91
9-19-74

DL-947

UCRL-51631 Pt. 1

INTERACTION OF INTENSE MICROWAVE RADIATION AND FULLY IONIZED HYDROGEN PLASMA

Part 1: High-Density System

Donald L. Ensley

July 26, 1974

Prepared for U.S. Atomic Energy Commission under contract No. W-7405-Eng-48

**LAWRENCE
LIVERMORE
LABORATORY**

University of California/Livermore

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

NOTICE

"This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights."

Printed in the United States of America
Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151

Price: Printed Copy \$ *; Microfiche \$1.45

<u>* Pages</u>	<u>NTIS Selling Price</u>
1-50	\$4.00
51-150	\$5.45
151-325	\$7.60
326-500	\$10.60
501-1000	\$13.60

LAWRENCE LIVERMORE LABORATORY
University of California, Livermore, California, 94550

UCRL-51631 Pt. 1

**INTERACTION OF INTENSE MICROWAVE RADIATION
AND FULLY IONIZED HYDROGEN PLASMA
Part 1: High-Density System**

Donald L. Ensley

MS. date: July 26, 1974

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, make any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

66

Contents

Abstract	1
Introduction.	1
List of Relevant Equations	1
Symbols	2
Plasma-Microwave Interactions	3
Individual Particle Behavior	3
Stimulated Bremsstrahlung	3
Stochastic or Transit-Time Effects	3
Collective Effects	3
"Shock" or Collective Collisionless Heating Due to Physically Displacing the Plasma Corona at Twice Field Frequency	3
Plasma-Wave Excitation and Instabilities	4
Hydroradiation Surface Instabilities	4
High-Density Plasma	4
Self-Consistent Fields -- High-Frequency Skin	4
Role of Fluctuations on the Skin Depth	5
Role of Collisions on the Skin Field	6
Self-Consistent Solutions	7
Application to High-Density Plasma Heating	7
Stimulated Inverse Bremsstrahlung (Skin Effect)	8
Wave-Particle Heating	9
Stochastic Heating	9
Application to Bootstrap Burn Conditions	10
Appendix A: Stimulated Bremsstrahlung	12
I. A Classical Derivation of Stimulated Bremsstrahlung	12
II. Relation to the Quantum Treatment of Stimulated Bremsstrahlung	19
III. Collisional Microwave Skin Heating	22
Appendix B: Collective Collisionless Ion Heating	24
Appendix C: Relativistic Quasi-Potentials	28
Appendix D: Plasma Hydroradiation Stability	30
Field Perturbation	30
Plasma Particle Pressure	33
Dispersion Relation for Surface Waves	34
Ablative Stability	35

Appendix E: Derivation of the Self-Consistent Field Equation for a Microwave Corona	37
Acknowledgments	40
Bibliography	40

INTERACTION OF INTENSE MICROWAVE RADIATION AND FULLY IONIZED HYDROGEN PLASMA

Part 1: High-Density System

Abstract

The interaction of intense standing-wave radiation and a fully ionized plasma is investigated for $\omega/\omega_{pe} < 1$. The high-density case is treated here, for which $\nu_e/\omega \gtrsim 1$ at corona densities satisfying the confinement relation, but for which there is no average confinement. It is demonstrated that under certain realizable circumstances the energy-absorption time of the plasma is less than the inertial disassembly time, and that the radiation/plasma interface is stable for perturbations less than the external field radiation wavelength.

Introduction

The application of microwave confinement and heating to controlled fusion has been historically the subject of much research in the Soviet Union and Europe, but until recently it has been studied only to a very limited extent in the U.S.A. The work presented in this report is representative of the current American research in this area.

Two approaches are being followed in this study of microwave fusion. In one system, operating in a pulsed mode with high-density plasma, the microwave radiation fields are used only for heating, and the plasma core expands inertially. In the other system, using low-density plasma, the average forces arising from the heating fields also confine the plasma during burn. The low-density system may have better characteristics for long-term, widespread use for generating electricity, whereas the high-density system is somewhat simpler.

The present document, Part 1 of UCRL-51631, treats the interaction of intense microwave radiation and a high-density hydrogen plasma. Part 2, to be published later, will deal with the low-density case. A general discussion of the principles of these microwave fusion systems can be found in Ensley (1974).

For the purposes of this critique, a high-density plasma is one for which the electron collision frequency exceeds the microwave frequency for ~ 1 cm radiation at ~ 3.5 keV temperature, and thus the electron density is $\gtrsim 10^{20}$ cm $^{-3}$. Low-density plasmas are anything with $N_e < 10^{20}$ cm $^{-3}$. Also $T_e \sim T_i$.

List of Relevant Equations (Gaussian Units)

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad (1)$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}, \quad (2)$$

$$\nabla^2 \phi = 4\pi e \int (f_e - f_i) d\vec{v}, \quad (3)$$

$$\vec{J} = e \int \vec{v} (f_i - f_e) d\vec{v}, \quad (4)$$

$$\frac{\partial f_{i,e}}{\partial t} + \vec{v} \cdot \frac{\partial f_{i,e}}{\partial \vec{r}} \mp \frac{e}{M_{i,e}} \left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B} + \vec{\nabla} \phi \right]_j \cdot \frac{\partial f_{i,e}}{\partial \nu_j} = \left(\frac{\partial f_{i,e}}{\partial t} \right)_{\text{coll}}, \quad (5)$$

$$\sigma = \frac{N_e(\mathbf{r})e^2\nu_e}{M_e(\omega^2 + \nu_e^2)} \quad (6)$$

$$\nu_e = \frac{5N_i e^4 \pi^{3/2}}{(KT)^{3/2} M_e^{1/2}} \quad (7)$$

$$H_e(\mathbf{r}, t) = -e\phi(\mathbf{r}, t) + \frac{(\vec{P} + e\vec{A})^2}{2M_e} - Ze^2 \frac{\exp[-\gamma|\mathbf{R}_e(t) - \mathbf{R}_c|]}{|\mathbf{R}_e(t) - \mathbf{R}_c|} \quad (8)$$

Symbols

H_e = electron Hamiltonian with infinite mass ions

$\vec{A}(\mathbf{r}, t)$ = vector potential of the microwave field

B_A = peak field strength

$P(t)$ = canonical momentum

$-e$ = electron charge

Ze = nuclear charge

$\mathbf{R}_e(t)$ = vector position of electron at time t

$\mathbf{R}_c(\mathbf{r})$ = position of ion

$\gamma = 1/\lambda_D$ = cutoff due to Debye shielding

λ_D = Debye distance

σ = plasma conductivity

$T_{i,e}$ = ion or electron temperature $\equiv \int \frac{1}{2} M_{i,e} v_{i,e}^2 f_{i,e} d\vec{v}$

$v_{i,e}$ = ion or electron speed

$N_{i,e}$ = ion or electron density

ν_e = total electron collision frequency

δ = classical microwave skin depth

ϕ = high-frequency time average quasi-potential

$\omega_{pi,e}$ = ion or electron plasma frequency

$E_{b,s}$ = electron kinetic energy from bremsstrahlung or transit time

$f_{i,e}$ = ion or electron distribution

$N_{e,d}^c$ = plasma density such that $2N_{e,d}^c KT = \frac{B^2 + E^2}{8\pi}$

In general, the solution to Eqs. (1) through (5) must be carried out self-consistently, including $f_{i,e}$. As will become apparent later, the calculational problem for low-density plasmas, although solvable, is much more complex than for high-density plasmas. The reason for this is that phase memory of electrons causes surface microwave energy exchange to grossly affect f_e for low densities. In systems where a collision-dominated core exists, the interior f_e is nearly independent of the details of particle-field exchanges, except through the effective temperature.

Plasma-Microwave Interactions

There are two extensive general categories of interactions between a plasma and microwave field: 1) individual particle behavior, and 2) collective effects, plus interactions between 1) and 2).

All cases considered here will deal with $\omega/\omega_{pe} \ll 1$. Then $\delta \sim c/\omega_p \ll 1$. Further, $KT_e > 1/2 kT_e$; also, the radiation wavelength > 0.01 cm. Under these conditions, the dominant interactions reduce to those discussed below.

INDIVIDUAL PARTICLE BEHAVIOR

Stimulated Bremsstrahlung. The stimulated-bremsstrahlung classical cross section is (see Appendix A):

$$\text{energy change per path length of electron orbit} \equiv \frac{dE_b(t)}{dx} = \frac{4\pi Z^2 |e|^5 \vec{A} \cdot \vec{v}_e}{M_e^2 |v_e|^4} \ln \left(\frac{2M_e v_e}{\hbar \gamma} \right) N_i^e,$$

which is valid for arbitrary field strength and spatial dependence, except that:

$$\frac{\omega}{\gamma v_e} \cdot \frac{eA}{M_e v_e} \ll 1 \text{ and } \left| \frac{dA}{dx_j} \right| \lambda_D \lesssim 1,$$

and this condition is always met here.

Stochastic or Transit-Time Effects

$$\Delta E_s = -e \int_0^\tau \vec{E} \cdot \vec{v}_e dt' = \left\{ \begin{array}{l} \text{net energy change per "bounce" lasting} \\ \text{time } \tau \text{ through the skin field } \vec{E}, \text{ per electron.} \end{array} \right.$$

In the above equations, it is understood that the fields are self-consistent and the path integrals are taken along the orbits given by $H_e(\mathbf{r}, t)$, Eq. (8), using these fields.

COLLECTIVE EFFECTS

"Shock" or Collective Collisionless Heating Due to Physically Displacing the Plasma Corona at Twice Field Frequency (see Appendix B). This mechanism arises if the radiation field in the corona is not circularly polarized. Then the fluctuating pressure on the plasma electrons, along the direction of propagation of the field, drives the corona.

More work is done by the field when the plasma is "compressed" than the plasma does on the field when it "expands." The heating rate produced, averaged over a period, is:

$$\dot{\omega} = \frac{1}{T} \int_0^T P_f(t') v_p(t') dt',$$

where

$$P_f(t') = \frac{B_A^2}{8\pi} \sin^2 \omega t',$$

v_p = corona velocity,

$$\dot{\omega} \sim N_i^e kT \langle v_z^2 \rangle,$$

$\langle v_z^2 \rangle$ = average ion velocity normal to the electrostatic charge wave (along $-\vec{\nabla}\phi$).

Plasma-Wave Excitation and Instabilities. Under the conditions described earlier, none of the classical instabilities which depend for growth on either electron or ion plasma wave resonances are strongly excited. The reason for this is twofold: First, the spatial region with high-frequency field available to electrons for which the density is reasonably constant is only about one Debye distance wide and thus any group of electrons only exists under conditions where a plasma wave could grow for about one plasma period. Secondly, the orbits of any group of electrons in the corona self-consistently always exist such that $eE/M\omega v_e = 1$ and hence no significant amplification of the two-stream instability could take place.

Another potential mechanism is parametrically driven electron and ion plasma oscillations in the drift direction caused by the fluctuating field "pump" wave at the boundary. In the low-density system, standing plasma wave modes might be excited and be driven to large amplitudes and feed instabilities. This is avoided if ω is chosen to be well above ω_{pi} but below ω_{pe} . In the high-density, collision-dominated case, $\omega \ll \omega_{pie}$ and $T_e \sim T_i$.

Even so, it may still be that nonlinear effects are so large with very intense fields in the "pump" microwave field that some nontrivial energy exchange will take place. No adequate theoretical treatment exists for saturation under these conditions, and further theoretical work is needed to understand the saturation of sound waves with $\sim \lambda_D$ wavelengths in the plasma.

Hydroradiation Surface Instabilities. The interface between the external field and plasma is not in general stable (see Appendix D). Certain plasma surface waves may grow spontaneously, depending upon the initial modes which are excited. If not counteracted with feedback, these may lead to uneven disassembly of the plasma. For the low-density system, perturbations where the surface wavelength exceeds the radiation wavelength may grow.

As will be shown, there are some mode combinations which may exhibit stability to all perturbations, even for long-wavelength surface waves. In a high-density system, the dimensions of which are approximately equal to or less than the radiation, and also for which a hydrodynamic disassembly time is comparable to moving through a radiation wavelength, this is less of a problem.

High-Density Plasma

In order to produce a useful answer, an estimate of the fields $\phi(\vec{r}, t)$, $\vec{A}(\vec{r}, t)$ must be made. This in turn means the high-frequency "skin" region of the plasma boundary must be described. This can be done by solving the self-consistent Maxwell and Poisson equations together with the appropriate form of the Liouville equation. Generally, when the skin region extends only over a few interior Debye distances and energy exchange rates are so large that even approximate integral invariants do not exist, it is essential to do numerical self-consistent field solutions.

SELF-CONSISTENT FIELDS - HIGH-FREQUENCY SKIN

In the high-plasma-density case, a great simplification results: all phase memory of a particular particle relative to the field is erased by many collisions in crossing the plasma core interior. It is thus really only necessary to solve for the density and current in the low-density, "collisionless" corona region and join this to the collisional region in the core as a Maxwellian, as previously mentioned.

The simplest model for conductivity under these conditions is Eqs. (6) and (7).

This conductivity can be somewhat in error due to gross distortions of the electron cloud due to the drift velocity dependence of the Coulomb scattering cross section (see Appendix A, Part III), but it will be used here to keep the argument simple. In actual orbit calculations, the collision cross section is used, and the simple model is used only for rough comparison.

Next, solve the Hamiltonian for the "collisionless" ($v_e < \omega$) region: this leads in the usual way to the high-frequency quasi-potential and charge separation in the corona resulting from solving,

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = (\omega/c)^2 \left[1 - \frac{\omega_{pe}^2}{\omega^2} \exp\left(\frac{-\tilde{E}^2}{2 M_e \omega^2 K T_e} + \frac{e\phi}{K T_e}\right) - \frac{\omega_{pi}^2}{\omega^2} \exp\left(\frac{-\tilde{E}^2}{2 M_i K T_i \omega^2} - \frac{e\phi}{K T_i}\right) \right] \tilde{\mathbf{E}},$$

or

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = (\omega/c)^2 \left[1 - \frac{4\pi e^2}{M_e \omega^2} N_e(L) - \frac{4\pi e^2}{M_i \omega^2} N_i(L) \right] \tilde{\mathbf{E}},$$

$$\nabla^2 \phi = 4\pi q [N_i(L) - N_e(L)],$$

which are Eqs. (1), (2), (3), and (4).

Generally, although $\nabla^2 \phi$ exists and there is a considerable difference in N_i and N_e , one can estimate N_e alone by making the quasi-neutral approximation: $N_i \sim N_e$. This is found numerically to give the correct order of magnitude of $\phi(L)$.

$$e\phi(L) \approx \left(\frac{1}{K T_e} + \frac{1}{K T_i} \right)^{-1} \frac{e^2 \tilde{E}^2}{2\omega^2} \left(\frac{1}{M_e K T_e} - \frac{1}{M_i K T_i} \right),$$

and if $T_i \sim T_e$ and ignoring the term with the ion mass, you have

$$e\phi(L) = \frac{e^2 \tilde{E}^2}{4 M_e \omega^2},$$

and the charge separation field locally is equal to half the quasi-potential at that point. The quasi-potential itself,

$$\phi_e = \frac{e^2 \tilde{E}^2}{2 M_e \omega^2},$$

is the nonrelativistic result, valid whenever $\phi_e < M_e c^2$. In strong field regions, for $\phi_e > M_e c^2$, this is

$$\phi_e = \frac{\sqrt{2}}{2} \frac{ce|E|}{\omega} \text{ linear in } E \text{ (see Appendix C).}$$

The problem with the above analysis is that ignoring the fluctuations in the electron orbits is equivalent to guiding center theory. This procedure fails badly when the time spent by an electron in the field region is less than a field cycle, so $\omega\tau < 1$. This is always true if $(\omega/\omega_{pe}) < 1$ for the case at hand. The next section shows how to replace guiding center theory for large fluctuations in orbits.

ROLE OF FLUCTUATIONS ON THE SKIN DEPTH

The previous analysis summarizes the theoretical treatment of the average high-frequency behavior. This leads to the classical skin depth $\delta_e \sim c/\omega_{pe}$. There are, of course, fluctuations. Let us review the requirements of self-consistency for $\tilde{\mathbf{J}}(L)$ and $N_e(L)$ carefully. Ignoring ion motion we had, from Maxwell's field equations ($\tilde{\mathbf{J}}_e$ = electron current, and N_0 = interior density):

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = \left(\frac{\omega}{c} \right)^2 \left[\tilde{\mathbf{E}} - \frac{4\pi}{\omega^2} \frac{\partial}{\partial t} \tilde{\mathbf{J}}_e \right].$$

and we tried

$$N_e(\mathbf{r}, v) = N_0 \left(\frac{M}{2\pi K T} \right)^{3/2} \int_0^\infty \exp \left(\frac{-Mv^2}{2KT} - \frac{e^2 \tilde{E}^2}{2MK T \omega^2} + \frac{e\phi}{KT} \right) dv$$

as a solution for the electron distribution. The quasi-neutral approximation was

$$N_e(t) = N_0 \exp \left(\frac{-e^2 \tilde{E}^2}{4M_e K T \omega^2} \right),$$

and the current from Eq. (4) is:

$$\frac{\partial}{\partial t} \tilde{J}_e = \frac{e^2}{M_e} N_0 \tilde{E} \exp \left(\frac{-e^2 \tilde{E}^2}{4M_e K T \omega^2} \right).$$

Then you have the nonlinear wave equation for $\tilde{E}$, the high-frequency field,

$$\nabla \times \nabla \times \tilde{E} = \left(\frac{\omega}{c} \right)^2 \left[1 - \frac{\omega_{pe}^2}{\omega^2} \exp \left(\frac{-e^2 |\tilde{E}|^2}{4M_e K T \omega^2} \right) \right] \tilde{E}.$$

If you now check this equation for self-consistency by summing over a large number of orbit calculations with the Hamiltonian using this $\tilde{E}$ field and numerically determine $N(t)$ and $\tilde{J}(t)$, Maxwell's field equations are not in general satisfied. The reason is that electrons arriving at one phase of the field have their turning point pulled outward and other phases pushed inward by the fluctuating $\tilde{v} \times \tilde{B}$ forces, which have no time average. One then iterates with the new $N(t)$, $\tilde{J}(t)$ until self-consistency is achieved in Maxwell's equations. This procedure replaces guiding center theory for $\omega_{pe}/\omega \gg 1$.

In some cases, the quasi-neutral (guiding center) solution is found to reasonably represent the skin-field behavior, but typically the self-consistent solutions yield skin depths several times the classical depth. The fluctuations thus form the physical limit to the minimum skin depth. The steeper the density gradient from the classical solution, the greater the counteracting fluctuations become.

ROLE OF COLLISIONS ON THE SKIN FIELD

We have seen that collisionless self-consistent fields can be estimated, even for very steep density gradients, when fluctuation effects are important. To see the physics involved when $\nu_e > \omega$, notice that the orbit from $H(\mathbf{r}, t)$ is dominated by collisions, and the quasi-potential, based on interactions only with the $\mathbf{A}$ field, is drastically reduced, because you now time average over something close to a random motion rather than a sine wave. The field is then approximately given by

$$\nabla \times \nabla \times \tilde{E} = (\omega/c)^2 \left[1 - \frac{\omega_{pe}^2}{(\omega^2 + \nu^2)} \right] \tilde{E},$$

with collisional skin depth $\delta \sim \left(\frac{\nu}{\omega} \right)^{1/2} \frac{c}{\omega_{pe}}$.

SELF-CONSISTENT SOLUTIONS

Choose the coordinate system at rest with respect to the average ion velocity, since the high-frequency skin will form on a time scale $\tau \sim \delta/v_i$; also, it is assumed that the high frequency field relaxation time is short compared to hydrodynamic relaxation time.

For convenience in discussion, the quasi-neutral equation will be used. For the region $\nu < \omega$:

$$\nabla \times \nabla \times \underline{E} = \left(\frac{\omega}{c}\right)^2 \left[1 - \frac{\omega_{pe}^2 e^{-\phi/KT}}{(\omega^2 + \nu_e^2)} \right] \underline{E}.$$

and $\nu > \omega$:

$$\nabla \times \nabla \times \underline{E} = \left(\frac{\omega}{c}\right)^2 \left[1 - \frac{\omega_{pe}^2}{(\omega^2 + \nu_e^2)} \right] \underline{E} : \phi = \frac{v^2 E^2}{2M\omega^2}.$$

Solutions to the first of the above equations are propagating if

$$\frac{\omega_{pe}^2}{\omega^2} e^{-\phi/KT} < 1.$$

If the field intensity is such that the above inequality is reversed at a self-consistent density, N_e^C , such that $\nu_e < \omega$ still holds, then the field dies exponentially with $\delta \sim c/\omega\rho$, and the corona is "collisionless." For the usual field strengths encountered, ϕ is small and this is the case. However, if the "collisionless" solution is tried for intense fields such that $\nu_e > \omega$, before the inequality is reversed, then the nonlinear penetration has delivered high-frequency field into the collisional regime, and while the first equation holds approximately to the point $\nu_e = \omega$, the field dies exponentially after this with

$$\delta \sim \frac{c}{\omega_p} \left(\frac{\nu_e}{\omega} \right)^{1/2},$$

as it would in a metal.

A further point should also be noted: numerical solutions of the nonlinear wave equation always yield self-consistent electric field gradients which peak in space near the point where $\frac{1}{2} M_e v_{th}^2 \approx \phi$, the turning point for a thermal electron. In this region the resulting self-consistent peak magnetic field satisfies

$$\frac{E^2 + B^2}{8\pi} \approx 2N_e^C KT : |B^2| \gg |E^2|.$$

Now if you sum the two-component Vlasov equation, the well-known result is just the plasma pressure balance above. In other words a microwave field under these conditions always nonlinearly penetrates to produce a plasma corona at least as dense as that which it could collisionlessly confine at the prevailing temperature. This N_e^C in turn can be large enough so that $\nu_e > \omega$ at say, 3.5 keV, through Eq. (7).

APPLICATION TO HIGH-DENSITY PLASMA HEATING

Heating rates are estimated and scaled according to the hydro disassembly time $\tau_h \sim r_0/v^1$ for a spherical plasma with $r_0 < \lambda_R$. It is shown that, because the quasi-potential energy well for individual particles $\gg KT_e$, stochastic electron heating time potentially can be much shorter than τ_h . The wave-particle and skin-effect heating both have energy doubling times $\sim \tau_h$ when $N_e/N_{core} \sim 1$, while the stochastic time $< \tau_h$ even for $10^{-1} < N_e/N_{core} < 10^{-2}$. If $N_{core} \sim 10^{22}$ and $KT \sim 3.5$ keV, then fields of about > 10 MG are needed. These

in turn may be stored efficiently in a superconducting resonator, probably a pulsed spherical cavity with wall fields ~ 50 kG. The dense plasma is then visualized to Q-switch the cavity by an intense rf breakdown near the center.

Stimulated Inverse Bremsstrahlung (Skin Effect). As outlined earlier, with $\nu_e \gtrsim \omega$,

$$\nu_e = \frac{5N_e e^4 \pi^{3/2}}{(KT)^{3/2} M_e^{1/2}},$$

where ω = radiation frequency, with N_e given by

$$2N_e KT_e = \frac{B_\lambda^2}{8\pi},$$

then in order to stay coupled, with $\omega \sim 10^{11}$ at 3.5 keV, requires $N_e \sim 10^{20}$, so $B_\lambda \gtrsim 10$ MG. Under these conditions ($\nu_e \gtrsim \omega$), the approximate plasma conductivity from the simple model is

$$\sigma = \frac{N_e e^2}{M \nu_e}$$

and the Poynting energy flux into the plasma surface is

$$\vec{S} = \left(\frac{B_\lambda^2}{8\pi} \right) \cdot \vec{c} \cdot \frac{(\nu_e \omega)^{1/2}}{\omega_p},$$

$$\omega_p^2 = \frac{4\pi N_e e^2}{M}$$

where the collisional skin

$$\delta = \frac{c}{\omega_p} \frac{(\nu_e \omega)^{1/2}}{\omega},$$

The well-known vacuum solutions for the eigenfields of a spherical resonator are:

$$\vec{E} = \vec{r} \times \vec{\nabla} \psi,$$

$$\psi(r, \theta, t) = e^{im\phi + i\nu t} (A_e j_e + b_e n_e) P_e \cos \theta,$$

and j_e, n_e are the spherical Bessel and Neumann functions. One property of these solutions is that for $e = 1$, a nodal "apple-core" from $\cos \theta$ appears, but also the field near the central conductor is greater than the cavity wall by the factor $\sim \sqrt{\frac{2}{3}} \cdot R/\lambda_R$, where R is the cavity radius. Hence a cavity one meter in radius with a 50-kG wall field would contain $\sim 10^8$ J and with $\lambda_R \sim 1/4$ to 1 cm. $B_\lambda \gtrsim 10$ MG. The expression for $\vec{S}$ can be written

$$\vec{S} = P_\omega \frac{(2\pi)^2}{3} \left(\frac{R}{\lambda} \right)^2 \omega \delta,$$

where P_ω = wall pressure. If you express

$$P_\omega \frac{(2\pi)^2}{3} \left(\frac{R}{\lambda} \right)^2 = 2N_e KT$$

and compute the energy-doubling time τ_s , with $N_s =$ central density, then $2N_c K T \omega \delta A \tau_s = 3N_s K T V$ for a sphere $v/s = r_0/3$, so

$$\tau_s = \frac{1}{2} \cdot \frac{N_s}{N_c} \frac{r_0}{\omega \delta}$$

so

$$\tau_s/\tau_h = \frac{v_i}{2(\omega \delta)} \frac{N_s}{N_c}$$

hence $\tau_s/\tau_h \sim 1$ requires

$$\boxed{N_c/N_s \sim \frac{v_i}{2\omega \delta}} \quad (\text{valid only if } v_e \sim \omega)$$

Typical values are $\delta \sim c/\omega_p \sim 2 \times 10^{-4}$ cm, $\omega \sim 10^{11}$, so $N_c/N_s \sim 1$ for fast heating. Orbit averages using the bremsstrahlung cross section confirm these heating rates.

Wave-Particle Heating (See Appendix B). This process heats ions. The rate is $\dot{E}_{\omega p} = N_c K T \langle v^i \rangle$, where $\langle v^i \rangle =$ average radial ion speed. Hence,

$$\tau_{\omega-p} = r_0 \frac{N_s}{N_c} \frac{1}{v_z^i}$$

so

$$\frac{\tau_{\omega-p}}{\tau_h} = \frac{N_s}{N_c}$$

so for fast ion heating

$$\boxed{N_c \sim N_s}$$

Stochastic Heating. Stochastic (transit time) heating can be understood by considering the passage of electrons through a powerful rf field. The rate at which the field does work on an electron entering the field skin region at angle θ and velocity v_e when it takes a time $\tau(\theta)$ to exit the field is, per crossing, in the region just outside the collisional skin,

$$\Delta E = -e \int_0^{\tau(\theta)} \vec{E} \cdot \vec{v}_e dt$$

It is understood that all particles are reflected back into the plasma. Because of the strong field-dominated motion, the exit time is a function of the phase of the particle relative to the $\vec{E}$ field. This action is analogous to energy exchange in a klystron, except that in this case there is no monoenergetic electron beam with efficient bunching; rather, the bunching is small and electrons impinge on the field skin gap from all angles and a Maxwellian speed distribution. Nevertheless, extensive computer studies summing over the electron distribution show strong, net, phase-average heating. Typically for a stochastically tuned system an individual thermal

electron can be heated by as much as its own energy in one passage in the skin, on the phase average. For this to occur, however, requires the radiation frequency, ω , to approximately satisfy $(\omega\tau) \sim 10^{-2}$. τ = average transit time $= \delta/v_e$. Typical values are, at 3.5 keV, $\delta \sim 2 \times 10^{-4}$ cm, $\omega \sim 10^{11}$, $v_e \sim 3 \times 10^9$, and $\omega\tau \sim 10^{-2}$. Hence the desired numerical relationship can be approximately satisfied. The overall heating rate is then $N_e v_e KTA$, and the energy doubling time, τ_E , is

$$\tau_E = \frac{2}{3} \tau_0 \frac{N_s}{N_e} \frac{1}{v_e},$$

so for rapid heating,

$$\frac{\tau_E}{\tau_h} = \frac{2 v_i}{3 v_e} \frac{N_s}{N_e} = 1,$$

so $N_e \sim 1 \times 10^{-2} N_s$, which only requires one tenth as much B field as the other mechanisms. The interesting thing about stochastic heating is that it is just the normal, radially inward component of electron energy which is increased, and so coupling is very fast.

APPLICATION TO BOOTSTRAP BURN CONDITIONS

It is well known that to achieve efficient, better than break-even yield in a pulse thermonuclear burn using the plasma, say, D-T or D-D, you need to "bootstrap." Since the fusion cross section increases up to ~ 100 keV, you need to couple some of the charged burn products into the ion temperature so the burn rate will exponentiate and require only enough initial energy to start the process.

There are several methods potentially available to accomplish this objective. Usually, the process involves only Coulomb scattering. For D-T, for instance, the well-known range at, say, solid density, $\sim 2 \times 10^{22}$, is $\lambda_\alpha \sim 2$ cm for α particles. Then, in order to have just enough stopping power within the fuel radius, you need a mass $\sim N_s M_i \frac{4\pi}{3} (\lambda_\alpha)^3$ and an energy supply to an initial fast heating value of, say, 3.5 keV for the ions, which is $E \sim 3 N_s KT$ volume and $\lambda_\alpha \sim 1/N$. Hence the usual relationship for required energy at a given temperature is $E \sim 1000 \left(\frac{N_s}{N}\right)^2$ in megajoules. If one is somewhat limited in available energy, then there is an obvious advantage to increasing N .

There is another way the bootstrap condition can be achieved with much smaller E and $N \sim N_s$. Instead of relying entirely on Coulomb scattering in a "one-shot" process, the charged burn products can be effectively coupled to the matter if they are trapped at the surface electrostatically. The charge-separation field at the boundary which confines ions also will reflect, for instance, any charged burn product with energy $\lesssim 20$ MeV.

In the case of α particles, for instance, $v_\alpha/v_i \sim 100$, so an α particle can make about 100 reverberations per τ_h . The initial energy goes with the physical mass, so you now require $\tau_d \sim \frac{v_i}{v_\alpha} \lambda_\alpha$:

$$E \sim 3 N_s KT \frac{4}{3} \pi \lambda_\alpha^3 (v_i/v_\alpha)^3$$

or $E \sim \left(\frac{N_s}{N}\right)^2$ in kilojoules.

We have seen that because of the quasi-potential for electrons a powerful charge-separation electric field arises and can confine individual charged burn products. Another logical question is the collective behavior. The average hot α component, for instance, is

$$N_\alpha \approx \frac{N_i^2}{4} \frac{\lambda_\alpha}{v_\alpha}$$

where λ_α/v_α is the mean alpha lifetime. Since the energy is ~ 3.5 MeV, the momentum flux into the charge-separation field will equal the cold ions at $N_\alpha \approx 10^{18}$, since we are assuming the conditions for $KT_e \sim 10$ keV and $N_e \sim 10^{20}$. Thus the alpha particles are electrostatically coupled to the ion core because in order to pull collectively the electrons through the field they need a density $N_e \sim N_\alpha$ for pressure balance, and this would imply a large density excess ($N_e/N_\alpha \sim 10^2$) for the electron which would then be bound to the core. Further numerical studies with the three-component equation coupled through Poisson's equation are being undertaken to try to better understand the collective behavior of the charged burn products, including wave-particle coupling.

At present it would appear that under the conditions outlined above, definite bootstrap to about 10 keV in D-T is expected, in solid density ($\bar{\sigma}_{D-T} \sim 10^{-16}$ at 10 keV). Then the α 's are expected to remain coupled collectively to the (ion + α) core hydroexpansion speed in order to preserve over-all charge neutrality, and continue the process until burn is mostly over. Of course, such small drops might disassemble too fast for large fractional burn-up if N does not increase.

It is interesting to consider the subject of very-high-efficiency burn resulting from strong burn coupled wave particle interaction which will be considered as a portion of a forthcoming report on methods of direct conversion.

Appendix A STIMULATED BREMSSTRAHLUNG

I. A CLASSICAL DERIVATION OF STIMULATED BREMSSTRAHLUNG

Consider the motion of an electron past an infinitely heavy stationary ion of charge Ze , while in a circularly polarized standing wave radiation field. The Hamiltonian of the electron is:

$$H = \frac{[\vec{P}(t) - q\vec{A}(z,t)]^2}{2M_e} - \frac{Ze^2 \exp \left[-\gamma |\vec{R}_e(t) - \vec{R}_c| \right]}{|\vec{R}_e(t) - \vec{R}_c|},$$

where

$\vec{P}(t)$ is the canonical momentum of the electron,

q is the electron charge, equal to $-e = -4.803 \times 10^{-10}$ esu,

$\vec{A}(z,t)$ is the vector potential, which I take equal to $\vec{A}(z,t) = A(z) (\sin \omega t, \cos \omega t, 0)$,

$A(z)$ = self-consistent field,

$\vec{R}_e(t)$ is the vector position of the electron at time t ,

$\vec{R}_c$ is the position of the ion, $\vec{R}_c = (X_c, Y_c, Z_c)$,

γ is an exponential cutoff due to Debye shielding $\gamma = \frac{1}{\lambda_D}$.

I will henceforth consider the problem of $A(z) = A$, a constant at any point in the skin. If it were not for the small perturbation of the particle orbit due to the ion, the solution of Hamilton's equations would be

$$\vec{R}_0(t) \equiv [x_0(t), y_0(t), z_0(t)] ,$$

$$x_0(t) = x + \frac{qA}{M_e \omega} (\cos \omega t - \cos \omega t_0) + \frac{P_{x0}(t - t_0)}{M_e} ,$$

$$y_0(t) = y - \frac{qA}{M_e \omega} (\sin \omega t - \sin \omega t_0) + \frac{P_{y0}(t - t_0)}{M_e} ,$$

$$z_0(t) = z + \frac{P_{z0}(t - t_0)}{M_e} ,$$

where (x, y, z) is the position of the particle at the initial time t_0 and $\frac{\vec{P}_0}{M_e} - \frac{q\vec{A}(t_0)}{M_e}$ is the velocity at this initial time.

Choose t_0 to be the time the unperturbed particle passes through that plane through the ion such that the velocity of the particle is perpendicular to the plane (Fig. A1).

Fig. A1.

This point $\vec{R} \equiv (x, y, z)$ is the point of closest approach of the unperturbed particle, and very near the point of closest approach of the perturbed particle. For the derivation to be unambiguously followed, the field strength, field frequency, electron velocity, and plasma parameters must be such that there is only one such point $\vec{R}$ within the Debye sphere of the ion, but this is hardly a restrictive assumption.

To solve for the motion of the particle in the electromagnetic field and Coulomb field, treat the Coulomb field as a perturbation and use the following iteration loop.

Using the unperturbed position $\vec{R}_0(t)$ and canonical momentum $\vec{P}_0$, and the full Hamiltonian of p. 12 on the right-hand side (only), write down and solve (by integrating explicitly with respect to time) Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i},$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i},$$

for the first-order position and canonical momentum. Plug these first-order values into the right-hand side of Hamilton's equations to obtain the second-order values, etc. Repeat as many times as it takes to obtain the desired accuracy.

The first-order position vector is

$$\begin{aligned} \vec{R}_1(t) = \vec{R}_0(t) - \frac{Ze^2}{M_e} \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \frac{[\vec{R}_0(t'') - \vec{R}_c] \exp[-\gamma |\vec{R}_0(t'') - \vec{R}_c|]}{|\vec{R}_0(t'') - \vec{R}_c|^3} \\ - \frac{\gamma Ze^2}{M_e} \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \frac{[\vec{R}_0(t'') - \vec{R}_c] \exp[-\gamma |\vec{R}_0(t'') - \vec{R}_c|]}{|\vec{R}_0(t'') - \vec{R}_c|^2}. \end{aligned}$$

It turns out to be sufficient for present purposes to have the second-order expression for the time derivative of the canonical momentum:

$$\frac{d\vec{P}}{dt} = \frac{-Ze^2[\vec{R}_1(t) - \vec{R}_c] \exp[-\gamma |\vec{R}_1(t) - \vec{R}_c|]}{|\vec{R}_1(t) - \vec{R}_c|^3} - \frac{\gamma Ze^2[\vec{R}_1(t) - \vec{R}_c] \exp[-\gamma |\vec{R}_1(t) - \vec{R}_c|]}{|\vec{R}_1(t) - \vec{R}_c|^2}.$$

Just how big is the perturbation in the position of the particle? The minimum impact parameter the electron can have is $R_{min} = \frac{h}{M_e v}$ cm. (This condition clearly comes from quantum mechanics -- witness the h -- so in this way, the calculation isn't really classical. For a justification of this minimum impact parameter, see, for example,

Jackson, *Classical Electrodynamics*, sections 13.3 and 15.2). The maximum deflection the particle can make from its unperturbed path is (impulse approximation)

$$\approx \frac{Ze^2}{M_e R_{\min}^2} \Delta T^2.$$

where ΔT is the interaction time. $\Delta T \approx \frac{R_{\min}}{v}$.

This deflection as a percentage of the minimum impact parameter is

$$\frac{Ze^2}{M_e v^2} \frac{M_e v}{h} = \frac{Ze^2}{h v} \approx \frac{2.5 Z \times 10^8}{v}.$$

This supplies a pretty lax condition on $\frac{Z}{v}$ for the previous and following approximations to be valid:

$$2.5 \times 10^8 \frac{Z}{v} \ll 1.$$

Having shown that the terms of $\vec{R}_1(t)$ proportional to Ze^2 are much smaller than $\vec{R}_0(t)$, expand the expression

for $\frac{d\vec{P}}{dt}$, dropping terms of higher than second order in (Ze^2) .

Define

$$\begin{aligned} \vec{\alpha}(Ze^2) &= \frac{Ze^2}{M_e} \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \frac{[\vec{R}_0(t'') - \vec{R}_c] \exp[-\gamma |\vec{R}_0(t'') - \vec{R}_c|]}{|\vec{R}_0(t'') - \vec{R}_c|^3}, \\ \vec{\beta}(Ze^2) &= \frac{Ze^2 \gamma}{M_e} \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \frac{[\vec{R}_0(t'') - \vec{R}_c] \exp[-\gamma |\vec{R}_0(t'') - \vec{R}_c|]}{|\vec{R}_0(t'') - \vec{R}_c|^2}, \\ \frac{d\vec{P}}{dt} &= -Ze^2 \exp[-\gamma |\vec{R}_0(t) - \vec{R}_c|] [\vec{R}_0(t) - \vec{R}_c] [\vec{R}_0(t) - \vec{R}_c] \cdot \left[\vec{\alpha}(Ze^2) + \vec{\beta}(Ze^2) \right] \\ &\quad \cdot \left[\frac{3}{|\vec{R}_0(t) - \vec{R}_c|^5} + \frac{3\gamma}{|\vec{R}_0(t) - \vec{R}_c|^4} + \frac{\gamma^2}{|\vec{R}_0(t) - \vec{R}_c|^3} \right] \\ &\quad + Ze^2 \exp[-\gamma |\vec{R}_0(t) - \vec{R}_c|] [\vec{\alpha}(Ze^2) - \vec{\beta}(Ze^2)] \left[\frac{1}{|\vec{R}_0(t) - \vec{R}_c|^3} + \frac{\gamma}{|\vec{R}_0(t) - \vec{R}_c|^2} \right] \\ &\quad - Ze^2 \exp[-\gamma |\vec{R}_0(t) - \vec{R}_c|] [\vec{R}_0(t) - \vec{R}_c] \left[\frac{1}{|\vec{R}_0(t) - \vec{R}_c|^3} + \frac{\gamma}{|\vec{R}_0(t) - \vec{R}_c|^2} \right]. \end{aligned}$$

Now, make a pair of further, somewhat related assumptions. First, assume that the field period of the radiation is very long compared to the interaction time of the electron with the ion. This amounts to $\frac{\omega}{v} \ll 1$.

Alternately, this can be rewritten (where λ is the wavelength of the radiation) $2\pi \frac{\lambda_D}{\lambda} \frac{c}{v} \ll 1$.

For electron velocities high enough to satisfy the previous assumptions and plasmas of densities of interest, this condition is easily met.

The explicit meaning of the second assumption can be seen by noting that $\vec{R}_1(t)$ can be rewritten as

$$\begin{aligned}\vec{R}_1(t) = & \left\{ x + \left(\frac{P_{x0}}{M_e} - \frac{eA}{M_e} \sin \omega t_0 \right) (t - t_0) + \left[\frac{eA}{M_e \omega} (\cos \omega t - \cos \omega t_0) + \frac{eA}{M_e} \sin \omega t_0 (t - t_0) \right] \right. \\ & y + \left(\frac{P_{y0}}{M_e} - \frac{eA}{M_e} \cos \omega t_0 \right) (t - t_0) + \left[\frac{eA}{M_e \omega} (\sin \omega t_0 - \sin \omega t) + \frac{eA}{M_e} \cos \omega t_0 (t - t_0) \right] \\ & \left. z + \frac{P_{z0} (t - t_0)}{M_e} \right\}.\end{aligned}$$

$$\text{Define } \vec{v} \equiv \left(\frac{P_{x0}}{M_e} - \frac{eA}{M_e} \sin \omega t_0, \frac{P_{y0}}{M_e} - \frac{eA}{M_e} \cos \omega t_0, \frac{P_{z0}}{M_e} \right),$$

$$\vec{R}(t) \equiv \vec{R} - \vec{R}_c + \vec{v}(t - t_0).$$

$\vec{v}$ is just the velocity of the (unperturbed) electron at the position $\vec{R}$.

$$\vec{v} \cdot (\vec{R} - \vec{R}_c) = 0,$$

and therefore,

$$|\vec{R}(t)|^2 = |\vec{R} - \vec{R}_c|^2 + v^2 (t - t_0).$$

$$\text{Since } \vec{E}(t) = -\frac{\partial \vec{A}(t)}{\partial t},$$

$$\vec{R}_0(t) - \vec{R}_c = \vec{R}(t) + \frac{e}{M_e} \vec{A}(t_0) (t - t_0) + \frac{e\vec{E}(t_0)}{M_e \omega^2} - \frac{e\vec{E}(t)}{M_e \omega^2}.$$

The exact value of $\vec{R}_0(t) - \vec{R}_c$ is important only when its magnitude is less than several Debye lengths. I have already assumed that the field turns very little while the electron is in this region. The field function can thus be accurately expanded around t_0 :

$$\begin{aligned}\vec{R}_0(t) - \vec{R}_c &= \vec{R}(t) + \frac{e}{M_e} \vec{A}(t_0) (t - t_0) + \frac{e\vec{E}(t_0)}{M_e \omega^2} - \frac{e\vec{E}(t)}{M_e \omega^2} \\ &= \frac{e\vec{E}(t_0) (t - t_0)^2}{2M_e} + \frac{e\vec{A}(t_0)\omega^2 (t - t_0)^3}{6M_e}\end{aligned}$$

This piece of $\vec{R}_0(t) - \vec{R}_c$ represents the deviation from straight-line motion of the electron in the vicinity of the ion. I henceforth assume this deviation to be small while the particle is interacting with the ion. The maximum relevant value of $|t - t_0|$ is $\frac{1}{\gamma v}$, so this assumption is equivalent to $\frac{eA\omega}{M_e \gamma^2 v^2} \ll \frac{1}{\gamma}$, or $\frac{\omega}{\gamma v} \frac{eA}{M_e v} \ll 1$.

Note that this is the only restriction on the strength of the field made in this derivation. Thus, the result will be of much wider validity than that found, for example, by the standard perturbation methods of quantum electrodynamics. The field can be strong enough to completely turn the particle around, as long as the field frequency is low enough so that the particle is not deflected a sizeable fraction of a Debye length while traveling across a Debye sphere.

It remains to simplify the expression for $\frac{d\vec{p}}{dt}$ and use it to evaluate the energy change of the electron. As this is merely a mathematical procedure with little further physical content, it will be brief.

Everywhere in the expression for $\frac{d\vec{p}}{dt}$, expand

$$|\vec{R}_0(t) - \vec{R}_c|^2 \approx |\vec{R}(t)|^2 \left[1 + \frac{\vec{R}(t) \cdot \left[\frac{e\vec{E}(t_0)(t-t_0)^2}{2M_e} + \frac{e\vec{A}(t_0)\omega^2(t-t_0)^3}{6M_e} \right]}{|\vec{R}(t)|^2} \right]$$

It turns out, though I won't demonstrate it here, that the

$$\frac{\vec{R}(t) \cdot \left[\frac{e\vec{E}(t-t_0)^2}{2M_e} + \frac{e\vec{A}(t_0)\omega^2(t-t_0)^3}{6M_e} \right]}{|\vec{R}(t)|^2}$$

piece leads to a contribution to the energy change of the electron which is down by at least

$$\frac{\omega}{\gamma v} \frac{eA}{M_e v} \left[\ln \left(\frac{2M_e v}{\hbar \gamma} \right) \right]^{-1}$$

from other contributions. Therefore I drop this piece now, and the expression for $\frac{d\vec{p}}{dt}$ on p. 14 is valid with $\vec{R}_0(t) - \vec{R}_c$ replaced everywhere by $\vec{R}(t)$. The time average energy of the unperturbed electron is a constant. Therefore the energy change due to scattering from the ion is

$$\Delta E = \int_{-\infty}^{\infty} \frac{dH}{dt} dt.$$

Since the electron motion is a solution of Hamilton's equations, accurate to sufficient order for present needs, $\frac{\partial H}{\partial t} = \frac{dH}{dt}$.

$$\begin{aligned} \Delta E &= \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} \left\{ \frac{|\vec{p} - e\vec{A}(t)|^2}{2M_e} \right\} \\ &= \frac{-eA\omega}{M_e} \int_{-\infty}^{\infty} dt [(P_x \cos \omega t) - (P_y \sin \omega t)]. \end{aligned}$$

Integrate by parts, ignoring the term evaluated at the limits as having time average zero.

$$\Delta E = + \frac{q}{M_e} \int_{-\infty}^{\infty} dt \vec{A}(t) \cdot \frac{d\vec{p}}{dt}$$

where $\frac{d\vec{p}}{dt}$ is given by the p. 14 expression.

Actually, one is interested in the energy change of the electron as it moves by *all* the ions down its path. Therefore, integrate over all the ions in a plane perpendicular to the velocity v at a given time, and then later integrate along the path of the electron to get the contribution of *all* the ions in the plasma. Rely once more on

the "small bending condition," $\frac{\omega}{\gamma v} \frac{eA}{M_e v} \ll 1$, to assure that any ions which get double-counted by this procedure have impact parameters much larger than $\frac{1}{\gamma}$, and hence are of little significance (see Fig. A2).

Fig. A2. Only the ions between the dotted lines matter much, so the fact that the ions at point A get double-counted is unimportant.

$$\frac{\Delta E}{\Delta x} = N_i \int d\vec{R}_c \left[\frac{q}{M_e} \int_{-\infty}^{\infty} dt \vec{A}(t) \cdot \frac{d\vec{P}}{dt} \right],$$

where the $d\vec{R}_c$ integral is over $|\vec{R} - \vec{R}_c| > R_{\min}$.

$$\Delta E = N_i \int ds \frac{dE}{ds},$$

N_i = ion density.

and the integral ds is along the path of the electron.

The integral over impact parameters eliminates the linear in Ze^2 terms of $\frac{d\vec{P}}{dt}$, as they are antisymmetric with respect to the impact parameter. (Or rather the resultant piece of ΔE is linearly proportional to $R_{\min}$ and small. Physically speaking, this term is proportional to the average field of the uniform-density ions, which is zero, rather than the rms field of the uniform density ions, which is not. See, for example, Boyd and Sanderson, *Plasma Dynamics*, pp. 234-235.)

Define the symbol

$$\begin{bmatrix} N & M \\ 2 & 2 \end{bmatrix}$$

$AR^J|KL$

to mean

$$A \int_{R_{\min}}^{\infty} R^{J+1} dR \int_{-\infty}^{\infty} dt \frac{t^K \exp[-\gamma(R^2 + v^2 t^2)^{1/2}]}{(R^2 + v^2 t^2)^{N/2}} \int_{-\infty}^t dt' \frac{t'^L \exp[-\gamma(R^2 + v^2 t'^2)^{1/2}]}{(R^2 + v^2 t'^2)^{M/2}}$$

The expression for $\frac{\Delta E}{\Delta x}$ above becomes, upon plugging in the expression for $\frac{d\vec{P}}{dt}$ on p. 14 and doing the dt integral by parts:

$$\frac{\Delta E}{\Delta x} = \frac{2\pi q}{M_e} \frac{(Ze^2)^2}{M_e} N_1$$

times terms listed below. Terms which are proportional to $\vec{A} \cdot \vec{R}$ and $\vec{E} \cdot \vec{R}$ have been dropped as antisymmetric in the impact parameter.

$\vec{A} \cdot \vec{v}$

$\begin{array}{ c c } \hline 2 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 4 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 4 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 5 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 5 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$
$-\gamma^2 11$	$+\gamma 22$	$+\gamma 11$	$+ 11$	$-3\gamma^2 R^2 20$	$-3\gamma R^2 20$	$-3\gamma R^2 20$	$-3R^2 20$
$-\gamma^2 02$	$-\gamma 02$	$-\gamma 02$	$- 02$	$+3\gamma^2 R^2 11$	$+3\gamma R^2 11$	$+3\gamma R^2 11$	$+3R^2 11$
		$-\gamma^3 R^2 20$	$-\gamma^2 R^2 20$	$-3\gamma^2 v^2 31$	$-3\gamma v^2 31$	$-3\gamma v^2 31$	$-3v^2 31$
		$-\gamma^3 R^2 11$	$+\gamma^2 R^2 11$	$+3\gamma^2 v^2 22$	$+3\gamma v^2 22$	$+3\gamma v^2 22$	$+3v^2 22$
		$-\gamma^3 v^2 31$	$-\gamma^2 v^2 31$				
		$-\gamma^3 v^2 22$	$+\gamma^2 v^2 22$				

$\vec{E} \cdot \vec{v}$

$\begin{array}{ c c } \hline 2 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 4 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 4 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 5 & 2 \\ \hline 2, & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 5 & 3 \\ \hline 2, & 2 \\ \hline \end{array}$
$-\gamma^2 21$	$-\gamma 21$	$-\gamma 21$	$- 21$	$+3\gamma^2 R^2 30$	$+3\gamma R^2 30$	$+3\gamma R^2 30$	$+3R^2 30$
$+\gamma^2 12$	$+\gamma 12$	$+\gamma 12$	$+ 12$	$-3\gamma^2 R^2 21$	$-3\gamma R^2 21$	$-3\gamma R^2 21$	$-3R^2 21$
		$+\gamma^3 R^2 30$	$+\gamma^2 R^2 30$	$+3\gamma^2 v^2 41$	$+3\gamma v^2 41$	$+3\gamma v^2 41$	$+3\gamma v^2 41$
		$-\gamma^3 R^2 21$	$-\gamma^2 R^2 21$	$-3\gamma^2 v^2 32$	$-3\gamma v^2 32$	$-3\gamma v^2 32$	$-3\gamma v^2 32$
		$+\gamma^3 v^2 41$	$+\gamma v^2 41$				
		$-\gamma^3 v^2 32$	$-\gamma^2 v^2 32$				

The $\vec{E} \cdot \vec{v}$ terms turn out to lead to contributions which are $\approx \frac{1}{25} \frac{\omega}{\gamma v}$ times the magnitude of the leading $\vec{A} \cdot \vec{v}$ term. The final result is (ignoring terms with the same angular dependence and the same coefficient without the logarithm):

$$\frac{dE}{dx} = \frac{4\pi Z^2 |e|^5}{M_e^2 v^4} \vec{A} \cdot \vec{v} \ln \left(\frac{2M_e v}{\hbar \gamma} \right) N_1$$

II. RELATION TO THE QUANTUM TREATMENT OF STIMULATED BREMSSTRAHLUNG

Stimulated processes are usually thought of as being inherently quantum effects. In the case of low-frequency $\hbar\omega \ll \frac{M_e v^2}{2}$ stimulated bremsstrahlung, however, there is no particular reason for quantum mechanics, because:

1. There are no bound states.
2. Interaction with a single photon does not change the momentum or energy of the electron by a significant fraction of the initial values.
3. The photon field in problems of interest is clearly strong enough to be treated as an externally imposed classical field (i.e., changing the photon population by a few photons has no effect on the problem).
4. The wave nature of the particles is unimportant other than in determining the minimum classical impact parameter of the electron, and this is easily fudged into the classical calculation.

The standard quantum-mechanical treatment of a problem of this type approximates the electron wave function by a plane wave. Thus, to recover the quantum result, it is necessary to construct the classical analogue of a plane wave.

Consider an ion imbedded in a half space of uniform $\vec{A} = A[\sin \omega t, \cos \omega t, 0]$ field, with the surface of the half-space perpendicular to the $\vec{v}$ of interest. Rain ions with velocity, $\vec{v}_0$, random impact parameters, and random times of arrival at the field edge down on the surface of the half plane (Fig. A3):

Fig. A3.

$$\vec{v} = v_0 (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

Perform a coordinate rotation to put the velocity along the z axis:

$$\vec{v} = v_0 \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi \sin \phi & 0 \\ -\sin \phi \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} = v_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The $\vec{A}$ field is now:

$$\vec{A}(t) = A \begin{bmatrix} \sin \omega t \cos \phi \cos \theta + \sin \phi \cos \theta \cos \omega t \\ -\sin \phi \sin \omega t + \cos \phi \cos \omega t \\ \sin \theta \cos \phi \cos \omega t + \sin \theta \sin \phi \cos \omega t \end{bmatrix}$$

Let z_E be the z coordinate of the field edge, and t_E the time of arrival at the field edge. Treating everything from now on to lowest order in the field, obtain:

Outside the field:

$$\left. \begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 \\ z(t) &= z_E + v_0(t - t_E) \end{aligned} \right\} t < t_E$$

Inside the field:

$$\vec{R}(t) = [x(t), y(t), z(t)] ,$$

where

$$x(t) = \frac{-qA}{M_e \omega} [-(\cos \omega t - \cos \omega t_E) \cos \phi \cos \theta + (\sin \omega t - \sin \omega t_E) \sin \phi \cos \theta] + x_0 ,$$

$$y(t) = \frac{-qA}{M_e \omega} [+\sin \phi (\cos \omega t - \cos \omega t_E) + \cos \phi (\sin \omega t - \sin \omega t_E)] + y_0 ,$$

$$z(t) = z_E + v_0(t - t_E) - \frac{qA}{M_e \omega} [-(\cos \omega t - \cos \omega t_E) \sin \theta \cos \phi + (\sin \omega t - \sin \omega t_E) \sin \theta \sin \phi] .$$

The zero-order time of closest approach is

$$T = t_E + \frac{(z_c - z_E)}{v} .$$

The first-order time of closest approach T_1 can be found from

$$[\vec{R}(T) - \vec{R}_c] \cdot \vec{v}_0(T) = 0 .$$

This yields $T_1 = T + \Delta T$, where

$$\begin{aligned} \Delta T = \frac{qA}{M_e v_0^2} \left\{ \frac{v_0}{\omega} \{ -(\cos \omega T - \cos \omega t_E) \sin \theta \cos \phi + (\sin \omega T - \sin \omega t_E) \sin \theta \sin \phi \} \right. \\ \left. + (x_0 - x_c)(\sin \omega T \cos \phi \cos \theta + \sin \phi \cos \theta \cos \omega T) \right. \\ \left. + (y_0 - y_c)(\sin \phi \sin \omega T + \cos \phi \cos \omega T) \right\} . \end{aligned}$$

The impact-parameter-dependent pieces of ΔT turn out not to affect the result. (It is not necessary to go back and do all the impact-parameter integrals to show that this is true. Examine the time average of the impact terms where ΔT is used below.)

Use these results to evaluate the energy change of the electron:

$$\begin{aligned} \frac{dE}{ds} &= \frac{4\pi Z^2 e^5}{M_e^2 v^4} \vec{A}(T_1) \cdot \vec{v} \ln \left(\frac{2M_e v}{h\gamma} \right) N_i \\ &= \frac{4\pi Z^2 e^5}{M_e^2 |\vec{v} + \vec{\epsilon}|^4} \left[\vec{A}(t) + \frac{\Delta T \partial \vec{A}}{\partial T} \right] \cdot |\vec{v}_0 + \vec{\epsilon}| \ln \left(\frac{2M_e |\vec{v} + \vec{\epsilon}|}{h\gamma} \right) N_i, \end{aligned}$$

where

$$\vec{\epsilon} = \frac{-qA}{M_e} \begin{bmatrix} \sin \omega T \cos \phi \cos \theta + \sin \phi \cos \theta \cos \omega T \\ \sin \phi \sin \omega T + \cos \phi \cos \omega T \\ \sin \theta \cos \phi \sin \omega T + \sin \theta \sin \phi \cos \omega T \end{bmatrix}.$$

The expression for ΔE to first order in $\vec{A}$ is zero when averaged over impact times. Average the various second order in $\vec{A}$ pieces over impact times:

$$\langle \vec{A} \cdot \vec{\epsilon} \rangle = \frac{-qA^2}{2M_e} [\cos^2 \phi \cos^2 \theta + \sin^2 \theta \cos^2 \phi + \sin^2 \phi \cos^2 \theta + \sin^2 \phi + \cos^2 \phi + \sin^2 \theta \sin^2 \phi] = \frac{-qA^2}{M_e},$$

$$\frac{1}{|\vec{v}_0 + \vec{\epsilon}|^4} = \frac{1}{v_0^4} \left(1 - \frac{4\vec{\epsilon} \cdot \vec{v}_0}{v_0^2} \right),$$

$$\langle \vec{A} \cdot \vec{v}_0 \vec{\epsilon} \cdot \vec{v}_0 \rangle = \frac{-qA^2 v_0^2}{2M_e} (\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi) = \frac{-qA^2 v_0^2 \sin^2 \theta}{2M_e},$$

$$\langle \Delta T \frac{\partial \vec{A}(T_0)}{\partial T_0} \cdot \vec{v}_0 \rangle = \frac{-qA^2}{2M_e} (\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi) = \frac{-qA^2 \sin^2 \theta}{2M_e},$$

$$\ln \left(\frac{2M_e |\vec{v} + \vec{\epsilon}|}{h\gamma} \right) = \ln \left(\frac{2M_e v_0}{h\gamma} \right) + \frac{\vec{\epsilon} \cdot \vec{v}_0}{v_0^2},$$

yields no non-zero averaged log terms.

Putting all the pieces together,

$$\frac{\partial E}{\partial x} = \frac{4\pi Z^2 e^5 \ln \left(\frac{2M_e v}{h\gamma} \right)}{M_e^2 v^4} \left(\frac{+qA^2}{M_e} \right) \left(-1 + 2 \sin^2 \theta - \frac{\sin^2 \theta}{2} \right) N_i,$$

$$\frac{\partial E}{\partial x} = \frac{-2\pi Z^2 e^6 A^2}{M_e^3 v^4} \ln \left(\frac{2M_e v}{h\gamma} \right) (3 \sin^2 \theta - 2) N_i,$$

where $q = -e$. This is, finally, the dominant term of the quantum result in the $\omega/\gamma v \ll 1$ case.*

*D. Marcuse, *Bell System Tech. J.* 41, 1557 (1962) and 42, 415 (1963). Note that his result is for a linearly polarized $\vec{A}$ field. Putting in a circularly polarized field yields the above result.

III. COLLISIONAL MICROWAVE SKIN HEATING

We are dealing with a self-consistent plasma-field boundary region. This represents an equilibrium between a fully ionized hydrogen plasma and a circularly polarized microwave field. We want to calculate the rate at which the ordered, field-driven motion of the electrons, in this "skin" field region, is transformed into heat due to ion collision when the thermal motion of the electrons is considered.

The approach of equivalent fields will be used. (See Trubnikov for a general theoretical treatment, especially pp. 175-181).

1. Average Momentum Exchange Between Electrons and Ions. Trubnikov noticed that, for a Coulomb interaction, the mean velocity change for a test particle, α , moving through a cloud of field particles, β , could be written as

$$\langle \Delta v_i \rangle^{\alpha/\beta} = \left(1 + \frac{m_\alpha}{m_\beta} \right) \lambda (4\pi e_\alpha e_\beta / m_\alpha)^2 \frac{\partial}{\partial v_i} \left(\frac{1}{4\pi} \int f_\beta \frac{(\mathbf{v}') d\mathbf{v}'}{|\mathbf{v} - \mathbf{v}'|} \right),$$

by exploiting the fact that the form of this function is mathematically identical in velocity space to the problem in ordinary space of finding the electrostatic field at an observation point due to an arbitrary "source" distribution of charge at points, with $\mathbf{r}' \rightarrow \mathbf{v}'$. The force of dynamic friction along the test particle velocity can thus be written

$$\vec{F}_i = \frac{d\vec{p}_\alpha}{dt} = -q_\alpha \left(1 + \frac{m_\alpha}{m_\beta} \right) \vec{E}_\alpha,$$

where

$$\vec{E}_\alpha = -\nabla \phi_\alpha,$$

and

$$\phi_\alpha = \int \frac{f_\beta(\mathbf{v}') d\mathbf{v}'}{|\mathbf{v} - \mathbf{v}'|}.$$

Now by analogy with electrostatics, the $\vec{E}_\alpha$ can be written down for any distribution $f_\beta(\mathbf{v}')$. From Gauss's theorem, any spherically isotropic velocity distribution will produce an equivalent "field" at field point $\mathbf{v}$ of

$$\vec{E}_\alpha = \frac{4\pi \int_0^v f_\beta(\mathbf{v}') d\mathbf{v}'}{4\pi v^2}; \quad \vec{F}_i = - \left(\frac{\vec{v}_\alpha}{|\vec{v}_\alpha|} \right) q_\alpha \left(1 + \frac{m_\alpha}{m_\beta} \right) E_\alpha^{\alpha}.$$

2. Heating and Comparison with Stimulated Bremsstrahlung. Using the simple "moving Maxwellian" model for electrons in the skin, normalized to n_β ,

$$\int_0^{|\vec{v}(E)|} f_\beta(\mathbf{v}') d\mathbf{v}' = n_\beta \frac{4}{\sqrt{\pi}} \int_0^{|\vec{v}|} e^{-v_\beta^2} v_\beta^2 dv_\beta,$$

where we have transformed to the coordinate frame at rest with respect to the average electron velocity

$$|\vec{v}(E)| \text{ and } |\vec{v}_\beta| = |\vec{v}(E)| \left(\frac{m_\beta}{2KTe} \right)^{1/2}.$$

$\vec{v}(E)$ is a solution using local self-consistent fields $\vec{E}$ of the equation of motion.

$$M_e \dot{\vec{v}}(E) = -e\vec{E} + \nu_e M_e \vec{v}(E)$$

defining

$$\mu(\nu_i^2) = \frac{2}{\sqrt{\pi}} \int_0^{\nu_i^2} e^{-\eta} \sqrt{\eta} d\eta.$$

The "test particle" force, or, in the present case, the mean force on an ion is, $n_g \equiv N_e$; $m_p \equiv M_e$:

$$F_i = -\nu_i q_i \left(1 + \frac{M_i}{M_e} \right) \frac{N_e M_e}{2KT_e \nu_i^2} \cdot \mu(\nu_i^2) \hat{\nu}_i \equiv \frac{\vec{v}(E)}{|\vec{v}(E)|}.$$

where

$$q_i \equiv 4\pi\lambda \frac{e_i^2 e_p^2}{M_i}$$

is the effective "charge" of the test particle (ion). Hence, one has the heating rate $\vec{F}_i \cdot \vec{v}(E) N_i$, which, after dropping unity compared to M_i/M_e , is

$$\dot{\omega} = \frac{N_i(t) N_e(t) 4\pi\lambda z^2 e^4}{(2KT_e M_e)^{1/2}} \frac{\mu(\nu^2)}{\nu_i}.$$

When orbit calculations are run for stimulated bremsstrahlung in the microwave skin, a value near the above is obtained. Further if $\nu_e > \omega$, the usual "skin effect" heating rate for a metal applies, and the value obtained from it,

$$\dot{\omega}_{\text{skin effect}} = \frac{B_1^2}{8\pi} C \cdot \left(\frac{\nu_e \omega}{\omega_{pe}} \right)^{1/2},$$

is also comparable to the other two, so that in this case,

$$\dot{\omega}_{\text{skin effect}} \approx \dot{\omega}_{\text{collisions}} \approx \dot{\omega}_{\text{stim. bremsstrahlung}}.$$

Appendix B

COLLECTIVE COLLISIONLESS ION HEATING

Under conditions of interest, $\nu_e/\omega_{pe} \ll 1$, and effects of interparticle collisions on orbits can be ignored while particles are turning around in the skin region. We wish to calculate the energy exchange between the time dependent A^2 potential and the plasma. Since the corona satisfies the conditions for self-consistent confinement on the average, the pressure fluctuations are at least $\sim 2N_i^0 kT$ and cannot be considered a small perturbation. Average hydrodynamic motion is ignored over one cycle of the high-frequency field. Here N_i^0 satisfies

$$\frac{B^2 + E^2}{8\pi} = 2N_i^0 kT$$

First of all, to simplify the discussion, consider radii of curvature, physical dimensions, etc., to correspond to a plane radiation front. Secondly, we are considering plane polarized radiation since the effect vanishes for a perfectly circular A field. ($A^2(t) = \text{constant}$ in time, then.) Hence take $\tilde{A} = \hat{A}(z) \sin \omega t$. For most problems of interest, $\omega < \omega_{pe,i}$. Since we are considering self-consistent solutions, we will work with $B^2/8\pi$ rather than A^2 .

$$\frac{B^2 + E^2}{8\pi} \equiv P_f = \text{field pressure exerted on the skin electrons.}$$

When P_f rises from zero, the electron pressure, which is in equilibrium with the field, has two components: one kinetic and one electrostatic due to the ions. (See Fig. B1.)

- η = relative coordinate between electrons and ions.
- $\equiv \frac{1}{N_i^0} \int (N_e - N_i) dz = \text{average turning point for electrons relative to the turning point for ions,}$
- z_p = inertial coordinate of the average classical turning point for ions.
- $\dot{z}_p \equiv V_p$
- $\dot{\eta} = \frac{dn}{dt}$
- $e\nabla\phi$ = ion accelerating force,
- $\nabla^2\phi = 4\pi e \int (f_e - f_i) dV_i$
- $N_{e,i} = \int f_{e,i} dV_{e,i}$

The electrostatic component depends upon η ; the kinetic component depends upon $\dot{\eta}$, V_p .

Fig. B1.

Since $2\omega \ll \omega_{pi}$, the ions will follow the phase of the accelerating field generated by η . There is a purely electrostatic potential associated with η for which the field energy stored by the P_I is given back to P_I every cycle. The irreversible work rate kinetically per cycle is

$$\dot{W} = \frac{-1}{\tau} \int_0^\tau (P_I^e V_p + P_I^e \eta) dt' = \frac{-1}{\tau} \int_0^\tau P_I(t') V_p(t') dt'.$$

We now compute $V_p(t')$.

From the Lorentz force equation, you must have continuity of mechanical and electromagnetic forces at each phase of the electromagnetic field

$$\begin{aligned} P_I(t) &= 2N_i^e M_i \int_{V_z = V_p^e \text{ if } V_z^e > 0}^{\infty} f_i V_z^i (V_z^i - V_p^e)^2 dV_z^i \\ &= 0 \text{ if } V_z^e < 0 \\ &+ 2N_e^e M_e \int_{V_z^e = (\tilde{\eta} + V_p^e) \text{ if } (\tilde{\eta} + V_p^e) > 0}^{\infty} f_e V_z^e [V_z^e - (\tilde{\eta} + V_p^e)]^2 dV_z^e \\ &= 0 \text{ if } (\tilde{\eta} + V_p^e) < 0 \end{aligned}$$

This equation comes about because the flux of particles $V_{e,i}^e$ into a moving wall is $\alpha(V_{e,i}^{e,i} - V_p^e)$ (inertial frame).

V_p^e = velocity of the wall, and the corresponding elastic change in momentum is $\alpha(V_{e,i}^e - V_p^e)$ per particle, where $V_{e,i}^e$ is the initial velocity of the particles and hence > 0 only, but V_p^e is ≤ 0 . We are simply calculating the self-consistent pressure against a moving self-consistent electrostatic "wall."

First, it is shown that $\tilde{\eta}/\langle V_e \rangle$ is negligible. We know that

$$e\phi_{\max} \sim \left(\frac{e^2 A^2}{2M} \right)_{\max} \approx 4KT,$$

hence,

$$\eta_{\max} \sim \lambda_D, \text{ then } \tilde{\eta}_{\max} \sim \lambda_D \frac{2\omega}{\pi} \text{ or } \frac{\tilde{\eta}_{\max}}{\langle V_e \rangle} \approx \frac{\omega}{\omega_{pe}} \ll 1.$$

Expanding the right-hand side (RHS) of the above expression, leaving out the superscripts on $N_{e,i}$ where it is understood from now on that these refer to average electron and ion densities corresponding to the confinement relation (the lower integration limits for V_z^i and V_z^e are as given above):

$$\begin{aligned} \text{RHS} &= 2N_i M_i \int_{f_i(V_z^i)^2}^{\infty} dV_z^i - 4N_i M_i V_p^e \int_{f_i V_z^i}^{\infty} dV_z^i \\ &+ 2N_i M_i (V_p^e)^2 \int_{f_i}^{\infty} dV_z^i \\ &+ 2N_e M_e \int_{f_e(V_z^e)^2}^{\infty} dV_z^e - 4N_e M_e (\tilde{\eta} + V_p^e) \int_{f_e V_z^e}^{\infty} dV_z^e \\ &+ 2N_e M_e (V_p^e + \tilde{\eta})^2 \int_{f_e}^{\infty} dV_z^e. \end{aligned}$$

As stated earlier, the above are the initial distribution functions. Because of phase memory loss in crossing the system, the initial distribution function of a displaced Maxwellian is the distribution at any time.

Taking note of the above magnitudes,

$$\text{RHS} = 2N_i M_i \langle (V_z^i)^2 \rangle + 2N_e M_e \langle (V_z^e)^2 \rangle + N_i M_i (V_p^e)^2 - 4N_i M_i V_p^e \langle V_z^i \rangle,$$

where

$$\langle V_z^2 \rangle = \sqrt{\frac{KT}{2\pi M_i}}$$

$$\langle (V_z^2)^2 \rangle = \frac{KT}{2M_i}$$

so

$$\overline{P_f(t')} = 2N_iKT + 2N_iM_i\overline{(V_p^z)^2} \text{ since } \int_0^{\frac{2\pi}{\omega}} V_p^z(t') dt = 0$$

$$\frac{B_A^2}{16\pi} = 2N_iKT + N_iM_i\overline{(V_p^z)^2} \quad B_A = \text{peak field strength.}$$

For the time-dependent motion, rewrite

$$P_f(t') = 2N_iM_i \left(\frac{M_i}{2\pi KT} \right)^{1/2} \int_{z=0 \text{ if } V_p^z > 0}^{\infty} \exp \left[\frac{-M_i}{2KT_i} (z + V_p^z)^2 \right] z^2 dz$$

$$= -V_p^z \text{ if } V_p^z < 0$$

$V_p^z < 0$ means z_p is increasing, a "compression," which means $P_f(t) > \overline{P_f}$ and $\frac{\pi}{4} < \omega t' < \frac{3\pi}{4}$.

The solution to the above equation for this phase range is

$$V_p(t') \approx 2\langle V_z^2 \rangle (1 - \sqrt{1 - \pi \cos 2\omega t'})$$

so $|V_p|$ has a negative peak $\sim 3\langle V_z^2 \rangle$. $V_p > 0$: "expansion" with $P_f(t) < \overline{P_f}$

$$0 < \omega t' < \frac{\pi}{4} \text{ and } \frac{3}{4} < \omega t' < \pi$$

$$P_f(t') = 2N_iM_i \left(\frac{M_i}{2\pi KT} \right)^{1/2} \int_0^{\infty} \exp \left[\frac{-M_i}{2KT_i} (z + V_p^z)^2 \right] z^2 dz$$

which must be evaluated numerically. However, when $\omega t = 0, \pi$, we must have $V_p \sim \sqrt{\frac{KT}{M_i}}$, since this corresponds to no field pressure and "free" expansion. This part of the cycle does not contribute much work back on the field, however, in any case, because $P_f \sim 0$. A sketch of $\dot{W}$, $P_f(t')$, $V_p(t')$ is shown in Fig. B2.

Fig. B2.

An estimate for $\dot{W}$ then is simple, and

$$\dot{W} \sim \langle v_z^i \rangle \frac{2\pi}{\omega} \frac{B_0^2}{8\pi} \int_{\pi/4\omega}^{\pi/2\omega} \sin 2\omega t' \left(1 - \sqrt{1 - \frac{\pi}{2} \cos \omega t'} \right) dt'.$$

Using a triangle from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, you have

$$\dot{W} \sim \frac{4\omega}{\pi} \frac{B_0^2}{8\pi} \frac{\pi}{16\omega} \langle v_z^i \rangle = \frac{B_0^2}{8\pi} \frac{\langle v_z^i \rangle}{4} = \frac{4N_i K T_i}{4} \langle v_z^i \rangle.$$

so

$$\dot{W} \approx NKT \langle v_z^i \rangle.$$

This represents the work done directly on the ions due to ion kinetic friction.

The above calculation can also be looked upon as a wave-particle interaction in which a strong time-dependent field produces an electrostatic wave which couples to the particles, when you have trapping from one direction. The above method is used because with large nonlinear behavior and steep field and density gradients in the skin, a purely formal mathematical solution to the coupled Vlasov-Maxwell-Poisson set is unknown.

It should also be borne in mind that any periodic motion of the surface of a hot plasma satisfying the confinement (and other) conditions always requires irreversible expenditure of energy.

Generalization to other geometry eigenfields $\vec{A}(\underline{r}, t)$ and arbitrary polarization is straightforward.

Appendix C RELATIVISTIC QUASI-POTENTIALS

Ignoring collisions, the relativistic Hamiltonian for an electron is

$$H = e\phi + c \left[(m_0 c)^2 + p^2 + e^2 A^2 + 2e(\vec{p} \cdot \vec{A}) \right]^{1/2}.$$

Writing $m\vec{v}_1 = \vec{P}_1 + e\vec{A}_1$ and choosing $\vec{A} = \vec{A}_y(z) + \vec{A}_x(z)$ for plane radiation,

$$P_L \equiv P_x + P_y,$$

$$A_L \equiv A_x + A_y,$$

$$(1 - \beta) \equiv 1 - (\dot{z}/c)^2,$$

$\dot{z}$ = speed along field gradient,

$$\alpha \equiv \frac{[P_L^2 - 2eP_x A_x - 2eP_y A_y + e^2 A_L^2]}{(m_0 c)^2}.$$

Then you can write

$$-\dot{P}_z = \frac{\partial H}{\partial z} = e \frac{\partial \phi}{\partial z} + \frac{m_0 c^2}{2} \left(\frac{1 - \beta}{1 + \alpha} \right)^{1/2} \frac{\partial \alpha}{\partial z}.$$

For $\alpha \gg 1, \beta \ll 1$,

$$|\alpha| = \frac{e^2 A^2}{2m_0^2 c^2}, \quad \sqrt{\alpha} = \frac{eA}{\sqrt{2} m_0 c},$$

so

$$-F_z = \pm e \frac{\partial \phi}{\partial z} + \frac{\sqrt{2}}{4} c e \cdot \frac{1}{A} \frac{\partial}{\partial z} A^2,$$

or

$$-F_z = \frac{\partial}{\partial z} \left(\pm e \phi \pm \frac{\sqrt{2}}{2} \frac{c e E}{\omega} \right).$$

So the quasi-potential is, for $|\vec{v}| \rightarrow c$,

$$\boxed{\frac{\phi}{\omega} = \frac{\sqrt{2}}{2} \frac{c e E}{\omega}}.$$

It might be noted that you can write

$$m = m_0 \cdot \left(\frac{1 + \alpha}{1 - \beta} \right)^{1/2},$$

and it should be shown that in this formulation $\vec{v} \cdot \vec{v} \leq c^2$, since we have field-driven motion that is valid for arbitrary field strength. Let $P_{\perp} = 0$; therefore,

$$v_{\perp} = \frac{eA_{\perp}}{m_0} \left(\frac{1-\beta}{1+\alpha} \right)^{1/2} = \left(\frac{eE}{m_0 \omega} \right) \left(\frac{1-\beta}{1+\alpha} \right)^{1/2} ,$$

so

$$|v_{\perp}|^2 = \left(\frac{eE}{m_0 \omega} \right)^2 \frac{1}{(1+\alpha)} - \left(\frac{eE}{m_0 \omega} \right)^2 \frac{1}{(1+\alpha)} \frac{z^2}{c^2} ,$$

so

$$v_{\perp}^2 + \left(\frac{eE}{m_0 \omega} \right)^2 \frac{1}{(1+\alpha)} \frac{z^2}{c^2} = \left(\frac{eE}{m_0 \omega} \right)^2 \frac{1}{1+\alpha} .$$

For very intense fields, $(1+\alpha) \rightarrow \alpha$; hence, in the limit of infinitely strong fields,

$$\boxed{v_{\perp}^2 + z^2 = c^2} .$$

Appendix D

PLASMA HYDRORADIATION STABILITY

Solutions to self-consistent field equations generally assume shape stability for the plasma-radiation eigenfield interface. We now consider the question of stability for such an interface. To simplify the discussion, a planar geometry will be assumed, with no loss in generality.

The general method will involve the basic assumption that the microwave field relaxes much faster than the plasma surface, so that any hydrodynamic motion of the plasma as a whole can be ignored in computing the fields.

Perturbation theory can be used to evaluate changes in the field intensity resulting from any particular Fourier surface-wave component. The resulting forces either tend to restore the surface to the original uniformity ("stable") or re-enforce the perturbation ("unstable").

FIELD PERTURBATION

A surface shape $\xi = \xi_0 \cos k_z z$ is considered at $\tau = 0$.

The plasma surface normal is:

$$\hat{n}_s = -\sqrt{\frac{\eta^2 - 1}{\eta^4 - 1}} (\eta \hat{x} + \hat{z})$$

where

$$\eta = 1/\xi_0 k_z \sin k_z z.$$

For $\eta \gg 1$,

$$\hat{n}_s \sim -(\hat{x} + \hat{z} \xi_0 k_z \sin k_z z).$$

Separation of the wave equation gives for the perturbation field E' (or B')

$$E'(B') = g'h'e^{i\omega t}.$$

(The surface wavelength $\lambda_z = 2\pi k_z^{-1}$ need not be considered smaller than the collisionless skin depth, because in that case the skin acts as a dielectric lens and washes out the assumed perturbations.)

$$\frac{d^2 g'}{dz^2} + k_z^2 g'(z) = 0, \quad \frac{d^2 h'}{dx^2} + h'(k_0^2 - k_z^2) = 0,$$

with solutions:

$$\gamma = \sqrt{k_0^2 - k_z^2},$$

$$\gamma^2 > 0: E'_i = (A_i \sin \gamma x + B_i \cos \gamma x) e^{ik_z z} e^{i\omega t},$$

$$\gamma^2 < 0: E_i = (A_i \sinh \gamma x + B_i \cosh \gamma x) e^{ik_z z} e^{i\omega t},$$

k_0 = original wavenumber of the field without the perturbation.

We now do a first-order perturbation in which the eigenfrequency remains essentially unchanged. Treating the plasma as a perfect conductor as well as the wall, the fields must satisfy

$$\begin{aligned}\nabla \cdot \vec{B} &= 0 & \text{interior} \\ \nabla \cdot \vec{E} &= 0 & \text{interior charge-free region} \\ \hat{n}_s \cdot \vec{B} &= 0 \\ \left. \frac{\partial B}{\partial \hat{n}} \right|_s &= 0 \end{aligned} \left. \vphantom{\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \cdot \vec{E} &= 0 \\ \hat{n}_s \cdot \vec{B} &= 0 \end{aligned}} \right\} \text{at surface}$$

$$(\hat{n}_s \times \vec{E}) = 0$$

$$\vec{E} = \vec{E}^0 + \vec{E}'$$

$$\vec{B} = \vec{B}^0 + \vec{B}'$$

E^0, B^0 are the eigenfields with no perturbation. We first consider the case where the surface troughs lie along E^0 . So the zero fields are

$$E_y^0 = \hat{y} B_0 \sin\left(\frac{n\pi}{\ell}\right) x,$$

$$B_z^0 = \hat{z} B_0 \cos\left(\frac{n\pi}{\ell}\right) x,$$

$$B'_z = (A \sin \gamma x + B \cos \gamma x) e^{ik_z z} e^{i\omega t},$$

$$B'_x = (C \sin \gamma x + D \cos \gamma x) e^{ik_z z} e^{i\omega t},$$

Since no normal $\vec{B}$ components are allowed at $x=0$, $D=0$. Also, at $x=\ell$,

$$B_t = (B'_z + B_z^0)^2,$$

so from

$$\frac{\partial B_t}{\partial \hat{n}_s} = 0$$

you have $A=0$, so

$$B'_z = B \cos \gamma x e^{ik_z z} e^{i\omega t};$$

$$B'_x = C \sin \gamma x e^{ik_z z} e^{i\omega t}.$$

At the plasma boundary,

$$\hat{n}_s = -(\hat{x} + 2\xi_0 k_z \sin k_z z),$$

so for $\hat{n}_s \cdot \vec{B} = 0$ you have

$$(\hat{x} + 2\xi_0 k_z \sin k_z z) \cdot \{(B_z^0 + B'_z)^2 + B'_x \hat{x}\}|_{x=\ell+\xi} = 0$$

Expanding the appropriate functions, you have

$$B'_x|_{x=\ell+\xi} \approx B'_x|_{x=\ell} + \left. \frac{\partial B'_x}{\partial x} \right|_{x=\ell} \cdot \xi.$$

$$B_z^0|_{x=\ell+\xi} \approx B_z^0|_{x=\ell} + \left. \frac{\partial B_z^0}{\partial x} \right|_{x=\ell} \cdot \xi.$$

$$B'_z|_{x=\ell+\xi} \approx B'_z|_{x=\ell} + \left. \frac{\partial B'_z}{\partial x} \right|_{x=\ell} \cdot \xi.$$

Hence, to first order in ξ you have

$$B'_x = -\xi_0 k_z \sin k_z z (-1)^n B_0.$$

or

$$C = -\frac{\xi_0 k_z \sin k_z z}{\sin \gamma \ell} B_0 (-1)^n.$$

Hence you have, from $\frac{\partial B'_x}{\partial x} = -\frac{\partial B'_z}{\partial z}$, that the two fields are:

$$B'_x = -\xi_0 k_z \sin k_z z B_0 (-1)^n \frac{\sin \gamma x}{\sin \gamma \ell},$$

$$B'_z = -\gamma \xi_0 e^{ik_z z} B_0 (-1)^n \frac{\cos \gamma x}{\sin \gamma \ell}.$$

For physical reasons cited earlier, we are considering

$$\xi_0 k_z = 2\pi \frac{\xi_0}{\lambda_z} \ll 1$$

and the variation in field intensity is at $x = \ell$, to first order in ξ .

$$\frac{(B)^2}{8\pi} \approx \frac{(\bar{B}_z^0 + B'_z)^2}{8\pi}.$$

or

$$\delta p_i = -\frac{B_0^2}{4\pi} \gamma \xi \cot \gamma \ell,$$

for

$$E_z^0 = 0,$$

$$B_z^0 = 2B_0 \cos\left(\frac{n\pi}{\ell}x\right),$$

$$E_y^0 = \gamma B_0 \sin\left(\frac{n\pi}{\ell}x\right).$$

The other case, where the troughs of the perturbation are perpendicular to E^0 , corresponds to choosing

$$B_z^0 = 0 \text{ and } E_x^0 = 0,$$

$$E_z^0 = \hat{y} E_0 \sin\left(\frac{n\pi}{\ell} x\right),$$

$$B_y^0 = \hat{y} B_0 \cos\left(\frac{n\pi}{\ell} x\right).$$

Then following the same procedure, but with

$$(\hat{n}_s \times \vec{E})|_{x=\ell+\xi} = 0,$$

and vanishing tangential field at $x=0$ and $\nabla \cdot \vec{E} = 0$ in interior, gives

$$E_z' = -E_0 \left(\frac{n\pi}{\ell}\right) (-1)^n \xi_0 e^{ik_z z} \frac{\sin \gamma x}{\sin \gamma \ell},$$

$$E_x' = \frac{k_z}{\gamma} \left(\frac{n\pi}{\ell}\right) (-1)^n \xi_0 \sin k_z z \frac{\cos \gamma x}{\sin \gamma \ell}.$$

So the B -field perturbation is along y and given by

$$\vec{B}' = -\frac{c}{\omega} \nabla \times \vec{E}', \quad \left[k_0 = \frac{\omega}{c} = \frac{n\pi}{\ell} \right]$$

$$B_y' = -\hat{y} B_0 (-1)^n \xi_0 e^{ik_z z} \frac{\cos \gamma x}{\sin \gamma \ell} \frac{k_0^2}{\gamma}, \quad k_z^2 < k_0^2$$

$$B_y' = -\hat{y} B_0 (-1)^n \xi_0 \cdot e^{ik_z z} \frac{\cosh \gamma x}{\sinh \gamma \ell} \frac{(2k_z^2 - k_0^2)}{\gamma}, \quad k_z^2 > k_0^2.$$

Again, $\delta p_f = 2B^0 B_y'$ and the sign of the pressure change is the same as in the previous case.

PLASMA PARTICLE PRESSURE

In the case of a skin density much less than core density, sound waves are restricted to small amplitudes because the large density gradient to the core provides a large impedance mismatch. In the case of the average confinement to core density, potentially large surface waves can exist as modes of the plasma as a whole.

In cases where the time scale of the surface motion is faster than the collision time for the ions, the plasma behaves more like a cloud of noninteracting particles raining onto a moving boundary.

As will now be demonstrated, the stability criteria are the same for all the above cases.

Case A: Sound-Wave Generation. In order that a plasma have a surface wave present at frequency ν ,

$$\xi = \xi_0 e^{ik_z z} e^{i\nu t}$$

of small amplitude ξ , no larger than $\sim \delta$, at the average confinement density N_0 , the pressure variations at the surface must be given as solutions of the linearized hydrodynamic equation, the velocity potential for which satisfies the wave equation, with solutions, for an infinite cylinder of plasma:

$$\phi = -i\xi_0 e^{ik_z z} e^{i\nu t} \frac{\nu J_0(\gamma r_0)}{\gamma_s J_0(\gamma r_0)}.$$

where $J'_0 = \frac{d}{dr} J_0$.

and hence pressure variations at the surface:

$$\bar{p} = -\rho_0 \xi_0 e^{ik_z z} \frac{\nu^2}{\gamma_s} \left[\frac{J_0(\gamma_s r_0)}{J'_0(\gamma_s r_0)} \right]_{\lim r_0 \rightarrow \infty} e^{i\nu t}, \quad k_z^2 < k_s^2,$$

where, to recover the simpler geometry of the slab, we let $\gamma_s r_0 \rightarrow \infty$. However, the J_0 are to be replaced by the I_0 if $k_z^2 > k_s^2$.

$$\gamma_s = \sqrt{k_z^2 - k_s^2},$$

$$k_s^2 = \left(\frac{\nu}{c_s} \right)^2,$$

c_s = sound speed in the gas.

ρ_0 = plasma density.

Case B: Noninteracting Particles. The pressure variation at the density N_i^0 is

$$P(t) - \bar{P} = 4N_i^0 M_i \langle v_i^2 \rangle \xi + N_i^0 M_i \xi^2.$$

We will ignore the square of the surface-wave velocity $\bar{P}$ = average pressure = $2N_i^0 k t$ and take the positive direction of ξ away from the wall into the plasma.

DISPERSION RELATION FOR SURFACE WAVES

For Case A, continuity of pressure across the plasma-radiation interface gives, for example,

$$-\frac{B_0^2}{4\pi} \gamma_s^2 e^{ik_z z + i\nu t} \cot \gamma \ell \quad = \quad -\rho_0 \xi_0 e^{ik_z z + i\nu t} \frac{\nu^2}{\gamma_s} \left[\frac{J_0(\gamma_s r_0)}{J'_0(\gamma_s r_0)} \right]_{\lim r_0 \rightarrow \infty}$$

(or $\coth \gamma \ell$
if $k_z^2 > k_0^2$)

The only important point here is that stability requires ν^2 to be real. The left side of the dispersion relation is all that affects this, with the requirement that

$$\cot \gamma \ell > 0, \quad k_z^2 < k_0^2$$

or

$$\coth \gamma \ell > 0, \quad k_z^2 > k_0^2.$$

The coth function does not change sign, so it is always true if $k_z^2 > k_0^2$. The other branch requires then that

$$\gamma \ell < \pi/2$$

or

$$N\pi \sqrt{1 - \frac{k_z^2}{k_0^2}} < \frac{\pi}{2}$$

or

$$\frac{k_z^2}{k_0^2} > 1 - \frac{1}{4N^2}$$

To the extent that a plasma can be represented as a collisional gas, characterized by a sound speed, perturbations with $\lambda_z = 2\pi/k_z < \lambda_{\text{radiation}}$ are stable. Longer wavelengths are unstable. λ_z are modes of the bulk plasma at frequency ν .

For case B, the dispersion relation is, ignoring the square of the surface velocity,

$$\frac{-B_0^2}{4\pi} \gamma \zeta(z, t) \cot \gamma \ell = 4N_i^2 M_i \langle v_z^i \rangle \frac{d\zeta}{dt}$$

so

$$\zeta = \zeta_0(0, z) e^{-\alpha t}$$

with

$$\alpha = \frac{B_0^2}{4\pi} \frac{\gamma \cot \gamma \ell}{4N_i^2 M_i \langle v_z^i \rangle}$$

which clearly decays for $\alpha > 0$, which is the same criterion as in case A.

ABLATIVE STABILITY

Before it can be concluded that a plasma with linear dimensions $< \lambda_{\text{radiation}}$ is stable for all perturbations, it must be determined whether the rate of differential energy absorption due to the perturbation can exceed the diffusion of energy away from the heated region. This requires a knowledge of the energy absorption of the plasma as a function of field intensity.

Stochastic heating is not a strong function of average pressure, but depends mostly upon the transit time of electrons through the skin. Skin-effect heating depends directly upon the energy density at the corona, as does wave-particle heating.

One can make the following simple plausibility arguments for heating and diffusing.

First consider the skin effect: For a skin in which $v_e > \omega$, the energy flux is, per unit area,

$$\dot{S} \ll P_f \omega \delta \quad P_f = \frac{B^2}{8\pi}$$

Hence energy is gained for the "stable" surface perturbation in regions protruding from the average surface away from the plasma, compared to those protruding inward, by the rate

$$\Delta \dot{S} \sim \Delta P_f \omega \delta ,$$

but any perturbation satisfying the stability criterion ($\lambda_z < \lambda_{\text{radiation}}$) has a time constant, ignoring local energy storage during the motion, of

$$\tau_S = \left(\frac{\Delta P_f}{\{4N_i^c M_i \langle v_z^i \rangle\}} \right)^{-1} = \frac{4N_i^c M_i \langle v_z^i \rangle}{B_0^2 \gamma \cot \gamma l} 4\pi ,$$

or, since $k_z > k_0$, $\gamma \approx k_z$ so the time available for depositing heat is

$$\tau_S' \approx \lambda_z / v_z^i .$$

which is therefore always about the time to diffuse the heat across $\lambda_z/2$ even by ion conduction.

We conclude that the particle pressure does not rise faster than the field pressure in the perturbation. Since the wave-particle heating also only changes linearly with intensity, the same conclusion holds.

Appendix E DERIVATION OF THE SELF-CONSISTENT FIELD EQUATION FOR A MICROWAVE CORONA

First derive the guiding-center constant of the motion E_c .

$$E_c = \frac{1}{2} M_e \dot{\mathbf{R}}^2 + \psi_e - e\phi = \text{constant} ,$$

where the actual coordinates are $(\vec{r}, \vec{v})$:

$$\vec{r} = \vec{R} + \vec{\rho} ,$$

$$\frac{d\vec{r}}{dt} = -e\vec{E}(\vec{r}, t) - \frac{\vec{v}}{c} \times \vec{B}(\vec{r}, t) ,$$

$$\psi_e = \frac{e^2 |\vec{E}|^2}{4M_e \omega^2} = \psi_e(\vec{R}) ,$$

$|\vec{E}|$ = magnitude of the peak $\vec{E}$ field.

$\vec{R}$ is a slowly varying part (guiding center) and $\vec{\rho}$ has time dependence ω^{-1} .

Letting the distribution function be, for electrons,

$$f_{e,i}(\epsilon_{e,i}) = f_{e,i} \left(\frac{1}{2} M_{e,i} \dot{\mathbf{R}}^2 + \frac{e^2 |\vec{E}|^2}{2M_e \omega^2} - e\phi \right) ,$$

and re-expressing this argument in terms of the orbit variables,

$$f_e(\epsilon_e) = f_e \left\{ \frac{1}{2} M_e \left(\dot{\vec{v}} + \frac{e}{M_e} \int^t \vec{E} dt' \right)^2 + \psi_e \left[\vec{r} + \frac{e}{M_e} \int^t \int^{t'} (\vec{E} dt'' dt') \right] - e\phi \right\} ,$$

where the approximate value for $\vec{\rho}$ of

$$\vec{\rho} = \frac{-e}{M_e} \int^t \int^{t'} \vec{E}(\vec{R}, t'') dt'' dt'$$

from the guiding-center expansion in $|\vec{v}|/c$ has been used, which is then neglecting terms of order $|\vec{v}|^2/c^2$. (See Motz and Watson, p. 164.)

From

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \text{ and } \nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

obtain for time harmonic $\vec{E}$,

$$\nabla \times \nabla \times \vec{E} = \left(\frac{\omega}{c} \right)^2 \left(\vec{E} - \frac{4\pi}{\omega^2} \frac{\partial}{\partial t} \vec{J}_e \right) ,$$

where

$$\vec{J}_e = e \int_{-\infty}^{+\infty} (f_i - f_e) \vec{v} d\vec{v} .$$

Now look at f_e (or f_i) term in $\vec{J}$,

$$\vec{J}_e = -e \int_{-\infty}^{+\infty} \vec{v} f_e \left[\frac{1}{2} M_e \left(\vec{v} - \frac{e}{M_e} \int^t \vec{E} dt' \right) + \psi_e \left(\vec{r} - \frac{e}{M_e} \int^t \int^{t'} \vec{E} dt'' dt' \right) - e\psi \right] d\vec{v}.$$

The essential step of the guiding-center theory is to remove the assumed small, time-dependent part of the argument for ψ_e so the equation for $\vec{J}_e$ can be written, after making the variable transformation

$$\vec{\eta} = \vec{v} + \frac{e}{M_e} \int^t \vec{E} dt',$$

$$\vec{J}_e = -e \int_{-\infty}^{+\infty} \left(\vec{\eta} - \frac{e}{M_e} \int^t \vec{E} dt' \right) f_e \left[\frac{1}{2} M_e \eta^2 + \psi_e(\vec{r}) - e\phi \right] d\vec{\eta},$$

which, from obvious symmetry properties, yields

$$\frac{\partial \vec{J}_e}{\partial t} = \frac{e^2}{M_e} \vec{E} \int_{-\infty}^{+\infty} f_e \left[\frac{1}{2} M_e \eta^2 + \psi_e(\vec{r}) - e\phi \right] d\vec{\eta},$$

which, after normalizing f_e to N_0 in the plasma interior, yields finally

$$\frac{\partial \vec{J}_e}{\partial t} = \frac{e^2}{M_e} N_0 \vec{E} \int_{-\infty}^{+\infty} f_e d\vec{\eta}.$$

After doing the same for ions, this gives

$$\nabla \times \nabla \times \vec{E} = \left(\frac{\omega}{c} \right)^2 \left[1 - \frac{\omega_{pe}^2}{\omega^2} \int_{-\infty}^{+\infty} f_e d\vec{\eta} - \frac{\omega_{pi}^2}{\omega^2} \int_{-\infty}^{+\infty} f_i d\vec{\eta} \right] \vec{E}.$$

Finally, if there is any reason to choose $f_{e,i}$ as Maxwellians, and after making the quasi-neutral approximation, $N_e \sim N_i = N_0$, so that

$$e\phi = \frac{e^2 |\vec{E}|^2}{8M_e \omega^2} = \frac{\psi_e}{2},$$

ignoring the M_e/M_i term. Then, for instance

$$\frac{i}{N_0} \int_{-\infty}^{+\infty} f_e d\vec{\eta} = \exp\left(-\frac{c^2 E^2}{8M_e \omega^2 kT}\right) = \frac{N_e}{N_0}$$

and ignoring the ion current, you obtain quasi-neutral solution,

$$\nabla \times \nabla \times \vec{E} = \left(\frac{\omega}{c} \right)^2 \left[1 - \frac{\omega_{pe}^2}{\omega^2} \exp\left(-\frac{c^2 E^2}{8M_e \omega^2 kT}\right) \right] \vec{E}.$$

We now briefly consider the solutions without the use of guiding-center approximation or quasi-neutrality.

The quasi-neutral equation above is used as the beginning point, but more generally,

$$\nabla \times \nabla \times \vec{E} = \left(\frac{\omega}{c} \right)^2 \left(\vec{E} - \frac{4\pi}{\omega^2} \frac{\partial \vec{J}}{\partial t} \right). \quad (A)$$

$$\nabla^2 \phi = 4\pi e \int (f_e - f_i) d\vec{v} \approx 4\pi e (N_e - N_i), \quad (B)$$

$$H_e = -e\phi + \frac{(\vec{P} + e\vec{A})^2}{2M_e} \quad (\text{neglecting interparticle scattering in the skin}), \quad (C)$$

$$\frac{\partial \vec{f}}{\partial t} = \vec{E} \left[\omega_{pe}^2 \exp \left(\frac{-e^2 E^2}{4M_e \omega^2 K T_e} + \frac{e\phi}{K T_e} \right) + \omega_{pi}^2 \exp \left(\frac{-e^2 E^2}{4M_i \omega^2 K T_i} - \frac{e\phi}{K T_i} \right) \right]. \quad (A')$$

$$\nabla^2 \phi = 4\pi N_0 e \left[\exp \left(\frac{-e^2 E^2}{4M_e \omega^2 K T_e} + \frac{e\phi}{K T_e} \right) - \exp \left(\frac{-e^2 E^2}{4M_i \omega^2 K T_i} - \frac{e\phi}{K T_i} \right) \right]. \quad (B')$$

Using for Cycle 1 $N_0^{e,i}$ and $\frac{\partial \vec{f}}{\partial t}$ from the Maxwellian quasi-neutral solution above, with corresponding E_0, ϕ_0 , solve for $N_1^{e,i}, \vec{J}_1^{e,i}, \vec{E}_1, \phi_1$, by computing from (C) the $N_1^{e,i}, \vec{J}_1^{e,i}$ using E_0, ϕ_0 in H_e with a FASOR code. Then compute ϵ_1, ϕ_1 by solving (A) and (B) using $N_1^{e,i}, \vec{J}_1^{e,i}$. The values for ϵ_1 and ϕ_1 should now be used in H_e for the second cycle, with $N_2^{e,i}, \vec{J}_2^{e,i}$ now coming from FASOR, and then used in (A) and (B) to obtain E_2, ϕ_2 . This procedure should converge to the self-consistent fields, and has been observed to do so several times. (See

Fig. E1.

Fig. E1.) FASOR is an acronym for Fast Orbit Integrator, which is a finite-difference code solving the electron motion in a given field and averaging the density according to transit time in a given zone.

Acknowledgements

Much of this work was performed by the author in association with Controlled Fusion, Inc., San Ramon, California. The author would like to thank A. D. MacKay of CFI for carrying out detailed integrations in Appendix A, part I; thanks also to L. D. Howard of CFI for machine calculations on particle orbits.

Bibliography

- Boyd, T. J. M., and J. J. Sanderson, *Plasma Dynamics* (Barnes & Noble, New York, 1969).
- Ensley, D. L., *Research and Development Proposal for Microwave Confinement of Plasma*, Lawrence Livermore Laboratory Rept. UCIR-824 (July 12, 1974).
- Jackson, J. D., *Classical Electrodynamics* (John Wiley & Sons, New York, 1962).
- Marcuse, D., "Stimulated Emission of Bremsstrahlung," *Bell System Tech. J.* 41, 1557 (1962).
- , "A Further Discussion of Stimulated Emission of Bremsstrahlung," *Bell System Tech. J.* 42, 415 (1963).
- Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), pp. 1052-1056.
- Motz, H., and C. J. H. Watson, "The Radio Frequency Confinement and Acceleration of Plasmas," in *Advances in Electronics and Electron Physics*, L. Marton, Ed., vol. 23 (Academic Press, New York, 1967).
- Sagdeev, R. Z., "Containment of a Plasma by the Pressure of a Standing Electromagnetic Wave," in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, vol. 3, M. A. Leontovich, Ed. (Pergamon Press, London, 1959), p. 406.
- Shafranov, V. D., "The Stability of a Cylindrical Gaseous Conductor in a Magnetic Field," *Sov. J. At. Energy* 1, 709 (1956).
- Trubnikov, B. A., "Particle Interactions in a Fully Ionized Plasma," in *Reviews of Plasma Physics*, vol. 1, M. A. Leontovich, Ed. (Consultants Bureau, New York, 1965), pp. 105-204.
- Volkov, T. F., "Oscillations and Stability of the Surface of a Plasma in the Field of a Travelling Electromagnetic Wave," in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, vol. 4, M. A. Leontovich, Ed. (Pergamon Press, London, 1960), p. 127.
- Weibel, E. S., "Dynamic Stabilization of a Plasma Column," *Phys. Fluids* 3, 946 (1960).