ON THE POINT AND BLOCK FACTORIZATION ITERATIVE
METHODS FOR ARBITRARY MATRICES AND THE
CHARACTERIZATION OF M-MATRICES

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January 1974

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PREPARED FOR THE U. S. ATOMIC ENERGY COMMISSION UNDER
CONTRACT NO. AT(11-1)-2280

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ON THE POINT AND CLOCK FACTORIZATION ITERATIVE METHODS
FOR ARBITRARY MATRICES AND THE CHARACTERIZATION
OF M-MATRICES^{*}

by

R. Beauwens

Abstract

A new formulation of the point Buleev method is proposed. Factorizability conditions and convergence criteria of composite methods including the general Oliphant-Buleev-Woznicki method are obtained for arbitrary matrices. The generalization of these results to the block Oliphant-Buleev-Woznicki method is performed and a further generalization of these results to the new class of block-M-matrices is presented.

As auxiliary results, the refinement of previous convergence criteria for essentially non-positive matrices leads to a new formulation of some classical characterization properties of M-matrices, due to Fiedler and Ptak, and these properties are also generalized to partitioned M-matrices and block-M-matrices.

Finally, using an extension of a theorem of Woznicki, strict comparisons between the asymptotic convergence rates of some factorization iterative methods are obtained.

*Work supported by a NATO fellowship and in part by Brussels University, Northwestern University and the U.S. Atomic Energy Commission under Contract No. AT(11-1)-2280.

1. Introduction

Block factorization iterative methods have been introduced recently [1]; however, the picture was not complete because the Buleev method and composite methods including the Buleev scheme could not be generalized. We show in the present report that it is possible to perform this generalization with the help of a new formulation of the point Buleev method.

It turns out that the new formulation of the point Buleev method has several other advantages: it simplifies the proofs of factorizability conditions and convergence criteria considered in [1] and it makes possible to establish convergence criteria for arbitrary matrices, a problem which was left unsolved for schemes including the Buleev method, even in the case of diagonally dominant matrices [2].

Some auxiliary results will be established which give refinements of the factorizability conditions obtained in [1]; as a consequence, a new form of classical characterization properties of M-matrices, due to Fiedler and Pták [3], is obtained.

Finally, a new class of matrices will be introduced, the class of block-M-matrices and the extension of our analysis of block methods to this class of matrices will be indicated.

All notations used in the subsequent sections are taken from section 2 of ref. [1] and their definitions will not be reproduced here. Therefore we shall also keep the terminology defined in [1]; we learned however recently, and after completion of most parts of the present work, that the Woznicki methods have first been considered by Varga [14] who, moreover, provided a convergence analysis for Stieltjes matrices. It would thus be more appropriate to call these methods the Varga or Varga-Woznicki methods.

2. Point methods

2.1 The point Buleev method

The point Buleev method has been defined in ref. [1] by

$$E_1 = E \quad F_1 = F \quad (1)$$

$$P = (I + \alpha) D - \text{diag}(EP^{-1}FS) \quad (2)$$

where α is a diagonal matrix of arbitrary parameters, and S is the matrix whose entries are all equal to 1. Other notations are defined in section 2 of [1]. This definition is a generalization of the method used by Dupont Kendall and Rachford [4] for the multidimensional diffusion equation and the equivalence between this method and the method first introduced by Buleev [5] has been shown in [6].

If we define the diagonal matrices α' and β by

$$\alpha' = (\text{diag}(EP^{-1}FS) - \text{diag}(EP^{-1}F))D^{-1} \quad (3)$$

$$\beta = I + \alpha - \alpha' . \quad (4)$$

We obtain from Eq.(2),

$$P = \beta D - \text{diag}(EP^{-1}F) . \quad (5)$$

Therefore, an equivalent definition of the point Buleev method may be written

$$E_1 = E \quad F_1 = F \quad (6)$$

$$P = \beta D - \text{diag}(EP^{-1}F) \quad (7)$$

where $\beta = (\beta_{ii})$ is a diagonal matrix of given iteration parameters.

We shall henceforth use this latter definition. The corresponding splitting is given by

$$M = (P - E)P^{-1}(P - F) \quad (8)$$

$$N = (\beta - I)D + \text{offdiag}(EP^{-1}F) . \quad (9)$$

We have intentionally used the notation β in conjunction with the similar notation introduced in [1] for the Oliphant method. No confusion

can arise since, here, β is defined only for the diagonal entries while for the Oliphant method, β is defined only for the off-diagonal entries. In this way, the Oliphant and Buleev techniques appear as complementary relaxation procedures applied to the EWA method of Woznicki.

2.2 Composite methods

The definitions of the composite methods of Oliphant-Buleev, Buleev-Woznicki and Oliphant-Buleev-Woznicki are accordingly modified. We shall simply quote here the new definitions for reference purposes, using the notations of [1].

The Oliphant-Buleev method

$$E_1 = E_\beta \quad F_1 = F_\beta \quad (10)$$

$$P = D_\beta - \text{diag}(E_\beta P^{-1} F_\beta) \quad (11)$$

$$D_\beta = (\beta_{ii} a_{ii}) \quad i \in [1, n] \quad (12)$$

$$E_\beta = -(\beta_{is} a_{is}) \quad (i, s) \in V_E \quad (13)$$

$$F_\beta = -(\beta_{sj} a_{sj}) \quad (s, j) \in V_F \quad (14)$$

$$M = (P - E_\beta) P^{-1} (P - F_\beta) \quad (15)$$

$$N = (D_\beta - D) + (E - E_\beta) + (F - F_\beta) + \text{offdiag}(E_\beta P^{-1} F_\beta) \quad (16)$$

The Buleev-Woznicki method

$$E_1 = E + G \quad F_1 = F + H \quad (17)$$

$$P = D_\beta - \text{diag}((E + G) P^{-1} (F + H)) \quad (18)$$

$$G + H = \text{rest}_W((E + G) P^{-1} (F + H)) \quad (19)$$

$$D_\beta = (\beta_{ii} a_{ii}) \quad i \in [1, n] \quad (20)$$

$$M = (P - E - G) P^{-1} (P - F - H) \quad (21)$$

$$N = (D_\beta - D) + (\text{offdiag}((E + G) P^{-1} (F + H)) - (G + H)) \quad (22)$$

The Oliphant-Buleev-Woznicki method

$$E_1 = E_\beta + G \quad F_1 = F_\beta + H \quad (23)$$

$$P = D_\beta - \text{diag}((E_\beta + G)P^{-1}(F_\beta + H)) \quad (24)$$

$$G + H = \text{rest}_W((E_\beta + G)P^{-1}(F_\beta + H)) \quad (25)$$

$$D_\beta = (\beta_{ii} a_{ii}) \quad i \in [1, n] \quad (26)$$

$$E_\beta = -(\beta_{is} a_{is}) \quad (i, s) \in V_E \quad (27)$$

$$F_\beta = -(\beta_{sj} a_{sj}) \quad (s, j) \in V_F \quad (28)$$

$$M = (P - E_\beta - G)P^{-1}(P - F_\beta - H) \quad (29)$$

$$N = (D_\beta - D) + (E - E_\beta) + (F - F_\beta) \quad (30)$$

$$+ (\text{offdiag}((E_\beta + G)P^{-1}(F_\beta + H)) - (G + H))$$

3. Factorizability conditions and convergence criteria for point methods

3.1 Essentially non-positive matrices

Essentially non-positive matrices have been considered in [1] where sufficient criteria have been obtained. Necessary conditions will be considered here. For the sufficient criteria, it may be observed that the proofs given in [1] are much simplified by the use of the new definition of the Buleev method, and that the analysis presented there for composite methods including the Buleev scheme can be based on lemma 1 and lemma 3 of [1]. For comparisons with the new definition, it must be noted that the condition $\alpha \geq \alpha^0$ used in [1] has now to be replaced by $\beta_{ii} \geq 1$ for $i \in [1, n]$.

For the necessary conditions, some of our proofs will be based on a strengthened form of theorem 3.13 of Varga [7], first given by Ortega and Rheinboldt [8]. A similar theorem, slightly more general, was published a year later by Price [9]. A still more general formulation has been given by Vandergraft [10].

Summarizing the results which will be needed here we introduce

Definition 1

For $n \times n$ real matrices A , M and N , $A = M - N$ is a weak regular splitting of A if M is non-singular with $M^{-1} \geq 0$ and (at least) $M^{-1}N \geq 0$ or $NM^{-1} \geq 0$.

Then we have [8], [9], [10]:

Theorem 1

Let $A = M - N$ be a weak regular splitting of the (real) matrix A and $T = M^{-1}N$ be the associated iteration matrix. Then $\rho(T) < 1$ if and only if A is monotone.

31.1 The Oliphant method

According to Varga [7], an essentially non-positive matrix is a

real $n \times n$ matrix whose off-diagonal elements are non-positive.

A general criterion which insures that the diagonal matrix P of the Oliphant method is positive¹ has been obtained in [1] for this class of matrices: if A is an M -matrix and $0 \leq \beta_{ij} \leq 1$ for $(i,j) \in V$, then $P > 0$.

The same condition is not necessary as shown by the following example

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 6 \end{bmatrix} \quad \beta = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (31)$$

It is, however, a necessary condition for convergence and we have

Theorem 2

Let A be essentially non-positive and $0 \leq \beta_{ij} \leq 1$ for $(i,j) \in V$; then $P > 0$ and the Oliphant method converges if and only if A is an M -matrix.

Proof

For the sufficient condition, we refer to [1]; for the necessary condition, assume that $P > 0$; then

$$M = P(I - P^{-1}E_{\beta})(I - P^{-1}F_{\beta}) \quad (32)$$

$$M^{-1} = (I - P^{-1}F_{\beta})^{-1} (I - P^{-1}E_{\beta})^{-1} P^{-1} \quad (33)$$

and

$$N = (E - E_{\beta}) + (F - F_{\beta}) + \text{offdiag}(E_{\beta} P^{-1} F_{\beta}) \quad (34)$$

Thus, it is seen that $M^{-1} \geq 0$ and $N \geq 0$ when $P > 0$ and $0 \leq \beta_{ij} \leq 1$ for $(i,j) \in V$; in other words, the splitting $A = M - N$ is a regular splitting and the conclusion follows from theorem 1.

31.2 Composite methods

The same method of proof applies to composite methods and we have

¹We use here the convention of ref. [1]: for diagonal matrices, $P > 0$ means that $p_{ii} > 0$ for $i = 1, 2, \dots, n$.

the following result which completes theorem 6 of [1].

Theorem 3

Let A be essentially non-positive and $0 \leq \beta_{ij} \leq 1$ for $(i,j) \in V$ while $\beta_{ii} \geq 1$ for $i \in [1,n]$; then $P > 0$ and the Oliphant-Buleev-Woznicki method converges if and only if A is an M-matrix.

3.2 The Choleski factorization and the characterization of M-matrices

32.1 The Choleski factorization of M-matrices

As stated above, the positivity of P is generally not a necessary condition for the monotonicity of A . There exists, however, an exception, when the Woznicki method reduces to the Choleski factorization. More precisely we have:

Lemma 1

Let $A = K - L - U$ where $K \geq 0$, $L \geq 0$ and $U \geq 0$ are respectively diagonal, strictly lower and upper triangular matrices and let the diagonal, strictly lower and upper triangular matrices P , G , and H respectively be defined by

$$P = K - \text{diag}((L + G)P^{-1}(U + H)) \quad (35)$$

$$G + H = \text{offdiag}((L + G)P^{-1}(U + H)). \quad (36)$$

Then $P > 0$ if and only if A is an M-matrix.

Proof

For the sufficient condition, we refer to [1]; for the necessary condition, assume that $P > 0$; then, from (35) and (36), we have

$$G \geq 0 \quad \text{and} \quad H \geq 0$$

$$A = K - L - U = (P - L - G)P^{-1}(P - U - H) \quad (37)$$

and

$$A^{-1} = (P - U - H)^{-1} P (P - L - G)^{-1}. \quad (38)$$

But

$$(P - U - H)^{-1} = P \sum_{k=0}^{n-1} (P^{-1}(U + H))^k \quad (39)$$

$$(P - L - G)^{-1} = P \sum_{k=0}^{n-1} (P^{-1}(L + G))^k \quad (40)$$

and, therefore, $A^{-1} \geq 0$, which completes the proof.

An auxiliary consequence of this result is the possibility of establishing upper bounds for the leading principal minors of A:

Theorem 4

Let $A = K - L - U$ where $K \geq 0$, $L \geq 0$ and $U \geq 0$ are respectively diagonal, strictly lower and upper triangular matrices be an M-matrix and let D_i be its leading principal minors; then

$$D_i \leq \prod_{s=1}^i p_{ss} \quad (41)$$

where p_{ss} are the entries of a diagonal matrix P defined by

$$P = K - \text{diag}((L + G)P^{-1}(U + H)) \quad (42)$$

$$G + H = \text{rest}_W((L + G)P^{-1}(U + H)) \quad (43)$$

G and H are respectively strictly lower and upper triangular and W is an arbitrary set of off-diagonal indices. Moreover if W contains the Choleski set of A, then

$$D_i = \prod_{s=1}^i p_{ss} \quad (44)$$

Proof

Let A_i , P_i , L_i , U_i , G_i and H_i be the leading principal submatrices of A, P, L, U, G and H; if W is the complete set of off-diagonal indices, we have from Eq. (37)

$$D_i = \det(A_i) = \det(P_i - L_i - G_i) \det(P_i^{-1}) \det(P_i - U_i - H_i) = \prod_{s=1}^i p_{ss} \quad (45)$$

From lemma 3 of [1], we know that P is a non-increasing function of the set W; therefore

$$D_i \leq \prod_{s=1}^i p_{ss} \quad (46)$$

with equality when W is greater or equal to the Choleski set of A.

Corollary

Let A and A' be essentially non-negative matrices such that $A \leq A'$ and let D_i and D'_i be their leading principal minors.

If A is monotone, then A' is monotone and

$$D_i \leq D'_i \quad i = 1, 2, \dots, n \quad (47)$$

Proof

It follows from theorem 1 of [1] that A' is monotone; on the other hand, we have

$$D_i = \prod_{s=1}^i p_{ss} \quad \text{and} \quad D'_i = \prod_{s=1}^i p'_{ss} \quad (48)$$

where P and P' are defined by Eq. (35) and (36) applied to A and A' and, from lemma 6 of [1],

$$p_{ss} \leq p'_{ss} \quad s = 1, 2, \dots, n \quad (49)$$

which completes the proof.

32.2 The characterization of M-matrices

The subject of finding criteria which insure that an essentially non-positive matrix is an M-matrix, has been considered in much detail by Fiedler and Pták [3] and the criteria given by theorem 2, theorem 3 and lemma 1 can be added to their list.

Actually a closer connection can be established and it is possible to give, on the basis of lemma 1 and theorem 4, new forms of two criteria first obtained by Fiedler and Pták.

Let N denote the set of indices $1, 2, \dots, n$; consider matrices as real functions on $P \times P$ where P is some index set; if $M \subset N$ and if A is a matrix on $N \times N$, then let $A(M)$ denote the partial function of A on $M \times M$: it is the principal submatrix of A corresponding to M and $\det A(M)$ is the principal minor of A corresponding to M .

With these notations, we have from the work of Fiedler and Pták [3]:

Theorem 5

Let A be an essentially non-positive matrix. Then the following conditions are equivalent to each other:

- 1°/ there exists a vector $x \geq 0$ such that $Ax > 0$;
- 2°/ there exists a strictly increasing sequence $0 \neq M_1 \subset M_2 \subset \dots \subset M_n = N$ such that the principal minors $\det A(M_i)$ are positive;
- 3°/ there exists a permutation matrix P such that PAP^{-1} may be written in the form RS where R and S are respectively lower and upper essentially non-positive triangular matrices with positive diagonal elements.

Using the same notations, we have

Theorem 6

Let A be an essentially non-positive matrix. Then the following conditions are equivalent to each other:

- 1°/ there exists a vector $x \geq 0$ such that $Ax > 0$;
- 2°/ the leading principal minors of A are positive;
- 3°/ the matrix A may be written in the form RS where R and S are respectively lower and upper essentially non-positive triangular matrices with positive diagonal elements.

Proof

Implication 1° to 2° follows from theorem 4. Implication 3° to 1° was done in the proof of lemma 1 (and, actually, the lemma 1 states the equivalence between 1° and 3°). It remains to prove the implication 2° to 3° and this is the subject of the proposition (3.1) of Fiedler and Pták [3].

Remark

By using the method of proof of Fiedler and Pták, a direct proof of theorem 6 is obtained thus providing an alternate and much simpler proof of lemma 3 of [1].

3.3 Criteria for general matrices

Let A be an arbitrary complex matrix. Following Fiedler and Pták, we shall introduce the essentially non-negative matrix $P(A)$ defined as

$$P(A) = |K| - |L| - |U| \quad (50)$$

where

$$A = K - L - U \quad (51)$$

is the decomposition of A into its diagonal, strictly lower and strictly upper triangular parts respectively and $|K|$, $|L|$, $|U|$ denote the matrices whose entries are the moduli of the corresponding entries of K , L , U respectively.

Some of the following results generalize criteria first obtained by Beauwens and Borysiewicz [2] for diagonally dominant matrices. The generalization is twofold: first, the condition of strict or irreducible diagonal dominance is replaced by the more general condition of monotonicity; secondly, composite methods including the Buleev scheme are covered.

As in ref. [2], the proof of the convergence theorem for general matrices is based on theorem 2.8 of Varga [7] and we quote it here for reference purpose:

Theorem 7 (Varga)

Let A and B be two $n \times n$ matrices with $0 \leq |B| \leq A$. Then

$$\rho(B) \leq \rho(A) \quad (52)$$

Lemma 2

Let $A = K - L - U$ where K , L and U are respectively diagonal, strictly lower and upper triangular matrices, be an arbitrary complex matrix and let the diagonal, strictly lower and upper triangular matrices P , G and H respectively be defined by

$$P = K - \text{diag}((L + G)P^{-1}(U + H)) \quad (53)$$

$$G + H = \text{rest}_W((L + G)P^{-1}(U + H)) \quad (54)$$

where W is an arbitrary set of offdiagonal entries, and let P' , G' and H' be defined by the same relations applied to $|K|$, $|L|$ and $|U|$.

If $P(A)$ is an M -matrix, then P is non-singular; we have moreover

$$|P| \geq P' > 0 \quad |G| \leq G' \quad |H| \leq H' \quad (55)$$

Proof

Let p_{ii} , k_{ii} , l_{is} , g_{is} , u_{sj} and h_{sj} denote the entries of P , K , L , G , U and H respectively and let p'_{ii} , k'_{ii} , l'_{is} , g'_{is} , u'_{sj} and h'_{sj} denote the entries of P' , $|K|$, $|L|$, G' , $|U|$ and H' respectively.

From Eq. (53) and (54) we have

$$p_{ii} = k_{ii} - \sum_{s=1}^{i-1} \frac{(l_{is} + g_{is})(u_{si} + h_{si})}{p_{ss}} \quad (56)$$

$$g_{ki} = \begin{cases} \sum_{s=1}^{i-1} \frac{(l_{ks} + g_{ks})(u_{si} + h_{si})}{p_{ss}} & \text{if } (k,i) \in W \\ 0 & \text{if } (k,i) \notin W \end{cases} \quad k > i \quad (57)$$

$$h_{ik} = \begin{cases} \sum_{s=1}^{i-1} \frac{(l_{is} + g_{is})(u_{sk} + h_{sk})}{p_{ss}} & \text{if } (i,k) \in W \\ 0 & \text{if } (i,k) \notin W \end{cases} \quad k > i \quad (58)$$

and the same relations hold between p'_{ss} , k'_{ii} , l'_{is} , g'_{is} , u'_{sj} and h'_{sj} .

We prove by induction that

$$|p_{ss}| \geq p'_{ss} \quad (59)$$

$$|g_{ks}| \leq g'_{ks} \quad \text{and} \quad |h_{sk}| \leq h'_{sk} \quad \text{for } k > s \quad (60)$$

For $s = 1$, these relations are evident. Assume that they hold for $s \leq i - 1$ and consider $k > i$. If $(k, i) \notin W$, we have

$$|g_{ki}| = g'_{ki} = 0 \quad (61)$$

while if $(k, i) \in W$

$$|g_{ki}| \leq \sum_{s=1}^{i-1} \left| \frac{(l'_{ks} + g'_{ks})(u'_{si} + h'_{si})}{p_{ss}} \right| \leq \sum_{s=1}^{i-1} \frac{(l'_{ks} + |g_{ks}|)(u'_{si} + |h_{si}|)}{|p_{ss}|} \quad (62)$$

and, therefore, by the induction hypothesis

$$|g_{ki}| \leq \sum_{s=1}^{i-1} \frac{(l'_{ks} + g'_{ks})(u'_{si} + h'_{si})}{p'_{ss}} = g'_{ki} \quad (63)$$

Similarly, we have

$$|h_{ik}| \leq h'_{ik} \quad (64)$$

for $k > i$.

For p_{ii} , we have from Eq. (56)

$$\begin{aligned} |p_{ii}| &\geq |k_{ii}| - \left| \sum_{s=1}^{i-1} \frac{(l'_{is} + g'_{is})(u'_{si} + h'_{si})}{p_{ss}} \right| \geq k'_{ii} - \sum_{s=1}^{i-1} \left| \frac{(l'_{is} + g'_{is})(u'_{si} + h'_{si})}{p_{ss}} \right| \\ &\geq k'_{ii} - \sum_{s=1}^{i-1} \frac{(l'_{is} + |g_{is}|)(u'_{si} + |h_{si}|)}{|p_{ss}|} \end{aligned} \quad (65)$$

and, by the induction hypothesis

$$|p_{ii}| \geq k'_{ii} - \sum_{s=1}^{i-1} \frac{(l'_{is} + g'_{is})(u'_{si} + h'_{si})}{p'_{ss}} = p'_{ii} \quad (66)$$

The conclusion follows from lemma 3 of [1].

Corollary

Let A be an arbitrary complex matrix; if $P(A)$ is an M -matrix, then

A is non-singular and

$$|A^{-1}| \leq (P(A))^{-1} \quad (67)$$

Proof

Let the diagonal, strictly lower and upper triangular matrices P, G and H respectively be defined by

$$P = K - \text{diag}((L + G)P^{-1}(U + H)) \quad (68)$$

$$G + H = \text{offdiag}((L + G)P^{-1}(U + H)) \quad (69)$$

and let P' , G' and H' be defined by the same relation applied to $|K|$, $|L|$ and $|U|$.

From lemma 2, P is not singular so that these relations are meaningful. From lemma 3 of [1], the same conclusion holds for P' .

Thus we have successively

$$A = (P - L - G)P^{-1}(P - U - H) \quad (70)$$

$$A^{-1} = (I - P^{-1}(U + H))^{-1}(I - P^{-1}(L + G))^{-1}P^{-1} \quad (71)$$

$$A^{-1} = \left(\sum_{k=0}^{n-1} (P^{-1}(U + H))^k \right) \left(\sum_{l=0}^{n-1} (P^{-1}(L + G))^l \right) P^{-1} \quad (72)$$

and, therefore

$$|A^{-1}| \leq \left| \sum_{k=0}^{n-1} (P^{-1}(U + H))^k \right| \cdot \left| \sum_{l=0}^{n-1} (P^{-1}(L + G))^l \right| \cdot |P^{-1}| \quad (73)$$

thus also

$$|A^{-1}| \leq \left[\sum_{k=0}^{n-1} (|P^{-1}|(|U| + |H|))^k \right] \left[\sum_{l=0}^{n-1} (|P^{-1}|(|L| + |G|))^l \right] |P^{-1}| \quad (74)$$

and by lemma 2

$$|A^{-1}| \leq \left[\sum_{k=0}^{n-1} (P^{-1}(|U| + |H|))^k \right] \left[\sum_{l=0}^{n-1} (P^{-1}(|L| + |G|))^l \right] P^{-1} = (P(A))^{-1} \quad (75)$$

where the notations of lemma 2 have been used

Theorem 8

If $P(A)$ is an M-matrix, $0 \leq \beta_{ij} \leq 1$ for $(i,j) \in V$ and $\beta_{ii} \geq 1$ for $i \in [1,n]$, then the Oliphant-Buleev-Woznicki method is convergent.

Moreover its asymptotic convergence rate is not less than the asymptotic convergence rate of the same method applied to $P(A)$.

Proof

Let $A = M - N$ and $P(A) = M' - N'$ be the splittings of A and $P(A)$ relative to the Oliphant-Buleev-Woznicki scheme.

We have successively

$$M = P(I - P^{-1}(E_{\beta} + F))(I - P^{-1}(F_{\beta} + H)) \quad (76)$$

$$M^{-1} = (I - P^{-1}(F_{\beta} + H))^{-1} (I - P^{-1}(E_{\beta} + G))^{-1} P^{-1} \quad (77)$$

$$M^{-1} = \left[\sum_{k=0}^{n-1} (P^{-1}(F_{\beta} + H))^k \right] \left[\sum_{l=0}^{n-1} (P^{-1}(E_{\beta} + G))^l \right] P^{-1} \quad (78)$$

thus

$$|M^{-1}| \leq \left[\sum_{k=0}^{n-1} (|P^{-1}|(|F_{\beta}| + |H|))^k \right] \left[\sum_{l=0}^{n-1} (|P^{-1}|(|E_{\beta}| + |G|))^l \right] |P^{-1}| \quad (79)$$

and, therefore, by lemma 2

$$|M^{-1}| \leq \left[\sum_{k=0}^{n-1} (P^{-1}(F'_{\beta} + H'))^k \right] \left[\sum_{l=0}^{n-1} (P^{-1}(E'_{\beta} + G'))^l \right] P^{-1} = M'^{-1} \quad (80)$$

where the following notations are used

$$E'_{\beta} = |E|_{\beta} \quad (81)$$

$$F'_{\beta} = |F|_{\beta} \quad (82)$$

and P' , G' and H' are the matrices P , G and H of the Oliphant-Buleev-Woznicki method applied to $P(A)$.

On the other hand,

$$N = (D_\beta - D) + (E - E_\beta) + (F - F_\beta) + (\text{offdiag}((E_\beta + G)P^{-1}(F_\beta + H)) - (G + H)) \quad (83)$$

$$|N| \leq |D_\beta - D| + |E - E_\beta| + |F - F_\beta| + |\text{offdiag}((E_\beta + G)P^{-1}(F_\beta + H)) - (G + H)| \quad (84)$$

But, since $\beta_{ii} \geq 1$ and $0 \leq \beta_{ij} < 1$ for $(i, j) \in V$,

$$|D_\beta - D| = |D|_\beta - |D| = D'_\beta - D' \quad (85)$$

$$|E - E_\beta| = |E| - |E|_\beta = E' - E'_\beta \quad (86)$$

$$|F - F_\beta| = |F| - |F|_\beta = F' - F'_\beta \quad (87)$$

where

$$D' = |D| \quad E' = |E| \quad F' = |F| \quad (88)$$

From lemma 2, we have

$$|\text{offdiag}((E_\beta + G)P^{-1}(F_\beta + H))| \leq \text{offdiag}((E'_\beta + G')P'^{-1}(F'_\beta + H')) \quad (89)$$

and therefore also

$$|\text{offdiag}((E_\beta + G)P^{-1}(F_\beta + H)) - (G + H)| \leq \text{offdiag}((E'_\beta + G')P'^{-1}(F'_\beta + H')) - (G' + H') \quad (90)$$

since the two members of inequality (90) are the restrictions of the two members of inequality (89) to the complementary set of W with respect to the whole set of off-diagonal entries.

Thus

$$|N| \leq D'_\beta - D' + E' - E'_\beta + F' - F'_\beta + \text{offdiag}((E'_\beta + G')P'^{-1}(F'_\beta + H')) - (G' + H') = N' \quad (91)$$

From the relations (80) and (91), we obtain

$$|M^{-1}N| \leq |M^{-1}| |N| \leq M'^{-1}N' \quad (92)$$

Therefore, by theorem 7,

$$\rho(M^{-1}N) \leq \rho(M'^{-1}N') \quad (93)$$

and, by theorem 6 of [1],

$$\rho(M^{-1}N') < 1 \quad (94)$$

which completes the proof.

Remarks

1^o/ For practical applications, this theorem brings us back to the consideration of sufficient conditions which insure that a given essentially non-positive matrix is an M-matrix. As stated above, this problem has been considered in much detail by Fiedler and Pták and we refer the reader to ref. [3].

2^o/ The results given by theorem 7 for general matrices are weaker than the corresponding results given by theorem 6 of [1] for M-matrices since no intercomparison between the convergence rates is given. However combining these results together, it is seen that for general matrices the spectral radius of the associated iteration matrix has an upper bound which is non-increasing with β_{ij} for $(i, j) \in V$, non-decreasing with β_{ii} and non-increasing with W , when $(P(A))^{-1} \geq 0$, $0 \leq \beta_{ij} \leq 1$ for $(i, j) \in V$ and $\beta_{ii} \geq 1$.

3.4 Strict comparisons of asymptotic convergence rates

Comparisons of the asymptotic convergence rates of two factorization iterative methods were based in [1] on the following theorem of Woznicki [11]:⁽¹⁾

Theorem 9

Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$ equality excluded, then

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1 \quad (95)$$

the inequality being strict when $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$.

We have the following extension of this result:

Theorem 10

Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$ equality excluded, then

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1 \quad (96)$$

the inequality being strict when $A^{-1} > 0$ and

$$N_1(M_1^{-1} - M_2^{-1})N_2 \neq 0 \quad (97)$$

or

$$N_2(M_1^{-1} - M_2^{-1})N_1 \neq 0 \quad (98)$$

or

$$N_2 \neq N_1 = 0 \quad (99)$$

Proof

As observed by Woznicki [1], the relation

$$A^{-1}(N_2 - N_1)A^{-1} > 0 \text{ (resp } \geq \text{)} \quad (100)$$

implies

$$\rho(T_1) < \rho(T_2) < 1 \text{ (resp } \rho(T_1) \leq \rho(T_2) < 1 \text{)} \quad (101)$$

where $T_i = M_i^{-1}N_i$.

If $N_2 \neq N_1 = 0$, we have

$$A^{-1}(N_2 - N_1)A^{-1} = A^{-1}N_2A^{-1} > 0 \quad (102)$$

(1) This theorem was incorrectly reported in [1]; theorem 3 to 8 of [1] must be corrected accordingly.

when $A^{-1} > 0$, thus $\rho(T_1) < \rho(T_2) < 1$.

On the other hand, the following identity [11]

$$\begin{aligned} A^{-1}(N_2 - N_1)A^{-1} &= (M_1^{-1} - M_2^{-1}) + A^{-1}N_1(M_1^{-1} - M_2^{-1}) + (M_1^{-1} - M_2^{-1})N_2A^{-1} \\ &\quad + A^{-1}N_1(M_1^{-1} - M_2^{-1})N_2A^{-1} \end{aligned} \quad (103)$$

shows that, if $N_1(M_1^{-1} - M_2^{-1})N_2 \neq 0$, then $A^{-1}(N_2 - N_1)A^{-1} > 0$; thus, again, $\rho(T_1) < \rho(T_2) < 1$.

The last case is proved similarly, using the identity

$$\begin{aligned} A^{-1}(N_2 - N_1)A^{-1} &= (M_1^{-1} - M_2^{-1}) + A^{-1}N_2(M_1^{-1} - M_2^{-1}) + (M_1^{-1} - M_2^{-1})N_1A^{-1} \\ &\quad + A^{-1}N_2(M_1^{-1} - M_2^{-1})N_1A^{-1} \end{aligned} \quad (104)$$

In the applications of the latter result, we shall also need the following property of monotone matrices

Theorem 11

If A is monotone with positive inverse, then A is irreducible.

Proof

Indeed, if A is reducible, then using its normal form, it is immediately seen that A^{-1} is also reducible, thus not positive.

As it is well known (see Varga [7]), an irreducible M -matrix has a positive inverse; thus we have

Corollary

Let A be an M -matrix; then $A^{-1} > 0$ if and only if A is irreducible.

The following result is a complement of theorem 6 of [1]:

Theorem 12

Let A be an M -matrix, let β and T be the matrix of parameters and the iteration matrix of the Oliphant- Buleev-Woznicki method.

If $0 \leq \beta_{ij} < 1$ for $(i, j) \in V$, $\beta_{ii} \geq 1$ for $i \in [1, n]$, and $A^{-1} > 0$, then $\rho(T)$ is a decreasing function of β_{ij} for $(i, j) \in V$, an increasing function

of β_{ii} and a decreasing function of W .

Proof

Assume that

$$A = M - N = M' - N' \quad (105)$$

are the splittings corresponding to the Oliphant-Buleev-Woznicki methods applied to A with β, β' and W, W' as matrices of parameters and sets of off-diagonal entries respectively.

As observed in [1] (with $0 \leq \beta_{ij}, \beta'_{ij} \leq 1$ for $(i, j) \in V$ and $\beta_{ii}, \beta'_{ii} \geq 1$) if

$$\beta_{ij} \leq \beta'_{ij} \quad \text{for} \quad (i, j) \in V \quad (106)$$

$$\beta_{ii} \geq \beta'_{ii} \quad \text{for} \quad i \in [1, n] \quad (107)$$

$$W \subset W' \quad (108)$$

and if one of these relations holds equality excluded, then

$$M'^{-1} \geq M^{-1} \text{ equality excluded} \quad (109)$$

On the other hand, since $A^{-1} > 0$, A is irreducible; thus, if $0 \leq \beta_{ij} \leq \beta'_{ij} < 1$ for $(i, j) \in V$, then N and N' are also irreducible; therefore

$$N'(M'^{-1} - M^{-1})N \neq 0 \quad (110)$$

since the graphs of N and N' are strongly connected; thus $\rho(T') < \rho(T) < 1$ by theorem 10.

Remark

This property does not extend to the complete range $0 \leq \beta_{ij} \leq 1$ for $(i, j) \in V$ without additional assumptions, as shown by the following example

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \quad \beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \beta' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We have $A^{-1} > 0$, $\beta \leq \beta'$ with $\beta \neq \beta'$; however

$$\rho(T) = \rho(T') = \frac{1}{2}$$

4. Block methods

In this and the next sections, all matrices are assumed to be partitioned into submatrices indexed i, j with $i, j = 1, 2, \dots, n$, according to one and the same partitioning, the diagonal submatrices being square.

4.1 The block Buleev method

With the help of the new definition of the point Buleev method, it becomes a straightforward matter to generalize it as well as composite methods including the Buleev scheme to block factorization iterative processes.

The block Buleev method is defined by

$$E_1 = E \quad F_1 = F \quad (111)$$

$$P = D_\beta - \text{blockdiag} (EP^{-1} F) \quad (112)$$

$$D_\beta = (\beta_{ii} \cdot D_{ii}) \quad (113)$$

where β_{ii} denotes a block of parameters of the same dimensions as D_{ii} and $\beta_{ii} \cdot D_{ii}$ denotes the element by element product.⁽¹⁾ Other notations are taken from section 2 of ref. [1].

The corresponding splitting is given by

$$M = (P - E)P^{-1} (P - F) \quad (114)$$

$$N = D_\beta - D + \text{offblockdiag} (EP^{-1} F) \quad (115)$$

4.2 Composite methods

The composite block methods of Oliphant-Buleev, Buleev-Woznicki, and Oliphant-Buleev-Woznicki are introduced in the same way. It will be sufficient to quote their definitions, using the notations of ref [1]

The block Oliphant-Buleev method

$$E_1 = E_\beta \quad F_1 = F_\beta \quad (116)$$

$$P = D_\beta - \text{blockdiag} (E_\beta P^{-1} F_\beta) \quad (117)$$

(1) better known in the literature as the Hadamard product (cf Styan [15]).

$$D_{\beta} = (\beta_{ii} \cdot A_{ii}) \quad i \in [1, n] \quad (118)$$

$$E_{\beta} = -(\beta_{is} \cdot A_{is}) \quad (i, s) \in V_E \quad (119)$$

$$F_{\beta} = -(\beta_{sj} \cdot A_{sj}) \quad (s, j) \in V_F \quad (120)$$

$$M = (P - E_{\beta})P^{-1}(P - F_{\beta}) \quad (121)$$

$$N = (D_{\beta} - D) + (E - E_{\beta}) + (F - F_{\beta}) + \text{offblockdiag}(E_{\beta} P^{-1} F_{\beta}) \quad (122)$$

The block Buleev-Woznicki method

$$E_1 = E + G \quad F_1 = F + H \quad (123)$$

$$P = D_{\beta} - \text{blockdiag}((E + G)P^{-1}(F + H)) \quad (124)$$

$$G + H = \text{rest}_W((E + G)P^{-1}(F + H)) \quad (125)$$

$$D_{\beta} = (\beta_{ii} \cdot A_{ii}) \quad i \in [1, n] \quad (126)$$

$$M = (P - E - G)P^{-1}(P - F - H) \quad (127)$$

$$N = (D_{\beta} - D) + (\text{offblockdiag}((E + G)P^{-1}(F + H)) - (G + H)) \quad (128)$$

The block Oliphant-Buleev-Woznicki method

$$E_1 = E_{\beta} + G \quad F_1 = F_{\beta} + H \quad (129)$$

$$P = D_{\beta} - \text{blockdiag}((E_{\beta} + G)P^{-1}(F_{\beta} + H)) \quad (130)$$

$$G + H = \text{rest}_W((E_{\beta} + G)P^{-1}(F_{\beta} + H)) \quad (131)$$

$$D_{\beta} = (\beta_{ii} \cdot A_{ii}) \quad i \in [1, n] \quad (132)$$

$$E_{\beta} = -(\beta_{is} \cdot A_{is}) \quad (i, s) \in V_E \quad (133)$$

$$F_{\beta} = -(\beta_{sj} \cdot A_{sj}) \quad (s, j) \in V_F \quad (134)$$

$$M = (P - E_{\beta} - G)P^{-1}(P - F_{\beta} - H) \quad (135)$$

$$N = (D_{\beta} - D) + (E - E_{\beta}) + (F - F_{\beta}) + (\text{offblockdiag}((E_{\beta} + G)P^{-1}(F_{\beta} + H)) - (G + H)) \quad (136)$$

5. Factorizability conditions and convergence criteria for block methods

In order to avoid tedious repetitions, the notations of section 4 will be used in this and the next section without explicit reference.

5.1 Essentially non-positive matrices

We have the following extension of theorem 8 of [1].

Theorem 13

Let A be essentially non-positive, let the entries of β_{ij} for $(i,j) \in V$ belong to $[0,1]$, the diagonal entries b_{kk}^{ii} of β_{ii} belong to $[1,\infty]$ and the off-diagonal entries b_{kl}^{ii} of β_{ii} belong to $[0, b_{kk}^{ii}]$. Then, the following conditions

- (1) P is an M-matrix;
- (2) the block Oliphant-Buleev-Woznicki method is convergent;

are simultaneously satisfied if and only if A is an M-matrix.

Under the same assumptions, if A is an M-matrix and $T = M^{-1}N$ is the associated iteration matrix, then $\rho(T)$ is

- (1) a non-increasing function of β_{ij} for $(i,j) \in V$;
- (2) a non-decreasing function of the diagonal entries of β_{ii} ;
- (3) a non-increasing function of the off-diagonal entries of β_{ii} when its diagonal entries are kept constant;
- (4) a non-decreasing function of θ_i when all entries of β_{ii} are set equal to θ_i ;
- (5) a non-increasing function of W .

If, moreover, $A^{-1} > 0$, $(\beta_{ii} \cdot D_{ii})^{-1} > 0$ for $i \in [1,n]$, and $\{1\}$ is deleted from the ranges of the off-diagonal parameters, then $\rho(T)$ is:

- (1) a decreasing function of β_{ij} for $(i,j) \in V$;
- (2) an increasing function of the diagonal entries of β_{ii} ;
- (3) a decreasing function of the off-diagonal entries of β_{ii} when its diagonal entries are kept constant;
- (4) an increasing function of θ_i when all entries of β_{ii} are set equal to θ_i ;

(5) a decreasing function of W .

Proof

We first show that under the general assumptions of the theorem, if P is an M -matrix, then the splitting

$$A = M - N \quad (137)$$

which describes the block Oliphant-Buleev-Woznicki method is a weak regular splitting.

We have successively

$$M = (I - (E_\beta + G)P^{-1})(I - (F_\beta + H)P^{-1})P \quad (138)$$

$$M^{-1} = P^{-1}(I - (F_\beta + H)P^{-1})^{-1}(I - (E_\beta + G)P^{-1})^{-1} \quad (139)$$

$$M^{-1} = P^{-1} \left(\sum_{k=0}^{n-1} ((F_\beta + H)P^{-1})^k \right) \left(\sum_{\ell=0}^{n-1} ((E_\beta + G)P^{-1})^\ell \right) \quad (140)$$

If P is an M -matrix, we have

$$P^{-1} \geq 0 \quad (141)$$

and this relation, together with

$$G + H = \text{rest}_W((E_\beta + G)P^{-1}(F_\beta + H)) \quad (142)$$

and

$$E_\beta \geq 0 \quad F_\beta \geq 0 \quad (143)$$

entails

$$G \geq 0 \quad H \geq 0 \quad (144)$$

(by induction on the successive columns of G and rows of H). Thus

$$M^{-1} \geq 0 \quad (145)$$

On the other hand, the relation

$$P = D_\beta - \text{blockdiag}((E_\beta + G)P^{-1}(F_\beta + H)) \quad (146)$$

together with (141), (143) and (144) entails

$$P \leq D_\beta \quad (147)$$

Thus D_β is also an M -matrix (see theorem 1 of [1]) and

$$D_\beta P^{-1} \geq I \quad (148)$$

Consider now

$$N = (D_\beta - D) + (E - E_\beta) + (F - F_\beta) + \left(\text{off block diag}((E_\beta + G)P^{-1}(F_\beta + H)) - (G + H) \right) \quad (149)$$

and let B be the purely diagonal matrix (b_{kk}^{ii}) :

$$B = \text{diag}(\beta) \quad (150)$$

We have

$$D_\beta \geq BD \quad (151)$$

$$I \geq B^{-1} \geq DD_\beta^{-1} \quad (152)$$

Therefore, by (148),

$$(D_\beta - D)P^{-1} = (I - DD_\beta^{-1})D_\beta P^{-1} \geq 0 \quad (153)$$

and, thus

$$(D_\beta - D)M^{-1} \geq 0 \quad (154)$$

Since the entries of β_{ij} for $(i,j) \in V$ belong to $[0,1]$, we have

$$(E - E_\beta) \geq 0 \quad (F - F_\beta) \geq 0 \quad (155)$$

thus also

$$(E - E_\beta)M^{-1} \geq 0 \quad (F - F_\beta)M^{-1} \geq 0 \quad (156)$$

and it follows from (141), (142), (143) and (144) that the last term of the right hand side of (149) is also nonnegative; thus

$$NM^{-1} \geq 0 \quad (157)$$

and the splitting (137) is a weak regular splitting.

Therefore, if P is an M -matrix and the block Oliphant-Buleev-Woznicki method is convergent in the assumed range for the iteration parameters, then A is an M -matrix by theorem 1.

Conversely, if A is an M -matrix, then

$$A_\beta = D_\beta - E_\beta - F_\beta \geq BD - E - F \geq BA \quad (158)$$

is also an M -matrix (see theorem 1 of [1]), therefore P is an M -matrix by lemma 6 of [1] and the splitting (137) is a weak regular splitting; thus $\rho(T) < 1$ by theorem 1.

For the second part of the theorem, we observe that, if A is an M -matrix, then

$$AD^{-1} = MD^{-1} - ND^{-1} \quad (159)$$

is a regular splitting of AD^{-1} .

We have indeed

$$DM^{-1} = \left(\sum_{k=0}^{n-1} (DP^{-1}(F_{\beta}D^{-1} + HD^{-1}))^k \right) \left(\sum_{\ell=0}^{n-1} (DP^{-1}(E_{\beta}D^{-1} + GD^{-1}))^{\ell} \right) DP^{-1} \quad (160)$$

and

$$PD^{-1} = D_{\beta}D^{-1} - \text{blockdiag}((E_{\beta}D^{-1} + GD^{-1})DP^{-1}(F_{\beta}D^{-1} + HD^{-1})) \quad (161)$$

By (158), we have

$$A_{\beta}D^{-1} \geq B(AD^{-1}) \quad (162)$$

Thus $A_{\beta}D^{-1}$ is an M-matrix by theorem 1 of [1] and so is PD^{-1} by lemma 6 of [1]. Therefore, using (143) and (144),

$$DM^{-1} \geq 0 \quad (163)$$

On the other hand, we have, by (151)

$$D_{\beta}D^{-1} \geq B \geq I \quad (164)$$

$$D_{\beta}D^{-1} - I \geq 0 \quad (165)$$

and, therefore, using (149)

$$ND^{-1} \geq 0 \quad (166)$$

(1) Let β and β' satisfy the general assumptions of the theorem with

$\beta_{ii} = \beta'_{ii}$ and

$$\beta_{ij} \leq \beta'_{ij} \quad \text{for all } (i,j) \in V \quad (167)$$

Then

$$E_{\beta}D^{-1} \geq E_{\beta'}D^{-1} \quad F_{\beta'}D^{-1} \geq F_{\beta}D^{-1} \quad (168)$$

and, by lemma 6 of [1]

$$DP'^{-1} \geq DP^{-1}, \quad G'D^{-1} \geq GD^{-1}, \quad H'D^{-1} \geq HD^{-1} \quad (169)$$

thus, by (160),

$$DM'^{-1} \geq DM^{-1} \quad (170)$$

Therefore, by theorem 9,

$$\rho(DM'^{-1}N'D^{-1}) = \rho(M'^{-1}N') \leq \rho(DM^{-1}ND^{-1}) = \rho(M^{-1}N) \quad (171)$$

We have similarly, by (140),

$$M'^{-1} \geq M^{-1} \quad (172)$$

and, moreover if equality is excluded in (167) and if $(\beta_{ii} \cdot D_{ii})^{-1} > 0$ for $i \in [1, n]$, then equality is excluded in (172) since $p_{ii}^{-1} > 0$ for $i \in [1, n]$ by (147).

On the other hand $A^{-1} > 0$ implies that A is irreducible by theorem 11; if the off-diagonal parameters (the entries of β_{ij} for $(i, j) \in V$ and the elements $b_{k\ell}^{ii}$ for $k \neq \ell$) are $\neq 1$, then N and N' are also irreducible, therefore,

$$N'(M'^{-1} - M^{-1})N \neq 0 \quad (173)$$

thus also

$$N'D^{-1}(DM'^{-1} - DM^{-1})ND^{-1} \neq 0 \quad (174)$$

and

$$\rho(DM'^{-1}N'D^{-1}) = \rho(M'^{-1}N') < \rho(DM^{-1}ND) = \rho(M^{-1}N) \quad (175)$$

by theorem 12.

(2) and (3) Let β and β' satisfy the general assumptions of the theorem with $\beta_{ij} = \beta'_{ij}$ for all $(i, j) \in V$ and

$$\beta_{ii} \cdot D_{ii} \geq \beta'_{ii} \cdot D_{ii} \text{ for } i \in [1, n] \quad (176)$$

i.e.

$$D_{\beta} \geq D_{\beta'} \quad (177)$$

Then

$$D_{\beta}D^{-1} \geq D_{\beta'}D^{-1} \quad (178)$$

$$A_{\beta}D^{-1} \geq A_{\beta'}D^{-1} \quad (179)$$

and, by lemma 6 of [1],

$$DP'^{-1} \geq DP^{-1}, \quad G'D^{-1} \geq GD^{-1}, \quad H'D^{-1} \geq HD^{-1} \quad (180)$$

As in the first case, we have then

$$\rho(M'^{-1}N') \leq \rho(M^{-1}N) \quad (181)$$

We have similarly

$$M'^{-1} \geq M^{-1} \quad (182)$$

and, moreover, if equality is excluded in (176), then

$$P' \leq P \quad \text{with } P' \neq P \quad (183)$$

and, thus, equality is excluded in (182); the proof is concluded as in the first case.

(4) In this case we have

$$PD^{-1} = B - \text{blockdiag}((E_{\beta}D^{-1} + GD^{-1})DP^{-1}(F_{\beta}D^{-1} + HD^{-1})) \quad (184)$$

Again, by lemma 6 of [1], if

$$I \leq B' \leq B \quad (185)$$

we have

$$DP'^{-1} \geq DP^{-1}, \quad G'D^{-1} \geq GD^{-1}, \quad H'D^{-1} \geq HD^{-1} \quad (186)$$

thus also (170) and (171).

On the other hand

$$P = BD - \text{blockdiag}((E_{\beta} + G)P^{-1}(F_{\beta} + H)) \quad (187)$$

and

$$G + H = \text{rest}_W \left((E_{\beta} + G)P^{-1} (F_{\beta} + H) \right) \quad (188)$$

Then it is seen, by induction on the block entries of P , G and H that

$$B' \leq B \quad \text{with } B' \neq B \quad (189)$$

implies

$$P' \leq P \quad \text{with } P' \neq P, \quad G' \geq G, \quad H' \geq H \quad (190)$$

and thus

$$M'^{-1} \geq M^{-1} \quad \text{with } M'^{-1} \neq M^{-1} \quad (191)$$

The proof is concluded as in the first case.

(5) Let

$$W \subset W' \quad (192)$$

Then, by lemma 6 of [1],

$$DP'^{-1} \geq DP^{-1}, \quad G'D^{-1} \geq GD^{-1}, \quad H'D^{-1} \geq HD^{-1} \quad (193)$$

thus also (170) and (171).

We have similarly, by (140)

$$M'^{-1} \geq M^{-1} \quad (194)$$

and, moreover, if equality is excluded in (192) and if $(\beta_{ii} \cdot D_{ii})^{-1} > 0$ for $i \in [1, n]$, then equality is excluded in (194) since $P_{ii}^{-1} > 0$ for $i \in [1, n]$ by (147).

The proof is concluded as in the first case.

Remark

We may equivalently assume, in theorem 9 that the off-diagonal entries b_{kl}^{ii} of β_{ii} belong to $[0, b_{\ell\ell}^{ii}]$; the proof is similar, showing that under the general assumptions of the theorem, if P is an M-matrix then $M^{-1} \geq 0$ and $M^{-1}N \geq 0$. The last part of the theorem is proved in this case by showing that if A is an M-matrix,

$$D^{-1}A = D^{-1}M - D^{-1}N \quad (195)$$

is a regular splitting of $D^{-1}A$, and by applying a similar reasoning.

5.2 Further results on the characterization of M-matrices

52.1 The block Choleski factorization of M-matrices

We have the following extension of lemma 1:

Lemma 3

Let $A = K - L - U$ where K , L and U are respectively block-diagonal, strictly lower and upper block-triangular matrices, be essentially non-positive and let the block-diagonal, strictly lower and upper block-triangular matrices P , G and H respectively be defined by

$$P = K - \text{block-diag}((L + G)P^{-1}(U + H)) \quad (196)$$

$$G + H = \text{offblock-diag}((L + G)P^{-1}(U + H)) \quad (197)$$

Then P is an M-matrix if and only if A is an M-matrix.

Proof

For the sufficient condition, we refer to [1]; the necessary condition is a straightforward extension of the proof of lemma 1 and we refer to section 32.1.

Similarly, we have the following extension of theorem 4:

Theorem 14

Let $A = K - L - U$ where K , L and U are respectively block-diagonal, strictly lower and upper block-triangular matrices be essentially non-positive⁽¹⁾ and let D_i be its leading principal minors relative to the same partitioning; let the block-diagonal, strictly lower and upper block triangular matrices P , G and H respectively be defined by

$$P = K - \text{block-diag}((L + G)P^{-1}(U + H)) \quad (198)$$

$$G + H = \text{rest}_W((L + G)P^{-1}(U + H)) \quad (199)$$

where W is an arbitrary set of off-diagonal indices.

If A is an M -matrix, P is an M -matrix and

$$D_i \leq \prod_{s=1}^i \det(P_{ss}) \quad (200)$$

with equality if W contains the block-Choleski set of A , for the same partitioning.

Proof

From lemma 6 of [1], P is an M -matrix and, as in the proof of theorem 4, we have

$$D_i = \prod_{s=1}^i \det(P_{ss}) \quad (201)$$

when W contains the block-Choleski set of A .

From lemma 6 of [1] we know that P is a non-increasing function of the set W and the conclusion follows from the corollary of theorem 4.

(1) i. e., $L \geq 0$, $U \geq 0$ and K essentially non-positive.

52.2 A further characterization of M-matrices

We have the following extension of theorem 6:

Theorem 15

Let A be an essentially non-positive matrix. Then A is an M-matrix if and only if A may be written in the form RS where R and S are respectively lower and upper block-triangular M-matrices according to one and the same (but otherwise arbitrary) partitioning.

Proof

If A is an M-matrix, the conclusion follows from lemma 3 by putting

$$R = P - L - G \quad (202)$$

$$S = P^{-1}(P - U - H) \quad (203)$$

If, conversely, we have

$$A = RS \quad (204)$$

where R and S are M-matrices, then

$$A^{-1} = S^{-1} R^{-1} \quad (205)$$

is monotone since $S^{-1} \geq 0$ and $R^{-1} \geq 0$.

5.3 Criteria for general matrices

We have the following extension of lemma 2 to partitioned matrices:

Lemma 4

Let $A = K - L - U$ where K , L and U are respectively block-diagonal, strictly lower and upper block-triangular be an arbitrary partitioned matrix; let the block-diagonal, strictly lower and upper block-triangular matrices P , G and H respectively be defined by

$$P = K - \text{block-diag}((L + G)P^{-1}(U + H)) \quad (206)$$

$$G + H = \text{rest}_W((L + G)P^{-1}(U + H)) \quad (207)$$

where W is an arbitrary set of off-block-diagonal entries; let

$P(A) = K' - L' - U'$ be the decomposition of $P(A)$ in its block-diagonal, strictly lower and upper block-triangular parts according to the same partitioning and let P' , G' and H' be defined by (206) and (207) applied to K' , L' and U' .

If $P(A)$ is an M-matrix, then P is non-singular; we have moreover

$$|P^{-1}| \leq P'^{-1} \quad |G| \leq G' \quad |H| \leq H' \quad (208)$$

Proof

Let P_{ii} , K_{ii} , L_{is} , G_{is} , U_{sj} and H_{sj} denote the block-entries of the partitioned matrices P , K , L , G , U and H respectively; the relation defining P , G and H may be written

$$P_{ii} = K_{ii} - \sum_{s=1}^{i-1} (L_{is} + G_{is}) P_{ss}^{-1} (U_{si} + H_{si}) \quad (209)$$

$$G_{ki} = \begin{cases} \sum_{s=1}^{i-1} (L_{ks} + G_{ks}) P_{ss}^{-1} (U_{si} + H_{si}) & \text{if } (k,i) \in W \\ & k > i \\ 0 & \text{if } (k,i) \notin W \end{cases} \quad (210)$$

$$H_{ik} = \begin{cases} \sum_{s=1}^{i-1} (L_{is} + G_{is}) P_{ss}^{-1} (U_{sk} + H_{sk}) & \text{if } (i,k) \in W \\ & k > i \\ 0 & \text{if } (i,k) \notin W \end{cases} \quad (211)$$

and similar relations hold between P'_{ii} , K'_{ii} , L'_{is} , G'_{is} , U'_{sj} and H'_{sj} , the block-entries of P' , K' , L' , G' , U' and H' .

We show by induction that

$$P(P_{ss}) \geq P'_{ss} \quad (212)$$

$$|G_{ks}| \leq G'_{ks} \quad \text{and} \quad |H_{sk}| \leq H'_{sk} \quad \text{for } k > s \quad (213)$$

For $s = 1$, the proposition is evident since

$$P(P_{11}) = P(K_{11}) = P'_{11} \quad (214)$$

$$G_{k1} = G'_{k1} = 0 \quad k > 1 \quad (215)$$

$$H_{1k} = H'_{1k} = 0 \quad k > 1 \quad (216)$$

Assume that these relations hold for $s \leq i - 1$; then applying the corollary of lemma 2 to the induction hypothesis (212), we conclude that P_{ss} is not singular and

$$|P_{ss}^{-1}| \leq (P(P_{ss}))^{-1} \quad (217)$$

and, by theorem 1 of [1],

$$(P(P_{ss}))^{-1} \leq P_{ss}^{-1} \quad (218)$$

Thus, the relation (210) is meaningful and we have successively, for $k > i$, $(k, i) \in W$

$$|G_{ki}| \leq \sum_{s=1}^{i-1} (|L_{ks}| + |G_{ks}|) |P_{ss}^{-1}| (|U_{si}| + |H_{si}|) \quad (219)$$

or

$$|G_{ki}| \leq \sum_{s=1}^{i-1} (L'_{ks} + |G_{ks}|) |P_{ss}^{-1}| (U'_{si} + |H_{si}|) \quad (220)$$

and, applying (217), (218) and (213),

$$|G_{ki}| \leq \sum_{s=1}^{i-1} (L'_{ks} + G'_{ks}) P_{ss}^{-1} (U'_{si} + H'_{si}) = G'_{ki} \quad (221)$$

On the other hand, if $k > i$ and $(k, i) \notin W$, we have

$$G_{ki} = G'_{ki} = 0 \quad (222)$$

The same method of proof applies to H_{ik} for $k > i$, and to the off-diagonal elements of P_{ii} , thus showing that

$$\text{offdiag } |P_{ii}| \leq \text{offdiag } |P'_{ii}| \quad (223)$$

For the diagonal elements of P_{ii} , we have

$$\text{diag} |P_{ii}| \geq \text{diag} |K_{ii}| - \sum_{s=1}^{i-1} \text{diag} |(L_{is} + G_{is}) P_{ss}^{-1} (U_{si} + H_{si})| \quad (224)$$

and the same method of proof applies to the second term of the right hand side of this inequality; thus

$$\text{diag} |P_{ii}| \geq \text{diag} (P'_{ii})$$

and, from (223) and (224) we conclude that

$$P(P_{ii}) \geq P'_{ii} \quad (225)$$

Therefore we have

$$P(P) \geq P' \quad (226)$$

and, applying the corollary of lemma 2 and theorem 1 of [1], we have

$$|P^{-1}| \leq (P(P))^{-1} \leq P'^{-1} \quad (227)$$

Theorem 16

If $P(A)$ is an M-matrix, if the entries of β_{ij} for $(i,j) \in V$ belong to $[0,1]$, the diagonal entries b_{kk}^{ii} of β_{ii} belong to $[1,\infty]$ and the off-diagonal entries b_{kl}^{ii} of β_{ii} belong to $[0,1]$, then the block-matrix P of the block Oliphant-Buleev-Woznicki method is non-singular and the method is convergent. Moreover its convergence rate is not less than the convergence rate of the same method applied to $P(A)$.

Proof

Let $A = M - N$ and $P(A) = M' - N'$ be the splitting of A and $P(A)$ relative to the block Oliphant-Buleev-Woznicki scheme.

As in the proof of theorem 8, but with $D' = P(D)$ and $D'_\beta = P(D_\beta) = (P(D))_\beta$ it is seen that

$$|M^{-1}| \leq M'^{-1} \quad (228)$$

$$|N| \leq N' \quad (229)$$

Therefore

$$|M^{-1} N| \leq M^{-1} N' \quad (230)$$

and, by theorem 7

$$\rho(M^{-1} N) \leq \rho(M^{-1} N') \quad (231)$$

The conclusion follows from theorem 13.

Remark

A similar property holds if we assume that the entries of each block β_{ij} are all equal to one and the same number θ_{ij} with $0 \leq \theta_{ij} \leq 1$ for $(i,j) \in V$ and $\theta_{ii} \geq 1$ for $i \in [1,n]$. (cf theorem 21 below).

6 Block-M-matrices

6.1 An extension of a theorem of Ky Fan

We investigate here the possible extension of our results for block-methods to a larger class of matrices, the block-M-matrices, whose definition follows.

We refer to [1] for the definition and necessary properties of monotone matrices, M-matrices and regular splittings.

By a $n \times n$ partitioned matrix, we mean any matrix $A = (A_{ij})$ partitioned into submatrices A_{ij} with $i, j = 1, 2, \dots, n$, the diagonal submatrices being square.

Definition

A real $n \times n$ partitioned matrix $A = (A_{ij})$ with $A_{ij} \leq 0$ for $i \neq j$ is a block-M-matrix if A and A_{ii} are monotone for $i = 1, 2, \dots, n$.

The following theorem gives a characterization of block-M-matrices; it is an extension of a theorem of Ky Fan [12] and Vandergraft [10].

Theorem 17

Let $A = (A_{ij})$ be a real $n \times n$ partitioned matrix such that $A_{ij} \leq 0$ for $i \neq j$. Then A is a block-M-matrix if and only if:

- 1°/ there exists a non-singular, non-negative, block-diagonal matrix B (according to the same $n \times n$ partitioning) such that $BA \leq I$;
- 2°/ there exists $x \geq 0$ such that $Ax > 0$.

Remark

Under the hypotheses of the theorem, it is easily seen that the 2° is equivalent to: there exists $x > 0$ such that $Ax > 0$.

Proof

If A is a block-M-matrix, let $B = K^{-1}$ where

$$K = \text{block-diag}(A) \quad (232)$$

On the other hand, let $x = A^{-1}y$ for any $y > 0$; since $A^{-1} \geq 0$, we have then $x \geq 0$ and $Ax > 0$.

Conversely, if there exists a non-singular $B \geq 0$ such that $BA \leq I$ and a vector $x \geq 0$ such that $Ax > 0$, then by applying the interior mapping principle (see [13], chap. II, section 2.1), we have also $BAx > 0$; thus

$$0 \leq (I - BA)x < x \quad (233)$$

Since $(I - BA) \geq 0$, there exists a non-negative eigenvector x_0 of $(I - BA)$ associated with $\rho(I - BA)$, by theorem 2.7 of Varga [7].

We now prove that $\rho(I - BA) < 1$; therefore we may assume that $\rho(I - BA) > 0$. Then, following Vandergraft's method of proof [10], let

$$t_0 = \sup(t > 0; \quad tx_0 \leq x) \quad (234)$$

Since x is strictly positive by (170), such a number t_0 exists and is finite. Then

$$\rho(I - BA)t_0 x_0 = t_0(I - BA)x_0 \leq (I - BA)x < x \quad (235)$$

But t_0 is the supremum of the $t > 0$ such that $tx_0 \leq x$; thus

$$\rho(I - BA)t_0 < t_0 \quad (236)$$

and

$$\rho(I - BA) < 1 \quad (237)$$

Therefore, the Neumann series in powers of $(I - BA)$ is convergent and defines $(BA)^{-1}$

$$(BA)^{-1} = \sum_{k=0}^{\infty} (I - BA)^k \quad (238)$$

Since each term of the series is non-negative, we have $(BA)^{-1} \geq 0$ and, multiplying by $B \geq 0$,

$$A^{-1} = (BA)^{-1}B \geq 0 \quad (239)$$

Thus A is monotone.

Since $A_{ij} \leq 0$ for $i \neq j$ and B is block-diagonal and non-negative,

our initial assumptions on A are also true for $K = \text{block-diag}(A)$; therefore, the same conclusion holds: K is monotone.

Thus A is a block-M-matrix.

Corollary

Let $A = (A_{ij})$ and $A' = (A'_{ij})$ be real $n \times n$ partitioned matrices such that $A'_{ij} \leq 0$ for $i \neq j$, $A \leq A'$ and A is monotone.

Then A' is a block-M-matrix if and only if there exists a non-singular, non-negative, block-diagonal matrix B (according to the same partitioning) such that $BA' \leq I$.

Proof

The necessary condition is part of theorem 17; on the other hand, if A is monotone, then, by applying theorem 17 to the matrix A considered as one block, there exists a vector $x \geq 0$, such that $Ax > 0$.

Since $A \leq A'$ we have also

$$A'x \geq Ax > 0 \quad (240)$$

and the conclusion follows from theorem 17.

Remark

As a consequence, if $A = (A_{ij})$ is a monotone, $n \times n$ partitioned matrix with $A_{ij} \leq 0$ for $i \neq j$, then A itself is a block-M-matrix if and only if there exists a non-singular, non-negative, block-diagonal matrix B such that $BA \leq I$.

6.2 Factorizability conditions and convergence criteria

We have the following extension of lemma 6 of [1] to the class of block-M-matrices.

Lemma 5

Let $A = K - L - U$ where $K^{-1} \geq 0$, $L \geq 0$ and $U \geq 0$ are respectively block-diagonal, strictly lower and upper block-triangular matrices and let the block-diagonal, strictly lower and upper block-triangular matrices P , G and H be defined by

$$P = K - \text{blockdiag}((L + G)P^{-1}(U + H)) \quad (241)$$

$$G + H = \text{rest}_W((L + G)P^{-1}(U + H)) \quad (242)$$

where W is a given set of offblockdiagonal block-entries (deleting zero entries).

If A is a block-M-matrix, P is monotone and, thus, G and H are non-negative; moreover P is a non-decreasing function of A (as far as A remains a block-M-matrix) and a non-increasing function of W ; G and H are non-increasing function of A (as far as A remains a block-M-matrix) and $G + H$ is an increasing function of W .

Proof

The proof is very similar to the proof of lemma 6 of [1] and will not be reproduced. The only difference consists of showing that the first condition of theorem 17 is satisfied for P whenever it is satisfied for A , and this is obvious since $P \leq K$.

As a consequence, we have the following convergence theorem for the block Oliphant-Woznicki method.

Theorem 18

Let $A = D - E - F$ be a real $n \times n$ partitioned matrix where D , E and F are respectively block-diagonal, strictly lower and upper block-triangular with $E \geq 0$ and $F \geq 0$. Let the entries of the blocks of parameters β_{ij} , $(i, j) \in V$ of the block Oliphant-Woznicki method belong to $[0, 1]$.

Then, the following conditions

(1) D and P are monotone;

(2) the block Oliphant-Woznicki method converges;

are simultaneously satisfied if and only if A is a block-M-matrix.

Under the same assumptions, if A is a block-M-matrix and T is the iteration matrix associated with the block Oliphant-Woznicki method, then $\rho(T)$ is a

non-increasing function of β_{ij} , $(i, j) \in V$ in this range and a non-increasing function of W . If moreover $A^{-1} > 0$, $D_{ii}^{-1} > 0$ for $i \in [1, n]$ and the entries of β_{ij} , $(i, j) \in V$ belong to $[0, 1[$, then $\rho(T)$ is a decreasing function of β_{ij} and of W .

Proof

The proof is similar to the proof of theorem 13; therefore we shall simply indicate the main steps

It is first shown that if the condition (1) is satisfied, then the splitting

$$A = M - N \quad (243)$$

associated with the block Oliphant-Woznicki method is a regular splitting.

The first part of the theorem is then a consequence of theorem 1, and the last parts are proven by applying theorems 9 and 10.

The extension of this theorem to block methods including the block Buleev scheme will be helped by introducing the following notations. Let D_{ii}^+ , D_{ii}^- , β_{ii}^+ and β_{ii}^- be defined by

$$D_{ii} = D_{ii}^+ - D_{ii}^- \quad , \quad D_{ii}^+ \geq 0 \quad , \quad D_{ii}^- \geq 0 \quad (244)$$

$$\beta_{ii} = \beta_{ii}^+ + \beta_{ii}^- \quad (245)$$

$$\beta_{ii} \cdot D_{ii} = \beta_{ii}^+ \cdot D_{ii}^+ - \beta_{ii}^- \cdot D_{ii}^- \quad (246)$$

In other words, the non-zero entries of D_{ii}^+ (D_{ii}^-) are the positive (negative) entries of D_{ii} ; the non-zero entries of β_{ii}^+ (β_{ii}^-) are the entries of β_{ii} which multiply the positive (negative) entries of D_{ii} . We have then:

Theorem 19

Let $A = D - E - F$ be a real $n \times n$ partitioned matrix where D is block-diagonal, $E \geq 0$ and $F \geq 0$ are respectively strictly lower and upper block-triangular; let β and W be the matrix of parameters and the set of off-diagonal

block-entries associated with the block Oliphant-Buleev-Woznicki method.

Let the entries of the blocks β_{ij} , $(i, j) \in V$ belong to $[0, 1]$; let the entries of β_{ii}^+ belong to $[1, \infty]$ and for any line k of the block β_{ii} , let b_k^i be a minimal non-zero entry of β_{ii}^+ on that line; let the entries of the k^{th} line of β_{ii}^- belong to $[0, b_k^i]$ and assume that $(\beta_{ii} \cdot D_{ii})^{-1} \geq 0$.

Then the following conditions

(1) P is monotone

(2) the block Oliphant-Buleev-Woznicki method is convergent;

are simultaneously satisfied if and only if A is a block-M-matrix.

Under the same assumptions, if A is a block M-matrix and $T = M^{-1}N$ is the associated iteration matrix, then $\rho(T)$ is

(1) a non-increasing function of β_{ij} for $(i, j) \in V$;

(2) a non-decreasing function of β_{ii}^+ ;

(3) a non-increasing function of β_{ii}^- when b_k^i is kept constant;

(4) a non-decreasing function of θ_i when all the entries of β_{ii} are set equal to θ_i ;

(5) a non-increasing function of W .

If moreover $A^{-1} > 0$, $(\beta_{ii} \cdot D_{ii})^{-1} > 0$ for $i \in [1, n]$ and $\{1\}$ is deleted from the ranges of the entries of β_{ij} for $(i, j) \in V$ and of β_{ii}^- , then $\rho(T)$ is

(1) a decreasing function of β_{ij} for $(i, j) \in V$;

(2) an increasing function of β_{ii}^+ ;

(3) a decreasing function of β_{ii}^- when b_k^i is kept constant;

(4) an increasing function of θ_i when all the entries of β_{ii} are set equal to θ_i ;

(5) a decreasing function of W .

Proof

Let

$$A = M - N \quad (247)$$

be the splitting associated with the block Oliphant-Buleev-Woznicki method;
we first show that if P is monotone then

$$AD_{\beta}^{-1} = MD_{\beta}^{-1} - ND_{\beta}^{-1} \quad (248)$$

is a regular splitting of AD_{β}^{-1} .

We have indeed

$$D_{\beta} M^{-1} = \left(\sum_{k=0}^{n-1} (D_{\beta} P^{-1} (F_{\beta} D_{\beta}^{-1} + HD_{\beta}^{-1}))^k \right) \left(\sum_{\ell=0}^{n-1} (D_{\beta} P^{-1} (E_{\beta} D_{\beta}^{-1} + GD_{\beta}^{-1}))^{\ell} \right) D_{\beta} P^{-1} \quad (249)$$

$$GD_{\beta}^{-1} + HD_{\beta}^{-1} = \text{rest}_W \left((E_{\beta} D_{\beta}^{-1} + GD_{\beta}^{-1}) D_{\beta} P^{-1} (F_{\beta} D_{\beta}^{-1} + HD_{\beta}^{-1}) \right) \quad (250)$$

$$P \leq D_{\beta} \quad (251)$$

and, since $D_{\beta}^{-1} \geq 0$

$$E_{\beta} D_{\beta}^{-1} \geq 0 \quad F_{\beta} D_{\beta}^{-1} \geq 0 \quad (252)$$

Therefore, if P is monotone,

$$D_{\beta} P^{-1} \geq 0 \quad (253)$$

and then, by the preceding relations

$$D_{\beta} M^{-1} \geq 0 \quad (254)$$

On the other hand

$$\begin{aligned} ND_{\beta}^{-1} &= (I - DD_{\beta}^{-1}) + (E - E_{\beta})D_{\beta}^{-1} + (F - F_{\beta})D_{\beta}^{-1} \\ &\quad + \left(\text{off block diag} \left((E_{\beta} D_{\beta}^{-1} + GD_{\beta}^{-1}) D_{\beta} P^{-1} (F_{\beta} D_{\beta}^{-1} + HD_{\beta}^{-1}) \right) - (GD_{\beta}^{-1} + HD_{\beta}^{-1}) \right) \end{aligned} \quad (255)$$

and, letting B be the purely diagonal matrix whose entry on line k of block (i, i) is b_k^i , we have

$$D_{\beta} \geq BD \quad (256)$$

thus

$$I \geq B^{-1} \geq DD_{\beta}^{-1} \quad (257)$$

$$I - DD_{\beta}^{-1} \geq 0 \quad (258)$$

Therefore

$$ND_{\beta}^{-1} \geq 0 \quad (259)$$

As a consequence,

$$NM^{-1} = ND_{\beta}^{-1} D_{\beta} M^{-1} \geq 0 \quad (260)$$

and the splitting (247) is a weak regular splitting.

Therefore, if P is monotone and $\rho(M^{-1}N) < 1$, then A is monotone by theorem 1. On the other hand, if A is monotone, there exists $x \geq 0$ such that $Ax > 0$ by theorem 17; since

$$A \leq D \quad (261)$$

and, by (256)

$$D_{\beta}^{-1}BD \leq I \quad (262)$$

We have also

$$D_{\beta}^{-1}BA \leq D_{\beta}^{-1}BD \leq I \quad (263)$$

thus A is a block-M-matrix by theorem 17.

Conversely, if A is a block-M-matrix, then

$$A = D - (E + F) \quad (264)$$

is a regular splitting and, by theorem 1,

$$\rho(D^{-1}(E + F)) < 1 \quad (265)$$

But

$$A_{\beta} = D_{\beta} - (E_{\beta} + F_{\beta}) \quad (266)$$

is also a regular splitting with

$$E_{\beta} + F_{\beta} \leq E + F \quad (267)$$

and, by (256) since $D^{-1} \geq 0$,

$$D_{\beta}^{-1} \leq D^{-1} \quad (268)$$

Therefore

$$\rho(D_{\beta}^{-1}(E_{\beta} + F_{\beta})) \leq \rho(D^{-1}(E + F)) < 1 \quad (269)$$

and A_{β} is monotone (thus a block-M-matrix) by theorem 1. Therefore P is monotone by lemma 5 and $\rho(M^{-1}N) < 1$ by theorem 1.

For the last parts of the proof, we observe that, if A is a block-M-matrix, then A_{β} is also a block-M-matrix as observed above; thus

$$A_{\beta} D^{-1} \quad (270)$$

is a block-M-matrix and

$$PD^{-1} \quad (271)$$

is monotone by lemma 5; then it is seen that

$$AD^{-1} = MD^{-1} - ND^{-1} \quad (272)$$

is a regular splitting of AD^{-1} .

The proof is concluded as that of theorem 13.

Remarks

1. As in the case of theorem 13, the ranges of the entries of β_{ii} may also be defined by using the columns of β_{ii} in place of the lines.
2. If all the entries of β_{ii} are equal to one and the same number $\theta_i \geq 1$, then clearly the condition $(\beta_{ii} \cdot D_{ii})^{-1} \geq 0$ (or > 0) is equivalent to $D_{ii}^{-1} \geq 0$ (or > 0).

6.3 The characterization of block-M-matrices

The following results will be stated without proofs whenever these are similar to the proofs of the corresponding results of M-matrices.

Lemma 6

Let $A = K - L - U$ where $K^{-1} \geq 0$, $L \geq 0$ and $U \geq 0$ are respectively block-diagonal, strictly lower and upper block-triangular matrices and let the block-diagonal, strictly lower and upper block-triangular matrices P , G and H respectively be defined by

$$P = K - \text{block-diag}((L + G)P^{-1}(U + H)) \quad (273)$$

$$G + H = \text{off block-diag}((L + G)P^{-1}(U + H)) \quad (274)$$

Then P is monotone if and only if A is a block-M-matrix.

Theorem 19

Let $A = K - L - U$ where $K^{-1} \geq 0$, $L \geq 0$ and $U \geq 0$ are respectively block-diagonal, strictly lower and upper block-triangular matrices and let D_i , $i = 1, 2, \dots, n$, be the leading principal minors of A relative to the same partitioning; let the block-diagonal, strictly lower and upper block-triangular matrices P , G and H respectively be defined by

$$P = K - \text{block-diag}((L + G)P^{-1}(U + H)) \quad (275)$$

$$G + H = \text{rest}_W((L + G)P^{-1}(U + H)) \quad (276)$$

where W is an arbitrary set of off-diagonal indices.

If A is a block-M-matrix, P is monotone and

$$D_i \leq \prod_{s=1}^i \det(P_{ss}) \quad (277)$$

with equality if W contains the block-Choleski set of A , for the same partitioning.

Theorem 20

Let $A = K - L - U$ where $K^{-1} \geq 0$, $L \geq 0$ and $U \geq 0$ are respectively block-diagonal, strictly lower and upper block-triangular matrices.

Then the following conditions are equivalent to each other:

1^o/ there exists a vector $x \geq 0$ such that $Ax > 0$;

2^o/ there exists a vector $x > 0$ such that $Ax > 0$;

3^o/ the inverse A^{-1} of A exists and $A^{-1} \geq 0$;

4^o/ the matrix A may be written in the form RS where R and S are respectively lower and upper block-triangular block-M-matrices according to the given partitioning;

5^o/ the matrix $K^{-1}A$ is an M-matrix.

Remark

From theorem 20, 5^o, it is seen that all the criteria set up by Fiedler and Pták [3] for M-matrices can possibly be used for identifying block-M-matrices.

6.4 Criteria for arbitrary complex matrices

On the basis of theorem 20, 5^o, we have the following extension of lemma 4 to the case of block-M-matrices:

Lemma 7

Let $A = K - L - U$ be the decomposition of the arbitrary $n \times n$ partitioned complex matrix A into its block-diagonal, strictly lower and upper block-triangular parts; let the block-diagonal, strictly lower and upper block-triangular matrices P , G and H respectively be defined by

$$P = K - \text{blockdiag}((L + G)P^{-1}(U + H)) \quad (278)$$

$$G + H = \text{rest}_W((L + G)P^{-1}(U + H)) \quad (279)$$

where W is an arbitrary set of off-block-diagonal entries; let K be non singular and

$$P(K^{-1}A) = I - L_1 - U_1 \quad (280)$$

be the decomposition of $P(K^{-1}A)$ into its block-diagonal, strictly

lower and upper block-triangular parts, according to the same partitioning; let P_1 , G_1 and H_1 be defined by (278) and (279) applied to I , L_1 and U_1 .

If $P(K^{-1}A)$ is an M-matrix, then P is non-singular; we have moreover

$$|P^{-1}K| \leq P_1^{-1} \quad |K^{-1}G| \leq G_1 \quad |K^{-1}H| \leq H_1 \quad (281)$$

Proof

From Eq. (278) and (279), it is seen that

$$\tilde{P} = I - \text{block-diag}((\tilde{L} + \tilde{G})\tilde{P}^{-1}(\tilde{U} + \tilde{H})) \quad (282)$$

$$\tilde{G} + \tilde{H} = \text{rest}_W((\tilde{L} + \tilde{G})\tilde{P}^{-1}(\tilde{U} + \tilde{H})) \quad (283)$$

where

$$\tilde{L} = K^{-1}L \quad \tilde{U} = K^{-1}U \quad (284)$$

$$\tilde{P} = K^{-1}P \quad \tilde{G} = K^{-1}G \quad \tilde{H} = K^{-1}H \quad (285)$$

Since we have

$$L_1 = |\tilde{L}| \quad U_1 = |\tilde{U}| \quad (286)$$

the proposition follows from lemma 4.

Corollary

Let A be an arbitrary partitioned complex matrix with non-singular block-diagonal K ; if $P(K^{-1}A)$ is an M-matrix, then A is non-singular and

$$|A^{-1}K| \leq (P(K^{-1}A))^{-1} \quad (287)$$

Theorem 21

Let A be an arbitrary $n \times n$ partitioned complex matrix with non-singular block-diagonal D ; if $P(D^{-1}A)$ is an M-matrix, if the entries of each block β_{ij} are all equal to one and the same number θ_{ij} with $0 \leq \theta_{ij} \leq 1$ for $(i, j) \in V$ and $\theta_{ii} \geq 1$ for $i \in [1, n]$, then the block-diagonal matrix P of the block Oliphant-Buleev-Woznicki method is non-singular and the method is convergent.

Moreover its convergence rate is not less than the convergence rate

of the same method applied to $P(D^{-1}A)$.

Proof

Let $A = M - N$ be the corresponding splitting; let

$$P(D^{-1}A) = M_1 - N_1 \quad (288)$$

be the splitting corresponding to the same method, applied to $P(D^{-1}A)$.

As in the proof of theorem 8, we have, by lemma 7,

$$|M^{-1}D| \leq M_1^{-1} \quad (289)$$

and

$$|D^{-1}N| \leq N_1 \quad (290)$$

Therefore

$$|M^{-1}N| = |M^{-1}D D^{-1}N| \leq |M^{-1}D| |D^{-1}N| \leq M_1^{-1}N_1 \quad (291)$$

and, by theorem 7,

$$\rho(T) \leq \rho(T_1) \quad (292)$$

where $T = M^{-1}N$ and $T_1 = M_1^{-1}N_1$.

The conclusion follows from theorem 13.

7 Concluding remarks

In this report we have obtained factorizability conditions and convergence criteria of point and block factorization iterative methods for general matrices.

These developments have been obtained by theoretical extensions of our previous results for essentially non-negative matrices [1] and they give, as auxiliary results, some new forms of characterization properties of M-matrices obtained by Fiedler and Ptak [3].

For general matrices, these developments lead to consider criteria which insure that a given essentially non-negative matrix is an M-matrix. In this respect, we hope that the sufficient conditions set up by Fiedler and Ptak will open the field to applications.

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