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GLOBAL ESTIMATES FOR NONLINEAR REACTION AND DIFFUSION

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1. INTRODUCTION

We shall obtain gradient bounds and some global estimates for the solution $u(x)$ of the nonlinear problem of combined diffusion and reaction

$$(1) \qquad -\Delta u = f(u) \quad , \quad x \in D \quad ; \quad \frac{\partial u}{\partial \nu} + hu = 0 \quad , \quad x \in \partial D \quad .$$

Here D is a bounded domain in R_n with boundary ∂D and outward normal ν, Δ is the n-dimensional Laplacian, h is a positive constant ($h = \infty$ corresponds to vanishing Dirichlet data), and f is such that (1) has a unique positive solution $u(x)$. We assume throughout that $f(z)$ is continuous for $z \geq 0$ and that $f(0) = 0$; the forced case $f(0) > 0$ and the nonlinear boundary condition $\partial u/\partial \nu + hu = p(u)$ will be treated elsewhere.

Problem (1) arises in a variety of applied contexts such as: a) steady operation of a homogeneous, monoenergetic nuclear reactor with feedback - here u is the neutron density; b) nonlinear heating such as Joule heating in a homogeneous medium (with u being the temperature); c) nonlinear chemical reaction combined with diffusion in a biochemical setting - here u is the concentration of a reactant. It should perhaps be noted that the equation $- \operatorname{div}(k(v) \operatorname{grad} v) = q(v)$ corresponding to a nonlinear diffusion coefficient can be transformed to equation (1) by the change of variable $u = \int_0^v k(z)dz$.

In the analysis of (1) an important role is played by the linear problem

$$(2) \qquad -\Delta\phi = \lambda\phi \quad , \quad x \in D \quad ; \quad \frac{\partial\phi}{\partial\nu} + h\phi = 0 \quad , \quad x \in \partial D \quad .$$

The fundamental eigenvalue $\lambda_1 = \lambda_1(h)$ of (2) is simple, positive, and increases with h, while the corresponding eigenfunction does not vanish in D. We shall let $\phi_1(x)$ be the <u>positive</u> fundamental eigenfunction whose <u>maximum value</u> is 1. If $h < \infty$, ϕ_1 is positive on the closure of D.

By setting $f(z) = \lambda g(z)$, we can regard (1) as a branching problem. Under suitable conditions on g, (1) will then have a branch of positive solutions emanating from the trivial solution at $\lambda = \lambda_1$. Preliminary results on existence and uniqueness of positive solutions will be obtained by using monotone methods (see KELLER [2], SIMPSON and COHEN [7], SATTINGER [6], STAKGOLD and PAYNE [8]). There is little that is new here apart from slight improvements in some of the proofs. Next we derive gradient bounds by an indirect use of Hopf's second maximum principle [1]. We find that $J(x) = |\text{grad } u|^2 + \int_0^u f(z)dz$ obeys an elliptic inequality except where grad u vanishes. A calculation shows that $\frac{\partial J}{\partial\nu} \leq 0$ on the boundary so that the maximum of J must occur at an exceptional point, leading to the desired bound. We then introduce level surface coordinates for u, enabling us to derive a number of isoperimetric norm estimates. Finally, by using the volume enclosed by a level surface as a new independent variable, we obtain an upper bound for the total flux of u through the boundary, this quantity having special importance in applications. In general outline the approach is similar to that in some of our previous papers (PAYNE and STAKGOLD [4,5,8]) but additional difficulties - both technical and conceptual - arise in virtue of the nonlinearity of the equation and the nature of the boundary condition. At the same time we are able to extend and deepen some of our earlier results, at the sacrifice of restricting ourselves to convex domains.

2. EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS

A <u>positive</u> solution of (1) is understood to be a solution $u(x)$ which is nonnegative but does not vanish identically in D. Let us set

(3a) $$f(z) = \lambda g(z) \quad , \quad \lambda > 0 \quad ,$$

where $g(z)$ is continuous for $z \geq 0$ and

(3b) $g'(z)$ is strictly decreasing for $z > 0$,

(3c) $$g'(0) = 1 \quad , \; g(0) = 0 \quad .$$

We then define

(4) $$\mu = \begin{cases} \lim\limits_{z \to \infty} z/g(z) & \text{if this limit is positive} \\ + \infty & \text{otherwise} \quad . \end{cases}$$

The boundary value problem (1) takes the form

(5a) $-\Delta u = f(u)$ or, equivalently $-\Delta u = \lambda g(u) \quad , \quad x \in D,$

(5b) $$\frac{\partial u}{\partial \nu} + hu = 0 \quad , \quad x \in \partial D \quad .$$

<u>Remarks</u>. 1. We do not exclude the possibility that g might become negative as z increases.

2. For each λ, $\lambda_1 < \lambda < \mu\lambda_1$, the curves $\lambda_1 z$ and $f(z)$ intersect at exactly one positive value of z. For any other λ, these curves do not intersect for $z > 0$.

Theorem 1. If f satisfies conditions (3), the boundary value problem (5) has one and only one positive solution for $\lambda_1 < \lambda < \mu\lambda_1$ and no positive solution for any other λ; moreover, for $\lambda \leq \lambda_1$ a solution of (5) can not be positive anywhere in D.

Proof A. Nonexistence. For (5) to have a positive solution, $f(z)-\lambda_1 z$ must change sign for $z > 0$. Indeed, adding $-\lambda_1 u$ to both sides of (5a) and using the Fredholm alternative we find that $\int_D [f(u)-\lambda_1 u]\phi_1 dx = 0$ which implies that either $f(z)-\lambda_1 z$ changes sign for $z > 0$ or that u is a constant - a positive zero of $f(z)-\lambda_1 z$. This latter possibility is eliminated because a nonzero constant does not obey the boundary condition. Thus positive solutions can occur only for $\lambda_1 < \lambda < \mu\lambda_1$.

To prove that solutions for $\lambda \leq \lambda_1$ can not be positive anywhere, suppose the contrary to be true. Then there exists a proper subdomain $D' \subset D$ with $u > 0$ in D', $\frac{\partial u}{\partial \nu} + hu = 0$ on $\partial D' \cap \partial D$ and $u = 0$ on the rest of D'. Let λ_1' and ϕ_1' be the fundamental eigenvalue and positive eigenfunction of $-\Delta$ for D' with the boundary conditions just described. For $\lambda \leq \lambda_1$, the hypothesis gives $f(z)-\lambda_1 z \leq 0$ for $z > 0$ and hence $f(z)-\lambda_1' z < 0$ since $\lambda_1' > \lambda_1$. Subtracting $\lambda_1' u$ from both sides of (5a) and applying the Fredholm alternative, we obtain $\int_{D'} [f(u)-\lambda_1' u]\phi_1' dx = 0$, which is a contradiction.

B. Existence. We shall use monotone iteration schemes to construct maximal and minimal positive solutions. Recall that an upper solution $\bar{u}(x)$ satisfies the inequalities

$$-\Delta\bar{u}-f(\bar{u}) \geq 0 \quad , \quad x \in D \quad ; \quad \frac{\partial \bar{u}}{\partial \nu} + h\bar{u} \geq 0 \quad , \quad x \in \partial D \quad ,$$

whereas for a lower solution $\underline{u}(x)$ both inequalities are reversed.

For $A > 0$, the function $v = A\phi_1$ satisfies

$$-\Delta v - f(v) = \lambda A\phi_1\left[\frac{\lambda_1}{\lambda} - \frac{g(A\phi_1)}{A\phi_1}\right] \quad ; \quad \frac{\partial v}{\partial \nu} + hv = 0 \text{ on } \partial D \quad .$$

Since $g'(0) = 1$, we can, for each $\lambda > \lambda_1$, choose A sufficiently small so that v is a lower solution. We also know that $\phi_1 \geq \delta > 0$ on D so that we can, for each $\lambda < \mu\lambda_1$, choose A so large that v is an upper solution. (The argument must be modified in the Dirichlet case because ϕ_1 now vanishes on ∂D; an upper solution can then be found in the form $A\tilde{\phi}_1$ where $\tilde{\phi}_1$ is the positive eigenfunction of (2) corresponding to a large positive value of h).

Starting from the lower solution just described we use a standard iteration procedure to construct a monotonically increasing sequence $\underline{u}_n$ which converges to the minimal positive solution u_* of (5). Similarly, from the upper solution, we construct a decreasing sequence $\bar{u}_n$ converging to the maximal positive solution u^* of (5).

C. Uniqueness. Let u_* and u^* be the minimal and maximal positive solutions constructed in part B. Since u_* exceeds $A\phi_1$ for some positive A, we have $u_* > 0$ in D. We know that $u^* \geq u_*$; suppose the strict inequality occurs on D' whereas equality holds on the remainder of D. Applying Green's theorem to D', we find

$$\int_{D'} u_* u^* \left[\frac{f(u^*)}{u^*} - \frac{f(u_*)}{u_*}\right] dx = \int_{\sigma_1+\sigma_2} \left[u^* \frac{\partial u_*}{\partial \nu} - u_* \frac{\partial u^*}{\partial \nu}\right] ds \quad ,$$

where $\sigma_1 = \partial D' \cap \partial D$ and σ_2 is the remainder of $\partial D'$.
The integral over σ_1 vanishes by the boundary condition (5b). On σ_2, we have $u^* = u_*$ and $\dfrac{\partial u_*}{\partial \nu} - \dfrac{\partial u^*}{\partial \nu} \geq 0$. By (3b), $f(z)/z$ is strictly decreasing which implies that the integral over D' is negative. This contradiction then shows that D' has zero measure; by contin-

uity it follows that $u_* = u^*$ in D, completing the proof of Theorem 1.

We conclude this section with a simple result.

Lemma 1. Let f satisfy conditions (3) and let u(x) be a positive solution of $-\Delta u = f(u)$ on Ω (no boundary conditions specified). Then $f(u(x)) \geq 0$ on Ω.

Proof. At any point in Ω where the maximum is attained, $\Delta u \leq 0$ so that $f(u_m) \geq 0$. But $f(z_0) \geq 0$ implies $f(z) > 0$ for $0 < z < z_0$.

3. GRADIENT BOUNDS.

From here on we shall assume that f satisfies conditions (3) and therefore Lemma 1 is applicable. Although we wish to obtain gradient bounds for the unique positive solution of (5) when $\lambda_1 < \lambda < \mu\lambda_1$, we shall have to proceed by steps.

Lemma 2. Let u(x) be a positive solution of $-\Delta u = f(u)$ on a domain Ω and let

$$J = |\text{grad } u|^2 + 2F(u) \quad , \tag{6}$$

where

$$F(u) = \int_0^u f(z)dz \quad ; \tag{7}$$

then J satisfies the elliptic inequality

$$0 \leq \Delta J + \left[\sum_{k=1}^{n} a_k \frac{\partial J}{\partial x_k}\right] / |\text{grad } u|^2 , \tag{8}$$

where the coefficients a_k are continuous and bounded on Ω.

Proof. Straightforward calculation and use of the Schwarz inequality (see [8]). The fact that $f(u(x)) \geq 0$ on Ω plays an essential role.

Theorem 2. Let u(x) be a positive solution of $-\Delta u = f(u)$ in a convex domain Ω with boundary $\partial\Omega$ (of class $C^{2+\varepsilon}$ for the time being) on which u is constant. Then

(9) $$|\text{grad } u|^2 \leq 2\left[F(u_m) - F(u)\right] \quad .$$

Proof. Introduce a normal-tangential coordinate system in a neighbourhood of $\partial\Omega$. We have

(10) $$J = \left(\frac{\partial u}{\partial \nu}\right)^2 + |\text{grad}_t\, u|^2 + 2F(u) \quad ,$$

where $\text{grad}_t\, u$ is the tangential component of grad u. Then

(11) $$\frac{\partial J}{\partial \nu} = 2\left(\frac{\partial u}{\partial \nu}\right)\left(\frac{\partial^2 u}{\partial \nu^2}\right) + 2\text{grad}_t\, u.\ \frac{\partial}{\partial \nu}\text{grad}_t\, u + 2f(u)\frac{\partial u}{\partial \nu}$$

and, since $\text{grad}_t\, u = 0$ on $\partial\Omega$,

(12) $$\left(\frac{\partial J}{\partial \nu}\right)_{\partial\Omega} = 2\frac{\partial u}{\partial \nu}\left[\frac{\partial^2 u}{\partial \nu^2} + f(u)\right] \quad .$$

By the smoothness assumption on $\partial\Omega$ we may apply the differential equation at the boundary where it takes the form

(13) $$\frac{\partial^2 u}{\partial \nu^2} + (n-1)K\frac{\partial u}{\partial \nu} + \Delta' u + f(u) = 0 \quad ,$$

where Δ' is the surface Laplacian and K the mean curvature. Since u is constant on $\partial\Omega$, $\Delta' u = 0$, and, substituting for $\frac{\partial^2 u}{\partial \nu^2}$ in (12), we find

(14) $$\left(\frac{\partial J}{\partial \nu}\right)_{\partial\Omega} = -2(n-1)K\left(\frac{\partial u}{\partial \nu}\right)^2 \leq 0$$

because $K \geq 0$ for a convex domain.

Since J satisfies (8) its maximum occurs either where grad u = 0 (this includes the case $J \equiv$ constant) or at a point on $\partial\Omega$ where $\frac{\partial J}{\partial \nu} > 0$ (by Hopf's second maximum principle). This latter possibility is ruled out by (14) so that J has its maximum where grad u vanishes and hence (9) follows.

Remarks. 1. The bound (9) is exact for one-dimensional problems.

2. By approximating with smooth boundaries, we can extend Theorem 2 to a convex domain Ω with a Lipschitz boundary.

3. Consider (5) for a convex domain D and $\lambda_1 < \lambda < \mu\lambda_1$. If $h = \infty$, u vanishes on ∂D and Theorem 2 is immediately applicable with $\Omega = D$. If $h < \infty$, u is not necessarily constant on ∂D, but its maximum value τ on ∂D is certainly less than u_m since $\frac{\partial u}{\partial \nu}$ is negative on ∂D. Thus (9) holds for the domain $\Omega \subset D$ where $u > \tau$ under the reasonable (but unproved) assumption that Ω is convex if D is convex. For the purpose of deriving isoperimetric inequalities in the sequel it is sufficient to know that (9) is valid for $u > \tau$. Alternatively, for $n = 2$, we have been able to modify the proof of Theorem 2 to take into account the fact that for $h < \infty$ neither $\text{grad}_t\, u$ in (11) nor $\Delta' u$ in (13) vanishes on ∂D.

<u>Theorem</u> 3. Consider problem (5) with D convex, $n = 2$, $\lambda_1 < \lambda < \mu\lambda_1$. Then (9) holds in D.

<u>Proof</u>. Let s be the tangential coordinate which coincides with the arc length on ∂D. The corresponding metric coefficient $k(x)$ is then identically equal to 1 on ∂D. We then have from (10) that

$$J = \left(\frac{\partial u}{\partial \nu}\right)^2 + \left(\frac{1}{k}\frac{\partial u}{\partial s}\right)^2 + 2F(u) \quad , \tag{15}$$

and (12) becomes

$$\left(\frac{\partial J}{\partial \nu}\right)_{\partial D} = 2\,\frac{\partial u}{\partial \nu}\left(\frac{\partial^2 u}{\partial \nu^2} + f\right) - 2\left(h + \frac{\partial k}{\partial \nu}\right)\left(\frac{\partial u}{\partial s}\right)^2 \tag{16}$$

whereas (13) takes the form

$$\frac{\partial^2 u}{\partial \nu^2} + K\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial s^2} + f(u) = 0 \,. \tag{17}$$

Substituting for $\frac{\partial^2 u}{\partial \nu^2}$ in (16), we find, on observing that $\frac{\partial k}{\partial \nu} = K \geq 0$,

$$\left(\frac{\partial J}{\partial \nu}\right)_{\partial D} \leq 2\,\frac{\partial u}{\partial \nu}\left[-K\,\frac{\partial u}{\partial \nu} - \frac{\partial^2 u}{\partial s^2}\right] \quad , \tag{18}$$

which differs from (14) by the second derivative term. We also have

$$\frac{\partial J}{\partial s} = 2\,\frac{\partial u}{\partial \nu}\,\frac{\partial^2 u}{\partial s\partial \nu} + 2\left(\frac{1}{k}\,\frac{\partial u}{\partial s}\right)\frac{\partial}{\partial s}\left(\frac{1}{k}\,\frac{\partial u}{\partial s}\right) + 2f(u)\,\frac{\partial u}{\partial s}$$

which on ∂D reduces to

$$(19)\qquad \left(\frac{\partial J}{\partial s}\right)_{\partial D} = \frac{\partial u}{\partial s}\left(2h^2u + f + 2\,\frac{\partial^2 u}{\partial s^2}\right) .$$

A further calculation yields

$$(20)\qquad \frac{1}{2}\left(\frac{\partial^2 J}{\partial s^2}\right)_{\partial D} = \left(\frac{\partial u}{\partial s}\right)\left(h^2\frac{\partial u}{\partial s} + \frac{\partial^3 u}{\partial s^3} + f'\frac{\partial u}{\partial s}\right) + \left(\frac{\partial^2 u}{\partial s^2}\right)\left(h^2u + f + \frac{\partial^2 u}{\partial s^2}\right) .$$

Let P be the point on the boundary at which J is supposed to have a maximum. Then $\frac{\partial J}{\partial s} = 0$ and $\frac{\partial^2 J}{\partial s^2} \leq 0$ at P. If $\frac{\partial^2 u}{\partial s^2} > 0$ at P, (19) shows that $\frac{\partial u}{\partial s} = 0$ and (20) would give $\frac{\partial^2 J}{\partial s^2} > 0$, a contradiction. Therefore $\frac{\partial^2 u}{\partial s^2} \leq 0$ at P, and (18) gives $\left(\frac{\partial J}{\partial \nu}\right)_P \leq 0$ in conflict with Hopf's second maximum principle. Thus the maximum of J must occur where grad u = 0 which establishes (9) once more.

<u>Corollary</u>. Let τ be the maximum of u on ∂D and τ^* the unique positive root of

$$(21)\qquad h^2\tau^2 + 2F(\tau) = 2F(u_m) \quad ;$$

then

$$\tau \leq \tau^* .$$

<u>Proof.</u> At the point on the boundary where $u = \tau$, we have

$$h^2\tau^2 = \left(\frac{\partial u}{\partial \nu}\right)^2 \leq |\text{grad } u|^2 \quad ,$$

but this last term has the upper bound in (9), by Remark 3 following Theorem 2.

4. NORM ESTIMATES.

Using the preceding Corollary and the bound (9) we can obtain some rough global estimates by integrating (5a) over D. It is, however, more fruitful to introduce the level surfaces for $u(x)$. Let $D(t)$ be the domain where u exceeds t; its boundary, which may include part of ∂D, is denoted by $\partial D(t)$, and its volume by $v(t)$. Clearly $v(t)$ is a decreasing function of t with maximum value V and minimum value 0 at $t = u_m$.

We first note some elementary relations between $v(t)$ and u. For any continuous function $a(z)$, define

$$\alpha(t) = \int_{D(t)} a(u)dx \quad .$$

Then

(22) $$\alpha'(t) = a(t)v'(t) \quad , \quad \alpha(t) = -\int_t^{u_m} a(z)v'(z)dz \quad ,$$

and

(23) $$v'(t) = -\int_{\partial D(t)} |\text{grad } u|^{-1} ds \quad , \quad t \geq \tau \quad .$$

<u>Lemma</u> 3. Let

$$\phi(t) = \int_{D(t)} f(u)dx \quad .$$

Then, for $t \geq \tau$,

(24) $$\phi(t) \leq \phi(\tau)\left[F(u_m)-F(t)\right]^{\frac{1}{2}}\left[F(u_m)-F(\tau)\right]^{-\frac{1}{2}} \quad .$$

<u>Proof</u>. For $t \geq \tau$, $\partial D(t)$ is the level surface $u = t$ so that $-\dfrac{\partial u}{\partial \nu} = |\text{grad } u|$ on $\partial D(t)$. Integration of (5a) over $D(t)$ then gives

$$\phi(t) = \int_{\partial D(t)} |\text{grad } u| \, ds \quad ,$$

while (22) and (23) yield

$$\phi'(t) = f(t)v'(t) = -f(t)\int_{\partial D(t)} |\text{grad } u|^{-1} ds \quad \ldots$$

Combining these equations for ϕ and ϕ', and using (9), we find

$$-\phi'/\phi \geq \frac{f(t)}{\max\limits_{x\varepsilon\partial D(t)} |\text{grad } u|^2} = \frac{f(t)}{2\left[F(u_m)-F(t)\right]} \quad .$$

Integrating this inequality from τ to t then gives (24).

Lemma 3 is the basis for the following theorem which yields a variety of norm estimates.

Theorem 4. Let $a(z)$ be an arbitrary continuous function increasing for $z \geq 0$. Then

$$(25) \qquad \frac{\int_D f(u)a(u)dx}{\int_D f(u)dx} \leq \left[2F(u_m)-2F(\tau)\right]^{-\frac{1}{2}} I(\tau) \leq \frac{1}{h\tau^*} I(\tau^*) \quad ,$$

where

$$(26) \qquad I(z) = \int_z^{u_m} a(t)f(t)\left[2F(u_m)-2F(t)\right]^{-\frac{1}{2}}dt \quad .$$

Proof. Since $a(t)$ is increasing, we can write

$$(27) \qquad \int_D f(u)a(u)dx = a(\tau)\int_{D-D(\tau)} f(u)dx + \int_{D(\tau)} f(u)a(u)dx \quad .$$

To estimate the last term, let us multiply (24) by $a'(t)$ and integrate from τ to u_m, the integration on the left side being done by parts. Using (22), we find

$$(28) \qquad \int_{D(\tau)} f(u)a(u)dx \leq \phi(\tau)a(\tau) + \phi(\tau)\left[2F(u_m)-2F(\tau)\right]^{-\frac{1}{2}} \int_\tau^{u_m} a'(t)\left[2F(u_m)-2F(t)\right]^{\frac{1}{2}}dt \quad .$$

When substituting in (27) the first terms on the right of (28) and (27) combine to give $a(\tau)\int_D f(u)dx$; in the remaining term we replace $\phi(\tau)$ by its upper bound $\int_D f(u)dx$ and integrate by parts to obtain the first inequality in (25). The second inequality follows from (21) and the observation that $\left[2F(u_m)-2F(z)\right]^{-\frac{1}{2}}I(z)$ is an increasing function of z for $z \geq 0$.

Some consequences of (25) are worth noting. If $a(u) = u^p/f(u)$, with p a positive integer, we obtain

$$\int_D u^p dx \leq \frac{\int_D f(u)dx}{h\tau^*} \int_{\tau^*}^{u_m} t^p \left[2F(u_m)-2F(t)\right]^{-\frac{1}{2}} dt \quad . \tag{29}$$

As $h \to \infty$, τ^* tends to 0 and $h\tau^* \to \left[2F(u_m)\right]^{\frac{1}{2}}$, so that, we find for the Dirichlet problem

$$\int_D u^p dx \leq \int_D f(u)dx \int_0^{u_m} t^p \left[2F(u_m)\right]^{-\frac{1}{2}} \left[2F(u_m)-2F(t)\right]^{-\frac{1}{2}} dt \quad .$$

This last inequality can be applied in the limiting linear case $f(u) = \lambda_1 u$; we then recover, for $p = 2$, a result of [8]:

$$\int_D u^2 dx \leq \frac{\pi}{4} u_m \int_D u dx \quad .$$

We conclude this section by deriving a Payne-Rayner type of inequality complementary to (25), see [3]. We confine ourselves to the 2-dimensional problem. Multiplying the expressions for ϕ and ϕ' in the proof of Lemma 3, we find, by using the Schwarz inequality,

$$-\frac{d}{dt}\phi^2(t) \geq 2f(t)S^2(t) \geq 8\pi f(t)v(t) \quad , \quad t \geq \tau \quad ,$$

where $S(t)$ is the length of the boundary $\partial D(t)$, and the classical isoperimetric inequality was used in the last step. An integration from $t = \tau$ to $t = u_m$ then gives

$$\phi^2(\tau) \geq 8\pi \int_{D(\tau)} F(u)dx - 8\pi F(\tau)v(\tau) \quad ,$$

which is used in the chain of inequalities

$$\int_D F(u)dx \leq \int_{D(\tau)} F(u)dx + F(\tau)\left[V-v(\tau)\right]$$

$$\leq \frac{1}{8\pi}\,\phi^2(\tau) + F(\tau)V$$

$$\leq \frac{1}{8\pi}\left[\int_D f(u)dx\right]^2 + F(\tau^*)V \quad .$$

In the Dirichlet case the last inequality becomes

$$\int_D F(u)dx \leq \frac{1}{8\pi}\left[\int_D f(u)dx\right]^2 \quad . \tag{30}$$

It is perhaps worth noting that inequalities such as (21), (25) (29) all become equalities in the one-dimensional case.

5. ISOPERIMETRIC INEQUALITY FOR THE TOTAL FLUX

To simplify the calculations we confine ourselves in this Section to the Dirichlet problem. As in Section 4 we let $v(t)$ be the volume enclosed by the level surface $u = t$. Since $v(t)$ is a decreasing function of t, we may use v as a new independent variable.

With $\phi(t)$ as in Lemma 3 of the preceding Section, we define

$$\Phi(v) = \phi(t(v)) \quad ,$$

from which it follows that

$$\Phi'(v) = f(t(v)) \tag{31}$$

$$\Phi''(v) = f't' = \frac{f'}{v'} \quad , \tag{32}$$

where v' can be expressed in terms of $|\text{grad } u|$ from (23). Multiplying (32) and the equation obtained from integrating (5a) over $D(t(v))$, we find

$$-\Phi\Phi'' \leq 2f'(t(v))\left[F(u_m)-F(t(v))\right] .$$

We establish the inequality

$$2f'(z)\left[F(u_m)-F(z)\right] - 2f'(0)F(u_m) \leq -f^2(z) \tag{33}$$

by noting that both sides vanish at $z = 0$ and that the derivative on the left side is smaller than on the right. Hence

$$(\Phi')^2 - \Phi\Phi'' \leq 2f'(0)F(u_m) = \alpha^2 \quad . \tag{34}$$

If we multiply this inequality by the positive quantity Φ'/Φ^3, we find

$$-\left[(\Phi'/\Phi)^2\right]' \leq -\alpha^2\left[1/\Phi^2\right]' \quad ,$$

which we now integrate from v to V to obtain

$$\Phi'(v) \leq \alpha\left[1 - \frac{\Phi^2(v)}{\Phi^2(V)}\right]^{\frac{1}{2}} \quad .$$

Integrating once more, this time from 0 to V, we find

$$\Phi(V) \leq \frac{2}{\pi}\alpha V \quad ,$$

or

$$\frac{\int_D f(u)dx}{V} \leq \frac{2}{\pi}\left[2f'(0)F(u_m)\right]^{\frac{1}{2}} \quad . \tag{35}$$

In the one-dimensional problem the two sides can be computed explicitly, the ratio of the right side to the left being $\sqrt{\lambda/\lambda_1}$ which will be small if we are close to criticality (the usual situation in applications). For the linear case, $f(u) = \lambda_1 u$, $F(u) = \lambda_1 u^2/2$, and we recover the isoperimetric inequality of [4],

$$\frac{\int_D u dx}{V u_m} \leq \frac{2}{\pi} \quad . \tag{36}$$

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BIBLIOGRAPHY

1. E. HOPF, Proc. Amer. Math. Soc. 3 (1952), 291-293.

2. H.B. KELLER, Bull. Amer. Math. Soc. 74 (1968), 887-891

3. L.E. PAYNE & M.E. RAYNER, Z. Angew. Math. Phys. 23 (1972), 13-15.

4. L.E. PAYNE & I. STAKGOLD, Applicable Analysis, to appear

5. L.E. PAYNE & I. STAKGOLD, to appear

6. D.H. SATTINGER, Topics in Stability and Bifurcation Theory Lecture Notes in Mathematics, no. 309, Springer, 1973.

7. R.B. SIMPSON & D.S. COHEN, J. Math. Mech. 19 (1970), 895-910.

8. I. STAKGOLD & L.E. PAYNE, Nonlinear problems in nuclear reactor analysis, in Nonlinear Problems in the Physical Sciences and Biology, Lecture Notes in Mathematics, no. 322, Springer, 1973.

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