρ_g/ρ as Parameters, Corresponding to General Solution of the Dispersion Equation ($M = 1$). ANL Neg. No. 900-2485.

Fig. 11

The Variation of $\Gamma_\mu \kappa^{*2}/|\alpha^*|$ (Liquid-viscosity Condition), Corresponding to General Solution of the Dispersion Equation, as a Function of the Nondimensional Wave Number, with Γ_μ and Density Ratio ρ_g/ρ as Parameters ($M = 1$). ANL Neg. No. 900-2489.

Fig. 12. The Variation of Angle θ , Corresponding to General Solution of the Dispersion Equation, as a Function of the Nondimensional Wave Number, with Γ_μ and Density Ratio ρ_g/ρ as Parameters ($M = 1$). ANL Neg. No. 900-2493.

Fig. 13. The Variation of the Nondimensional Wave Velocity, Corresponding to General Solution of the Dispersion Equation, as a Function of the Nondimensional Wave Number, with Γ_μ and Density Ratio ρ_g/ρ as Parameters ($M = 1$). ANL Neg. No. 900-2490 Rev. 1.

Fig. 14

The Variation of the Nondimensional Amplification Factor as a Function of the Nondimensional Wave Number with $\tilde{\Gamma}_\mu$ as a Parameter, for a Subsonic Gas Jet Corresponding to a General Solution of the Dispersion Equation. ANL Neg. No. 900-2564.

Fig. 15

The Variation of $\tilde{\Gamma}_\mu \tilde{\kappa}^2 / |\tilde{\alpha}|$ (Liquid-viscosity Condition) as a Function of the Nondimensional Wave Number with $\tilde{\Gamma}_\mu$ as a Parameter for a Subsonic Gas Jet, Corresponding to a General Solution of the Dispersion Equation. ANL Neg. No. 900-2563.