RAMAN AND BRILLOUIN SCATTERINGS OF ELECTROMAGNETIC WAVES IN INHOMOGENEOUS PLASMAS

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ABSTRACT

Raman and Brillouin scattering of an electromagnetic wave in an inhomogeneous, expanding plasma are studied. Application to laser-pellet irradiation is considered.

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I. INTRODUCTION

In considering the possibility of fusion by intense laser irradiation on a solid deuterium pellet, it is important that a significant fraction of light is absorbed by the plasma. It is well-known that, near the region of critical density where the local plasma frequency $\omega_p = (4\pi ne^2/m)^{1/2}$ is equal to the laser frequency ω_0 , the parametric decay of the pump wave into electron plasma waves and ion waves constitutes an important process for the anomalous absorption of the laser light.¹ The laser light is nearly normally incident on a plasma profile of increasing density, and must pass through the underdense region, where the local plasma frequency ω_p is lower than the laser frequency ω_0 ,

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to reach the region of critical density. Although the underdense region is transparent to the light according to linear theory, non linear scattering instabilities exist if a certain threshold power is exceeded, and provided neither of the decay waves is strongly damped. Decay of the incident light wave into a scattered wave and an electron plasma wave² (Raman scattering) can become unstable for $2k_0\lambda_D < 1$ where k_0 is the wave number of the incident light and λ_D the Debye length of the plasma; decay into a scattered wave and an ion acoustic wave³ (Brillouin scattering) can be unstable for $2k_0\lambda_D < (T_e/T_i)^{\frac{1}{2}}$. A fraction of the incident energy can thus be anomalously scattered in the underdense region once a threshold power, determined principally by plasma inhomogeneity, is exceeded, tending to prevent the light wave from reaching the critical density where it can be anomalously absorbed.

In Section II we review the theory of Raman and Brillouin scattering instabilities in a homogeneous plasma. In Section III the general formalism of parametric instabilities in an inhomogeneous plasma is discussed, and a simple necessary condition for the existence of an absolute instability is determined, namely the wave vector mismatch $K(x) = k_{0x} - k_{1x} - k_{2x}$, which depends on x through the dispersion relations of the waves involved, must possess a second order zero. If the zero is first order only spatial amplification can occur, limited in amplitude by convection. In Section IV

this formalism is applied to Raman and Brillouin instabilities. In Section V we examine those points at which $K(x)$ possesses a second order zero and evaluate the growth rates and thresholds for the temporally growing instabilities. Section VI discusses Raman and Brillouin scattering for very large field intensities.

II RAMAN AND BRILLOUIN SCATTERING IN HOMOGENEOUS PLASMAS

Consider three waves $(\omega_i, \vec{k}_i)$ with subscript 0 denoting the pump laser wave, subscript 1 the scattered photon, and subscript 2 the electrostatic plasma waves (either Langmuir or ion acoustic), satisfying the resonance conditions: $\omega_0 = \omega_1 + \omega_2$ and $\vec{k}_0 = \vec{k}_1 + \vec{k}_2$. For a wave with energy density ϵ_i , the action amplitude, a_i , is defined by $|a_i|^2 = N_i = \epsilon_i / \omega_i$, which is the number of quanta (action) in the wave. The equation in a homogeneous plasma for the time evolution of a_i due to the resonant three-wave interaction is

$$\frac{\partial a_i}{\partial t} = V_{ij} a_j a_k \quad (1)$$

The conditions for detailed balance (Manley-Rowe relations) imply that the matrix V_{jk} is symmetric, and that the high frequency wave must be unstable to decay into lower frequency. The growth rate will then be given simply by $\gamma = |V| |a_0|$. To evaluate the

matrix element, we consider the two-fluid equations for the electrostatic waves:

$$\frac{\partial n_{\alpha}}{\partial t} + \nabla \cdot (n_{\alpha} \vec{v}_{\alpha}) = 0 \quad (2)$$

$$\frac{\partial \vec{v}_{\alpha}}{\partial t} + (\vec{v}_{\alpha} \cdot \nabla) \vec{v}_{\alpha} = \frac{\nabla p_{\alpha}}{m_{\alpha} n_{\alpha}} + \frac{e_{\alpha}}{m_{\alpha}} [\vec{E}_{\alpha} + \vec{v}_{\alpha} \times \vec{B}/c] \quad (3)$$

$$\nabla \cdot \vec{E}_2 = 4\pi \sum_{\alpha} e_{\alpha} n_{\alpha} \quad (4)$$

where the subscript α denotes the species (ion or electron). The nonlinear terms in Eq. (3) arise from the electromagnetic waves 0 and 1. For these waves ($j = 0, 1$), $\dot{n}_j = 0$, $\vec{v}_j = e\vec{E}_j/im\omega_j$ and $\nabla \times \vec{v}_j = -\frac{e}{m} \vec{B}_j$. Using the vector identity $(\vec{v} \cdot \nabla) \vec{v} = (\nabla \times \vec{v}) \times \vec{v} + \nabla(v^2/2)$, we may rewrite Eq. (3) as

$$\frac{\partial \vec{v}_2}{\partial t} + \frac{\nabla p_2}{nm} - \frac{e\vec{E}_2}{m} = -\nabla(\vec{v}_0 \cdot \vec{v}_1) \quad (5)$$

Thus the coupling is such that wave 2 is purely electrostatic. Taking the divergence of Eq. (5) and substituting into Eqs. (2) and (4), we obtain

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 + k^2 v_e^2\right) \left(\frac{\partial^2}{\partial t^2} + \omega_{pi}^2\right) n_i - \omega_p^2 \omega_{pi}^2 n_i = -\omega_{pi}^2 n_0 k^2 (\vec{v}_0 \cdot \vec{v}_1) \quad (6)$$

where we have set $T_i = 0$ since we are interested in the case $T_e \gg T_i$, and $\omega_{pi}^2 = (4\pi n e^2/M)^{\frac{1}{2}}$, $v_e = (T_e/m)^{\frac{1}{2}}$. The driving term in the right hand side is due to the beating of waves 0 and 1.

Without the pump wave, the right hand side of Eq. (6) vanishes and two eigenfrequencies can readily be found: $\omega_R^2 = \omega_p^2 + k^2 v_e^2$ (a

kinetic treatment would yield a factor 3 in the last term) for the Langmuir wave and $\omega_B^2 = k^2 c_s^2 / (1 + k^2 \lambda_D^2)$ for the ion acoustic wave, where $c_s^2 = T_e / M$. With the pump wave, we now look for solutions of the form $n_i = n_i(t) \exp(i\omega_2 t)$, where $\omega_2 = \omega_R$ or ω_B corresponding to Raman or Brillouin scattering which we denote by subscripts R and B respectively. For these cases we immediately find from Eq. (6)

$$\frac{\partial n_{iR}}{\partial t} = -\frac{i}{2} \frac{m}{M} \frac{n_0}{\omega_p} k^2 (\vec{v}_0 \cdot \vec{v}_1) \quad (7)$$

and

$$\frac{\partial n_{iB}}{\partial t} = -\frac{i}{2} \frac{m}{M} \frac{n_0}{\omega_B} k^2 (\vec{v}_0 \cdot \vec{v}_1) \quad (8)$$

It remains to express n , $\vec{v}_1$ in terms of the action amplitudes, a . For the electromagnetic waves with energy density $\epsilon = (E^2 + B^2)/8\pi$, $v_1 = \frac{e}{m} \left(\frac{2\pi}{\omega_1}\right)^{\frac{1}{2}} a_1$. For electrostatic waves of frequency Ω , the energy density is $\epsilon_\Omega = (\omega \frac{\partial \epsilon}{\partial \omega})_{\omega=\Omega} E^2/8\pi$, where $\epsilon(\omega, \vec{k})$ is the dielectric function. Thus, $\epsilon_R = E^2/4\pi$ and $\epsilon_B = (1 + \frac{1}{k^2 \lambda_D^2}) E^2/4\pi$ from which we find

$$n_{iR} = \frac{m}{M} \frac{k}{e} \left(\frac{\omega_p}{8\pi}\right)^{\frac{1}{2}} a_R \quad (9)$$

and

$$n_{iB} = \left(\frac{n_0 k^2}{2M\omega_B}\right)^{\frac{1}{2}} a_B \quad (10)$$

Substituting into the rate equations yields

$$\frac{\partial a_R}{\partial t} = \frac{i}{2} k_2 v_0 \left(\frac{\omega_p}{\omega_1}\right)^{\frac{1}{2}} (\vec{e}_0 \cdot \vec{e}_1) a_1 \equiv (V_R a_0) a_1 \quad (11)$$

and

$$\frac{\partial a_B}{\partial t} = \frac{i}{2} \frac{\omega_{pi} \cdot k_2 v_0}{2(\omega_1 \omega_B)^{\frac{1}{2}}} (\vec{e}_0 \cdot \vec{e}_1) \equiv (V_B^a)_1 a_1, \quad (12)$$

where $\vec{e}_1$ and $\vec{e}_0$ are the direction of polarization of photons.

The laser intensity enters through the induced electron velocity

$v_0 = eE_0/m\omega_0$. From the general discussion given earlier, we have a similar equation for a_1 : $\frac{\partial a_1}{\partial t} = a_0 \left\{ \frac{V_R^a}{V_B^a} \right\}$, for Raman and Brillouin scattering respectively. We then obtain the growth rate

for Raman scattering

$$\gamma_R^0 = \frac{k_2 v_0}{2} \sqrt{\frac{\omega_p}{\omega_1}} |\vec{e}_1 \cdot \vec{e}_0| \quad (13)$$

and for Brillouin scattering with $k_2 \lambda_D \ll 1$

$$\gamma_B^0 = \frac{\omega_{pi}}{2} \frac{k_2 v_0}{(\omega_1 k_2 c_s)^{\frac{1}{2}}} |\vec{e}_1 \cdot \vec{e}_0|. \quad (14)$$

Note that the growth rate for both processes attains its maximum

value: $\gamma_{RM}^0 \approx (v_0/c) \sqrt{\omega_0 \omega_p}$ and $\gamma_{BM}^0 = \omega_{pi} \frac{v_0}{\sqrt{2} c c_s}$ for backscattering

where $|\vec{k}_2| \approx 2k_0$ has its maximum. By the superscript zero we

indicate the growth rate for an infinite, homogeneous plasma in the

absence of damping. For Brillouin scattering, $k\lambda_D$, a kinetic

treatment⁴ yields the growth rate

$$\gamma_B = \frac{\omega_{pi}}{2} \frac{v_0}{\sqrt{c c_s}} \left(\frac{k_2}{k_0}\right)^{\frac{1}{2}} \frac{1}{(1 + k_2^2 \lambda_D^2)^{\frac{1}{2}}} \xrightarrow{k_2 \lambda_D > 1} \frac{\omega_{pi} v_0}{2 (k_0 k_2 c c_s \lambda_D^2)^{\frac{1}{2}}}$$

which maximizes for minimum k_2 so long as $k_2 \lambda_D \gg 1$.

For very high incident laser power, the frequency shift of the excited plasma waves can exceed their eigenfrequencies^{4,5} and

Eq. (6) then becomes, for Brillouin scattering, $(\frac{\partial^2}{\partial t^2} + \omega_B^2)a_B = -\frac{\omega_{pi}}{2} k_2 v_0 (\frac{\omega_B}{\omega_1})^{\frac{1}{2}} |\vec{e}_1 \cdot \vec{e}_0| a_1$, yielding the frequency and growth rate

$$\omega + i\gamma = [- (\gamma_B^0)^2 \omega_B]^{\frac{1}{3}} = (\frac{\sqrt{3} + i}{2}) \left(\frac{\omega_{pi}^2 k_2^2 v_0^2}{\omega_0} |\vec{e}_1 \cdot \vec{e}_0| \right)^{\frac{1}{3}} \quad (15)$$

valid for $\gamma_B^0 \gg \omega_B$.

The threshold due to wave damping may easily be obtained by letting $\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \nu$ in Eq. (1) where ν is the damping rate, collisional or Landau, of the waves. This leads to a threshold $(\frac{\partial}{\partial t} = 0)$ of $\gamma^0 = (\nu_1 \nu_2)^{\frac{1}{2}}$. As the reflected wave is subject only to collisional damping which is weak (neglecting the turbulent damping due to nonlinear interactions), this threshold power is generally very low and is of no concern to us.

We note however that for $k_2 \lambda_D \gtrsim 1/3$, the Langmuir waves become heavily Landau damped with consequent reduction in growth rate for the Raman mode. This implies that when electrons are heated such that $v_e/c > 0.2$ the Raman mode is effectively turned off. Of course, nonlinear Compton scattering due to the Landau damping of the beat mode persists with a smaller growth rate even for $k_0 \lambda_D > 1$.⁵ In the absence of an effective heating mechanism for ions, no such limiting effect on Brillouin scattering exists.

III. PARAMETRIC INSTABILITIES IN INHOMOGENEOUS PLASMAS

In an inhomogeneous plasma, the $\vec{k}$ -matching condition can hold only locally and the mismatch $K = k_{0x}(x) - k_{1x}(x) - k_{2x}(x)$ develops due to the spatially dependent quantities, e.g. density, temperature, which occur in the dispersion relations determining $k(\omega, x)$. This phase mismatch then localizes the region of resonant interaction.

For a given pump strength, the propagation of the decay waves out of this resonant region then provides an additional threshold for effective growth. This, in fact, is the major threshold for Raman and Brillouin scattering in which one of the decay waves is a light wave with its rapid speed of propagation. The general formalism for parametric instabilities in inhomogeneous media with the WKB approximation has been previously developed by one of us.⁶ Here, we extend the previous work by including the effect of wave damping. For simplicity, we consider a plasma slab with density gradient in the x direction. In the weak-pump case, the equations for the action amplitudes of the decay waves in an inhomogeneous medium are a simple extension of Eq. (1)

$$\begin{aligned} \frac{\partial a_1}{\partial t} + \nu_1 a_1 + V_1 \frac{\partial a_1}{\partial x} &= \gamma_0 a_2 \exp \left(i \int^x K dx' \right) \\ \frac{\partial a_2}{\partial t} + \nu_2 a_2 - V_2 \frac{\partial a_2}{\partial x} &= \gamma_0 a_1 \exp \left(- i \int^x K dx' \right) + \delta(x) \end{aligned} \quad (16)$$

where $\delta(x)$ is the source due to thermal excitation, V_1, V_2 are the magnitudes of the group velocities of the two decay waves assumed to be oppositely propagating along the density gradient and ν_1 and ν_2 their respective damping rates. Laplace transforming in time with $a(p) = \int_0^\infty dt \exp(-pt) a(t)$, where p is the Laplace transform variable and writing $a_1(p) = \hat{a}_1 \exp(i \int_0^x \frac{1}{2} K dx')$, $a_2(p) = \hat{a}_2 \exp(-i \int_0^x \frac{1}{2} K dx')$, we have

$$(p + \nu_1) \hat{a}_1 + i \frac{KV_1}{2} \hat{a}_1 + V_1 \frac{\partial \hat{a}_1}{\partial x} = \gamma_0 \hat{a}_2 + \hat{a}_1(0)$$

$$(p + \nu_2) \hat{a}_2 + i \frac{KV_2}{2} \hat{a}_2 - V_2 \frac{\partial \hat{a}_2}{\partial x} = \gamma_0 \hat{a}_1 + \hat{a}_2(0).$$

Choosing initial values $\hat{a}_1(0) = 0$ and $\hat{a}_2(0) = b\delta(x)$ and eliminating $\hat{a}_2$, we have, with $a = \hat{a}_1 \exp\left(\frac{p+\nu_1}{V_1} - \frac{p+\nu_2}{V_2}\right) \frac{x}{2}$:

$$\frac{\partial^2 a}{\partial x^2} + \frac{1}{4} \left[K - i \left(\frac{p+\nu_1}{V_1} + \frac{p+\nu_2}{V_2} \right) \right]^2 a + \left(\frac{\gamma_0^2}{V_1 V_2} + \frac{i}{2} \frac{dK}{dx} \right) a = \frac{\gamma_0 b}{V_1 V_2} \delta(x) \quad (17)$$

Without the source term on the right hand side, Eq. (17) is an eigenvalue equation for a with p its eigenvalue. If there is a well-behaved solution with $\text{Re } p_0 > 0$, then p_0 will correspond to the eigenvalue for a temporally growing mode with growth rate given by $\text{Re } p_0$.

This mode is generally localized within certain regions and grows exponentially in time until some nonlinear mechanism limits the growth. If no such p_0 exists, then only spatial amplification over a given source at the thermal level is possible. By spatial amplification, we mean that a thermal source, due to spontaneous emission of waves, can grow for only a limited period

of time and stops growing once a certain maximum level is reached. This level of growth eventually reached is limited principally by the duration of interaction, which in turn is determined by the wave propagation out of the interaction region where the phase matching conditions are approximately satisfied. Quite generally, since K increases with x , the possible asymptotic behavior at infinity is $a_I = \exp -[(p + \nu_1)/V_1] x$ and $a_{II} = \exp [i \int_0^x K dx + (p + \nu_2)x V_2^{-1}]$. For $V_1 V_2 < 0$ both solutions are badly behaved at either plus or minus infinity for $\text{Re } p > 0$ and no temporally growing modes are possible.

With $V_1 V_2 > 0$ (decay waves propagating oppositely) normal modes with $\text{Re } p > 0$ are possible, provided the solutions well-behaved at $\pm \infty$ can be joined at $x = 0$. Let us now consider the following cases.

(a) $K = K' x$. Changing the variable to $x' = \frac{x}{\sqrt{K'}} + \frac{i}{\sqrt{K'}} \left(\frac{p + \nu_1}{V_1} + \frac{p + \nu_2}{V_2} \right)$,

Eq. (17) reduces to the standard parabolic cylinder equation:

$$\frac{\partial^2 a}{\partial x'^2} + \frac{x'^2}{4} a + \left(\lambda + \frac{i}{2} \right) a = \frac{\lambda \bar{a}}{\gamma_0} \delta \left[x' + \frac{i}{\sqrt{K'}} \left(\frac{p + \nu_1}{V_1} + \frac{p + \nu_2}{V_2} \right) \right] \quad (18)$$

where $\lambda = \gamma_0^2 / K' V_1 V_2$ and we have chosen the normalization factor for the source $b = -\bar{a} / \sqrt{K'}$, $\bar{a}$ representing the thermal level wave amplitude. The homogeneous equation ($\bar{a} = 0$) has been shown to possess no well-behaved solution for $\text{Re } p > 0$.⁶ Thus only spatial amplification is possible. The spatial and temporal evolution of an initial fluctuation has been reported by us elsewhere.⁷ A pulse develops which grows to a maximum determined by convection and in the presence of wave damping, eventually damps away. Here we will confine ourselves to an exact calculation of the maximum

amplitude at $x = 0$ obtained (over all t) including damping. To this purpose we solve Eq. (16) with $\frac{\partial}{\partial t} = 0$, or equivalently Eq. (18) with $p = 0$. Assuming $\lambda \gg 1$ and using the WKB approximation, we find that the turning points of the resulting equation are on the imaginary axis for $V_1 V_2 > 0$:

$$x' = \pm i x_T = \pm i \frac{2\gamma_0}{K' (V_1 V_2)^{\frac{1}{2}}}$$

The real x axis is displaced above the real x' axis by an amount $d = (\nu_1/V_1 + \nu_2/V_2)/K'$ intersecting two anti-Stokes lines at some points Q at large $|x|$. See Fig. 1. Consider solutions to the homogeneous equation which are well behaved at $x = \pm \infty$, parabolic cylinder functions (ψ_I, ψ_{II} respectively), and normalize these solutions to $x^{-\frac{1}{2}} \exp(\pm \frac{K' x^2}{4})$ at $x = \pm \infty$. The solution to the inhomogeneous equation can then be written

$$\psi(x) = [\theta(x) \psi_I(x)/\psi_I(0) + \theta(-x) \psi_{II}(x)/\psi_{II}(0)] A$$

with θ the Heaviside step function and $A = \psi_I(0) \psi_{II}(0) / [\psi_I'(0) \psi_{II}(0) - \psi_{II}'(0) \psi_I(0)]$. Then using the fact that the amplitude of ψ is constant along an anti-Stokes line and thus $\psi(Q) = \psi(ix_T)$ where Q is the intersection of the anti-Stokes line with the real axis, we find, using the WKB solutions, that (aside from the slow $x^{-\frac{1}{2}}$ dependence)

$$\begin{aligned} \frac{\psi(id)}{\psi(Q)} &= \int_d^{x_T} dy \left[\frac{\gamma_0^2}{V_1 V_2} - \frac{1}{4} K' y^2 \right]^{\frac{1}{2}} \\ &= \exp \frac{2\gamma_0^2}{K' V_1 V_2} \int_{\phi}^{\pi/2} \cos^2 \theta d\theta, \text{ where } \sin \phi = \left(\frac{\nu_1}{V_1} + \frac{\nu_2}{V_2} \right) \frac{\sqrt{V_1 V_2}}{2\gamma_0} \end{aligned} \quad (19)$$

Thus the amplitude at $x = 0$ ($x' = id$) reaches a maximum determined by the pump through Eq. (19). This is the local parametric amplification of the normal background source level. The amplification following the convecting pulse can be larger or smaller than this value, depending on the wave damping and on the wave group velocities. For details see reference 7. In the limit of undamped waves, the peak amplification is (not only for $x = 0$ but for all x)

$$I = I_0 \exp (2\pi\gamma_0^2/V_1 V_2 K') . \quad (20)$$

Thus substantial amplification is obtained provided

$$\lambda \equiv \gamma_0^2/V_1 V_2 K' > 1 \quad (21)$$

and

$$2\gamma_0 > \left(\frac{\nu_1}{V_1} + \frac{\nu_2}{V_2} \right) \sqrt{V_1 V_2} . \quad (22)$$

These two conditions are that the pump be strong enough to overcome convective losses, and damping, respectively. The latter condition is also that for the existence of absolute instability in a homogeneous plasma.⁸

(b) $K = K''(0) x^2/2$. In many cases of interest $K(x)$ possesses a second order zero at particular points in the plasma and or for particular scattering angles. Temporally growing normal modes may then exist for $V_1 V_2 > 0$. For large γ_0 , so the term dK/dx in Eq. (17) can be neglected, we have

$$\frac{d^2 a}{dx^2} + \left\{ \frac{1}{4} \left[\frac{1}{2} K'' x^2 i \left(\frac{p+\nu_1}{V_1} + \frac{p+\nu_2}{V_2} \right) \right]^2 + \gamma_0^2/V_1 V_2 \right\} a = 0 . \quad (23)$$

There are four turning points at

$$x = \pm \left\{ \frac{2}{K''} \left[i \left(\frac{p+\nu_1}{V_1} + \frac{p+\nu_2}{V_2} \right) \pm 2i\gamma_0 \frac{1}{V_1 V_2} \right] \right\}^{\frac{1}{2}} \quad (24)$$

We look for normal modes localized near $x = 0$ i. e. a solution with eigen value p such that two of the turning points lie close $x = 0$:

$$p = 2\gamma_0 \left(\sqrt{\frac{V_1 V_2}{V_1 + V_2}} \right) (1 - \Delta) - \left(\frac{\nu_1 V_2 + \nu_2 V_1}{V_1 + V_2} \right) \quad (25)$$

where $\Delta \ll 1$ is assumed. Eq. (24) is thus reduced to

$$\frac{d^2 a}{dx^2} + \frac{1}{4} \left[\left(\frac{1}{2} K'' x^2 + \frac{4i\gamma_0}{V_1 V_2} \right) \left(\frac{1}{2} K'' x^2 - \frac{2i\gamma_0 \Delta}{\sqrt{V_1 V_2}} \right) \right] a = 0 \quad (26)$$

Near the inner turning points, this is just the harmonic oscillator equation whose solution falls off away from the turning points like $\exp(-\delta x^2/2)$ with $\delta = (-i\pi/4)(\gamma_0 K'')^{1/2} (V_1 V_2)^{-1/4}$ and

$$\Delta = \exp \left(-\frac{i\pi}{4} \right) 2^{-3/2} (K'')^{1/2} (\gamma_0^2 / V_1 V_2)^{-3/4} \ll 1 \quad (27)$$

Thus, a temporally growing mode exists for $\Delta \ll 1$ if the absolute threshold (Eq. (22)) is exceeded, and the growth rate is given by Eq. (25). For Brillouin scattering, which we treat in detail in Section IV, $V_1 \approx c \gg V_2 = c_s (T_e/M)^{\frac{1}{2}}$, and $\nu_2 \gg \nu_1$ for T_e not too much larger than T_i . Eq. (22) then provides a severe threshold.

IV. RAMAN AND BRILLOUIN INSTABILITIES IN INHOMOGENEOUS PLASMAS.

We now apply the analysis of Section III to Raman and Brillouin scattering. Suppose the density of the plasma slab to be given by $n(x) = n_0(1 + x/L_n)$ and let θ denote the scattering angle of the electromagnetic wave for normal incidence of the pump wave.

(a) Raman Scattering. From the dispersion relations for the three waves involved, $\omega_0^2 = \omega_p^2 + k_0^2 c^2$, $\omega_1^2 = \omega_p^2 + k_1^2 c^2$, $\omega_2^2 = \omega_p^2 + 3k_2^2 v_e^2$, we find for $K' =$

$$K' = \frac{d}{dx} (k_{0x} - k_{1x} - k_{2x}) = \frac{\omega_p^2}{2L_n} \left[-\frac{1}{k_0^2 c^2} + \frac{1}{k_1^2 c^2} + \frac{1}{3k_2^2 v_e^2} \right] \quad (28)$$

Using $\vec{k}_0 = \vec{k}_1 + \vec{k}_2$, $\omega_1 = \omega_2 - \omega_0$, and the dispersion relations we find that the exact phase matching requires

$$\left[\omega_p - \omega_0 + \frac{3}{2} \frac{v_e^2}{\omega_p} (\vec{k}_0 - \vec{k}_1)^2 \right]^2 = \omega_p^2 + k_1^2 c^2 \quad (29)$$

Solving for k_1 neglecting terms of order $(v_e/c)^4$, we find

$$k_1^2 c^2 = \omega_0^2 - 2\omega_0 \omega_p - 3 \frac{v_e^2}{c^2} \left(\frac{\omega_0 - \omega_p}{\omega_p} \right) [k_0^2 c^2 + \omega_0^2 - 2\omega_0 \omega_p - 2k_0 c (\omega_0^2 - 2\omega_0 \omega_p)^{\frac{1}{2}} \cos \theta] \quad (30)$$

Note that $k_1 = 0$ for $\omega_0 = 2\omega_p (1 + \frac{3}{4} k_0^2 \lambda_D^2)$. Substituting this into Eq. (25)

and using $k_{1x} = k_1 \cos \theta$, $k_{2x} = k_0 - k_1 \cos \theta$ gives K' as a function of plasma density through (ω_p/ω_0) and θ . In particular, at any point in the plasma there exists a value of θ for which K' vanishes, and a temporally growing mode can be expected from the analysis of Section III. To see this we find

upon setting $K' = 0$ from Eq. (28) $\frac{1}{k_0^2 c^2} = \frac{1}{k_{1x}^2 c^2} + \frac{1}{3k_{2x}^2 e}$

Substituting $k_{2x} = k_0 - k_{1x}$ and solving for k_{1x} we find to order $\frac{e}{2}$ that $k_{1x} = \frac{e}{c^2} k_0$

$$\cos \theta = - \frac{3v^2 e}{c^2} \quad (31)$$

and thus $K' = 0$ only for $\theta \approx \pi/2$, in the underdense region except for the point $\omega_p \approx \omega_0/2$. From the dispersion relation of the light wave $\omega^2 = \omega_p^2(x) + [k_{1x}^2(x) + k_1^2] c^2$, we obtain $\frac{dk_{1x}}{dx} = \frac{\omega_p^2}{2k_{1x} c^2 L_n}$

The validity of the WKB approximation requires that

$dk_{1x}/dx \ll 2k_{1x}^2$, or

$$\cos \theta \equiv \frac{k_{1x}}{k_1} > (\omega_p/2\omega_0)^{2/3} (k_0 L)^{-1/3} \quad (32)$$

Thus K' vanishes for values of θ , k_1 which invalidate the use of the WKB analysis for the scattered wave. In Section V the growth rate and threshold for this mode will be calculated without resorting to the WKB approximation. In the underdense region $\omega_p < \omega_0/2$ we have $k_2 \approx 2k_0 \sin \frac{\theta}{2}$, $k_{2x} \approx 2k_0 \sin^2 \frac{\theta}{2}$, $k_1 \approx k_0$, giving

$$K' = \frac{\omega_p^2}{6k_0 L_n v_e^2} \left[\frac{1}{2 \sin^2 \frac{\theta}{2}} + \frac{3v_e^2}{c^2} \left(\frac{1}{\cos \theta} - 1 \right) \right] \quad (33)$$

Using Eq. (13) for γ_0 and Eq. (33) we find for the exponential amplification factor λ

$$\lambda_R = (\gamma_R^0)^2 / V_1 V_2 K' = \frac{4 \left(\frac{v_0}{c}\right)^2 k_0 L_n \sin^2 \frac{\theta}{2}}{\left| \cos \theta \left[1 + 6 \left(\frac{v_e}{c}\right)^2 \left(1 - \frac{1}{\cos \theta}\right) \sin^2 \frac{\theta}{2} \right] \right|} \quad (34)$$

Here $\theta \approx \pi/2$ is not allowed because of the breakdown of the WKB approximation. The threshold power for Raman scattering in an inhomogeneous plasma is $\lambda_R > 1$. The backscattering threshold is $(v_0/c)^2 k_0 L > 1$. This threshold, previously obtained in a one-dimensional calculation,⁹ has been verified in computer simulations.¹⁰ For side scatter ($\theta \approx \pi/2$) the threshold is lower as discussed in Section V.

(b) Brillouin scattering in expanding plasmas:

In the case of scattering by an ion wave, the phase shift is primarily due to the nonuniform expansion velocity, $u(x)$.¹¹ We assume that the density profile is essentially stationary in the laboratory frame. The temporal variation of the local density constitutes an additional stabilizing effect, which however, is small compared with the effects of spatial gradient for Brillouin and Raman scattering. In the laboratory frame, the ion wave frequency is thus Doppler-shifted $\omega = \vec{k} \cdot \vec{u} - kc_s / (1 + k^2 \lambda_D^2)^{\frac{1}{2}}$. Then, the variation of the ion wave k is

$$\frac{dk_x}{dx} = \frac{k_x u'}{u - c_s \sin \left(\frac{\theta + \theta_0}{2} \right)} + \frac{k^2}{2k_x} \left(\frac{k^2 \lambda_D^2}{L_n} + \frac{1}{L_T} \right) \quad (35)$$

where $L_T = (d \ln T_e / dx)^{-1}$ is the scale length of the electron temperature gradient. The phase shift rate for the Brillouin scattering is therefore

$$K' = - \frac{2k_0}{L_u} \left\{ \frac{\sin^2 \frac{\theta}{2}}{1 - \mathcal{M}^{-1} \sin \frac{\theta}{2}} + \frac{L_u}{L_T} + \frac{\omega_p^2}{\omega_0^2} \frac{L_u}{L_n} \left[4 \left(\frac{\omega_0}{\omega_p} \right)^4 \frac{v_e^2}{c^2} \sin^2 \frac{\theta}{2} - \frac{1}{\cos \theta} + 1 \right] \right\} \quad (36)$$

whose $\mathcal{M} = u(0)/c_s$ is the Mach number. The exponential amplification factor, for $\mathcal{M} \gg 1$ then given by

$$\lambda_B = \frac{\gamma_B^2}{v_1 v_2 K'} = \frac{\frac{1}{8} \left(\frac{v_0}{v_e} \right)^2 \left(\frac{\omega_p}{\omega_0} \right)^2 k_0 L_u \cos^2 \phi}{\sin^2 \frac{\theta}{2} \left\{ \cos \theta + \frac{\omega_p^2}{2\omega_0^2} \frac{L_u}{L_n} \left[1 + 4 \cos \theta \left(\frac{\omega_0}{\omega_p} \right)^4 \frac{v_e^2}{c^2} \right] \right\} + \frac{L_u}{2L_T} \cos \theta} \quad (37)$$

provided $K' \neq 0$. Note that λ_B tends to peak for scattering at an angle θ for which $\cos \theta \approx -\omega_p^2 L_u / 2\omega_0^2 L_n$ and for forward scattering ($\theta \approx 0$). The amplification factor for side scattering with $\cos \theta < (L_u / 2L_n) (\omega_p / \omega_0)^2$ is

$$\lambda_B(\theta \approx \frac{\pi}{2}) = \frac{1}{4} \left(\frac{v_0}{v_e} \right)^2 \left(\frac{\omega_p}{\omega_0} \right)^2 k_0 L_n \cos^2 \phi \quad (38)$$

comparable to that for Brillouin scattering without blowoff. For forward scattering $[\sin \theta < (L_u / 2L_T)^{\frac{1}{2}}]$,

$$\lambda_B(\theta \approx 0) = \frac{1}{4} \left(\frac{v_0}{v_e} \right)^2 \left(\frac{\omega_p}{\omega_0} \right)^2 k_0 L_T \cos^2 \phi. \quad (39)$$

For backscattering ($\theta \approx \pi$)

$$\lambda_B(\theta \approx \pi) = \frac{1}{8} \left(\frac{v_0}{v_e} \right)^2 \left(\frac{\omega_p}{\omega_0} \right)^2 k_0 L_u \cos^2 \phi / \left| (-1 + \frac{\omega_p^2}{2\omega_0^2} \frac{L_u}{L_n}) \right| \quad (40)$$

(c) Temporally growing modes for Brillouin scattering: At a given point in space, K' as given by Eq. (36) in fact vanishes for certain angles. Setting $L_T \rightarrow \infty$ and $\mathcal{M} \gg 1$, we find that, for $(k_0 \lambda_D)^2 \ll 1$, $K' \approx 0$ at angles for which

$$\cos \theta_1 = - \frac{L_u}{L_n} \left(\frac{\omega_p}{\omega_0} \right)^2 \quad (41)$$

and the temporally growing modes exist if Eqs. (22) and (27) are satisfied, which for Brillouin scattering with the Landau damping rate of ion wave given by $\nu_B = \frac{kc}{2} \sqrt{\pi} (m/M)^{\frac{1}{2}}$ for $T_e \gg T_i$ and $K'' \approx -2k/L_u^2$, become

$$(v_0/v_e)^2 > (\omega_p/\omega_0)^4 (m/M)(L_u/L_n) \quad (42)$$

$$(v_0/v_e)^2 > 4(k_0 L_u)^{-4/3} (L_u/L_n) \quad (43)$$

respectively. The growth rate is

$$\gamma = \omega_{pi} \left(\frac{v_0}{c} \right) |\cos \theta_1|^{-\frac{1}{2}} = \omega_{pi} \left(\frac{v_0}{c} \right) \left(\frac{\omega_0}{\omega_p} \right) (L_n/L_u)^{\frac{1}{2}} \quad (44)$$

The region of the localization of the instability around the phase matching point has a width

$$l \approx (c c_s)^{1/4} L_u^{1/2} / (\omega_{pi} k_0 v_0)^{1/4} \quad (45)$$

In addition, if L_u, L_T both tend to infinity, then K' also vanishes in the low density region for

$$\cos \theta_2 = - \frac{1}{4} \left(\frac{c}{v_e} \right)^2 \left(\frac{\omega_p}{\omega_0} \right)^4 \quad (46)$$

The temporally growing mode again exists with growth rate

$$\gamma \approx 2\gamma_B^{(0)} (c_s/c)^{\frac{1}{2}} \quad \text{and threshold power}$$

$$P \equiv \frac{1}{2\pi} \left(\frac{m}{e}\right)^2 \left(\frac{v_0}{c}\right)^2 \omega_0^2 c^3 > \frac{m^2 \omega_0^2 c^4}{4\pi e^2 L} \left(\frac{v_e}{c}\right)^{5/3} (k_0 L)^{-1/3} \quad (47)$$

The localization width is $(c v_e / v_0^2)^{1/8} (c L / \omega_p)^{1/2}$. The short duration τ of the laser pellet irradiation experiments further means that the condition $\gamma\tau \gg 1$, which is necessary for effective growth during the experiment, may impose a "threshold" condition on these weakly growing modes. These modes are presumably saturated at a low level where the turbulent damping rate is comparable to their growth rate.

V. TEMPORALLY GROWING/RAMAN AND BRILLOUIN SIDE SCATTERING INSTABILITIES IN REGIONS WHERE THE WKB ANALYSIS FAILS.

From Section IV, we see that the threshold for Raman scattering in an inhomogeneous plasma is a minimum and K' may vanish for scattering at nearly 90° . The scattered electromagnetic wave is near its turning point and the WKB approximation breaks down. We therefore first develop the appropriate equations for these cases. Because the electron density fluctuation n_e coupled with $\vec{v}_0$ in the pump field serves as a nonlinear current source in the

Maxwell equation, it is convenient to use the equation for the electron density fluctuation instead of Eq. (6):

$$\frac{\partial^4 n_e}{\partial t^4} + \left[\omega_p^2 - v_e^2 \nabla^2 \right] \frac{\partial^2 n_e}{\partial t^2} - v_e^2 \omega_{pi}^2 \nabla^2 n_e = \left(\frac{\partial^2}{\partial t^2} + \omega_{pi}^2 \right) \nabla^2 (\vec{v}_0 \cdot \vec{v}_1). \quad (48)$$

For the Raman scattering, $\omega \approx \omega_p$, this becomes

$$\frac{\partial^2 n_e}{\partial t^2} + \left[\omega_{p(x)}^2 - v_e^2 \nabla^2 \right] n_e = \nabla^2 (\vec{v}_0 \cdot \vec{v}_1) \quad (49)$$

For Brillouin scattering, $\omega \ll \omega_p$, Eq. (48) becomes

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right)^2 n_e - c_s^2 \nabla^2 n_e = \frac{1}{\omega_p^2} \left(\frac{\partial^2}{\partial t^2} + \omega_{pi}^2 \right) \nabla^2 (\vec{v}_0 \cdot \vec{v}_1) \quad (50)$$

where we have included the flow velocity u of the plasma. The Maxwell equation for the scattered wave is

$$\left[c_s^2 \nabla^2 - \frac{\partial^2}{\partial t^2} - \omega_p^2 \left(1 + \frac{x}{L} \right) \right] v_1 = v_0 \omega_p^2 \frac{n_e}{n_0} (\vec{e}_1 \cdot \vec{e}_0) \quad (51)$$

These equations are used in the following to study the sidescattering.

- (a) Existence of a temporally growing Raman Mode. Neglecting the v_e^2 term in Eq. (49), and substituting in Eq. (51), we obtain the following eigenvalue equation (eigenvalue ω , complex) for the vector potential

$$A_- = A(\omega - \omega_0, \vec{k} - \vec{k}_0 = k_y \hat{y})$$

$$\left[\frac{d^2}{dx^2} - k_y^2 + \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \left(1 + \frac{x}{L} \right) \right] A_- = \frac{\Lambda}{\omega^2} A_- \quad (52)$$

$$1 - \frac{\omega_p^2}{\omega^2} \left(1 + \frac{x}{L} \right)$$

where $\omega_-^2 = (\omega - \omega_0)^2$, $\Lambda = \left(\frac{eE_0}{m\omega_0 c} \right)^2 \frac{\omega_p^2 k^2}{\omega^2}$ and k is the plasmon wave number.

Defining $\xi = x \left(\frac{\omega^2}{c^2 L} \right)^{1/3}$ we find

$$\left(\frac{\partial^2}{\partial \xi^2} - \xi + D - \frac{\lambda}{B - \xi} \right) A_- = 0 \quad (53)$$

where $D = \left(\frac{\omega^2}{c^2 L} \right)^{-2/3} \left[-k_y^2 + \frac{\omega^2}{c^2} - \frac{\omega^2}{c^2} \right]$, $B = \frac{\omega^2 - \omega_p^2}{\omega_p^2} \left(\frac{\omega^2 L^2}{c^2} \right)^{1/3}$,

and $\lambda = \left(\frac{v_0}{c} \right)^2 \left(\frac{kc}{\omega} \right)^{2/3} (kL)^{4/3}$, $\frac{v_0}{c} = \frac{eE_0}{m\omega_0 c}$. Letting $\xi' = \xi - \frac{(D+B)}{2}$

and simplifying, we find

$$\left[\frac{d^2}{d\xi'^2} - \frac{\xi'^2 - \alpha^2 - \lambda}{\alpha + \xi'} \right] A_- = 0 \quad (54)$$

where $\alpha = \frac{D-B}{2}$. This equation has two turning points at $\xi' = \pm \xi_T$,

$\xi_T^2 = \alpha^2 + \lambda$. Assuming that $|\xi_T| \ll \alpha$, which we must verify a

posteriori, the ξ' in the denominator can be ignored and Eq. (54)

is solvable in terms of a parabolic cylinder function. The

eigenvalue equation is $(\alpha^2 + \lambda)/2\sqrt{\alpha} = n + 1/2$, $n = 0, 1, 2, \dots$

Solving this equation for α iteratively, with $\lambda \gg 1$ we find

$$\alpha = i\sqrt{\lambda} \left(1 + (n + 1/2)(-\lambda)^{-3/4} \right) \quad (55)$$

Thus the condition $|\xi_T| \ll \alpha$ reduces to $\lambda^{1/4} \gg 2n + 1$, or

$$\frac{v_0}{c} \gg \frac{(2n+1)^2}{(kL)^{2/3}} \left(\frac{\omega}{kc} \right)^{1/3}$$

readily satisfied by the growing modes with growth rate given by

Eq. (57). Since we are examining the region where

the WKB analysis of Section IV is invalid the scattering is approxi-

mately at 90° . Using the phase matching condition, and the dispersion

relations for the waves and $\omega \approx \omega_p - \omega_0$, we find

$$c^2 k^2 = 2\omega_0^2 - 2\omega_0 \omega_p - \omega_p^2 \quad (56)$$

Substituting the expressions for λ and α into Eq. (55) and simplifying we find

$$\omega = \omega_p - i \left(\frac{v_0}{c} \right) \frac{k c \omega_p}{\omega_0} \left[1 - \frac{(n + \frac{1}{2}) i^{\frac{1}{2}} \left(\frac{\omega_p}{k c} \right)^{\frac{1}{2}}}{\frac{3/2}{\left(\frac{v_0}{c} \right) (k L)}} \right]$$

which gives for the growth rate $\gamma = -\text{Im } \omega$

$$\gamma = \left(\frac{v_0}{c} \right) \left(\frac{k c \omega_p}{\omega_0} \right) \left[1 - \frac{(n + \frac{1}{2}) \left(\frac{\omega_p}{k c} \right)^{\frac{1}{2}}}{\frac{3/2}{\sqrt{2} \left(\frac{v_0}{c} \right) k L}} \right] \quad (57)$$

where $k c = (2\omega_0^2 - 2\omega_0 \omega_p - \omega_p^2)^{\frac{1}{2}}$ for exact frequency matching as given by Eq. (56). As a special case, at the quarter-critical density point $k_y = 0$, $k = k_0$, $\omega_0 = 2\omega_p$, giving¹²

$$\frac{\omega}{\omega_0} = \frac{1}{2} - i \left(\frac{v_0}{c} \right) \frac{\sqrt{3}}{4} \left[1 - \frac{(n + \frac{1}{2}) \sqrt{i}}{3^{\frac{1}{4}} \left(\frac{v_0}{c} \right)^{\frac{3}{2}} k_0 L} \right] \quad (58)$$

From the absolute value of the turning point we find that the width of the large amplitude region is $\frac{\Delta x}{L} \approx \left(\frac{v_0}{c} \right)^{\frac{1}{4}} (k_0 L)^{-1/2}$. Although these modes have been shown to be absolutely growing in x , i. e. localized eigenmodes, it remains to investigate their behavior in y . In the following it will be shown that the except at the quarter-critical density point, where $k_y = 0$, the modes are in fact convective in y . These modes exist throughout the underdense

region from the quarter-critical density point to where $k\lambda_D \sim 0(1)$ and Landau damping suppresses the Raman mode.

(b) Threshold. Eq. (57) cannot be used to predict threshold values since if ω is real α^2 is real and positive and therefore the singularity in Eq. (54) at $\xi' = -\alpha$ is located within the turning points of that equation, in violation of our assumption. In fact if this singularity is located well within the turning points we can write Eq. (53) as, for $\omega = \omega_p$, $\gamma = 0$,

$$\left(\frac{d^2}{d\xi^2} - \xi + D + \frac{\lambda}{\xi} \right) A_- = 0 \quad (59)$$

where λ is to be regarded as the eigenvalue chosen such that with D real A_- vanishes at $+\infty$ and is outgoing, i.e. proportional to $\exp(-i \frac{2}{3} (-\xi)^{3/2})$ at $-\infty$. We solve Eq. (59) analytically by noting that $1/\xi = i\pi \delta(\xi) + P \frac{1}{\xi}$ and treating the principal-value part as a perturbation. The threshold condition thus obtained and checked numerically is

$$\lambda_{\text{crit}} \equiv \left(\frac{v_0}{c} \right)^2 \left(\frac{c}{\omega_0} \frac{d \ln n}{dx} \right)^{-4/3} \left(\frac{\omega_0}{\omega_p} \right)^{2/3} \left(1 - \frac{\omega_p^2}{\omega_0^2} \right)^{-1/2} \left[2 - 2 \frac{\omega_p}{\omega_0} - \frac{\omega_p^2}{\omega_0^2} \right] \\ = 0.24 \quad (60)$$

where we used the definition of λ and substituted $d \ln n / dx$ for L^{-1} to allow for various profiles, and the factor $(1 - \omega_p^2 / \omega_0^2)^{-1/2}$ is to account for the wave swelling approaching critical density, $\frac{v_0}{c}$ now refers to the incident field. Eq. (60) thus gives the critical value for (v_0/c) for side scattering at various points in the

density profile below the quarter critical point as well as at the $\frac{1}{4}$ -critical-density point. For an exponential density profile it is the low density part of the profile which has the lowest threshold for instability. For $\frac{dn}{dx} = \text{const.}$, which is used in many simulations $(v_0/c)^2 \sim n^{-1}$ and scattering at the high-density point where $\omega_p = \omega_0/2$ has the lowest threshold. The experimental profiles are sometimes fit by $n \sim r^{-3}$ in which case $d \ln n / dx \approx n^{1/3}$ and the low density region is strongly favored for side scattering. Of course as we are considering Raman scattering, it should be borne in mind that at low densities and high temperatures Landau damping quickly becomes important and suppressing the Raman mode for $k_0 \lambda_D \sim 0(1)$.

(c) The convective and absolute Raman Instability. The instability near $\omega_0 \simeq 2\omega_p$ is absolute as the scattered wave is trapped in the x-direction and non-convective in the y-direction because $k_y = 0$. However at the quarter critical point there is also present a competing instability, the decay into two plasma waves. The relative importance of the two instabilities in various parameter regions remains to be evaluated. When the Raman absolute mode dominates, the radiation trapped between the turning points can be so intense that its pondermotive force can expel the plasma below the quarter critical point, as evidenced in some numerical simulations. In the low density region, the

scattered wave has finite k_y and propagates in the y -direction, though the region of large amplitude is localized between the turning points in the x -direction. The instability is in fact convective in the y -direction, as shown below.

Suppose a pulse is initiated at $x = y = 0$, and its space-time evolution is to be followed by the proper superposition of the eigenmodes of Eq. (52). As an initial value problem, the initial-value term from the Laplace transform, representing the pulse $\delta(x) \delta(y)$ at the initial moment, is now introduced into Eq. (52) as a source term on the right hand side of Eq. (52):

$$\left[\frac{d^2}{dx^2} - k_y^2 + \frac{\omega^2}{c^2} (1 + 2i\gamma/\omega) - \frac{\omega^2}{c^2} (1 + \frac{x}{L}) - \frac{\omega^2 \Lambda}{1 - \frac{p}{2} (1 + \frac{x}{L})} \right] \psi_{k_y}(x) = a \delta(x) \quad (61)$$

where a is constant because of the Fourier transform of $\delta(y)$. First we examine the asymptotic behavior of the solution of the homogeneous equation, Eq. (52):

$$A_t(x) \rightarrow \exp + i \left[\int^x K_x dx' \right] \quad (62)$$

where $K_x = + \left[-\frac{p}{2} \frac{x'}{L} + \frac{2i\gamma\omega}{c^2} \right]^{\frac{1}{2}}$ with the positive sign in front of the square root so chosen as to ensure the proper asymptotic behaviors: outgoing and well-behaved for the growing mode ($\gamma > 0$) at large negative x ; evanescent at large positive x . To find the solutions $A_{k_y}(x)$ for different k_y , we note that, for $k_y + \delta k$ and $\omega + \delta\omega$, Eq. (52) can be rendered invariant by transforming

to a new variable $x' = x - \delta x$ such that

$$-2k_y \delta k + \frac{2\omega - \delta\omega}{c^2} - \frac{\omega_p^2}{c^2} \frac{\delta}{L} = 0$$

$$2 \frac{\delta\omega}{\omega} - \frac{\delta x}{L} = 0$$

From these we obtain

$$\frac{\delta\omega}{\delta k} = \frac{-k_y c^2}{\omega_0}, \quad \frac{\delta x}{\delta k} = -\frac{2k_y c^2}{\omega_0 \omega_p} \quad (63)$$

for $\omega = \omega_p$. Therefore, if $\exp(ik_y y - i\omega_- t + \gamma t) A(x)$ is the eigenfunction for k_y with growth rate (eigenvalue) γ , then $\exp[i(k_y + \delta k)y - i(\omega_- + \delta\omega)t + \gamma t] A(x - \delta x)$ is the eigenfunction for $k_y + \delta k$ with essentially the same growth rate γ , aside from the dependence of A on k . With the source at $\delta(x) = \delta(x' + \delta x)$, the solution to the inhomogeneous Eq. (61) for $k_y + \delta k$ is

$$\psi_{k_y + \delta k}(x') = \frac{A_+(x')A_-(-\delta x) + A_-(x')A_+(-\delta x)}{W} \quad (64)$$

Constructing the wave packet consisting of eigenmodes with mean wave number k_y and growth rate γ , which is the least dispersive packet consisting of maximally growing modes, we have, at $x = 0$

$$(x' = -\delta x = -\frac{\partial x}{\partial k_y} \delta k)$$

$$\begin{aligned} \Psi(x=0, y, t) &= \exp(ik_y y - i\omega_- t) \int d\delta k \exp(i\delta k y - i\delta\omega t + \gamma t) \psi_{k_y + \delta k}(-\delta x) \\ &= \frac{\exp(ik_y y - i\omega_- t)}{W} \int d\delta k \exp \left[(i\delta k y - i\delta\omega t + \gamma t) - 2iK_x \frac{\delta x}{\delta k} \delta k \right] \end{aligned} \quad (65)$$

where we have used Eq. (62). The integral in Eq. (65) can be

evaluated by the method of Steepest descent with the saddle point at

$$y - \frac{\delta\omega}{\delta k} - 2 \frac{\delta x}{\delta k} \left(-\frac{p}{Lc} \frac{\omega^2 x'}{2} + \frac{2i\omega \gamma}{c^2} \right)^{\frac{1}{2}} = 0 \quad (66)$$

where $x' = -(\partial x / \partial k) \delta k$, and $\delta\omega / \delta k$, $\delta x / \delta k$ are given by Eq. (63). Because $\delta x / \delta k$, $\delta\omega / \delta k$ are both negative, saddle point exists only for

$$y < \frac{\delta\omega}{\delta k} = -\frac{ck^2}{\omega_0} \quad (67)$$

Within the range of validity of Eq. (68), we may solve Eq. (67) to find the saddle point location.

$$\delta k = -\frac{2i\omega \gamma L}{\omega_p^2 (\delta x / \delta k)} + \text{real part} \quad (68)$$

Substituting Eq. (68) into Eq. (65) and neglecting the rapidly varying phases, we find the amplitude at $x = 0$ as a function of y and t as

$$\psi(x=0, y, t) = \exp \gamma \left[t \left(2 - \frac{\omega_0}{\omega_p} \right) + \frac{\omega_0 (\omega_0 - \omega_p)}{\omega_p k c^2} y \right] \quad (69)$$

Thus the growth rate is maximum and equal to γ in the convecting frame with $y = \frac{\delta\omega}{\delta k} t$ is bounded below by

$$y > -\frac{k c^2}{(\omega_0 - \omega_p)}$$

which is essentially the causality condition, in addition to the upper bound given by Eq. (67). Thus the Raman side scattering instability

in the low-density region (below the quarter critical point) is convective in y in the sense that it grows only in the narrow region moving with the convection velocity. The absolute nature of the instability in the x -direction (the existence of an eigenmode) still manifests itself in the fact that the growth following the convecting pulse in y is not limited by convection, as is the growth of an instability convective in x . The instability will saturate when the wave packet propagates out of the plasma. For a plasma with finite extent in the y -direction, L_y , such a convective loss presumably gives rise to an additional threshold $\gamma L_y (\partial\omega/\partial k_y)^{-1} \approx (v_0/c) [\omega_p/\omega_0 - \omega_p] (k_0 L_y) > 1$ where γ is given by Eq. (58). This additional threshold, however, is not very stringent once the threshold given by Eq. (60), which can be approximately expressed as

$$(v_0/c) (k_0 L)^{2/3} > 1 \quad (70)$$

is satisfied, provided the plasma is not too underdense.

(d) Brillouin sidescattering. In Section IV(b), it is shown that temporally growing modes exist for Brillouin scattering at angles where K' vanishes. Here we wish to show that the temporally growing instability in fact exists for the Brillouin sidescattering. Letting $n = \tilde{n}(x, t) \exp(-i\vec{k} \cdot \vec{r} - i\omega t)$ with $\omega = kc_s - \vec{k} \cdot \vec{u}$ in Eq. (50) we obtain

$$\frac{\partial \tilde{n}}{\partial t} + \frac{k_x}{k} (u + c_s) \frac{\partial \tilde{n}}{\partial x} - \frac{iku}{L_u} (1 + 2 \frac{\omega}{\omega_0} \frac{L_u}{L_n}) x \tilde{n} = i \frac{m}{M} \frac{k}{2c_s} (\vec{v}_0 \cdot \vec{v}_1) \quad (71)$$

Neglecting convective (second) term in Eq. (70) and substituting into eq. (51) we find

$$\left[\frac{\partial^2}{\partial \xi^2} - \xi + 2i\beta\xi + \Delta \right] A_1 = \frac{i\lambda' A_1}{2\Gamma - i\xi\delta} \quad (72)$$

where $\lambda' = \lambda \left(\frac{m}{M}\right)^{\frac{1}{2}} (k\lambda_D)^{-1}$, $\delta = (ku/2\omega_p)(L_n/L_u)(1 + 2 \frac{\omega_p^2}{\omega_0^2} \frac{L_u}{L_n})$ and $\xi, \lambda, \beta, \Delta$ are the same as defined in Eq. (53). Eq. (71) is formally identical with Eq. (53). Temporally growing instability exists with the growth rate

$$\gamma = \sqrt{2} k_0 v_0 (m/M)^{\frac{1}{2}} (L_n/L_u)^{\frac{1}{2}} \left(1 + 2 \frac{\omega_p^2}{\omega_0^2} \frac{L_u}{L_n}\right)^{\frac{1}{2}} \quad (73)$$

which is the same as that given in Sec. IV for $\cos \theta_1 = \frac{-L_u}{L_n} (\omega_p/\omega_0)^2$. Thus it appears to be the same instability extended to more oblique angles. The threshold condition is

$$\left(\frac{v_0}{v_e}\right)^2 (k_0 L_n)^{4/3} \left(1 + 2 \frac{\omega_p^2}{\omega_0^2} \frac{L_u}{L_n}\right) \frac{L_n}{L_u} = 0.12 \quad (74a)$$

subject to the condition of validity $\gamma \gg c_s/\Delta x$ or

$$(v_0/v_e)^{5/4} (k_0 L_n)^{1/2} (\omega_p/\omega_0)^{5/6} \gg 1 \quad (74b)$$

where $\Delta x = (v_0/v_e)^{1/4} (\omega_p/\omega_0)^{1/6} (c^2 L/\omega_p^2)^{1/3} (k_0 L)^{1/6}$ is the width of the localized region around the point of phase match. Near the threshold where Eq.(74a) is just satisfied, the condition given by Eq.(74b) is very stringent as it requires $(\omega_0/\omega_p)^{4/3} (k_0 L_n)^{-1/3} (L_n/L_u) (u/c_s) > 1$.

VI. BRILLOUIN SCATTERING BY QUASI MODES

For a sufficiently strong pump: $v_B^0 > k c_s$ or $(v_0/c)^2 > (\omega_{pi}/\omega_0) (k \lambda_D)^3$, the frequency shift and the growth rate of the ion mode by the pump is greater than the acoustic frequency, as given by Eq. (15). For these quasi-modes, not the eigenmode of the system, there is no linear dispersion relation from which the spatial rate of change of k can be determined in an inhomogeneous medium. Of course, the WKB approximation in the frequency domain also breaks down for the ion wave s . We therefore use Eq. (47) for

the ion mode with Eq. (51) for the light wave. We make the following ansatz for the density fluctuation:

$$\tilde{n}(\mathbf{x}, t) = n(\mathbf{x}, t) \exp(i \int \vec{k} \cdot d\vec{r}) \quad (75)$$

where $\vec{k}(\mathbf{x}) = \vec{k}_0(\mathbf{x}) - \vec{k}_1(\mathbf{x})$, the difference of the wave vectors of the incident and scattered waves. Laplace transforming Eqs. (47) and (51) in time with transform variable p , and assuming $p \gg k c_s$, $c_s d \ln n / dx$ by the nature of the quasi mode, we obtain

$$(p + c \cos \theta \frac{\partial}{\partial x}) \tilde{A} = - \frac{i \gamma_B^2 k c_s}{p^2} \tilde{A} + \delta(x) \quad (76)$$

where γ_B is given by Eq. (14). For $\text{Re } p > 0$, the solution is

$$\begin{aligned} \tilde{A} &= C \exp \left[\left(p + \frac{i \gamma_B^2(0) k(0) c_s}{p^2} \right) \frac{x}{c \cos \theta} + \frac{i \gamma_B^2(0) k'(0) c_s x^2}{p^2 c \cos \theta} \right] \text{ for } x > 0 \\ &= 0 \quad \text{for } x < 0. \end{aligned} \quad (77)$$

Inverting the Laplace transform by the method of steepest descent

$$\begin{aligned} \tilde{A}(x, t) &= \int dp \exp \left[p \left(t - \frac{x}{c \cos \theta} \right) - \frac{i \gamma_B^2(0) k(0) c_s}{p^2 c \cos \theta} \left(x + \frac{k'(0)}{k(0)} x^2 \right) \right] \\ &= \begin{cases} C \exp \left\{ 2^{-5/2} (\sqrt{3} + i) [\gamma_B^2(0) k(0) c_s]^{1/3} \left(x + \frac{d \ln k}{dx} x \right)^{2/3} \frac{(ct \cos \theta - x)^{2/3}}{c \cos \theta} \right\} & \text{for } t c \cos \theta > x \\ 0 & \text{for } t c \cos \theta < x. \end{cases} \end{aligned} \quad (78)$$

Thus, the amplitude at a given x grows like $\exp(\gamma t)^{2/3}$ and the growth for the quasi-mode stops when the condition for its existence fails.

We want to evaluate the maximum of

$$\lambda \equiv \left[\gamma_B^2(0) k(0) \frac{c_s}{c} \right]^{1/3} \left(x + \frac{d \ln k}{dx} x^2 \right)^{1/3} t^{2/3} \cos^{2/3} \theta \quad (79)$$

subject to the conditions that the logarithmic rate of change of A or

$\lambda/t > k(0) \left(1 + \frac{d \ln k}{dx} x \right) c_s$ and $ct \cos \theta > x$ (causality). From

these we obtain the limit of growth when $\left[\gamma_B^2(0) k' \frac{c_s}{c} \right]^{1/3} x^{2/3} t^{2/3} >$

$K' c_s x t$ or the maximum amplification

$$\lambda_{\max} = \gamma_B^2(0) / k'(0) c_s c \cos \theta. \quad (80)$$

This is the same form of amplification as the normal modes in the

WKB approximation. The major difference is that the phase-shift

$k'(0) = k'_0 - k'_1$ is entirely due to that of the electromagnetic waves.

The quasi-mode does not contribute to the phase shift. The same

result as Eq. (79) is obtained for a blow-off plasma where the phase

shift of the ion acoustic wave usually dominates that of the normal

mode. Thus, we conclude that, in a medium with a large gradient

in temperature or flow velocity, the threshold for Brillouin scattering

is significantly reduced with the pump power exceeding that for the

existence of quasi-modes.

In conclusion, we have derived the threshold conditions for the

Raman and Brillouin instabilities in an inhomogeneous plasma with

possible nonuniform flow velocity (Table 1). Once the threshold

conditions for the parametric instabilities are satisfied, a significant

fraction of the incident energy can be scattered, the total amount

depending on the nature of the non-linear saturation mechanisms. Thus, there is evidence, from simulation experiments, that at very high laser power, a significant fraction of the incident energy is reflected.

ACKNOWLEDGEMENTS

We would like to thank S. Bodner, J. Dawson, D. Forslund, B. D. Fried, A. N. Kaufman, J. M. Kindel, W. Kruer, G. Laval, C. Max, R. Morse, K. Nishikawa, J. NucKolls, T. O'Neil, R. Pellat, F. W. Perkins, E. Teller, and R. Sagdeev for helpful conversations. Research sponsored by the Atomic Energy Commission, Contract No. AT (11-1)-3237. Part of the work was performed at Plasma Physics Laboratory, Princeton, New Jersey 08540.

TABLE I

<u>Instability</u>	<u>Plasma Density</u>	<u>Growth Rate</u>	<u>Threshold</u>	<u>Nature</u>
Raman Backscatter	$\omega_0 > 2\omega_p > \left(\frac{v_e}{c}\right)^2 \omega_0$	$\left(\frac{v_0}{c}\right) \sqrt{\omega_0 \omega_p}$	$(v_0/c)^2 k_0 L > 1$	Convective in x
Raman Backscatter	$\omega_0 \approx 2\omega_p$	$\left(\frac{v_0}{c}\right) \omega_0$	$(v_0/c)^2 (k_0 L)^{4/3} > 1$	absolute
Raman Side Scatter	$\omega_0 > 2\omega_p > \left(\frac{v_e}{c}\right)^2 \omega_0$	$\left(\frac{v_0}{c}\right) \omega_p$	$(v_0/c)^2 (k_0 L)^{4/3} > 1$ and $(v_0/c)(\omega_p/\omega_0)k_0 L_y > 1$	Temporally growing but convective in y
Brillouin Backscatter	$\omega_p > \frac{T_i}{mc^2} \omega_0$	$\frac{v_0}{\sqrt{2} c c_s} \omega_{pi}$	$\left(\frac{v_0}{v_e}\right)^2 \left(\frac{p}{\omega_0}\right) k_0 L_u > 1$	Convective in x
Brillouin Side Scatter	$\omega_p > \frac{T_i}{mc^2} \omega_0$	$\left(\frac{v_0}{c}\right) \frac{\omega_0}{\omega_p} \sqrt{\frac{L_n}{L_u}} \omega_{pi}$	$\left(\frac{v_0}{v_e}\right)^2 \left(\frac{L_n}{L_u}\right) (k_0 L_u)^{4/3} > 1$ $(1 + L_u/L_n)^{2/3} > 1$	Temporally growing but convective in y for finite k_y
Brillouin Scatter (quasi mode)	$\omega_p > \frac{T_i}{mc^2} \omega_0$	$\frac{v_0}{\sqrt{2} c c_s} \omega_{pi}$	$\left(\frac{v_0}{c}\right)^2 > (k_0 \lambda_D)^3 (\omega_{pi}/\omega_0)$ and $\left(\frac{v_0}{v_e}\right)^2 (k_0 L_n) > 1$	Convective in x and y

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