INVARIANT QUANTIZATION, SCALE SYMMETRY
AND EUCLIDEAN FIELD THEORY

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I. Introduction

A method for extracting interesting results from field theoretic models, without explicitly solving the equations of motion which govern these models - an impossible task at the present time - is to study the algebraic relations which follow from canonical commutators. In the past, equal-time canonical commutators were used to derive the very important and successful algebra of currents at equal times.¹ More recently, canonical light-cone commutators led to current algebra on the light cone, which supplements the old equal time algebra and also gives a succinct description of Bjorken scaling in deep inelastic processes.²

The fortunate circumstance that canonical commutators can be determined a priori without solving the equations of motion, is a consequence of the quantization procedure. In the conventional approach, time is selected as the direction of evolution of the system, while the quantization conditions - the canonical commutators - are specified a priori on the space-like surface $t = \text{constant}$. In light-cone quantization, the system is allowed to develop along the x^+ direction,³ and commutators are given their a priori values on the surface $x^+ = \text{constant}$. It is natural to inquire whether yet other quantization procedures exist which provide useful information.

In these lectures I shall report on a new method of quantization which was developed by S. Fubini, A. Hanson and me.⁴ The surface of quantization is chosen to be the space-like hyperboloid $x^2 = \text{positive constant}$, and propagation of

the system proceeds along the normal direction, from one hyperboloid to the next. Of course the physical content of the theory is not changed by the choice of quantization surface, and we shall show that the formula for the S matrix in our approach, coincides with the conventional Feynman-Dyson expression. Nevertheless, as with light-cone quantization, the present technique organizes the theory in a novel fashion and offers new insights into the structure of field theory.

The results which have been obtained up to now are the following. In addition to exhibiting new canonical commutators, we shall give an unexpected interpretation for anomalous dimensions. An operator basis for Euclidean field theory has been discovered which differs from the one used in the past.⁵ Especially interesting is the application of our methods to 2-dimensional field theories where we find that the present ideas provide a bridge between the formalism of conventional field theory and that of the dual resonance model. Explicitly we can derive the Virasoro algebra for a large class of 2-dimensional field theories, and with further hypotheses, the Hagedorn spectrum is observed.

II. Review of Quantization Methods

Let me begin by giving a general discussion of our quantization procedure, comparing and contrasting it with other approaches. In the conventional method one considers the

planes $x^0 = \text{constant}$. The evolution operator, that is the operator which takes the physical system from one surface to the next, is the time translation generator P^0 (the Hamiltonian). Also there exist six kinematical operators, that is operators generating motions which leave the surface invariant. These are space translations generated by P^i , and space rotations generated by $M^{ij} = -M^{ji}$ ($i, j = 1, 2, 3$). The quantum conditions, that is the commutators, are specified on the selected surface; in the conventional case this is at $x^0 = \text{constant}$, i.e. at equal times.

In light-cone quantization, the selected surfaces are $x^+ = \text{constant}$. The evolution operator is P^- , while the six kinematical operators are P^+ , P^i , M^{ij} , M^{i+} ($i, j = 1, 2$). Canonical commutators are specified at equal x^+ .

A third possibility, which is the basis of our theory, is a family of hyperboloids $x^2 = \text{constant} > 0$. Clearly the six kinematical operators are the six Lorentz generators $M^{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$), while the evolution operator is the dilatation generator D . In other words if we wish to proceed from one hyperboloid $x^2 = \tau_1^2$ to another $x^2 = \tau_2^2$, this can be achieved by a dilatation $x^\mu \rightarrow px^\mu$. Canonical commutators are given their a priori values on surfaces of equal x^2 .

Some attractive and interesting aspects of our quantization are already apparent. First, note that the method is explicitly Lorentz invariant at every stage. Unlike in the previous cases, the quantization surface does not select

a preferred direction in space-time; clearly the surface $x^2 = \text{constant}$ transforms into itself under a Lorentz transformation. Second, and more striking, is the crucial role that dilatations play in the present discussion: the evolution of the system is governed by the dilatation generator. Currently dilatation transformations are the focus of much theoretical activity, as a consequence of the famous MIT-SLAC deep inelastic experiments. In the conventional approach to quantum field theory, scale transformations are somewhat peripheral to the main lines of development; thus we are pleased that these interesting transformations are central to the present theory.

An object of great importance in the discussion of the evolution of the system is the propagator function (Green's function). This is given by the vacuum expectation value of the ordered product of two fields, where the ordering is along the direction of propagation: time ordering or x^+ ordering in the familiar examples. For us the ordering will be along the surfaces $x^2 = \tau^2$.

When quantizing a field, it is convenient to expand it on the quantization surface in a complete set of functions associated with three of the six kinematical operators. In the usual equal time case, the three kinematical operators are chosen to be the translation generators P^i ($i = 1, 2, 3$), and the expansion functions are eigenfunctions of $\frac{\partial}{\partial x^i}$, i.e. they are $e^{-i k \cdot x}$. For a free theory, one further expands

in eigenfunctions associated with the evolution operator. This is P^0 in the conventional approach and one expands in eigenfunctions of $\frac{\partial}{\partial t} e^{ik^0 t}$. Thus a free field is expanded in e^{ikx} .

In the light-cone quantization, the three kinematical operators are P^+ , P^i ($i = 1, 2$) and the evolution operator is P^- . Hence a free field is again expanded in terms of e^{ikx} .

The present theory is characterized by the kinematical operators $M^{\mu\nu}$; hence we shall expand in eigenfunctions associated with three objects constructed from $M^{\mu\nu}$. The evolution operator is D ; thus the free field will further be expanded in eigenfunctions of $x^\mu \frac{\partial}{\partial x^\mu}$. Consequently our expansion functions, which replace the conventional exponentials are powers and 4-dimensional harmonics.

All this may appear extremely bizarre; yet in fact our approach is quite common in Schrödinger theory. The Schrödinger wave function, which satisfies

$$-\frac{1}{2} \nabla^2 \psi = (E - V) \psi,$$

can be expanded in momentum eigenfunctions.

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \phi(k)$$

$$\left(\frac{k^2}{2} - E\right) \phi(k) = - \int \frac{d^3q}{(2\pi)^3} V(k - q) \phi(q)$$

This would be the analogy to the conventional approach in field theory. Yet practical calculations are not performed

in momentum space. Rather one expands the wave function in terms of harmonics.

$$\psi(r) = \sum_{lm} \frac{R_{lm}(r)}{r} Y_{lm}(\Omega)$$

and furthermore, the radial wave function is frequently expanded in powers.

$$R_{lm}(r) = \sum_n C_{nlm} r^n$$

This is quite similar to the expansion procedure in our approach to field theory. Moreover, the Green's function for the Laplacian is given by

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \delta(r - r') \sum_{lm} \frac{1}{r} \left(\frac{r}{r'}\right)^l \frac{4\pi}{2l+1} Y_{lm}(\Omega) Y_{lm}^*(\Omega') \\ &+ \delta(r' - r) \sum_{lm} \frac{1}{r'} \left(\frac{r}{r'}\right)^l \frac{4\pi}{2l+1} Y_{lm}(\Omega) Y_{lm}^*(\Omega') \end{aligned}$$

It is seen that this has a representation as an ordered product, where the ordering is along the radial direction, in complete analogy to the ordering which we shall employ. Thus it is recognized that our method of quantization has already been widely used in Schrödinger theory, and one may view our work as an attempt to use in field theory techniques which proved themselves so successful in Schrödinger theory.

III. Euclidean Field Theory

A technical difficulty presents itself in carrying out quantization on hyperboloids. The problem is that the space-

like surfaces $x^2 = -\tau^2 > 0$ do not span all of space-time; as τ^2 varies from 0 to ∞ ; the region outside the light-cone $x^2 < 0$ is not reached. Thus the evolution of the system in that region must be discussed separately, a task made difficult by the fact that the surfaces $x^2 = -\tau^2$ are time-like.⁶ In order to overcome this obstacle, we simply continue our field theory to Euclidean space, where of course the surfaces $x^2 = \tau^2$ span all of space.

The continuation in no way excludes application of the general method to physical Minkowski-space theories. Indeed even in the conventional approach, all practical computations are always performed in Euclidean space. Furthermore I shall show explicitly that our formula for the S matrix, coincides with the conventional Feynman-Dyson expression, continued to Euclidean space.

Our definition of Euclidean field theory will now be presented since it differs in all respects from other approaches.⁵ We replace x^0 by ix^4 , and for every field in Minkowski space $\phi(x_\mu, x)$, we define a corresponding field in Euclidean space $\phi_E(x_\mu, x)$.

$$\phi_E(x_\mu, x) = \phi(ix_\mu, x) \quad (3.1)$$

Thus the free, massless equation in Minkowski space

$$\square \phi(x) = (\partial_\mu^2 - \partial^2) \phi(x) = 0 \quad (3.2)$$

goes over in Euclidean space to⁷

$$\square_E \phi_E(x) = (\partial_\mu^2 + \partial^2) \phi_E(x) = 0 \quad (3.3)$$

The Green's function for a free massless and spinless field in Minkowski space is

$$G(x, y) = \frac{1}{4\pi^2} \frac{1}{(x-y)^2 - i\epsilon} \quad (3.4)$$

In Euclidean space this becomes

$$G_E(x, y) = \frac{1}{4\pi^2} \frac{1}{(x-y)^2} \quad (3.5)$$

It is required that this Green's function be given by the ordered vacuum-expectation value of two Euclidean fields.

$$\begin{aligned} G_E(x, y) &= \langle 0 | T \phi_E(x) \phi_E(y) | 0 \rangle \\ &= \theta(x^2 - y^2) \langle 0 | \phi_E(x) \phi_E(y) | 0 \rangle \\ &\quad + \theta(y^2 - x^2) \langle 0 | \phi_E(y) \phi_E(x) | 0 \rangle \end{aligned} \quad (3.6)$$

Clearly the insistence on ordering makes sense only if the Euclidean fields are non-commuting operators.⁷ In the subsequent, I shall suppress the subscript E, since the discussion will be confined to Euclidean fields exclusively. Also vectors will be frequently written as

$$\begin{aligned} x^\mu &= (x, a^\mu) \\ x^2 &= x^2, \quad a^\mu = x^\mu / r \end{aligned}$$

IV. Quantization of Free Massless Field

The form of canonical field commutators can be deduced from the Green's function, (3.5) and (3.6), which satisfies

$$\square_x \langle 0 | R \phi(x) \phi(y) | 0 \rangle = -\delta^4(x-y) \quad (4.1)$$

However since the field itself satisfies $\square_x \phi(x) = 0$, the nonvanishing result (4.1) can arise only from the effect of $\square_x$ on the step functions which define the R ordering. The differentiation can be carried out with the help of the identity

$$\square = \frac{1}{x^2} \left[\left(\frac{d}{d \log r} \right)^2 + (L^2 + 1) \right] x \quad (4.2)$$

where L^2 is a purely angular derivative operator defined by

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \\ L^2 = \frac{1}{2} \epsilon^{\mu\nu}{}_{\mu\nu} \quad (4.3)$$

Eq. (4.1) may be regained, provided we set

$$[\chi(r, \alpha_1), \chi(r, \alpha_2)] = 0 \quad (4.4a)$$

$$[\chi(r, \alpha_1), \dot{\chi}(r, \alpha_2)] = -\delta^3(\alpha_1 - \alpha_2) \quad (4.4b)$$

$$[\dot{\chi}(r, \alpha_1), \chi(r, \alpha_2)] = 0 \quad (4.4c)$$

Here the notation is

$$\chi(r, \alpha) = r \phi(x)$$

$$\dot{\chi}(r, \alpha) = \frac{d}{d \log r} \chi(r, \alpha)$$

The δ function is defined on the sphere

$$\int d\alpha_1 f(\alpha_1) \delta^3(\alpha_1 - \alpha_2) = f(\alpha_2)$$

The commutators (4.4) can also be obtained from the action principle. The action is

$$I = \frac{1}{2} \int d^4x \partial_\mu \phi \partial^\mu \phi = -\frac{1}{2} \int d^4x \phi \square \phi \quad (4.5a)$$

With the help of (4.2), this may be rewritten

$$I = -\frac{1}{2} \int_0^\infty \frac{dr}{r} \int d\alpha \chi(r, \alpha) \left[\left(\frac{d}{d \log r} \right)^2 + (L^2 + 1) \right] \chi(r, \alpha) \\ = \frac{1}{2} \int_0^\infty d \log r \int d\alpha \left[\left(\frac{d}{d \log r} \right)^2 \chi + \chi (L^2 + 1) \right] \quad (4.5b)$$

Clearly the canonical coordinate should be identified with χ , while the canonical momentum is $\dot{\chi}$. The results (4.4) now follow.

Next I wish to discuss the properties of the field under Hermitian conjugation. We notice that the right-hand side of Eq. (4.4b) is real. This means that $\chi(r, \alpha)$ can not be an Hermitian operator, which is not surprising when it is recalled that the passage from Minkowski space to Euclidean space introduces factors of i . It is postulated that the field $\chi(r, \alpha)$ satisfies the relation

$$\chi^\dagger(r, \alpha) = \chi\left(\frac{1}{r}, \alpha\right) \quad (4.6)$$

which is consistent with (4.4b). Thus another transformation, which is currently much studied - inversion and its relative the conformal transformation - plays a central role in our theory. The Ansatz (4.6) will have far reaching consequences. As will be seen later, the momentum operator P^μ will be the negative of the Hermitian conjugate of the conformal operator K^μ .

Finally an expansion for the fields is postulated in terms of 4-dimensional harmonics, which are complete and orthonormal.⁸

$$\chi(x, a) = \sum_{l, n, m} \left\{ g_{l, n, m}^{(+)}(x) Y_{l, n, m}(a) + g_{l, n, m}^{(-)}(x) Y_{l, n, m}^*(a) \right\} \quad (4.7)$$

The dependence of the operators $g_{l, n, m}^{(\pm)}(x)$ on x is deduced from $\square \phi = 0$. This gives an expansion for $g_{l, n, m}^{(\pm)}(x)$ in terms of powers

$$\chi(x, a) = \sum_{l, n, m} \left\{ \frac{a_{l, n, m}^{(+)} x^{l+1}}{\sqrt{2l+2}} Y_{l, n, m}(a) + \frac{a_{l, n, m}^{(-)} x^{-(l+1)}}{\sqrt{2l+2}} Y_{l, n, m}^*(a) \right\} \quad (4.8)$$

The Hermiticity Ansatz (4.6) implies that

$$a_{l, n, m}^{(+)} = [a_{l, n, m}^{(-)}]^\dagger \quad (4.9)$$

Let us now determine the algebraic properties of our operators. We shall postulate the fundamental commutators

$$\begin{aligned} [a_{l, n, m}^{(+)} , a_{l', n', m'}^{(+)}] &= 0 \\ [a_{l, n, m}^{(-)} , a_{l', n', m'}^{(+)}] &= \delta_{ll'} \delta_{nn'} \delta_{mm'} \end{aligned} \quad (4.10)$$

It is now easy to verify that the field χ satisfies the commutation relations (4.4). We introduce a vacuum state $|0\rangle$, annihilated by $a_{l, n, m}^{(-)}$.

$$a_{l, n, m}^{(-)} |0\rangle = 0 \quad (4.11a)$$

The Hermiticity conditions (4.6) and (4.9) insure the existence of a dual vacuum state $\langle 0|$ annihilated by $a_{l, n, m}^{(+)}$

$$\langle 0| a_{l, n, m}^{(+)} = 0 \quad (4.11b)$$

We are now in a position to check whether the above formalism correctly reproduces the Green's function according to (3.5) and (3.6). From (4.8), (4.10) and (4.11) it follows that

$$\begin{aligned} \langle 0| R + (x_1) \phi(x_2) |0\rangle &= \sum_{l, n, m} \frac{1}{2l+2} Y_{l, n, m}^*(a_1) Y_{l, n, m}(a_2) \times \\ &\quad \left\{ \frac{\theta(x_1 - x_2)}{x_1^2} \left(\frac{x_2}{x_1} \right)^l + \frac{\theta(x_2 - x_1)}{x_2^2} \left(\frac{x_1}{x_2} \right)^l \right\} \end{aligned} \quad (4.12)$$

With the help of the addition formula for 4-dimensional spherical harmonics,⁹ (4.12) is recognised to be $\frac{1}{4\pi^2} \frac{1}{(x_1 - x_2)^2}$. This shows that we have succeeded in giving a consistent quantization scheme, and have provided a new and intuitively attractive operator basis for Euclidean field theory.

The expansion (4.12) can be used to give a spectral representation for the propagator. By use of the identity

$$\begin{aligned} \frac{1}{\pi} \int_0^\infty d\gamma \frac{z^{i\gamma}}{\gamma^2 + (l+1)^2} &= \frac{z^{l+1} \theta(1-z) + z^{-(l+1)} \theta(z-1)}{l+1} \\ z > 0, \quad l > -1 \end{aligned} \quad (4.13)$$

Eq. (4.12) can be cast in the form

$$G(x_1, x_2) = \frac{1}{4\pi^2} \frac{1}{(x_1 - x_2)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\gamma \sum_{l, n, m} \frac{P_{\gamma l n m}^*(x_1) P_{\gamma l n m}(x_2)}{\gamma^2 + (l+1)^2}$$

where the functions

$$P_{\gamma l n m}(x) = r^{i\gamma-1} Y_{l n m}(\alpha) \quad (4.14)$$

define the transformation between the coordinate space and the conjugate $(\gamma l n m)$ space in which the dimensionality γ and the angular momentum variables $(l n m)$ are diagonalized. In this new space, particularly convenient for dilatation invariant problems, the propagator is diagonal. It is

$$D(\gamma, l, n, m) = \frac{1/2\pi}{\gamma^2 + (l+1)^2} \quad (4.15)$$

This 4 dimensional $(\gamma l n m)$ space plays the same role in the present context as 4 dimensional momentum space in the usual quantization schemes.

Although the above has been derived for the free massless case, the introduction of masses poses no special difficulty. The main difference is that the expansion functions are no longer simple powers; rather they are Bessel functions.⁹ How to deal with interactions will be explained below.

V. Conserved Quantities and Generators of Motion

As always, symmetries of the action lead to currents J^μ , which are conserved according to Noether's theorem.

$$\partial_\mu J^\mu = \frac{\partial}{\partial t} J^0 + \vec{\nabla} \cdot \vec{J} = 0 \quad (5.1)$$

In order to obtain the "charge" associated with this symmetry, the current is integrated over the quantization surface.

$$Q = \int d\tau^\mu J_\mu(x) \quad (5.2)$$

where $d\tau^\mu$ is the infinitesimal surface element. For us, the surface is given by $x^2 = \tau^2$, and

$$d\tau^\mu = d^4x \delta(x^2 - \tau^2) 2x^\mu \quad (5.3)$$

(The normal to the surface is $2x^\mu$). Hence the charge is

$$Q = \int d^4x \delta(x^2 - \tau^2) 2x_\mu J^\mu(x) \quad (5.4a)$$

Of course, as a consequence of current conservation, Q does not depend on τ , which may be set equal to 1. Thus (5.4a) may be cast into a purely angular integral

$$Q = \int d\alpha \alpha_\mu J^\mu(1, \alpha) \quad (5.4b)$$

An especially important object in our theory is the dilatation generator

$$D = i \int d\tau^\mu x^\nu \theta_{\mu\nu} \quad (5.5a)$$

where $x_\nu \theta^{\mu\nu}(x)$ is the dilatation current, expressed in terms of the new improved energy momentum tensor.¹⁰ For the free massless theory governed by the action (4.5) this current is conserved. Upon expressing $\theta^{\mu\nu}$ in terms of the field ϕ , expanding ϕ in creation and annihilation operators as in (4.8), and performing the integral indicated in (5.5a), we get

$$D = -i \sum_{l,n,m} (l+1) a_{l,n,m}^{(+)} a_{l,n,m}^{(-)} \quad (5.5b)$$

Thus the dilatation generator is diagonal and anti-Hermitian. Frequently it will be convenient to define a Hermitian generator.

$$\Delta = i D \quad (5.6)$$

(If the theory is not scale invariant, D will no longer be independent of the surface over which $x_\mu \theta^{\mu\nu}$ is integrated; i.e. it will depend on τ . This in no way changes the fact that $D(\tau)$ describes the evolution of the system. The analogous situation in the ordinary theory occurs when time-dependent interactions are present. The Hamiltonian is then time-dependent, yet it still governs evolution in time.)

The remaining generators of the conformal group can also be constructed

$$M^{\mu\nu} = i \int d\tau_\alpha (x^\mu \theta^{\nu\alpha} - x^\nu \theta^{\mu\alpha}) \quad (5.7a)$$

$$P^\mu = i \int d\tau_\nu \theta^{\mu\nu} \quad (5.7b)$$

$$K^\mu = i \int d\tau_\nu (2x^\nu x_\alpha - g_\alpha^\nu x^i) \theta^{\nu\alpha} \quad (5.7c)$$

Their expression in terms of creation and annihilation operators is quite complicated; I shall not exhibit it here. However, the Hermiticity properties can be easily deduced. We find

$$(M^{\mu\nu})^\dagger = M^{\mu\nu} \quad (5.8a)$$

$$(P^\mu)^\dagger = -P^\mu \quad (5.8b)$$

$$(K^\mu)^\dagger = -P^\mu \quad (5.8c)$$

As promised, conformal transformations are as important in our theory as translations, since the two are related by Hermitian conjugation. The geometrical meaning of (5.8b) and (5.8c) is clear, once it is remembered that Hermitian conjugation is just coordinate inversion, see (4.6): a translation in the inverted coordinate system is equivalent to a conformal transformation.

VI. States of the System

A "single particle" state is defined by

$$|l,n,m\rangle = a_{l,n,m}^{(+)} |0\rangle \quad (6.1)$$

Naturally, this state does not coincide with the physical one particle state defined by Minkowski space quantization. Observe that $|l,n,m\rangle$ is an eigenstate of D with eigenvalue $-i(l+1)$. Furthermore, it is $(l+1)^2$ -fold degenerate: for fixed value of l , there are $(l+1)^2$ different values of n and m .⁸

The singularity structure of the dilatation propagator, (4.15), may now be understood. Note that the propagator has poles at $\tau = \pm i(l+1)$. This is just the dilatation eigenvalue of the single particle state. Thus the propagator in our conjugate (τ, l, n, m) space has poles at the eigenvalues of the evolution operator. This is in complete analogy with the conventional situation in momentum space, where the propagator has poles at $p^0 = \pm \sqrt{p^2 + m^2}$, i.e. at the eigenvalues of the Hamiltonian.

In the usual theory, interactions modify the position of the pole in the propagator - a phenomenon known as mass renormalization. We expect that in the present theory a similar shift should occur, once interactions are taken into account. Clearly this is the origin of anomalous dimensions in our formalism. Thus the intriguing possibility presents itself of computing anomalous dimensions as the poles of the complete dilatation propagator.

VII. Perturbation Theory and the S Matrix

The above considerations give a complete solution to the problem of quantizing a free theory. Now we must describe how interactions should be included. Suppose the equation of motion for the interacting field ϕ is

$$\square \phi = \eta \quad (7.1)$$

where η is the source. Free field canonical commutators (4.4) may still be imposed on $\phi(x)$ and $\frac{d}{d \log r} [r \phi(x)]$, at equal r .

To solve the equations of motion (7.1) we define the operator $U(r, r_0)$ by the equations

$$\begin{aligned} \frac{d}{d \log r} U(r, r_0) &= -i D^{\text{int}}(r) U(r, r_0) \\ U(r_0, r_0) &= 1 \end{aligned} \quad (7.2)$$

where $D^{\text{int}}(r)$ is the interaction part of the dilatation generator, constructed from the free fields ϕ . In a familiar

fashion, one proves that $U(r, r_0) \phi(x) U^{-1}(r, r_0)$ is a free field. Since we have solved the free-field problem, all that remains is to integrate (7.2).

Integration of (7.2) gives

$$U(r, r_0) = R \exp -i \int_{r_0}^r \frac{dx'}{x'} D^{\text{int}}(x') \quad (7.3)$$

where the R ordering is along the spheres. Recalling the definition of D^{int} , we see that

$$\begin{aligned} -i \int_{r_0}^r \frac{dx'}{x'} D^{\text{int}}(x') &= -i \int_{r_0}^r \frac{dx'}{x'} \int d^4x \frac{dx''}{x''} \int d^4y \frac{dx'''}{x'''} \delta(x^2 - x'^2) \delta(x'^2 - x''^2) \delta(x''^2 - x'''^2) \\ &= -i \int_{r_0}^r \frac{dx'}{x'} \int d^4x \delta(x^2 - x'^2) 2x^\mu (i x^\nu \phi_{\mu\nu}^{\text{int}}(x)) \\ &= - \int_{r_0}^r \frac{dx'}{x'} \int d^4x \delta(x^2 - x'^2) 2x^\mu L^{\text{int}}(x) \\ &= - \int_{r_0}^r d^4x L^{\text{int}}(x) \end{aligned}$$

$$U(r, r_0) = R \exp - \int_{r_0}^r d^4x L^{\text{int}}(x) \quad (7.4)$$

The integration is over all space bounded by the spheres $x^2 = x_0^2$.

We have regained the Schwinger-Tomonaga result that the evolution of the system from $x^2 = x_0^2$ to $x^2 = r^2$ is governed by the ordered exponential of the integral of the interaction Lagrangian. Clearly the S matrix is $U(\infty, 0)$.

$$\begin{aligned} S &= R \exp - \int d^4x L^{\text{int}}(x) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^4x_1 \dots d^4x_n R L^{\text{int}}(x_1) \dots L^{\text{int}}(x_n) \end{aligned} \quad (7.5)$$

Applying Wick's theorem to this, we regain the familiar Feynman-Dyson perturbation theory. Thus we see that our quantization procedure is entirely successful and consistent with the conventional approach.

The evaluation of the perturbation series may be carried out in position space, or in our conjugate (γ_{lm}) space. Calculations in the (γ_{lm}) space are especially convenient for conformally invariant interactions. We have not yet researched questions of divergences and renormalization in this new conjugate space.

VIII. Counting States, Asymptotic Degeneracy

It has already been observed that the single particle state $|lm\rangle$ is an eigenstate of $\Delta = iD$, with eigenvalue $l+1$ and degeneracy $(l+1)^2$. Consider now the N particle state

$$|N\rangle = a_{l_1 m_1}^{(+)} \cdots a_{l_N m_N}^{(+)} |0\rangle \quad (8.1)$$

This is an eigenstate of Δ with eigenvalue d_N , and degeneracy $g(N)$ ($d_1 = 2$, $g(1) = 4$). We wish to compute $g(N)$ for large N in terms of d_N .

The computation is facilitated by introducing a generating function

$$F(\beta) = \text{Tr} \exp -\beta \Delta \quad (8.2)$$

From its definition, two expressions for $F(\beta)$ can be given. Clearly from (8.2) it follows that

$$F(\beta) = \sum_N g(N) \exp -\beta d_N \quad (8.3)$$

An alternate expression is got for $F(\beta)$ by using the formulas (5.5) and (5.6) for Δ .

$$\begin{aligned} F(\beta) &= \text{Tr} \exp -\beta \sum_{lm} (l+1) a_{lm}^{(+)} a_{lm}^{(-)} \\ &= \text{Tr} \prod_{lm} \exp -\beta (l+1) a_{lm}^{(+)} a_{lm}^{(-)} \\ &= \prod_{lm} \text{Tr} \exp -\beta (l+1) a_{lm}^{(+)} a_{lm}^{(-)} \end{aligned} \quad (8.4a)$$

Recalling that for each mode $\text{Tr} x^{a^+ a} = (1-x)^{-1}$, and that each mode is $(l+1)^2$ -fold degenerate, it follows from (8.4a) that

$$\begin{aligned} F(\beta) &= \prod_l (1 - e^{-\beta(l+1)})^{-(l+1)^2} \\ \log F(\beta) &= - \sum_{l=0}^{\infty} (l+1)^2 \log [1 - e^{-\beta(l+1)}] \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{8\pi/2} + e^{-8\pi/2}}{[e^{8\pi/2} - e^{-8\pi/2}]}, \end{aligned} \quad (8.4b)$$

Our task is therefore to obtain a relationship between $g(N)$ and d_N for large N , which must hold if (8.3) and (8.4) are simultaneously true. This can be done by an elaborate analytic procedure based on the classic work of Hardy and Ramanujan.¹¹ However, it is more instructive, to obtain the answer by a thermodynamical analogy.

Observe that $F(\beta)$ defined by (8.2) is a partition function where β plays the role of inverse temperature, $\beta=1/T$, and Δ is the analog of the energy operator in the usual theory.

This is as it should be -- we have repeatedly emphasized that the dilatation generator replaces the Hamiltonian in our method. We may use this analogy to develop a thermodynamics based on the partition function $P(\beta)$. This thermodynamics is manifestly covariant. With the thermodynamical technique an expression for $g(N)$ is easily obtained.

We define the "free energy" by

$$A = -\frac{1}{\beta} \log P(\beta) \quad (8.5)$$

A is related to the "entropy" S and to the "internal energy" U by the well known equations

$$S = \beta^2 \frac{\partial}{\partial \beta} A \quad (8.6)$$

$$U = A + \frac{1}{\beta} S = -\frac{\partial}{\partial \beta} \log P(\beta) \quad (8.7)$$

Since we seek $g(N)$ only for large N , we are effectively in the high temperature limit $T \rightarrow \infty$ or $\beta \rightarrow 0$. In this region the "entropy" should be identified with $\log g(N)$ and the "internal energy" with the dilatation eigenvalue d_N . Furthermore the asymptotic form for $P(\beta)$ as $\beta \rightarrow 0$ is easy to obtain from (8.4b)

$$\log P(\beta) \sim \frac{\pi^d}{45\beta^d} \quad (8.8)$$

Hence we get from (8.5), (8.6), (8.7) and (8.8)

$$A \sim -\frac{\pi^d}{45\beta^d} \quad (8.9)$$

$$S \sim \frac{\pi^d}{45\beta^d} \quad (8.10)$$

$$U \sim \frac{\pi^d}{15\beta^d} \quad (8.11)$$

Eliminating the temperature $1/\beta$ between (8.10) and (8.11) gives a relationship between the entropy and the internal energy, valid in the high temperature limit.

$$S = c U^{\frac{1}{d}} \quad (8.12)$$

$$c = \frac{4\pi}{3(15)^{\frac{1}{d}}}$$

With our interpretation for S and U , we finally derive

$$g(N) \underset{N \rightarrow \infty}{\sim} \exp c d_N^{\frac{1}{d}} \quad (8.13)$$

The same answer emerges, after much effort, when the computation is performed analytically,⁴ which verifies the consistency and correctness of our thermodynamics.

The degeneracy, seen to be of the exponential variety, is reminiscent of the Hagedorn expression. To make contact with the latter formula, let us first recompute the degeneracy in d dimensions, generalizing the 4-dimensional derivation given above. The answer, simply obtained by the same method, is

$$g_d(N) \underset{N \rightarrow \infty}{\sim} \exp c_d d_N^{1-1/d} \quad (8.14)$$

$$c_d = \frac{d}{d-1} \left\{ 2(d-1) \prod_{n=1}^{\infty} \frac{1}{n} \right\}^{1/d}$$

In particular for 2-dimensions

$$g_2(N) \underset{N \rightarrow \infty}{\sim} \exp \frac{2\pi}{\sqrt{3}} d_N^{\frac{1}{2}} \quad (8.15)$$

Let us now postulate, that for reasons that are by no means apparent, there exists a "Law of Nature" relating physical mass

to scale-dimensionality in this 2-dimensional space.¹²

$$d_N = \frac{3\pi}{4\pi^2} m^2 \quad (8.16)$$

With this postulate, we find that the degeneracy of levels is $\exp \kappa m$, which is just the Hagedorn formula.

Since the derivation of the Hagedorn degeneracy is an open challenge to theoreticians, it is useful to elaborate on the possible circumstances that would lead to the new "Law of Nature" (8.16). Imagine that the correct, as yet unknown, physical theory is a field theory defined on some 2-dimensional, Euclidean parameter space. This parameter space is not the physical 4-dimensional space-time, though perhaps it is related to it. On the 2-dimensional space, dilatations can be defined; they are generated by D . In addition, since this hypothetical theory is to describe the 4-dimensional physical world, there must exist a momentum operator, generating physical 4-dimensional translations. Associated with the momentum operator will be the physical mass operator M^2 . The postulate (8.16) is then equivalent to the statement that the expression for M^2 in terms of the fundamental fields is proportional to the formula for D .

The observation that 2-dimensional models are in some way relevant to particle theory of the 4-dimensional physical world seems to be a very important though not understood feature of Nature. Historically the relevance of a 2-dimensional transverse space was noted first in the analysis of multi-particle production in the "pionization" region.¹³ A consistent, satisfactory description of single particle production in this region

is obtained by cutting off large, transverse momenta and using simply the longitudinal, 2-dimensional phase space factor dE/E . More recently, the success of the parton model in interpreting many of the important features of high energy reactions has called attention to 2-dimensional field theories as a means of describing the parton distribution functions.¹² Indeed it was in this context that the relation (8.16) was first suggested. Finally, recent investigations of dual resonance models make essential use of the general formalism of 2-dimensional field theories.¹⁴

This collection of facts leads me to exhibit further applications of our techniques to 2-dimensional field theories.

IX. Two-Dimensional Field Theories, The Virasoro Algebra

Field theories in 2 dimensions have been first introduced as an interesting theoretical laboratory where explicitly solvable models can be studied. More recently, however, the interest in such models has become much less academic. We apply our formalism to 2-dimensional field theories and demonstrate that our Lorentz-invariant quantization rules in Euclidean space lead directly to the operator structure present in dual models.

It has been shown that the higher moments of the energy-momentum tensor, in a large class of 2-dimensional field theories, obey essentially the Virasoro algebra.^{14,15} However, with a conventional approach, some difficulties persist in the inter-

pretation of various integrals. With our formalism, we can obtain a clear, unambiguous derivation of the Virasoro algebra, without the difficulties previously encountered.

Another point of contact between the present approach and dual models is seen in the structure of the R ordered product appearing in our Euclidean perturbation theory, Eq. (7.5). This is identical to that of the operator form for the integrals in the n -point Veneziano formulas.

I shall begin by considering a theory in d dimensions with a traceless, symmetric energy-momentum tensor (for the moment the metric is immaterial) and I construct the current

$$J_F^\mu(x) = \theta^{\mu\nu}(x) f_\nu(x) \quad (9.1)$$

where $f^\mu(x)$ is a function of x . Special cases of (9.1) are well known. These are the currents of the conformal group

$$\begin{aligned} \text{Translation current : } f^\mu(x) &= a^\mu \\ \text{Lorentz current : } f^\mu(x) &= \omega^{\mu\nu} x_{\nu}, \omega^{\mu\nu} = -\omega^{\nu\mu} \\ \text{Dilatation current : } f^\mu(x) &= ax^\mu \\ \text{Conformal current : } f^\mu(x) &= 2x^\mu a_\alpha x^\alpha - a^\mu x^2 \end{aligned} \quad (9.2)$$

Of course all these currents are conserved, when the energy-momentum tensor is traceless, symmetric and conserved. Are there any other forms for f^μ , such that $\partial_\mu J_F^\mu(x) = 0$? To answer this question, we differentiate (9.1)

$$\begin{aligned} \partial_\mu J_F^\mu &= \partial_\mu \theta^{\mu\nu} f_\nu + \theta^{\mu\nu} \partial_\mu f_\nu \\ &= \frac{1}{2} \theta^{\mu\nu} (\partial_\mu f_\nu + \partial_\nu f_\mu - \frac{2}{d} g_{\mu\nu} \partial_\alpha f^\alpha) \end{aligned}$$

Clearly the current will be conserved when

$$\partial_\mu f_\nu(x) + \partial_\nu f_\mu(x) - \frac{2}{d} g_{\mu\nu} \partial_\alpha f^\alpha(x) = 0 \quad (9.3)$$

Eq. (9.3) for $d > 2$ is solved only by the forms (9.2).¹⁶ However for $d=2$ a wider class of solutions exists. Thus if we wish to have the large symmetry which leads to the conserved currents J_F^μ , with f^μ satisfying (9.3), we are again led to 2-dimensional theories. Since, as will emerge presently, this symmetry is related to the Virasoro algebra,¹⁴ I shall now confine myself to 2-dimensional models.

Examples of field theories in 2 dimensions with a traceless, symmetric and conserved energy-momentum tensor are the following:

I. Free Boson Field

$$\begin{aligned} L &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\ \theta^{\mu\nu} &= \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi \end{aligned} \quad (9.4)$$

II. Free Fermion Field

$$\begin{aligned} L &= \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi \\ \theta^{\mu\nu} &= \frac{i}{4} (\bar{\psi} \gamma^\mu \partial^\nu \psi - \bar{\psi} \gamma^\nu \partial^\mu \psi) \end{aligned} \quad (9.5)$$

III. Gradient Coupling Model

$$\begin{aligned} L &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \lambda \bar{\psi} \gamma^\mu \phi \partial_\mu \phi \\ \theta^{\mu\nu} &= \partial^\mu \phi \partial^\nu \phi + \frac{i}{4} (\bar{\psi} \gamma^\mu \partial^\nu \psi + \bar{\psi} \gamma^\nu \partial^\mu \psi) \\ &\quad - \frac{\lambda}{2} (\bar{\psi} \gamma^\mu \phi \partial^\nu \psi + \bar{\psi} \gamma^\nu \phi \partial^\mu \psi) - \frac{1}{2} g^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi \end{aligned} \quad (9.6)$$

IV. Thirring Model

$$L = \frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \lambda \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma_\mu \psi$$

$$\theta^{\mu\nu} = \frac{1}{4} (\bar{\psi} \gamma^\mu \partial^\nu \psi + \bar{\psi} \gamma^\nu \partial^\mu \psi) - \lambda g^{\mu\nu} \bar{\psi} \gamma^\alpha \psi \bar{\psi} \gamma_\alpha \psi \quad (9.7)$$

V. The "Covariant String"

$$L = \frac{1}{2} \{ \epsilon_{\mu\nu} \partial^\mu \phi^A \partial^\nu \phi^B \} \{ \epsilon_{\alpha\beta} \partial^\alpha \phi^A \partial^\beta \phi^B \}^{\frac{1}{2}}$$

$$= (\det \partial_\mu \phi^A \partial_\nu \phi^A)^{\frac{1}{2}} \quad (9.8)$$

Here $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor, $\epsilon^{01}=1$. The indices A,B describe some additional degree of freedom and are summed over. (In applications to the dual model, A,B refer to physical space-time¹⁷ - one may of course allow models I-IV to possess such further degrees of freedom also.) In this model the energy-momentum tensor vanishes identically.

For all the above theories the currents J_ϵ^μ will be conserved, provided f^μ satisfies (9.3). I shall discuss first the situation in Minkowski space, and shall demonstrate that a consistent algebraic description of this symmetry is not possible. In Euclidean space, when the theory is quantized according to our method, no difficulties arise.

To find the solutions to (9.3) take the ++ component and the -- component of that equation. (The +- component vanishes identically.) In Minkowski space we thus find

$$\partial_+ f_+ = 0, \quad \partial_- f_- = 0 \quad (9.9a)$$

The integrals clearly are (recall that $f_\pm = f^\pm$)

$$f^+ = f^+(x^+)$$

$$f^- = f^-(x^-) \quad (9.9b)$$

The reason for the existence of all these conserved currents is that 2-dimensional theories with a traceless, symmetric and conserved energy-momentum tensor are invariant under the coordinate transformation

$$\delta_\epsilon x^\mu = f^\mu(x) \quad (9.10a)$$

where f^μ is of the form (9.9) and the field is transformed as follows

$$\delta_\epsilon \phi(x) = f^\mu(x) \partial_\mu \phi(x) + \frac{\partial f^\mu(x)}{2} [g^{\alpha\mu} d - \epsilon^{\alpha\mu} \Sigma] \phi(x) \quad (9.10b)$$

Here d is the scale dimension of the field ϕ and $\epsilon^{\alpha\mu} \Sigma$ is the spin matrix associated with ϕ . (In 2 dimensions we may set $\Sigma^{\alpha\mu} = \epsilon^{\alpha\mu} \Sigma$.) The symmetry can be seen directly in models I-IV above. For the string model, example V above, the currents J_ϵ^μ vanish identically since $\theta^{\mu\nu}$ is zero. The model is invariant under the transformation (9.10), even if f^μ does not satisfy (9.9). This is a gauge symmetry which does not lead to any conserved currents.¹⁷ In the following we consider only theories with non-vanishing $\theta^{\mu\nu}$.

The composition law for the transformation (9.10) is

$$\begin{aligned}\delta_g \delta_f x^\mu &= f^\mu(x) + g^\mu(x) + f^\nu(x) \partial_\nu g^\mu(x) \\ \delta_f \delta_g x^\mu &= g^\mu(x) + f^\mu(x) + g^\nu(x) \partial_\nu f^\mu(x)\end{aligned}\quad (9.11a)$$

It is easy to see that if f^μ and g^μ satisfy (9.9), so does

$$h^\mu = g^\nu \partial_\nu f^\mu - f^\nu \partial_\nu g^\mu \quad (9.11b)$$

and the transformations form a group. By choosing

$$\begin{aligned}f^+ &= 0 \\ f^- &= f(x^-)\end{aligned}\quad (9.12a)$$

$$\begin{aligned}f^+ &= f(x^+) \\ f^- &= 0\end{aligned}\quad (9.12b)$$

we recognize the fact that one is dealing with a direct product of two groups of transformation, and it suffices to consider only one of the factors. Henceforth, we make the choice (9.12a).

The charges can be constructed as follows.

$$\begin{aligned}Q_f &= \int_{-\infty}^{\infty} dx^1 \theta_{\mu}^{++}(x) f^\mu(x) \\ &= \int_{-\infty}^{\infty} dx^1 \theta^{++}(x) f(x^-) \\ &= \int_{-\infty}^{\infty} \frac{dx^1}{\sqrt{2}} \{ \theta^{++}(x) + \theta^{--}(x) \} f(x^-)\end{aligned}\quad (9.13a)$$

$\theta^{+-}(x)$ vanishes since the stress tensor is traceless. Also from the conservation of the stress tensor, $\partial_\mu \theta^{\mu+}(x) = 0 = \partial_\mu \theta^{++}(x)$, we learn that θ^{++} depends only on x^- . Hence the charge is

$$Q_f = \int_{-\infty}^{\infty} dx^- \theta^{++}(x) f(x^-) \quad (9.13b)$$

From the composition law (9.11) one would expect that the charges satisfy the commutation relations

$$\begin{aligned}i[Q_f, Q_g] &= Q_h \\ h &= f'g - g'f\end{aligned}\quad (9.14)$$

In order to verify (9.14), consider

$$\begin{aligned}\int_{-\infty}^{\infty} dx^- dy^- f(x^-) g(y^-) \{ \theta^{++}(x), \theta^{++}(y) \} &= \\ \int_{-\infty}^{\infty} dx^- dy^- f(x^-) g(y^-) \{ \theta^{++}(x), \theta^{++}(y) \} \Big|_{x^+=y^+}\end{aligned}\quad (9.15)$$

where the equality is true simply because θ^{++} does not depend on x^+ . The form of the local commutator of two energy momentum tensors which will insure the validity of (9.15) is

$$\{ \theta^{++}(x), \theta^{++}(y) \} \Big|_{x^+=y^+} = i \{ \theta^{++}(x) + \theta^{++}(y) \} \partial_- \delta(x^- - y^-) \quad (9.16)$$

and canonical evaluation in simple models yields this result.

However from general principles one can show that the commutator of the stress tensor with itself cannot be of the form given in (9.16). That the term proportional to the first derivative of the delta function is as indicated insures Poincaré invariance of the theory; indeed (9.16) is the analog of the Dirac-Schwinger commutator for the present problem. In addition, positivity and Lorentz covariance insure that a triple derivative is necessarily present. This is the analog of the

usual Schwinger term in current commutators, and the failure of canonical manipulations to expose it is of course familiar.¹⁸ The correct expression for the commutator which replaces (9.16) is

$$\begin{aligned}
 [\theta^{++}(x), \theta^{++}(y)]|_{x^+=y^+} &= i[\theta^{++}(x) + \theta^{++}(y)]\partial_- \delta(x^- - y^-) \\
 &\quad - \frac{ia_2}{2} \delta(x^- - y^-) \\
 a &> 0
 \end{aligned} \quad (9.17)$$

The commutator of the charges therefore is

$$i[Q_f, Q_g] = Q_h - \frac{a}{2} \int_{-\infty}^{\infty} dx^- f'''(x^-) g(x^-) \quad (9.18)$$

and for arbitrary f and g the charges do not satisfy the "classical" algebra.

The additional c number poses also the problem of convergence of the integral. If $f(x)$ or $g(x)$ are powers of x , as they are for the Virasoro algebra $\int_{-\infty}^{\infty} dx^- g(x^-) f'''(x^-)$ does not converge. Furthermore, one may question the convergence of the formula (9.13) for the charges, when f is a positive or negative power. Thus we conclude that the symmetry (9.10) and the associated algebra cannot be given an operator basis in a field theory defined on 2-dimensional Minkowski space.

The difficulties which have been encountered in giving a coherent field theoretic basis for the Virasoro algebra in a Minkowski space field theory do not exist if the field theory is quantized by our method, in Euclidean space. I shall now show this. The solution of Eq. (9.3) for f^u in Euclidean space is

$$\begin{aligned}
 f_1(x) + i f_2(x) &= f(re^{i\theta}) \\
 f_1(x) - i f_2(x) &= g(re^{-i\theta}) \\
 x_1 &= r \cos \theta, \quad x_2 = r \sin \theta.
 \end{aligned} \quad (9.19)$$

Here f and g are arbitrary functions of the argument. As before we may set g to zero, and confine our attention to a non-vanishing f . The charge is given by an angular integral, (see (5.4))

$$\begin{aligned}
 Q_f &= i \int_0^{2\pi} d\theta x_\mu J_f^{\mu} \\
 &= i \int_0^{2\pi} d\theta x_\mu \theta^{\mu\nu}(x) f_\nu(x) \\
 &= i \int_0^{2\pi} d\theta r e^{i\theta} f(re^{i\theta}) \theta(x)
 \end{aligned} \quad (9.20)$$

where

$$\theta(x) \equiv \frac{1}{2} [\theta^{11}(x) - i\theta^{12}(x)] \quad (9.21)$$

It is easy to show that θ is only a function of $x^1 + ix^2 = re^{i\theta}$, as a consequence of conservation tracelessness and symmetry of $\theta^{\mu\nu}$. Hence Q_f is also given by

$$Q_f = \oint_C dz f(z) \theta(z) \quad (9.22)$$

and the integration contour is over the circle of radius r . Clearly Q_f is independent of r , if $f(z)\theta(z)$ is analytic for $0 < |z| < \infty$.

To establish the algebraic properties of the charges, in our Euclidean space theory, we compute the $[Q_f, Q_g]$ commutator. Again what is needed is the $\{\theta, \theta\}$ commutator at equal r .

The Euclidean space analog of (9.17) is

$$i[\theta(x), \theta(x')] \Big|_{x=x'} = \frac{1}{x^2} \{ e^{-2i\theta} \theta(x) + e^{-2i\theta'} \theta(x') \} \delta_0(\theta-\theta') \\ + \frac{a}{2x^4} e^{-2i(\theta+\theta')} \delta_0(\theta-\theta') + 3^3 \delta_0(\theta-\theta') \quad (9.23)$$

$a > 0$

Consequently we find

$$i[Q_f, Q_g] = Q_h - \frac{ia}{2} \int_0^{2\pi} d\theta \, x e^{i\theta} g(x e^{i\theta}) \, f'''(x e^{i\theta}) \\ = Q_h - \frac{a}{2} \int_C dz \, g(z) \, f'''(z) \quad (9.24)$$

When f and g are analytic for $0 < |z| < \infty$, the integral is independent of contour. In contrast to the Minkowski space formula (9.18), the present result (9.24) is free from divergences. Also the expression for the charge, (9.20) or (9.22), involves a finite range integral and no question about convergence need be raised. Thus the Euclidean quantization provides a consistent algebraic description of the large symmetry present in many 2-dimensional field theories. The only anomaly is the c number addition proportional to a .

The discrete Virasoro algebra is obtained by setting

$$f(z) = iz^{1-n} \\ g(z) = iz^{1-m} \quad (9.25)$$

Then from (9.24) one gets

$$[Q_n, Q_m] = (m-n)Q_{n+m} - \pi a \delta_{n+m,0} (n^3-n) \quad (9.26)$$

Notice that the commutator anomaly is absent for n or $m = 0, \pm 1$. These cases correspond to a constant, linear or quadratic function f ; i.e. they give rise to the usual conformal group, Eq. (9.2). Thus the ordinary conformal algebra is realized in a conventional way, both for the Minkowski-space theory and for Euclidean-space theory. Only when $|n| > 1$ does the additional c number become relevant, and in that instance only the Euclidean method gives non-divergent results. The presence of a non-vanishing, c number anomaly shows that Q_n cannot annihilate the vacuum, for all n , and the symmetry is spontaneously broken.

Solutions of invariant equations, that is of equations arising in theories which possess this large symmetry such as the models I-IV given before, provide representations for the group. Explicit examples of spin 0 and spin $\frac{1}{2}$ fields exist, see (9.4) to (9.8). However, I have not been able to construct a spin 1 theory which leads to an invariant equation. The best I can do is the Lagrangian.

$$L = \frac{1}{2} \partial_\mu A^\nu \partial^\mu A_\nu - (\partial_\nu A^\nu)^2$$

The action is invariant under the conformal group, i.e. $n=0, \pm 1$ in Eq. (9.25), but the larger group of transformations (9.10) is not a symmetry of this model.

X. Conclusion

The aspects of this investigation that we find most

interesting are the following. The development of a manifestly covariant quantization procedure which can replace the conventional non-covariant method provides a wealth of covariant canonical commutators. It will be interesting to examine current commutators in this context and to see whether previous successes of equal-time and light-like current algebra can be extended.

We have given an operator basis for Euclidean quantum field theory. The quantization procedure follows closely the method of solving partial differential equations of the Laplacian type in Euclidean space, where the initial value data is specified on a sphere, rather than on a plane. Our approach gives special prominence to the dilatation operator. It makes contact with the explicitly conformally covariant formalism of Johnson and Adler.¹⁹ Furthermore it offers the intriguing possibility of studying anomalies of scale invariance as modifications of the dilatation propagator.

A covariant thermodynamics has put us closer to the goal of deriving the Hagedorn spectrum from first principles. What still is missing is an a priori justification of the use of 2-dimensional field theory and of the identification of mass squared with the dilatation eigenvalue. Nevertheless our approach to field theory in Euclidean space offers an attractive alternative to the conventional field theoretic formulations of the dual resonance model.

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investigations the non-interacting Euclidean field does not satisfy the free equation (3.3). Also, for us the fields are non-commuting operators, while previously they were taken to commute.

8. The properties of the harmonics are as follows.

$$Y_{l,n,m}(\alpha) = Y_{l,n,m}(\theta, \phi, \psi) \\ = N_{l,n,m} e^{im\psi} (\sin \theta)^n C_{l-n}^{n+1}(\cos \theta) (\sin \phi)^m C_{n-m}^{m+1}(\cos \phi)$$

The $C_\nu^\lambda(z)$ are Gegenbauer polynomials and $N_{l,n,m}$ is a normalization factor defined by

$$(N_{l,n,m})^{-1} = 2\pi \int_0^\pi \int_0^\pi \int_0^\pi Y_{l,n,m}(\alpha) Y_{l,n,m}(\alpha') d\alpha d\alpha' \\ E_k(l,n) = \frac{\pi 2^{k-2n} \Gamma(l+n-k+2)}{(2l+2-k)(l-n)! [\Gamma(n+1-k/2)]^2}$$

The range of the indices is

$$l = 0, 1, \dots, \infty$$

$$n = 0, 1, \dots, l$$

$$m = -n, -n+1, \dots, n$$

the harmonics are complete

$$\int_{\alpha} Y_{l,n,m}^*(\alpha) Y_{l,n,m}(\alpha') = \delta^3(\alpha-\alpha') \\ = \frac{1}{\sin^2 \theta \sin \psi} \delta(\theta-\theta') \delta(\phi-\phi') \delta(\psi-\psi')$$

and orthogonal

$$\int_0^\pi d\theta \sin^2 \theta \int_0^\pi d\phi \sin \phi \int_0^\pi d\psi Y_{l,n,m}^*(\alpha) Y_{l',n',m'}(\alpha') = \delta_{ll'} \delta_{nn'} \delta_{mm'}$$

They satisfy the addition theorem

$$\sum_{n=0}^l \sum_{m=-n}^n Y_{l,n,m}^*(\alpha) Y_{l,n,m}(\alpha') = \frac{l+1}{2\pi^2} C_l^1(\alpha \cdot \alpha')$$

The harmonic is an eigenfunction of the angular derivative defined in (4.3).

$$L^2 Y_{l,n,m} = l(l+2) Y_{l,n,m}$$

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